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Some Convergence Results for the Solution of New Type of Variational Inequalities Associated with Generalized Pseudo-Monotone Mappings in Complete $\mathcal{CAT}(0)$ Spaces

Maliha Rashid[1], Amna Kalsoom[1], Nida Masood[1], Ahmad Aloqaily[2], Nabil Mlaiki[2,*]

[1] *Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan*
[2] *Department of Mathematics and Sciences, Prince Sultan University, Saudi Arabia*

Abstract. The basic purpose of this article is to introduce a generalized version of pseudo-monotone variational inequality in the setting of complete $\mathcal{CAT}(0)$ spaces and to present some strong and Δ-convergence results for the existence of solutions for the respective variational inequality problem. Algorithm 1 and 2 are proposed in accordance with pseudo-monotone and α-strongly pseudo-monotone mappings to prove our results under some conditions. A numerical implication of our proposed algorithm is also presented.

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Key Words and Phrases: Projection type method, variational inequality, pseudo-monotone mapping

1. Introduction

Variational inequalities originated in the beginning of 1960s through the revolutionary work of the Italian mathematician Guido Stampacchia [1], who analyse free boundary problems occuring in elasticity theory and mechanics by using the variational inequality as an analytic tool. From 1960-1975 many foundational articles presented in the literature emphasizing the association between the complementarity problems and the variational inequalities. For the early advancement on variational inequalities readers are referred to [2–6].

*Corresponding author.
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Email addresses: maliha.rashid@iiu.edu.pk (M. Rashid),
amna.kalsoom@iiu.edu.pk (A. Kalsoom),
nida.msma673@iiu.edu.pk (N. Masood),
maloqaily@psu.edu.sa (A. Aloqaily),
nmlaiki@psu.edu.sa (N. Mlaiki)

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A large number of articles, proposed in second half of 1990s, was dedicated to the reformulation of the nonlinear complementarity problem in terms of the algorithms produced through a globally convergent Newton method. After that, many iterative schemes have been formulated for finding the solutions of variational inequalities and their relevant optimization problems (see [7, 8] and literature cited in).

One of the numerical methods for solving variational inequality problems (VIP's) is known as projection method [9–11] which is further expanded to gradient, extragradient and subgradient methods (see, e.g., [12–25] and the references therein).

Extragradient method is not practically useful for the solution of variational inequalities having non-Lipschitz mapping. In case of extragradient method, the reforms regarding [26, 27] mentioned in [28, 29] ensured the convergence without Lipschitz continuity. In [30], the authors discussed strong and weak convergence results for a VIP in the context of a pseudo-monotone, classical non-Lipschitzian, continuous mapping in Hilbert spaces over $\mathbb{R}$.

Korpelevich [11], introduced an extragradient method in finite dimensional Euclidean space to obtain solution of variational inequality problem under the mapping of monotone and Lipschitz continuous. The extragradient method has been further extended in infinite dimensional spaces by many researchers (see [12–15, 22–25] and the refferences therein). The modification in [28, 29], enables convergence in finite dimensional Euclidean space without Lipschitz continuity of the mappings associated variational inequality. in [30], the extragradient method has been expanded in infinite dimensional space to get weak and strong convergence results for VIP under the condition of classical non-lipschitz, pseudo-monotone and continuous mapping.

The following article is dedicated to the analysis of a pseudo-monotone VIP in the setting of $\mathcal{CAT}(0)$ space, which gives a clear modification of extragradient algorithm for strong and Δ-convergence.

$\mathcal{CAT}(0)$ spaces, established by Alexandrov in the 1950's, were given recognition by M. Gromov, who displayed that a great deal of the theory of manifolds of non-positive sectional curvature could be designed without using much more than the $\mathcal{CAT}(0)$ condition. Gromov described the key aspects of the global geometry of manifolds of non-positive curvature, primarily relying on the $\mathcal{CAT}(0)$ inequality(see [31]).

Let $(\mathcal{Y}, \varrho)$ be a metric space. A geodesic segment connecting $u_1 \in \mathcal{Y}$ to $u_2 \in \mathcal{Y}$ is a mapping $\Upsilon : [0, \varrho(u_1, u_2)] \to \mathcal{Y}$ such that $\Upsilon(0) = u_1, \Upsilon(\varrho(u_1, u_2)) = u_2$ and

$$\varrho(\Upsilon(g_1), \Upsilon(g_2) = |g_1 - g_2|), \quad \forall g_1, g_2 \in [0, \varrho(u_1, u_2)].$$

A geodesic segment linking any two different points $u_1, u_2 \in \mathcal{Y}$ is an isometry with $\Upsilon(0) = u_1, \Upsilon(u_1, u_2) = u_2$. A unique geodesic segment is expressed by $[u_1, u_2]$. The metric space $(\mathcal{Y}, \varrho)$ is known as a geodesic metric space if any two points are joined by a geodesic segment and the metric $(\mathcal{Y}, \varrho)$ is a uniquely geodesic if there is exactly one geodesic segment to link them. A subset $\mathcal{L} \subseteq \mathcal{Y}$ is called convex if any two points in $\mathcal{Y}$ can be joined by a geodesic and the image of every such geodesic is lying in $\mathcal{L}$.

Suppose $(\mathcal{Y}, \varrho)$ be the geodesic metric space. In a geodesic metric space, a geodesic triangle has three corners $u_1, u_2, u_3 \in \mathcal{Y}$ and three geodesic segments $([u_1, u_2], [u_2, u_3], [u_3, u_1])$ joining them. For this triangle there exist a comparison (Alexandrov) triangle $\overline{\Delta}(u_1, u_2, u_3) \subset \mathbb{R}^2$ such that

* $\varrho(\overline{u_1}, \overline{u_2}) = \varrho(u_1, u_2)$,
* $\varrho(\overline{u_2}, \overline{u_3}) = \varrho(u_2, u_3)$,
* $\varrho(\overline{u_3}, \overline{u_1}) = \varrho(u_3, u_1)$.

When all geodesic triangles in a geodesic metric space satisfy the following $\mathcal{CAT}(0)$ comparison axiom then geodesic metric space is known as $\mathcal{CAT}(0)$ space (this term is due to M.Gromov [32]) if . Let Δ and $\overline{\Delta}$ be a geodesic and comparison triangle in $\mathcal{Z}$, respectively. If the following inequality is satisfied for all $u_1, u_2 \in \Delta$ and all comparison points $\overline{u_1}, \overline{u_2} \in \overline{\Delta}$,

$$\varrho(u_1, u_2) \leq \varrho(\overline{u_1}, \overline{u_2}),$$

then Δ is said to satisify $\mathcal{CAT}(0)$ inequality.

The motivation for the conversion thus stems from the need to improve robustness and applicability of results. By transiting to $\mathcal{CAT}(0)$ spaces, researchers can better handle nonlinearities, ensuring more meaningful and reliable insights across a broad range of applications. We introduce two algorithms in Hadamard spaces that does not require to have previous knowledge of Lipschitz- like constants. Our proposed algorithms converges to a solution of VIP. Moreover, we present a numerical example in a Hadamard space to demonstrate the performance of our method.

Question 1. *Can we obtain convergence results for VIP using an Extragradient algorithm in Hadamard space under the condition of pseudo-monotone.*
our contributions in this paper are briefly highlighted as:
1. Our work extends algorithm of extragradient for pseudomonotone from linear spaces to nonlinear spaces.
2. Our algorithm not depends on the Lipschitz constant.

2. Preliminaries

In this section, we display some notations, familiar definitions, and relevent results that will be required in the proof of our main results.

Definition 1. *A geodesically connected metric space $\mathcal{Y}$ is known as $\mathcal{CAT}(0)$ space and every geodesic triangle in $\mathcal{Y}$ is at least as 'thin' as its comparison triangle in the Euclidean plane. For a systematic study of geodesic spaces and $\mathcal{CAT}(0)$ spaces the readers are referred to [31].*

According to Bruhat and Tits [33], the (CN) inequality is defined as follows:

Definition 2. *If $u, u_1, u_2 \in \mathcal{CAT}(0)$ space and if $u_0 \in [u_1, u_2]$ be the middle point of the segment, then the $\mathcal{CAT}(0)$ inequality yields*

$$\varrho(u, u_0)^2 \leq \frac{1}{2}\varrho(u, u_1)^2 + \frac{1}{2}\varrho(u, u_2)^2 - \frac{1}{4}\varrho(u_1, u_2)^2.$$

In recent past, $\mathcal{CAT}(0)$ spaces have appealed many mathematicians, due to their geometrical relevance in multiple directions. Hadamard spaces [34] are originally the complete $\mathcal{CAT}(0)$ spaces.
In 2008, the notion of quasilinearization was initiated by Berg and Nikolaev [35], given as follows:

Definition 3. *Denoting a vector as a pair* $(\xi, \eta) \in \mathcal{Y} \times \mathcal{Y}$ *by* $\overrightarrow{\xi\eta}$, *the quasilinearization is defined as a mapping* $\langle ., . \rangle : (\mathcal{Y} \times \mathcal{Y}) \times (\mathcal{Y} \times \mathcal{Y}) \to \mathbb{R}$ *satisfying*

$$\langle \overrightarrow{\xi\eta}, \overrightarrow{\gamma\delta} \rangle = \frac{1}{2} \left[\varrho^2(\xi, \delta) + \varrho^2(\eta, \gamma) - \varrho^2(\xi, \gamma) - \varrho^2(\eta, \delta) \right],$$

where $\xi, \eta, \gamma, \delta \in \mathcal{Y}$.

