EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6681
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Statistical Inference for the Epanechnikov-Burr XII Distribution with Simulation and Case Studies

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Abstract. In this article, the Epanechnikov kernel function and the Burr-XII distribution are combined to create the Epanechnikov-Burr-XII distribution (EBD). The mathematical characteristics of this novel distribution, such as moments, moment generating function, reliability analysis functions, and order statistics, are investigated. The maximum likelihood estimation (MLE) approach is used to estimate the EBD's parameters. The consistency of the MLE estimators is demonstrated by a simulation analysis, and real-world data applications reveal that the EBD fits data better than the conventional Burr-XII distribution. The outcomes demonstrate how adaptable and useful the EBD is for modeling lifespan data.

2020 Mathematics Subject Classifications: 60E05, 62-07

Key Words and Phrases: Burr XII distribution, epanechnikov, maximum likelihood, moment, moment generating function

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1. Introduction

The optimal mean square error features of the Epanechnikov kernel function (EKF) make it a popular non-parametric tool for density estimation. In this work, we suggest a novel distribution, the Epanechnikov-Burr-XII distribution (EBD), by combining the EKF and the Burr-XII distribution. When combined with the EKF, the Burr-XII distribution, which is well-known for its adaptability in modeling lifetime data, becomes much more versatile and applicable.

The employment of different transformation techniques, including kernel functions and quadratic transmutation maps, to generalize distributions has been the subject of recent

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6681

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study. These generalizations have been helpful in survival research, reliability analysis, and other areas. By embedding the EKF into the Burr-XII distribution, the EBD is obtained, producing a distribution with more complex mathematical characteristics. The EBD's moments, reliability functions, hazard rates, and order statistics are all examined in this work. Furthermore, MLE is used to estimate the EBD's parameters, and the consistency of these estimators is confirmed by a simulation analysis. The enhanced performance of the EBD over the conventional Burr-XII distribution is illustrated using real-world datasets.

A lot of researchers generalized some distributions using the distribution theory. [1] used Epanechnikov-pareto Distribution. [2] propose the transmuted Janardan distribution. The two-parameter Lindley distribution [3] and the Ishita distribution [4] are all generalized using the same map. Through her master's thesis, [5] combined the Epanechnikov kernel function with the Weibull distribution to produce the Epanechnikov-Weibull distribution. [6] calculated the two-parameter weighted exponential distributions and the transmuted reciprocal. In contrast, [7] proposed the transmuted Aradhana distribution using the quadratic transmutation map. [8] Provide new estimators for the moving extreme ranked set sampling scheme's simple linear regression parameters. The transmuted Gumbel-logistic distribution was introduced by [9] who used the inverse Weibull distribution as the kernel function. By applying a beta kernel function to a standard exponential distribution, [10] used the exponential distribution and the Epanechnikov kernel function to suggest a new distribution. [11] used the maximum likelihood estimator approach to estimate the reliability based on the Pareto distribution. In order to estimate the parameters and reliability, [12] presented a novel estimation method based on non-parametric kernel density estimation. [13] develop Mukherjee-Islam distribution into transmuted Mukherjee-Islam distribution. In this paper we will use the EKF and the Burr-XII distribution to generate a new distribution called the Epanechnikov-Burr-XII distribution (EBD).

2. Epanechnikov-Burr-XII Distribution

The **Burr-XII distribution** was first developed by Burr [14] in 1942 and has received considerable attention due to its wide range of applications in areas such as acceptance sampling plans, failure-time modeling, and reliability analysis.

The probability density function (PDF) of the Burr-XII is

$$f(x, \alpha, \theta) = \theta\alpha x^{\alpha-1}(1 + x^{\alpha})^{-\theta-1}, \quad x, \alpha, \theta \geq 0$$

and its cumulative distribution function (CDF) is

$$F(x) = 1 - (1 + x^{\alpha})^{-\theta}$$

The **Epanechnikov kernel function (EKF)** is defined as

$$K(u) = \frac{3}{4}(1 - u^2), \quad |u| \leq 1$$

The Epanechnikov kernel is defined for $|u| \leq 1$. In the construction of the EBD, the variable u is given by the CDF of the Burr-XII distribution, $F(x) = 1 - (1 + x^{\alpha})^{-\theta}$, which takes values in the interval $[0,1]$. This naturally satisfies the condition $|u| \leq 1$, ensuring the kernel truncation is properly enforced without requiring additional constraints on x. By embedding the Epanechnikov kernel into the Burr-XII distribution, we define a new class of distributions called the **Epanechnikov-Burr-XII distribution (EBD)**. Its probability density function is given by the following theorem.