Remark 1. *It can easily verify that for all* $\xi, \eta, \gamma, \delta, \zeta \in \mathcal{Y}$,

(i) $\langle \overrightarrow{\xi\eta}, \overrightarrow{\gamma\delta} \rangle = \langle \overrightarrow{\gamma\delta}, \overrightarrow{\xi\eta} \rangle$,

(ii) $\langle \overrightarrow{\xi\eta}, \overrightarrow{\gamma\delta} \rangle = -\langle \overrightarrow{\eta\xi}, \overrightarrow{\gamma\delta} \rangle$,

(iii) $\langle \overrightarrow{\xi\zeta}, \overrightarrow{\gamma\delta} \rangle + \langle \overrightarrow{\zeta\eta}, \overrightarrow{\gamma\delta} \rangle = \langle \overrightarrow{\xi\eta}, \overrightarrow{\gamma\delta} \rangle$,

(iv) $\mathcal{Y}$ *satisfies the Cauchy-Schwarz inequality if*

$$\langle \overrightarrow{\xi\eta}, \overrightarrow{\gamma\delta} \rangle \leq \varrho(\xi, \eta)\varrho(\gamma, \delta).$$

Remark 2. *A geodesically connected metric space is a* $\mathcal{CAT}(0)$ *space if and only if it satisfies the Cauchy-Schwarz inequality ([35], Corollary 3).*

In 2010 Kakavandi and Amini[36] develop dual space of Hadamard space $\mathcal{Z}$ by using the concept of quasilinearization and by initiating the concept of pseudometric space.

Definition 4. *To explain the conjugate space of Hadamard space* $\mathcal{Z}$, *consider the map* $\Phi : \mathbb{R} \times \mathcal{Z} \times \mathcal{Z} \to \mathcal{H}(\mathcal{Z}, \mathbb{R})$ *defined by*

$$\Phi(t, \xi, \eta) = t\langle \overrightarrow{\xi\eta}, \overrightarrow{\xi\alpha} \rangle \quad (t \in \mathbb{R}, \xi, \eta, \gamma \in \mathcal{Z}) \tag{1}$$

where $\mathcal{H}(\mathcal{Z}, \mathbb{R})$ *is the space of all continuous real-valued functions on* $\mathcal{Z}$. *Then the Cauchy-Schwartz inequality implies that* $\Phi(t, \xi, \eta)$ *is a Lipschitz function with Lipschitz semi-norm* $\mathcal{H}(\Phi(t, \xi, \eta)) = t\varrho(\xi, \eta)$, *for all* $t \in \mathbb{R}$ *and* $\xi, \eta \in \mathcal{Z}$, *where* $\mathcal{H}(\Psi) = sup\{\frac{\Psi(\xi) - \Psi(\eta)}{\varrho(\xi, \eta)}; \xi, \eta \in \mathcal{Z}, \xi \neq \eta\}$ *is the Lipschitz semi-norm, for any function* $\Psi : \mathcal{Z} \to \mathbb{R}$. *Now, we present the pseudometric* $\mathcal{D}$ *on* $\mathbb{R} \times \mathcal{Z} \times \mathcal{Z}$ *by*

$$\mathcal{D}((t, \xi, \eta), (s, \gamma, \delta)) = \mathcal{H}(\Phi(t, \xi, \eta) - \Phi(s, \gamma, \delta)) \quad (t, s \in \mathbb{R}, \xi, \eta, \gamma, \delta \in \mathcal{Z}). \tag{2}$$

Lemma 1. *[36]* $\mathcal{D}((t,\xi,\eta),(s,\gamma,\delta)) = 0$ *if and only if* $t\langle\overrightarrow{\xi\eta},\overrightarrow{er}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{er}\rangle$, *for all* $e, r \in \mathcal{Z}$.

Proof. By (1) and (2) and formulation of Lipschitz semi-norm, $\mathcal{D}((t,\xi,\eta),(s,\gamma,\delta)) = 0$ *if and only if there exist a constant* $\kappa \in \mathbb{R}$ *such that* $t\langle\overrightarrow{\xi\eta},\overrightarrow{\xi e}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{\gamma e}\rangle + \kappa$, *for all* $e \in \mathcal{Z}$. *Therefore, for all* $e, r \in \mathcal{Z}$

$$t\langle\overrightarrow{\xi\eta},\overrightarrow{er}\rangle = t\langle\overrightarrow{\xi\eta},\overrightarrow{\xi r}\rangle - t\langle\overrightarrow{\xi\eta},\overrightarrow{\xi e}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{\gamma e}\rangle - s\langle\overrightarrow{\gamma\delta},\overrightarrow{\gamma e}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{er}\rangle.$$

Conversely if $t\langle\overrightarrow{\xi\eta},\overrightarrow{er}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{er}\rangle$, *for all* $e, r \in \mathcal{Z}$, *then*

$$\Phi(t,\xi,\eta)(e) = t\langle\overline{\xi\eta},\xi e\rangle = s\langle\overline{\gamma\delta},\overline{\xi e}\rangle = \Phi(s,\gamma,\delta)(e) - s\langle\overline{\gamma\delta},\overline{\gamma\xi}\rangle,$$

for all $e \in \mathcal{Z}$, *which yields* $\mathcal{D}((t,\xi,\eta),(s,\gamma,\delta)) = 0$.

Definition 5. *For a Hadamard space* $(\mathcal{Z},\varrho)$, *the pseudometric space* $(\mathbb{R}\times\mathcal{Z}\times\mathcal{Z},\mathcal{D})$ *can be considered as a subspace of the pseudometric space* $(Lip(\mathcal{Z},\mathbb{R}),\mathcal{H})$ *of all real-valued Lipschitz functions. Also,* $\mathcal{D}$ *explain an equivalence relation on* $\mathbb{R}\times\mathcal{Z}\times\mathcal{Z}$, *where the equivalence class of* (t,ξ,η) *is*

$$[t\overrightarrow{\xi\eta}] = \{s\overrightarrow{\gamma\delta}; t\langle\overrightarrow{\xi\eta},\overrightarrow{\gamma\delta}\rangle = s\langle\overrightarrow{\gamma\delta},\overrightarrow{\gamma\delta}\rangle\ (\gamma,\delta \in \mathcal{Z})\}.$$

The set $Y^* = \{t\overrightarrow{\xi\eta}; (t,\gamma,\delta) \in \mathbb{R}\times\mathcal{Z}\times\mathcal{Z}\}$ *is a metric space with metric* $\mathcal{D}$, *which is called the dual metric space of* $(\mathcal{Z},\varrho)$.

In [37], Theorem 2.3, the projection operator is utilized for the existence of solution of the respective variational inequality in a Hilbert space over $\mathbb{R}$. By using the concept of quasilinearization, authors in [38] extended the above mentioned result in $\mathcal{CAT}(0)$ space that is as follows:

Theorem 1. *[38] Let* $(\mathcal{Z},\varrho)$ *be a complete* $\mathcal{CAT}(0)$ *space and* $\emptyset \neq \mathcal{L} \subseteq \mathcal{Z}$ *is convex. Then*

$$w = P_{\mathcal{L}}w_2 \quad \Leftrightarrow \quad \langle\overrightarrow{w_1 w},\overrightarrow{w w_2}\rangle \geq 0, \quad \forall\ w_1 \in \mathcal{L}, w_2 \in \mathcal{Z},$$

and $w \in \mathcal{L}$.

In the following, $(\mathcal{Y},\varrho)$ and $(\mathcal{Z},\varrho)$ will represent $\mathcal{CAT}(0)$ space and complete $\mathcal{CAT}(0)$ space respectively.

These are some lemma's taken from literature which are helpful in our main results.

Lemma 2. *[39] Let* $u_1, u_2, u \in \mathcal{Y}$ *and* $\tau \in [0,1]$. *Then*

(i) $\varrho(\tau u_1 \oplus (1-\tau)u_2, u) \leq \tau\varrho(u_1,u) + (1-\tau)\varrho(u_2,u)$,

(ii) $\varrho^2(\tau u_1 \oplus (1-\tau)u_2, u) \leq \tau\varrho^2(u_1,u) + (1-\tau)\varrho^2(u_2,u) - \tau(1-\tau)\varrho^2(u_1,u_2)$.

Lemma 3. *[39] Let* $u_1, u_2, u \in \mathcal{Y}$ *and* $\tau \in [0,1]$. *Then*

(i) $\varrho(\tau u_1 \oplus (1-\tau)u_2, \gamma u_1 \oplus (1-\gamma)u_2) = |\tau-\gamma|\varrho(u_1,u_2)$,

(ii) $\varrho(\tau u_1 \oplus (1-\tau)u_2, \tau u_1 \oplus (1-\tau)q) \leq (1-\tau)\varrho(u_2, u)$.

Lemma 4. *[40] In $(\mathcal{Z}, \varrho)$ space, every bounded sequence always has a Δ-convergent subsequence.*

Lemma 5. *[41] Assume $\{\aleph_n\}, \{\Im_n\}, \{c_n\}$ and $\{\sigma_n\}$ be nonnegative sequences such that*

$$\aleph_{n+1} \leq (1-\sigma_n)\aleph_n + \sigma_n \Im_n + c_n, \quad n \geq 0$$

with $\{\sigma_n\} \subset [0,1], \Sigma_{n=o}^{\infty}\sigma_n = \infty, \lim_{n\to\infty} \Im_n = 0$ and $\sigma_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} \aleph_n = 0$.

Lemma 6. *[42] For $(\mathcal{Z}, \varrho)$, the inequality stated below holds*

$$\varrho^2(u, w) \leq \varrho^2(r, w) + 2\langle \overrightarrow{ur}, \overrightarrow{uw} \rangle, \quad \forall\ u, r, w \in \mathcal{Z}.$$

Lemma 7. *[42] For any $\ell \in (0,1)$ and $s, t \in \mathcal{Y}$, assume*

$$s_\ell = \ell s \oplus (1-\ell)t.$$

Then, for all $u, v \in \mathcal{Y}$,

(i) $\langle \overrightarrow{s_\ell u}, \overrightarrow{s_\ell v} \rangle \leq \ell \langle \overrightarrow{su}, \overrightarrow{s_\ell v} \rangle + (1-\ell)\langle \overrightarrow{tu}, \overrightarrow{s_\ell v} \rangle$

(ii) $\langle \overrightarrow{s_\ell u}, \overrightarrow{sv} \rangle \leq \ell \langle \overrightarrow{su}, \overrightarrow{sv} \rangle + (1-\ell)\langle \overrightarrow{tu}, \overrightarrow{sv} \rangle$
and $\langle \overrightarrow{s_\ell u}, \overrightarrow{tv} \rangle \leq \ell \langle \overrightarrow{su}, \overrightarrow{tv} \rangle + (1-\ell)\langle \overrightarrow{tu}, \overrightarrow{tv} \rangle$.

Lemma 8. *[43] Assume a non-negative sequence $\{b_n\}$ of real numbers, such that there exist a subsequence $\{b_{n_l}\}$ of the sequence $\{b_n\}$ satisfying $b_{n_l} < b_{n_{l+1}}$, for all $l \in \mathbb{N}$. So there is a non-decreasing sequence $\{a_k\}$ of natural numbers in such a way that $a_k \to \infty$ as $k \to \infty$, and for all $k \in \mathbb{N}$ satisfy the conditions stated below:*

$$b_{a_k} \leq b_{a_{k+1}} \quad \text{and} \quad b_k \leq b_{a_{k+1}}.$$

Indeed, $a_k = \max\{l \leq k : b_l \leq b_{l+1}\}$.

Lemma 9. *[44] Consider a sequence $\{b_n\} \in \mathcal{Z}$ and if a nonempty subset $\mathcal{L} \subseteq \mathcal{Z}$ satisfying the following conditions:*

(i) for every $\omega \in \mathcal{L}, \lim_{n\to\infty} \varrho(b_n, \omega)$ exists;

(ii) if $\{b_{n_j}\}$ is a subsequence of $\{b_n\}$ which is Δ-convergent to v, then $v \in \mathcal{L}$.

Then $\{b_n\}$ Δ-converges to an element of $\mathcal{L}$.

3. Variational Inequality and Some Crucial Lemmas

In this section we introduce variational inequality and several lemmas which are essential for our main results.
Consider a closed, convex subset $\mathcal{L} \subseteq \mathcal{Z}$, and define a map $A_1 : \mathcal{L} \to \mathcal{Z}^*$, $A_2 : \mathcal{Z}^* \to \mathcal{L}$ and $A : \mathcal{L} \to \mathcal{L}$. Finding a point $w^* \in \mathcal{L}$ such that

$$\langle \overrightarrow{wAw^*}, \overrightarrow{ww^*} \rangle \geq 0, \quad \text{for all } w \in \mathcal{L}. \tag{3}$$

Problem (3) is referred as variational inequality and denoted by $VI(\mathcal{L}, A)$.

Definition 6. *The map $A : \mathcal{L} \to \mathcal{L}$ is known as*

(i) monotone if

$$\langle \overrightarrow{Aw_1 Aw_2}, \overrightarrow{w_1 w_2} \rangle \geq 0, \quad \forall\ w_1, w_2 \in \mathcal{L}.$$

(ii) pseudo-monotone if

$$\langle \overrightarrow{w_1 Aw_1^*}, \overrightarrow{w_1 w_1^*} \rangle \geq 0 \Rightarrow \langle \overrightarrow{w_1^* Aw_1}, \overrightarrow{w_1^* w_1} \rangle \geq 0, \quad \forall\ w_1, w_1^* \in \mathcal{L}.$$

Definition 7. *Consider the space $(\mathcal{Z}, \varrho)$. For $\alpha > 0$, map A is known as $\alpha-$strongly pseudo-monotone if*

$$\langle \overrightarrow{Aw_1 Aw_2}, \overrightarrow{w_1 w_2} \rangle \geq \alpha \varrho^2(w_1, w_2), \quad \forall\ w_1, w_2 \in \mathcal{L}.$$

The convergence of the approaches is assumed to meet the following conditions.

Condition 1. *The subset $\mathcal{L}$ of a Hadamard space $(\mathcal{Z}, \varrho)$ is nonempty, closed and convex.*

Condition 2. *The mapping $A : \mathcal{L} \to \mathcal{L}$ is a pseudo-monotone, uniformly continuous on $\mathcal{L}$.*

Condition 3. *The solution set of VI(3) is non-empty, that is $VI(\mathcal{L}, A) \neq \phi$.*

Condition 4. *Let $\varpi : \mathcal{L} \to \mathcal{Z}$ be a contraction map. Let's say there's a sequence $\{\xi_n\}$ of real numbers in an open interval $(0, 1)$ in such a way that*

$$\lim_{n \to \infty} \xi_n = 0, \quad \Sigma_{n=1}^{\infty} \xi_n = \infty.$$

Now we will discuss some lemmas which are crucial for our main results.
These lemmas has been established by authors in the framework of Hilbert space. Here, we explain these lemmas in a complete $\mathcal{CAT}(0)$ space setting and provide the proof.

Lemma 10. *Let $u_1 \in \mathcal{Z}$. Then*

$$\varrho^2(P_{\mathcal{L}}u_1, u_2) \leq \varrho^2(u_1, u_2) - \varrho^2(u_1, P_{\mathcal{L}}u_2), \quad \text{for all } u_2 \in \mathcal{L}.$$

Proof. Consider

$$\begin{aligned}
\langle \overrightarrow{u_1u_2}, \overrightarrow{u_1u_2} \rangle &= \langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{u_1u_2} \rangle + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{u_1u_2} \rangle, \\
&= \langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{u_1P_{\mathcal{L}}u_1} \rangle + \langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{u_1P_{\mathcal{L}}u_1} \rangle \\
&\quad + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle, \\
&= \langle \overrightarrow{u_1P_{\mathcal{L}}u_2}, \overrightarrow{u_1P_{\mathcal{L}}u_2} \rangle + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle + 2\langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle, \\
&= \langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{u_1P_{\mathcal{L}}u_1} \rangle + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle + 2\langle \overrightarrow{u_2P_{\mathcal{L}}u_1}, \overrightarrow{P_{\mathcal{L}}u_1u_1} \rangle.
\end{aligned}$$

By Theorem 1, we have $\langle \overrightarrow{u_2P_{\mathcal{L}}u_1}, \overrightarrow{P_{\mathcal{L}}u_1u_1} \rangle \geq 0$. *We have*

$$\begin{aligned}
\langle \overrightarrow{u_1u_2}, \overrightarrow{u_1u_2} \rangle &\geq \langle \overrightarrow{u_1P_{\mathcal{L}}u_1}, \overrightarrow{u_1P_{\mathcal{L}}u_1} \rangle + \langle \overrightarrow{P_{\mathcal{L}}u_1u_2}, \overrightarrow{P_{\mathcal{L}}u_1u_2} \rangle, \\
\varrho^2(u_1, u_2) &\geq \varrho^2(u_1, P_{\mathcal{L}}u_1) + \varrho^2(u_2, P_{\mathcal{L}}u_1), \\
\varrho^2(u_2, P_{\mathcal{L}}u_1) &\leq \varrho^2(u_1, u_2) - \varrho^2(u_1, P_{\mathcal{L}}u_1).
\end{aligned}$$

Lemma 11. *Consider a closed, convex subset $\mathcal{L} \subset \mathcal{Z}$ and defined $C := \{u \in \mathcal{Z} : \psi(u) \leq 0\}$. If $\mathcal{L}$ is nonempty and a real valued function ψ is Lipschitz continuous on $\mathcal{Z}$ with modulus $\Theta > 0$, then*

$$\varrho(u, C) \geq \Theta^{-1} \max\{\psi(u), 0\}, \quad \text{for all } u \in \mathcal{L}, \tag{4}$$

the distance from u to C is denoted by $d(u, C)$.

Proof. Clearly (4) holds for all $u \in C$ and we are left to proof that (4) holds for every $u \in \mathcal{L}/C$. Assume $u \notin C$ but $u \in \mathcal{L}$. Since C is closed, there exist $\omega(u) \in C$ such that $\varrho(u, \omega) = \varrho(u, C)$. Since ψ is Lipschitz continuous, we have

$$\begin{aligned}
\varrho(\psi(u), \psi(\omega(u))) &\leq \Theta\varrho(u, \omega), \\
&= \Theta\varrho(u, C).
\end{aligned}$$

Since $u \notin C$ and $\omega(u) \in \mathcal{L}$, we have $\psi(u) > 0$ and $\psi(y(u)) \leq 0$. Then

$$\begin{aligned}
\psi(u) &\leq \psi(u) - \psi(\omega(u)) \leq |\psi(u) - \psi(\omega(u))|, \\
&= \varrho(\psi(u), \psi(\omega(u))) \leq \Theta\varrho(u, C).
\end{aligned}$$

Lemma 12. *Let $\mathcal{L}$ be a nonempty, closed and convex subset of a complete $\mathcal{CAT}(0)$ space $\mathcal{Z}$ and A be a pseudo-monotone map. If*

$$\langle \overrightarrow{uAu}, \overrightarrow{uu^*} \rangle \geq 0, \quad \forall\ u \in \mathcal{L}. \tag{5}$$

Then u^* is the solution of $VI(\mathcal{L}, A)$.