Theorem 1. *A random variable X is said to have an EBD if its CDF and PDF are respectively given by*

$$G(x) = \frac{3}{2}\left[\frac{2}{3} - (1+x^{\alpha})^{-2\theta} + \frac{1}{3}(1+x^{\alpha})^{-3\theta}\right]$$

and

$$g(x) = 3\alpha\theta\left[x^{\alpha-1}(1+x^{\alpha})^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1+x^{\alpha})^{-3\theta-1}\right], \quad x \geq 0$$

Proof. We begin with the definition of the cumulative distribution function for kernel-based distributions:

$$G(x) = 2\int_0^{F(x)} K(u)du$$

Substituting the Epanechnikov kernel function $K(u) = \frac{3}{4}(1-u^2)$ and the Burr-XII CDF $F(x) = 1-(1+x^{\alpha})^{-\theta}$, we obtain:

$$G(x) = 2\int_0^{1-(1+x^{\alpha})^{-\theta}} \frac{3}{4}(1-u^2)du$$

Evaluating this integral yields:

$$G(x) = \frac{3}{2}\left[1-(1+x^{\alpha})^{-\theta} - \frac{1}{2}\left(1-(1+x^{\alpha})^{-\theta}\right)^2\right]$$

Expanding and simplifying the expression inside the brackets:

$$G(x) = \frac{3}{2}\left[\frac{2}{3} - (1+x^{\alpha})^{-2\theta} + \frac{1}{3}(1+x^{\alpha})^{-3\theta}\right] \tag{1}$$

To obtain the probability density function, we differentiate $G(x)$ with respect to x:

$$g(x) = \frac{d}{dx}G(x) = 3\alpha\theta\left[x^{\alpha-1}(1+x^{\alpha})^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1+x^{\alpha})^{-3\theta-1}\right] \tag{2}$$

This completes the derivation.

This completes the derivation of the EBD distribution functions. Thus, $g(x)$ is a probability density function, as the integral of $g(x)$ over the interval $[0,\infty)$ is equal to one. The Epanechnikov kernel imposes a bounded quadratic transformation $K(u) = \frac{3}{4}(1-u^2)$ on the Burr–XII distribution, which moderates the extreme tail behavior. While the standard Burr–XII has polynomial tails decaying as $x^{-\alpha\theta-1}$, the EBD incorporates additional terms with faster-decaying components $(1+x^{\alpha})^{-3\theta-1}$ that provide smoother transition to the tails and reduce the propensity for extreme outliers, making it more suitable for modeling real-world data where very extreme values are less common.

3. Moments of the Distribution

Let X be a random variable that follows the EBD with parameters $\alpha > 0$ and $\theta > 0$. The r^{th} moment of X is defined as

$$E(X^r) = \int_0^\infty x^r g(x) dx$$

Theorem 2. *Let $X \sim EBD(\alpha, \theta)$ then the r^{th} moment of X is given by*

$$E(X^r) = 3\theta \left[\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(2\theta - \frac{r}{\alpha}\right)}{\Gamma(2\theta+1)} - \frac{1}{2}\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(3\theta - \frac{r}{\alpha}\right)}{\Gamma(3\theta+1)} \right]$$

where the r^{th} moment exists if and only if $r < 2\theta\alpha$.

Proof. The r^{th} moment is:

$$E(X^r) = \int_0^\infty x^r g(x) dx$$

Substituting the EBD PDF from Equation 2:

$$E(X^r) = 3\alpha\theta \int_0^\infty x^r \left[x^{\alpha-1}(1+x^\alpha)^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1+x^\alpha)^{-3\theta-1} \right] dx$$

Combining terms:

$$E(X^r) = 3\alpha\theta \int_0^\infty \left[x^{\alpha+r-1}(1+x^\alpha)^{-2\theta-1} - \frac{1}{2}x^{\alpha+r-1}(1+x^\alpha)^{-3\theta-1} \right] dx$$

Expressing in terms of Beta functions:

$$E(X^r) = 3\theta \left[B\left(\frac{r}{\alpha}+1, 2\theta - \frac{r}{\alpha}\right) - \frac{1}{2}B\left(\frac{r}{\alpha}+1, 3\theta - \frac{r}{\alpha}\right) \right]$$

Converting to Gamma functions:

$$E(X^r) = 3\theta \left[\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(2\theta - \frac{r}{\alpha}\right)}{\Gamma(2\theta+1)} - \frac{1}{2}\frac{\Gamma\left(\frac{r}{\alpha}+1\right)\Gamma\left(3\theta - \frac{r}{\alpha}\right)}{\Gamma(3\theta+1)} \right]$$

The moment exists when $r < 2\theta\alpha$ to ensure positive Gamma arguments.