Proof. *Suppose* $\langle \overrightarrow{uAu}, \overrightarrow{uu^*} \rangle \geq 0$ *holds for all* $u \in \mathcal{L}$. *Thus*

$$\begin{array}{rcl} \langle \overrightarrow{{u_\lambda}^* A{u_\lambda}^*}, \overrightarrow{{u_\lambda}^* u^*} \rangle & \geq & 0, \quad {u_\lambda}^* \in \mathcal{L} \\ \langle \overrightarrow{{u_\lambda}^* u^*}, \overrightarrow{{u_\lambda}^* A{u_\lambda}^*} \rangle & \geq & 0. \end{array}$$

By using Lemma 7 and applying limit, we obtain

$$\begin{array}{rcl} \langle \overrightarrow{{u_\lambda}^* u^*}, \overrightarrow{{u_\lambda}^* A{u_\lambda}^*} \rangle & \leq & \lambda \langle \overrightarrow{uu^*}, \overrightarrow{{u_\lambda}^* A{u_\lambda}^*} \rangle + (1-\lambda) \langle \overrightarrow{u^* u^*}, \overrightarrow{{u_\lambda}^* A{u_\lambda}^*} \rangle. \\ & = & \lambda \langle \overrightarrow{uu^*}, \overrightarrow{{u_\lambda}^* A{u_\lambda}^*} \rangle \leq \langle \overrightarrow{{u_\lambda}^* A{u_\lambda}^*}, \overrightarrow{uu^*} \rangle, \\ & \leq & \lambda \langle \overrightarrow{uA{u_\lambda}^*}, \overrightarrow{uu^*} \rangle + (1-\lambda) \langle \overrightarrow{u^* A{u_\lambda}^*}, \overrightarrow{uu^*} \rangle, \\ & = & \lambda \langle \overrightarrow{uAu^*}, \overrightarrow{uu^*} \rangle + (1-\lambda) \langle \overrightarrow{u^* Au^*}, \overrightarrow{uu^*} \rangle, \\ & = & \lambda \langle \overrightarrow{uAu^*}, \overrightarrow{uu^*} \rangle + (1-\lambda) \langle \overrightarrow{u^* u}, \overrightarrow{uu^*} \rangle + (1-\lambda) \langle \overrightarrow{uAu^*}, \overrightarrow{uu^*} \rangle, \\ & = & \langle \overrightarrow{uAu^*}, \overrightarrow{uu^*} \rangle + (1-\lambda) \langle \overrightarrow{u^* u}, \overrightarrow{uu^*} \rangle, \\ & \leq & \langle \overrightarrow{bAu^*}, \overrightarrow{uu^*} \rangle + (1-\lambda) \varrho^2(u, u^*). \end{array}$$

This implies

$$\langle \overrightarrow{uAu^*}, \overrightarrow{uu^*} \rangle \geq 0.$$

Thus u^* *is a solution of (5).*

Now, we introduce our algorithm as follows:

Algorithm 1. ***Initialization:*** *Given* $\mu, \nu, \varsigma \in (0, 1)$. *Let* $b_1 \in \mathcal{L}$ *be arbitrary*
Iterative Steps: *For the given iteration* b_n, *we first calculate* b_{n+1} *as stated below:*
Step 1. *Compute*

$$s_n = P_{\mathcal{L}}(\xi_n b_n \oplus (1 - \xi_n) A b_n),$$

where $\xi_n := \varsigma \nu^{m_n}$, *with* m_n *is the minimal nonnegative integer satisfying*

$$\langle \overrightarrow{Ab_n As_n}, \overrightarrow{b_n s_n} \rangle \leq \mu \varrho^2(b_n, s_n).$$

If $As_n = 0$ *or* $b_n = s_n$ *holds then algorithm stops and* s_n *is a solution of VI. Else*
Step 2. *Calculate*

$$b_{n+1} = P_{\mathcal{L}_n}(b_n),$$

where

$$\mathcal{L}_n := \{\mathcal{L} \in \mathcal{Z} : h_n(b) \leq 0\}$$

and

$$h_n(b) = \langle \overrightarrow{bs_n}, \overrightarrow{(\xi_n b_n \oplus (1 - \xi_n) A b_n) A b_n} \rangle. \tag{6}$$

Place $n := n + 1$ *and repeat the* ***Step 1***.

Lemma 13. *Suppose that Conditions 1-3 hold. Let b^* be a solution of $VI(\mathcal{L}, A)$ and the function h_n be defined by (6). Then $h_n(b^*) \leq 0$ and $h_n(b_n) \geq (1-\mu)\varrho^2(b_n, s_n)$.*

Proof. *Since $b^* \in VI(\mathcal{L}, A)$, we have*

$$\langle \overrightarrow{s_n A b^*}, \overrightarrow{b^* s_n} \rangle \leq 0. \tag{7}$$

It is implied from Lemma 7 and (7) that

$$\begin{aligned} h_n(b^*) &= \langle \overrightarrow{b^* s_n}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n} \rangle, \\ &\leq \xi_n \langle \overrightarrow{b^* s_n}, \overrightarrow{b_n A b_n} \rangle + (1-\xi_n)\langle \overrightarrow{b^* s_n}, \overrightarrow{Ab_n A b_n} \rangle, \\ &\leq \xi_n \langle \overrightarrow{b^* s_n}, \overrightarrow{b_n s_n} \rangle + \langle \overrightarrow{b^* s_n}, \overrightarrow{s_n A b^*} \rangle + \xi_n \langle \overrightarrow{b^* s_n}, \overrightarrow{Ab^* A b_n} \rangle, \end{aligned}$$

by taking limit, we get

$$h_n(b^*) \leq 0.$$

Thus Claim 1 of Lemma 13 holds. Now, To prove Claim 2, from (6), we have

$$\begin{aligned} h_n(b_n) &= \langle \overrightarrow{b_n s_n}, \overrightarrow{s_n A b_n} \rangle = \langle \overrightarrow{b_n s_n}, \overrightarrow{s_n b_n} \rangle + \langle \overrightarrow{b_n s_n}, \overrightarrow{b_n A s_n} \rangle + \\ &\quad \langle \overrightarrow{b_n s_n}, \overrightarrow{A s_n A b_n} \rangle. \end{aligned}$$

As $\langle \overrightarrow{b_n s_n}, \overrightarrow{b_n A s_n} \rangle \geq 0$, we have

$$\begin{aligned} h_n(b_n) &\geq \langle \overrightarrow{b_n s_n}, \overrightarrow{s_n b_n} \rangle + \langle \overrightarrow{b_n s_n}, \overrightarrow{A s_n A b_n} \rangle, \\ &\geq -\varrho^2(b_n, s_n) - \mu\varrho^2(b_n, s_n), \\ &= (-1-\mu)\varrho^2(b_n, s_n). \end{aligned}$$

Lemma 14. *Consider a nonempty, closed and convex subset of a Hadamard space $\mathcal{Z}$ be $\mathcal{L}$. The map A is uniformly continuous and pseudo-monotone on $\mathcal{L}$. The solution set of the $VI(\mathcal{L}, A)$ is nonempty and suppose a sequence produced by Algorithm 1 is $\{b_n\}$. If there is a subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $\{b_{n_k}\}$ Δ-converges to $z \in \mathcal{Z}$ and $\lim_{k\to\infty} \varrho(b_{n_k}, s_{n_k}) = 0$, then $z \in VI(\mathcal{L}, A)$.*

Proof. *From $\Delta - \lim_{n\to\infty} b_{n_k} = z, \lim_{k\to\infty} \varrho(b_{n_k}, s_{n_k}) = 0$, and $s_n \subset \mathcal{L}$, we have $z \in \mathcal{L}$ and $s_{n_k} = P_{\mathcal{L}}(\xi_{n_k} b_{n_k} \oplus (1-\xi_{n_k})Ab_{n_k})$ thus,*

$$0 \leq \langle \overrightarrow{b s_{n_k}}, \overrightarrow{s_{n_k}(\xi_{n_k} b_{n_k} \oplus (1-\xi_{n_k})Ab_{n_k})} \rangle, \quad \forall b \in \mathcal{L}.$$

By Lemma 7 and Remark 1, we get

$$\begin{aligned} \langle \overrightarrow{(\xi_{n_k} b_{n_k} \oplus (1-\xi_{n_k})Ab_{n_k})s_{n_k}}, \overrightarrow{s_{n_k} b} \rangle &\leq \xi_{n_k} \langle \overrightarrow{b_{n_k} s_{n_k}}, \overrightarrow{s_{n_k} b} \rangle + (1-\xi_{n_k})\langle \overrightarrow{Ab_{n_k} s_{n_k}}, \overrightarrow{s_{n_k} b} \rangle, \\ &= \xi_{n_k} \langle \overrightarrow{b_{n_k} s_{n_k}}, \overrightarrow{s_{n_k} b} \rangle + (1-\xi_{n_k})\langle \overrightarrow{s_{n_k} b}, \overrightarrow{b s_{n_k}} \rangle + \\ &\quad (1-\xi_{n_k})\langle \overrightarrow{b A b_{n_k}}, \overrightarrow{b s_{n_k}} \rangle, \end{aligned}$$

$$\begin{aligned} &= \xi_{n_k}\langle \overrightarrow{b_{n_k}s_{n_k}}, \overrightarrow{s_{n_k}b}\rangle + (1-\xi_{n_k})\langle \overrightarrow{s_{n_k}b}, \overrightarrow{bs_{n_k}}\rangle + \\ &\quad (1-\xi_{n_k})\langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle + (1-\xi_{n_k})\langle \overrightarrow{bAb_{n_k}}, \overrightarrow{b_{n_k}s_{n_k}}\rangle, \\ &\leq \langle \overrightarrow{b_{n_k}s_{n_k}}, \overrightarrow{s_{n_k}b}\rangle - \varrho^2(s_{n_k}, b) + \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle + \langle \overrightarrow{bAb_{n_k}}, \\ &\quad \overrightarrow{b_{n_k}s_{n_k}}\rangle. \end{aligned}$$