For $r = 1$:

$$E(X) = \frac{3\theta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right) \left[\frac{\Gamma\left(2\theta - \frac{1}{\alpha}\right)}{\Gamma(2\theta+1)} - \frac{\Gamma\left(3\theta - \frac{1}{\alpha}\right)}{2\Gamma(3\theta+1)} \right]$$

For $r = 2$:

$$E(X^2) = \frac{6\theta}{\alpha}\Gamma\left(\frac{2}{\alpha}\right) \left[\frac{\Gamma\left(2\theta - \frac{2}{\alpha}\right)}{\Gamma(2\theta+1)} - \frac{1}{2}\frac{\Gamma\left(3\theta - \frac{2}{\alpha}\right)}{\Gamma(3\theta+1)} \right]$$

The variance is given by:

$$v(X) = E(X^2) - [E(X)]^2$$

$$v(X) = \frac{6\theta}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\left[\frac{\Gamma(2\theta - \frac{2}{\alpha})}{\Gamma(2\theta+1)} - \frac{1}{2}\frac{\Gamma(3\theta - \frac{2}{\alpha})}{\Gamma(3\theta+1)}\right] - \left[\frac{3\theta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left(\frac{\Gamma(2\theta - \frac{1}{\alpha})}{\Gamma(2\theta+1)} - \frac{\Gamma(3\theta - \frac{1}{\alpha})}{2\Gamma(3\theta+1)}\right)\right]^2.$$

4. Characteristic Function

The ordinary moment generating function $M_X(t) = E(e^{tX})$ does not exist for the Epanechnikov–Burr XII distribution when $t > 0$ due to its heavy-tailed nature. Instead, we characterize it using its *characteristic function*, which always exists. The characteristic function is defined as

$$Q_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} g(x)\, dx = 3\alpha\theta \int_0^{\infty} e^{itx}\left[x^{\alpha-1}(1+x^\alpha)^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1+x^\alpha)^{-3\theta-1}\right] dx,$$

where i is the imaginary unit.

Although this integral does not admit a closed-form solution in terms of elementary functions, it can be expressed using special functions or evaluated numerically for inference and simulation purposes. The moments of the distribution, within their existence domain $r < 2\theta\alpha$ as established in Theorem 3.1, can also be derived from the characteristic function.

5. Reliability Analysis

The reliability function is defined as:

$$R(t) = P(T \geq t) = 1 - G(t)$$

For the EBD:

$$R(t) = \left[\frac{3}{2}(1+t^\alpha)^{-2\theta} - \frac{1}{2}(1+t^\alpha)^{-3\theta}\right]$$

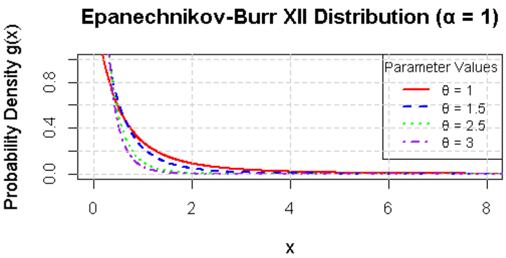
The hazard rate is the probability that the life of the item ends in the next moment if it remains alive till time t. It is defined as the ratio of the pdf to the reliability

$$h(t) = \frac{g(t)}{R(t)}$$

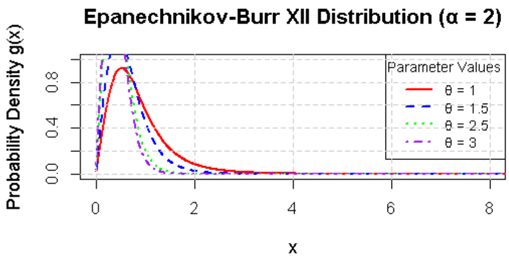
which simplifies to:

$$h(t) = \frac{6\alpha\theta t^{\alpha-1}}{1+t^\alpha} \cdot \frac{1 - \frac{1}{2}(1+t^\alpha)^{-\theta}}{3 - (1+t^\alpha)^{-\theta}}$$

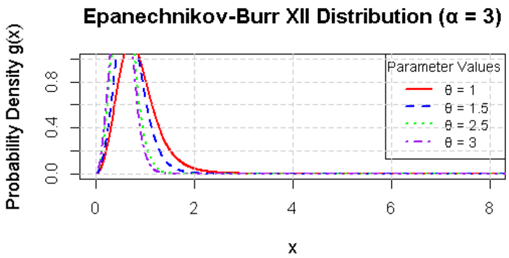
The EBD exhibits more flexible hazard rate shapes compared to the standard Burr-XII. While both distributions can capture increasing, decreasing, and bathtub-shaped hazard rates, the EBD's modified structure through the Epanechnikov kernel allows for smoother transitions between these phases and can model more complex failure patterns. The additional parameters introduced by the kernel provide enhanced flexibility in capturing the instantaneous failure rate evolution over time.