This implies that

$$\langle \overrightarrow{b_{n_k}s_{n_k}}, \overrightarrow{s_{n_k}b}\rangle + \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle + \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{b_{n_k}s_{n_k}}\rangle \geq 0, \quad \forall b \in \mathcal{L}. \tag{8}$$

Now, we will prove

$$\liminf \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle \geq 0. \tag{9}$$

Taking $k \to \infty$ in (8), we get

$$\liminf \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle \geq 0.$$

Since A is uniformly continuous map, thus we have

$$\varrho(Ab_{n_k}, As_{n_k}) \to 0 \text{ as } k \to \infty. \tag{10}$$

On the other, hand we have

$$\begin{aligned} \langle \overrightarrow{bAs_{n_k}}, \overrightarrow{bs_{n_k}}\rangle &= \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bs_{n_k}}\rangle + \langle \overrightarrow{Ab_{n_k}As_{n_k}}, \overrightarrow{bs_{n_k}}\rangle \\ &= \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{bb_{n_k}}\rangle + \langle \overrightarrow{bAb_{n_k}}, \overrightarrow{b_{n_k}s_{n_k}}\rangle + \langle \overrightarrow{Ab_{n_k}As_{n_k}}, \overrightarrow{bs_{n_k}}\rangle, \end{aligned}$$

which together with (9) and (10) gives

$$\liminf \langle \overrightarrow{bAs_{n_k}}, \overrightarrow{bs_{n_k}}\rangle \geq 0.$$

Now, we have to prove that $v \in VI(\mathcal{L}, A)$. Since A is pseudo-monotone, so we get

$$\langle \overrightarrow{vAb}, \overrightarrow{vb}\rangle = \lim_{k\to\infty} \langle \overrightarrow{s_{n_k}Ab}, \overrightarrow{s_{n_k}b}\rangle = \lim_{k\to\infty} \inf \langle \overrightarrow{s_{n_k}Ab}, \overrightarrow{s_{n_k}b}\rangle \geq 0.$$

Using Lemma 12, we get $v \in VI(\mathcal{L}, A)$ and the proof is finished.

4. convergence results

In this section we prove the results of strong and Δ-convergence.

Theorem 2. *Consider a nonempty, closed and convex subset of a Hadamard space $\mathcal{Z}$ be $\mathcal{L}$. The map $A : \mathcal{L} \to \mathcal{L}$ is a pseudo-monotone, uniformly continuous on $\mathcal{L}$. The solution set of the $VI(\mathcal{L}, A)$ is nonempty, that is $VI(\mathcal{L}, A) \neq \phi$. Then any sequence $\{b_n\}$ Δ-convergent in $VI(\mathcal{L}, A)$.*

Proof. **Claim 1.** The sequence $\{b_n\}$ is a bounded. To proof the claim, assume $v \in VI(\mathcal{L}, A)$, we have

$$\begin{aligned} \varrho^2(b_{n+1}, v) &= \varrho^2(P_{\mathcal{L}_n} b_n, v) \le \varrho^2(b_n, v) - \varrho^2(P_{\mathcal{L}_n} b_n, b_n), \\ \varrho^2(b_{n+1}, v) &\le \varrho^2(b_n, v) - \varrho^2(b_n, \mathcal{L}_n). \end{aligned} \tag{11}$$

This implies that

$$\varrho(b_{n+1}, v) \le \varrho(b_n, v).$$

Thus $\lim_{n\to\infty} \varrho(b_n, v)$ exists. Therefore, the sequence $\{b_n\}$ is bounded and implies that the sequence $\{s_n\}$ is also bounded.

Claim 2.

$$\left[\frac{1}{\mathcal{M}}(1-\mu)\varrho^2(b_n, s_n)\right]^2 \le \varrho^2(b_n, v) - \varrho^2(b_{n+1}, v), \quad \text{for some } \mathcal{M} > 0.$$

Indeed, the bounded sequences $\{b_n\}, \{s_n\}$ implies that $\{Ab_n\}, \{As_n\}$ are also bounded, so for all n there exists $\mathcal{M} > 0$ such that $\xi_n d(b_n, Ab_n) \le \mathcal{M}$. Therefore, for all $u, q \in \mathcal{Z}$, we have

$$\begin{aligned} \varrho(h_n(u), h_n(q)) &= |h_n(u) - h_n(q)|, \\ &= \Big|\langle \overrightarrow{us_n}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle \\ &\quad -\langle \overrightarrow{qs_n}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle\Big|. \end{aligned}$$

By using Remark 1, we have

$$\varrho(h_n(u), h_n(q)) = \left|\langle \overrightarrow{uq}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle\right|.$$

By Lemma 7 and Cauchy schwartz inequality, we get

$$\begin{aligned} \langle \overrightarrow{uq}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle &\le \xi_n \langle \overrightarrow{uq}, \overrightarrow{b_n Ab_n}\rangle + \Big[(1-\xi_n) \\ &\quad \times \langle \overrightarrow{uq}, \overrightarrow{Ab_n Ab_n}\rangle\Big], \\ &= \xi_n \langle \overrightarrow{uq}, \overrightarrow{b_n Ab_n}\rangle \le \xi_n d(u,q) d(b_n, Ab_n), \\ &\le \mathcal{M}\varrho(u,q). \end{aligned}$$

Thus we have $h_n(.)$ is $\mathcal{M}$-Lipschitz continuous on $\mathcal{Z}$ and by Lemma 11, we get

$$\varrho(b_n, \mathcal{L}_n) \ge \frac{1}{\mathcal{M}} h_n(b_n), \tag{12}$$

which, together with Lemma 13, we get

$$\varrho(b_n, \mathcal{L}_n) \ge \frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n, s_n).$$

Combining (11) and (12),

$$\varrho^2(b_{n+1}, v) \leq \varrho^2(b_n, v) - \left[\frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n, s_n)\right]^2.$$

Thus, Claim 2 is proven.

Claim 3 The sequence $\{b_n\}$ Δ-converges in $VI(\mathcal{L}, A)$. Indeed, by Lemma 4, there exists the subsequence $\{b_{n_k}\}$ of bounded sequence $\{b_n\}$ such that the subsequence $\{b_{n_k}\}$ Δ-converges to $v \in \mathcal{Z}$.

Using Claim 2, we can find

$$\lim_{n\to\infty} \varrho(b_n, s_n) = 0.$$

It is implied from Lemma 14 that $v \in VI(\mathcal{L}, A)$.

Therefore, we proved that:

(i) $\lim_{n\to\infty} \varrho(b_n, v)$ exists, for every $v \in V(\mathcal{L}, A)$;

(ii) Each Δ-limit of the sequence $\{b_n\} \in VI(\mathcal{L}, A)$.

Thus, by Lemma 9, the sequence $\{b_n\}$ is Δ-convergent in $VI(\mathcal{L}, A)$.

Now we introduce an algorithm for strong convergence:

Algorithm 2. ***Initialization:*** *Given $\mu, \nu, \varsigma \in (0, 1)$. Let b_1 be the arbitrary element of $\mathcal{L}$.*

Iterative Steps: *For the given iteration b_n, first calculate b_{n+1} as stated below:*

Step 1. *Compute*

$$s_n = P_{\mathcal{L}}(\xi_n b_n \oplus (1-\xi_n)Ab_n),$$

where $\xi_n := \varsigma\nu^{m_n}$, with m_n is the minimal nonnegative integer satisfying

$$\langle \overrightarrow{Ab_nAs_n}, \overrightarrow{b_ns_n}\rangle \leq \mu\varrho^2(b_n, s_n).$$

If $b_n = s_n$ or $As_n = 0$ then algorithm stops and s_n is a solution of VI. Otherwise

Step 2. *Calculate*

$$b_{n+1} = \xi_n\varpi(b_n) \oplus (1-\xi_n)P_{\mathcal{L}_n}(b_n),$$

where

$$\mathcal{L}_n := \{b \in \mathcal{Z} : h_n(b) \leq 0\}$$

and

$$h_n(b) = \langle \overrightarrow{bs_n}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle.$$

*Place $n := n+1$ and move to **Step 1**.*

Theorem 3. *Consider a nonempty, closed and convex subset of a Hadamard space $\mathcal{Z}$ be $\mathcal{L}$. The mapping $A : \mathcal{L} \to \mathcal{L}$ is a pseudo-monotone, uniformly continuous on $\mathcal{Z}$. The solution set of VI is nonempty, that is $VI(\mathcal{L}, A) \neq \emptyset$. Let $\{\xi_n\}$ be the sequences of real numbers in $(0, 1)$ such that*

$$\lim_{n\to\infty} \xi_n = 0, \Sigma_{n=1}^{\infty}\xi_n = \infty.$$

Then any sequence $\{b_n\}$ converges strongly to $v \in VI(\mathcal{L}, A)$, where $v = P_{VI(\mathcal{L},A)}\varpi(v)$.

Proof.

Claim 1. *The sequence $\{b_n\}$ is bounded. To prove the claim, assume $z_n = P_{\mathcal{L}_n}(b_n)$, using Lemma 10, we have*

$$\varrho^2(z_n, v) \quad = \quad \varrho^2(P_{\mathcal{L}_n} b_n, v) \le \varrho^2(b_n, v) - \varrho^2(P_{\mathcal{L}_n} b_n, b_n),$$

according to Claim 1 in Theorem 1, we get

$$\varrho^2(z_n, v) = \varrho^2(P_{L_n} b_n, v) \le \varrho^2(b_n, v) - \left[\frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n, s_n)\right]^2.$$

This implies that

$$\varrho(z_n, v) \le \varrho(b_n, v). \tag{13}$$

Consider $\varrho(b_{n+1}, v) = \varrho(\xi_n \varpi(b_n) \oplus (1-\xi_n) P_{\mathcal{L}_n}(b_n), v)$ and by using Lemma 2-(i),we get

$$\begin{aligned}
\varrho(\xi_n \varpi(b_n) \oplus & (1-\xi_n) P_{\mathcal{L}_n}(b_n), v) \\
& \le \xi_n \varrho(\varpi(b_n), v) + (1-\xi_n)\varrho(P_{\mathcal{L}_n} b_n, v), \\
& \le \xi_n(\varrho(\varpi(b_n), \varpi(v) + \varrho(\varpi(v), v)) + \Big[(1-\xi_n)\varrho(P_{\mathcal{L}_n} b_n, v)\Big], \\
& \le \xi_n \varrho(\varpi(b_n), \varpi(v)) + \xi_n \varrho(\varpi(v), v) + \Big[(1-\xi_n)\varrho(P_{\mathcal{L}_n} b_n, v)\Big], \\
& \le \xi_n \rho \varrho(b_n, v) + \xi_n \varrho(\varpi(v), v) + (1-\xi_n)\varrho(z_n, v), \\
& \le \xi_n \rho \varrho(b_n, v) + \xi_n \varrho(\varpi(v), v) + (1-\xi_n)\varrho(b_n, v), \\
& = (\xi_n \rho + (1-\xi_n))\varrho(b_n, v) + \xi_n \varrho(\varpi(v), v), \\
& = (1-\xi_n(1-\rho))\varrho(b_n, v) + \xi_n(1-\rho)\frac{\varrho(\varpi(v), v)}{(1-\rho)}, \\
& \le \max\left\{\varrho(b_n, v), \frac{\varrho(\varpi(v), v)}{(1-\rho)}\right\}, \\
& \quad \vdots \\
& \le \max\left\{\varrho(b_1, v), \frac{\varrho(\varpi(v), v)}{(1-\rho)}\right\},
\end{aligned}$$

for $0 \le \xi_n(1-\rho) \le 1$. Thus we proved the Claim 1.

Claim 2. *To prove*

$$\varrho^2(z_n, b_n) \le \varrho^2(b_n, v) - \varrho^2(b_{n+1}, v) + 2\xi_n \langle \overrightarrow{\varpi(b_n)v}, \overrightarrow{b_{n+1}v} \rangle.$$

Let $s_n = \xi_n v \oplus (1-\xi_n) z_n$. It follows from Lemmas 6 and 7 that

$$\begin{aligned}
\varrho^2(b_{n+1}, v) \quad &= \quad \varrho^2(\xi_n \varpi(b_n) \oplus (1-\xi_n) z_n, v) \le \varrho^2(s_n, v) + 2\langle \overrightarrow{b_{n+1}s_n}, \overrightarrow{b_{n+1}v} \rangle, \\
&= \quad [\varrho(\xi_n v + (1-\xi_n) z_n, v)]^2
\end{aligned}$$

$$+2\langle\overrightarrow{(\xi_n\varpi(b_n)\oplus(1-\xi_n)z_n))s_n},\overrightarrow{b_{n+1}v}\rangle,$$

$$\begin{aligned}
\varrho^2(b_{n+1},v) &\leq [\xi_n d(v,v)+(1-\xi_n)\varrho(z_n,v)]^2+2[\xi_n\langle\overrightarrow{\varpi(b_n)s_n},\overrightarrow{b_{n+1}v}\rangle\\
&\quad +(1-\xi_n)\langle\overrightarrow{z_ns_n},\overrightarrow{b_{n+1}v}\rangle],\\
&= (1-\xi_n)^2\varrho^2(z_n,v)+2[\xi_n\langle\overrightarrow{\varpi(b_n)(\xi_nv\oplus(1-\xi_n)z_n)},\overrightarrow{b_{n+1}v}\rangle\\
&\quad +(1-\xi_n)\langle\overrightarrow{z_n(\xi_nv\oplus(1-\xi_n)z_n)},\overrightarrow{b_{n+1}v}\rangle],\\
&\leq (1-\xi_n)^2\varrho^2(z_n,v)+2\Big[\xi_n^2\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle+\Big\{\xi_n(1-\xi_n)\\
&\quad \times\langle\overrightarrow{\varpi(b_n)z_n},\overrightarrow{b_{n+1}v}\rangle\Big\}+\xi_n(1-\xi_n)\langle\overrightarrow{z_nv},\overrightarrow{b_{n+1}}\rangle+\Big\{(1-\xi_n)^2\\
&\quad \times\langle\overrightarrow{z_nz_n},\overrightarrow{b_{n+1}v}\rangle\Big\}\Big],\\
&= (1-\xi_n)^2\varrho^2(z_n,v)+2\xi_n^2\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle\\
&\quad +2\xi_n(1-\xi_n)\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle,\\
&= (1-\xi_n)^2\varrho^2(z_n,v)+2\xi_n^2\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle\\
&\quad +2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle-2\xi_n^2\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle,\\
&= (1-\xi_n)^2\varrho^2(z_n,v)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle.\\
\varrho^2(b_{n+1},v) &\leq (1-\xi_n)\,d^2(z_n,v)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle\\
\varrho^2(b_{n+1},v) &\leq \varrho^2(z_n,v)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle. \qquad (14)
\end{aligned}$$

On the other hand, we have

$$\varrho^2(z_n,v) \leq \varrho^2(b_n,v)-\varrho^2(z_n,b_n),$$

by putting above in (4.3), we have

$$\begin{aligned}
\varrho^2(b_{n+1},v) &\leq \varrho^2(b_n,v)-\varrho^2(b_n,z_n)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle,\\
\varrho^2(b_n,z_n) &\leq \varrho^2(b_n,v)-\varrho^2(b_{n+1},v)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle.
\end{aligned}$$

Claim 3. *To prove*

$$(1-\xi_n)\left[\frac{1}{\mathcal{M}}\xi_n(-1-\mu)\varrho^2(b_n,s_n)\right]^2 \leq \varrho^2(b_n,v)-\varrho^2(b_{n+1},v)+\xi_n\varrho^2(\varpi(b_n),v).$$

Consider $\varrho^2(b_{n+1},v)=\varrho^2(\xi_n\varpi(b_n)\oplus(1-\xi_n)z_n,v)$ *and by using Lemma 2-(ii), we have*

$$\begin{aligned}
\varrho^2(b_{n+1},v) &\leq \xi_n\varrho^2(\varpi(b_n),v)+(1-\xi_n)\varrho^2(z_n,v)\\
&\quad -\xi_n(1-\xi_n)\varrho^2(\varpi(b_n),z_n),\\
&\leq \xi_n\varrho^2(\varpi(b_n),v)+(1-\xi_n)\varrho^2(z_n,v).
\end{aligned}$$

By using (13), we obtain

$$\varrho^2(b_{n+1},v) \leq \xi_n\varrho^2(\varpi(b_n),v)+(1-\xi_n)\varrho^2(b_n,v)$$

$$
\begin{aligned}
&-(1-\xi_n)\left[\frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n,s_n)\right]^2,\\
\leq\ & \xi_n\varrho^2(\varpi(b_n),v)+\varrho^2(b_n,v)-(1-\xi_n)\\
&\times\left[\frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n,s_n)\right]^2.
\end{aligned}
$$

This implies that

$$
(1-\xi_n)\left[\frac{1}{\mathcal{M}}(-1-\mu)\varrho^2(b_n,s_n)\right]^2 \leq \xi_n\varrho^2(\varpi(b_n),v)+\varrho^2(b_n,v)-\varrho^2(b_{n+1},v).
$$

Claim 4. *To prove*

$$
\varrho^2(b_{n+1},v) \leq \frac{(1-(1-\rho)\xi_n)}{1-\xi_n\rho}\varrho^2(b_n,v)+\frac{2\xi_n}{1-\xi_n\rho}\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle.
$$

Consider

$$
\varrho^2(b_{n+1},v) \leq (1-\xi_n)\varrho^2(z_n,v)+2\xi_n\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle. \tag{15}
$$

By Cauchy schwartz inequality, we have

$$
\begin{aligned}
\langle\overrightarrow{\varpi(b_n)v},\overrightarrow{b_{n+1}v}\rangle &= \langle\overrightarrow{\varpi(b_n)\varpi(v)},\overrightarrow{b_{n+1}v}\rangle+\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle,\\
&\leq d(\varpi(b_n),\varpi(v))d(b_{n+1},v)+\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle,\\
&\leq \rho d(b_n,v)d(b_{n+1},v)+\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1v}}\rangle,\\
&\leq \frac{\rho}{2}[\varrho^2(b_n,v)+\varrho^2(b_{n+1},v)]+\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle,
\end{aligned}
$$

by putting above in (15), we have

$$
\begin{aligned}
\varrho^2(b_{n+1},v) &\leq (1-\xi_n)\varrho^2(z_n,v)+\xi_n\rho\varrho^2(b_n,v)\\
&\quad+\xi_n\rho\varrho^2(b_{n+1},v)+2\xi_n\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle,\\
\varrho^2(b_{n+1},v)-\xi_n\rho\varrho^2(b_{n+1},v) &\leq (1-\xi_n+\xi_n\rho)\varrho^2(b_n,v)\\
&\quad+2\xi_n\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle\\
(1-\xi_n\rho)\varrho^2(b_{n+1},v) &\leq (1-(1-\rho)\xi_n)\varrho^2(b_n,v)+2\xi_n\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle,\\
\varrho^2(b_{n+1},v) &\leq \frac{(1-(1-\rho)\xi_n)}{1-\xi_n\rho}\varrho^2(b_n,v)\\
&\quad+\frac{2\xi_n}{1-\xi_n\rho}\langle\overrightarrow{\varpi(v)v},\overrightarrow{b_{n+1}v}\rangle.
\end{aligned}
$$

Claim 5. *The sequence $\varrho^2(b_n,v)$ converges to zero. There are two scenarios for the proof of convergence of sequence $\{\varrho^2(b_n,v)\}$.*
Case 1. *There exist a natural number $\mathbb{N}$ such that $\varrho^2(b_{n+1},v)\leq\varrho^2(b_n,v)$ for all $n\geq\mathbb{N}$. This implies that $\lim_{n\to\infty}\varrho^2(b_n,v)$ exist. From Claim 2, we have*

$$
\lim_{n\to\infty}\varrho^2(b_n,z_n)=0.
$$

Now, according to Claim 3,

$$\lim_{n\to\infty} \varrho^2(b_n, s_n) = 0.$$

Since the sequence $\{b_n\}$ is bounded. It is implied by Lemma 4, that every bounded sequence in $(\mathcal{Z}, \varrho)$ always has a Δ-convergent subsequence, say b_{n_k} Δ-converges to z such that

$$\lim_{n\to\infty} \sup\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_n v}\rangle = \lim_{n\to\infty} \langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n_k} v}\rangle = \langle \overrightarrow{\varpi(v)v}, \overrightarrow{zv}\rangle. \quad (16)$$

Since b_{n_k} Δ-converges to z and $\varrho(b_n, s_n) = 0$, it implies from Lemma14 that $z \in VI(\mathcal{L}, A)$. On the other hand,

$$\begin{aligned} \varrho(b_{n+1}, z_n) &= \varrho(\xi_n \varpi(b_n) \oplus (1-\xi_n) z_n, z_n) \\ &\leq \xi_n \varrho(\varpi(b_n), z_n) + (1-\xi_n)\varrho(z_n, z_n) \\ &= \xi_n \varrho(\varpi(b_n), z_n) \to 0 \quad \textit{as } n \to \infty. \end{aligned}$$

Thus,

$$\varrho(b_{n+1}, b_n) \leq \varrho(b_{n+1}, z_n) + \varrho(z_n, b_n) \to 0 \quad \textit{as } n \to \infty.$$

Since $v = P_{VI(\mathcal{L},A)}\varpi(v)$ and b_{n_k} Δ-converges to $z \in VI(\mathcal{L}, A)$, using (16), we get

$$\lim_{n\to\infty} \sup\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_n v}\rangle = \langle \overrightarrow{\varpi(v)v}, \overrightarrow{zv}\rangle \leq 0.$$

This implies that

$$\lim_{n\to\infty} \sup\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n+1} v}\rangle \leq \lim_{n\to\infty} \sup\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n+1} b_n}\rangle + \lim_{n\to\infty} \sup\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_n v}\rangle \leq 0.$$

From Claim 4

$$\begin{aligned} \varrho^2(b_{n+1}, v) &\leq 1 - \frac{(\xi_n - 2\xi_n\rho)}{1-\xi_n\rho}\varrho^2(b_n, v) + \frac{2\xi_n}{1-\xi_n\rho}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n+1} v}\rangle \\ &= 1 - \frac{(1-2\rho)\xi_n}{1-\xi_n\rho}\varrho^2(b_n, v) + 2\frac{(1-2\rho)\xi_n}{(1-\xi_n\rho)(1-2\rho)}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n+1} v}\rangle. \end{aligned}$$

Now, taking

$$\lambda_n = \frac{(1-2\rho)\xi_n}{1-\xi_n\rho}, \Im_n = \frac{2\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{n+1} v}\rangle}{1-2\rho},$$

by Lemma 5, we can conclude that

$$\varrho^2(b_n, v) = 0.$$

$\Rightarrow b_n \to v$ *as* $n \to \infty$.

Case 2. *There exist a subsequence $\{\varrho^2(b_{n_j}, v)\}$ of $\{\varrho^2(b_n, v)\}$ such that $\varrho^2(b_{n_j}, v) < \varrho^2(b_{n_{j+1}}, v)$, for all $j \in \mathbb{N}$. In this case, from Lemma 8 that there exist a nondecreasing sequence of natural numbers $\{a_k\}$ such that*

$$\lim_{k\to\infty} a_k = \infty,$$

and the inequalities stated below holds for all values of $k \in \mathbb{N}$:

(i) $\varrho^2(b_{a_k}, v) \leq \varrho^2(b_{a_{k+1}}, v)$,

(ii) $\varrho^2(b_k, v) \leq \varrho^2(b_{a_{k+1}}, v)$.

According to Claim 2, we have

$$\begin{aligned}
\varrho^2(z_{a_k}, b_{a_k}) &\leq \varrho^2(b_{a_k}, v) - \varrho^2(b_{a_{k+1}}, v) + 2\xi_{a_k}\langle \overrightarrow{\varpi(b_{a_k})v}, \overrightarrow{b_{a_{k+1}}v}\rangle, \\
&\leq \varrho^2(b_{a_{k+1}}, v) - \varrho^2(b_{a_{k+1}}, v) + 2\xi_{a_k}\langle \overrightarrow{\varpi(b_{a_k})v}, \overrightarrow{b_{a_{k+1}}v}\rangle, \\
&\leq \xi_{a_k}\langle \overrightarrow{\varpi(b_{a_k})v}, \overrightarrow{b_{a_{k+1}}v}\rangle, \\
&\leq \xi_{a_k}\varrho(\varpi(b_{a_k}), v)\varrho(b_{a_{k+1}}, v) \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

According to Claim 3, we have

$$\begin{aligned}
&(1-\xi_{a_k})\left[\frac{1}{\mathcal{M}}\xi_n(-1-\mu)\varrho^2(b_{a_k}, s_{a_k})\right]^2 \\
&\leq \varrho^2(b_{a_k}, v) - \varrho^2(b_{a_{k+1}}, v) + \xi_{a_k}\varrho^2(\varpi(b_{a_k}), v), \\
&\leq \varrho^2(b_{a_{k+1}}, v) - \varrho^2(b_{a_{k+1}}, v) + \xi_{a_k}d^2(\varpi(b_{a_k}), v), \\
&\leq \xi_{a_k}\varrho^2(\varpi(b_{a_k}), v) \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Using the same argument as in the proof of Case 1, we obtain

$$\varrho(b_{a_{k+1}}, b_{a_k}) \to 0$$

and

$$\limsup_{k\to\infty} \langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{a+1}v}\rangle \leq 0.$$

From Claim 4

$$\varrho^2(b_{m_{k+1}}, v) \leq \frac{(1-(1-\rho)\xi_{m_k})}{(1-\xi_{m_k}\rho)}\varrho^2(b_{m_k}, v) + \frac{2\xi_{m_k}}{1-\xi_{m_k}\rho}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{m_{k+1}}v}\rangle,$$

since $\varrho^2(b_k, v) \leq \varrho^2(b_{m_{k+1}}, v)$, *we have*

$$\varrho^2(b_{m_{k+1}}, v) \leq \frac{(1-(1-\rho)\xi_{m_k})}{1-\xi_{m_k}\rho}\varrho^2(b_{m_{k+1}}, v) + \frac{2\xi_{m_k}}{1-\xi_{m_k}\rho}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{m_{k+1}}v}\rangle,$$

$$\begin{aligned}
\xi_{m_k}(1-2\rho)\varrho^2(b_{m_{k+1}}, v) &\leq 2\xi_{m_k}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{m_{k+1}}v}\rangle, \\
\varrho^2(b_k, v) &\leq \frac{2}{1-2\rho}\langle \overrightarrow{\varpi(v)v}, \overrightarrow{b_{m_{k+1}}v}\rangle.
\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} \varrho^2(b_k, v) \leq 0,$$

that is, $b_k \to v$.

5. Some consequences and numerical illustrations

In this section, we first derive 1 and 2 corollaries of Theorem 2 and 3 respectively. We then illustrate a numerical experiment to demonstrate the performance of our method.

Corollary 1. *Consider a nonempty, closed and convex subset of a Hadamard space $\mathcal{Z}$ be $\mathcal{L}$. Let $\mathcal{Z}^*$ be a metric space and $A_2 : \mathcal{L} \to \mathcal{Z}^*$, $A_1 : \mathcal{Z}^* \to \mathcal{L}$. The map A_2 is a uniformly continuous on $\mathcal{L}$ and A_1 is uniformly continuous on $A_2(L)$, A is a pseudo-monotone on $\mathcal{L}$. The solution set of the $VI(\mathcal{L}, A)$ is nonempty, that is $VI(\mathcal{L}, A) \neq \phi$. Then any sequence b_n is Δ-convergent in $VI(\mathcal{L}, A)$.*

Corollary 2. *Let $\mathcal{L}$ be a nonempty, closed and convex subset of a Hadamard space $\mathcal{Z}$. Consider the mappings $A_1 : \mathcal{L} \to \mathcal{L}$ and $A_2 : \mathcal{L} \to \mathcal{L}$ such that $A_1 o A_2 = A$. The mapping A is a pseudo-monotone, uniformly continuous on $\mathcal{Z}$. The solution set of VI is nonempty, that is $VI(\mathcal{L}, A) \neq \emptyset$. Let $\{\xi_n\}$ be the sequences of real numbers in $(0, 1)$ such that*

$$\lim_{n\to\infty} \xi_n = 0, \Sigma_{n=1}^{\infty} \xi_n = \infty.$$

Then the sequence $\{b_n\}$ converges strongly to $v \in VI(\mathcal{L}, A)$, where $v = P_{VI(\mathcal{L},A)}\varpi(v)$.

Example 1. *Let $\mathcal{Z} = \mathbb{R}^2, \mathcal{L} = \{u \in \mathbb{R}^2 : -1 \leq u_i \leq 1, i = 1, 2\}$. Define the maps $A_1 : \mathcal{L} \to (\mathbb{R}^2)^*$, and $A_2 : (\mathbb{R}^2)^* \to \mathcal{L}$ such that $A_1(b) = \mathfrak{f}$, $\mathfrak{f} \in (\mathbb{R}^2)^*$, $A_2(b) = s'$ respectively. Let $b_1 = (0.1, 0.2) \in \mathcal{L}$ be arbitrary and define $g_i : \mathbb{R}^2 \to \mathbb{R}$ such that $g_1(b_1, b_2) = \cos(b_1), g_2(b_1, b_2) = \cos(b_2)$. Then we have*

$$\begin{aligned} Ab_1 &= \left(\frac{\cos(b_1) + \cos(b_2)}{2}, \frac{\cos(b_1) - \cos(b_2)}{2}\right) \\ &= \left(\frac{\cos(0.1) + \cos(0.2)}{2}, \frac{\cos(0.1) - \cos(0.2)}{2}\right) \\ &= (0.98754, 0.00747). \end{aligned}$$

Step 1
For $n = 1$, and $\xi_1 = 0.2$,

$$\begin{aligned} s_1 &= P_{\mathcal{L}}(\xi_1 b_1 \oplus (1 - \xi_1) Ab_1) \\ &= P_{\mathcal{L}}((0.2)(0.1, 0.2) \oplus (1 - 0.2)(0.98754, 0.00747)) \\ &= P_{\mathcal{L}}((0.02, 0.04) \oplus (0.790032, 0.005976)) \\ &= P_{\mathcal{L}}(0.810032, 0.045976). \end{aligned}$$

As $(0.810032, 0.045976) \in \mathcal{L}$, So it's unique nearest point in $\mathcal{L}$ is $(0.810032, 0.045976)$ implies $s_1 = (0.810032, 0.045976)$.

$$\begin{aligned} As_1 &= \left(\frac{0.68948 + 0.99894}{2}, \frac{0.68948 - 0.99894}{2}\right) \\ &= (0.84421, -0.15473). \end{aligned}$$

Now to prove

$$\langle \overrightarrow{Ab_1As_1}, \overrightarrow{b_1s_1}\rangle \leq \mu\varrho^2(b_1, s_1). \tag{17}$$

Consider left hand side of (17),

$$\begin{aligned}\langle \overrightarrow{Ab_1As_1}, \overrightarrow{b_1s_1}\rangle &= \frac{1}{2}\left[\varrho^2(Ab_1, s_1) + \varrho^2(As_1, b_1) - \varrho^2(Ab_1, b_1)\right.\\ &\quad \left.-\varrho^2(\Pi Ay_1, y_1)\right]\\ &= -0.0932945\end{aligned}$$

and

$$\begin{aligned}\varrho^2(b_1, s_1) &= (0.1 - 0.810032)^2 + (0.2 - 0.045976)^2\\ &= 0.527873\\ \mu\varrho^2(b_1, s_1) &= 0.2(0.527873) = 0.1055746\end{aligned}$$

Thus, we have

$$\langle \overrightarrow{Ab_1As_1}, \overrightarrow{b_1s_1}\rangle \leq \mu\varrho^2(b_1, s_1).$$

Step 2

Now, for $b_2 = P_{\mathcal{L}_n}(b_1)$, $h_n(b) = \langle \overrightarrow{bs_n}, \overrightarrow{(\xi_n b_n \oplus (1-\xi_n)Ab_n)Ab_n}\rangle$ *and* $\mathcal{L}_n = \{b \in \mathcal{Z} : h_n(b) \leq 0\}$. *Let* $b = (\mu_1, \mu_2)$, $h_1(b) = 0$. *Then*

$$\begin{aligned}h_1(b) = \langle \overrightarrow{bs_1}, \overrightarrow{(\xi_1 b_1 \oplus (1-\xi_1)Ab_1)Ab_1}\rangle &= 0\\ \frac{1}{2}\left[\begin{array}{c} d^2(b, Ab_1) + d^2(s_1, \xi_1 b_1 \oplus (1-\xi_1)Ab_1)\\ -d^2(b, \xi_1 b_1 \oplus (1-\xi_1)Ab_1) - d^2(s_1, Ab_1)\end{array}\right] &= 0\\ -0.355016\mu_1 + 0.77012\mu_2 + 0.2845414 &= 0. \end{aligned}\tag{18}$$

From (18), we have $\mu_1 = 0.21993\mu_2 + 0.801164$. *If* $\mu_1 = 0$, *then* $\mu_2 = 3.6428$ *and if* $\mu_2 = 0$, *then* $\mu_1 = 0.801164$. *Since* $b_2 = \{s' \in \mathcal{L}_1 : \varrho(b_1, s') = \varrho(b_1, \mathcal{L}_1)\}$ *and*

$$\begin{aligned}\varrho((0.1, 0.2), \mathcal{L}_1) &= \frac{-0.355016(0.1) + 0.77012(0.2) + 0.2845414}{\sqrt[2]{(-0.355016)^2 + (0.77012)^2}}\\ &= 0.727949\end{aligned}$$

Let $s'(\mu_1, \mu_2)^t$. *Then* s' *is point of intersection of line* $\mathcal{L}_1$ *with the perpendicular line. The equation of perpendicular line is*

$$\mu_1 = -\frac{1}{0.21993}\mu_2 + 0.801164 \tag{19}$$

If $\mu_1 = 0$, *then* $\mu_2 = 0.176199$ *and if* $\mu_2 = 0$, *then* $\mu_1 = 0.801164$. *To find point of intersection, subtracting (18) and (19), we get* $\mu_2 = 0$, $\mu_1 = 0.801164$ *and* $s' = b_2 = (0.801164, 0)$.

Example 2. *Let*

$$\mathcal{L} = \{(x_1, 0, 0, \dots) \in \ell^2 : 0 \le x_1 \le 1\},$$

which is convex (if $x = (x_1, 0, \dots)$, $y = (y_1, 0, \dots) \in \mathcal{L}$ *and* $\lambda \in [0,1]$, *then*

$$\lambda x + (1-\lambda)y = \big(\lambda x_1 + (1-\lambda)y_1, 0, \dots\big) \in \mathcal{L}.$$

For a concrete test, we work with a simple diagonal operator $A_1 : \mathcal{L} \to \ell^2$ *given by*

$$A_1(x) = \big(\alpha_n x_n\big), \qquad \alpha_n = \frac{1}{n+1} \ \ (\text{so } \alpha_1 = \tfrac{1}{2}),$$

and $A_2(y) = P_{\mathcal{L}}(y)$. *Choose* $\xi_n = \xi = 0.2$ *and an initial vector* $b_1 \in \mathcal{L}$ *with first coordinate* $x_1 = 0.8$.

The orthogonal projection $P_{\mathcal{L}}$ *onto set* $\mathcal{L}$ *can be described explicitly*

$$P_{\mathcal{L}}(v) = (\pi_{[0,1]}(v_1), 0, 0, \dots),$$

so the Algorithm 1 update $s_n = P_{\mathcal{L}}(\xi b_n + (1-\xi)Ab_n)$, *the entire iteration is governed solely by the first coordinate, leading to a simple one-dimensional recurrence relation. For the purpose of demonstration we will take the simplified update* $b_{n+1} = s_n$, *where* $s_n = P_{\mathcal{L}}(\xi b_n + (1-\xi)Ab_n)$. *With* $\xi = 0.2$ *and* $\alpha_1 = 1/2$ *one gets*

$$x_{n+1} = 0.6\, x_n, \qquad x_1 = 0.8,$$

hence $x_n = 0.6^{\,n-1} x_1$ *and* $b_n = (x_n, 0, 0, \dots) \to 0$ *strongly in* ℓ^2. *This worked-out example provides a straightforward demonstration in the infinite-dimensional framework (implemented by truncation for computations) validating the convergence behaviour asserted in Theorem 2.*

Conclusion 1. *The variational inequality is defined for asymptotically nonexpansive mapping in* $\mathcal{CAT}(0)$ *spaces. We acquire strong and* Δ*-convergence of two projection type methods for variational inequality problem using pseudo-monotone mapping. Numerical experiment performed for our perposed algorithm.*

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