(a) For $(\alpha, \theta) = (1, 1)$



(b) For $(\alpha, \theta) = (2, 1)$



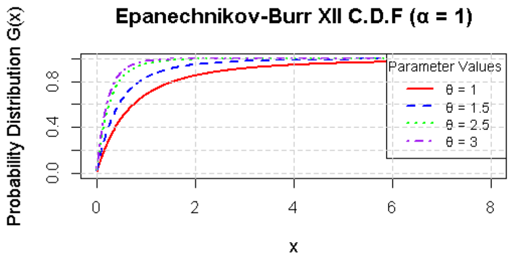
(c) For $(\alpha, \theta) = (2, 2)$

Figure 1: Plots for the PDF for different values of θ and α

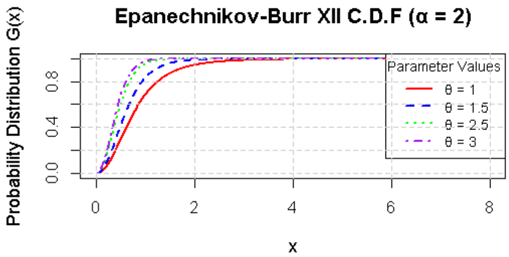
Figure 1 illustrates the shape of the probability density function $g(x)$ for various combinations of α and θ. The plots demonstrate the flexibility of the EBD in capturing different distributional shapes, including right-skewed, heavy-tailed, and nearly symmetric forms, depending on the parameter values. This versatility makes the EBD suitable for modeling diverse types of lifetime data.

The cumulative distribution function $G(x)$ is displayed for several parameter settings as shown in figure 2. The curves show the probability that a random variable X takes a value less than or equal to x. The smooth and monotonically increasing nature of the CDF curves confirms the validity of the EBD as a proper distribution function. The varying rates of increase reflect the influence of different parameter values on the distribution's spread and tail behavior.

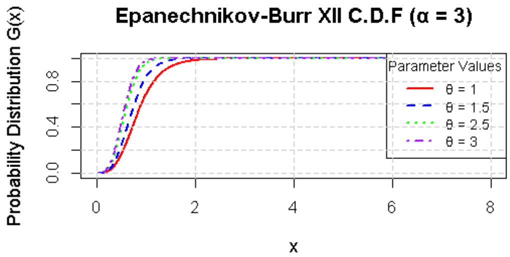
Figure 3 presents the reliability (or survival) function, which gives the probability that an item survives beyond time t. The declining trends of $R(t)$ across different parameter combinations highlight the distribution's applicability in reliability engineering and survival analysis. The rate of decrease varies with α and θ, indicating how these parameters influence the lifetime characteristics.



(a) For $(\alpha, \theta) = (1, 1)$



(b) For $(\alpha, \theta) = (2, 1)$



(c) For $(\alpha, \theta) = (2, 2)$

Figure 2: Plots for the PDF for different values of θ and α

From Figure 4, the hazard rate function, which represents the instantaneous failure rate at time t, is plotted for various parameter values. The shapes of the hazard curves, including increasing, decreasing, and upside-down bathtub forms, demonstrate the EBD's ability to model a wide range of failure patterns commonly observed in real-world systems. This flexibility enhances the distribution's utility in reliability modeling and risk analysis.

6. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the EBD (Equation 2), and let

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

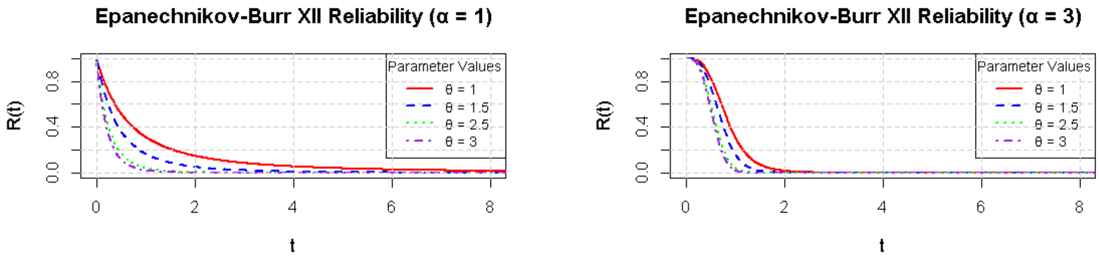
be the ordered statistics, where

$$X_{(1)} = \min(X_1, X_2, \ldots, X_n), \quad X_{(i)} \text{ is the } i\text{th ordered statistic } (i = 1, \ldots, n),$$

$$X_{(n)} = \max(X_1, X_2, \ldots, X_n).$$

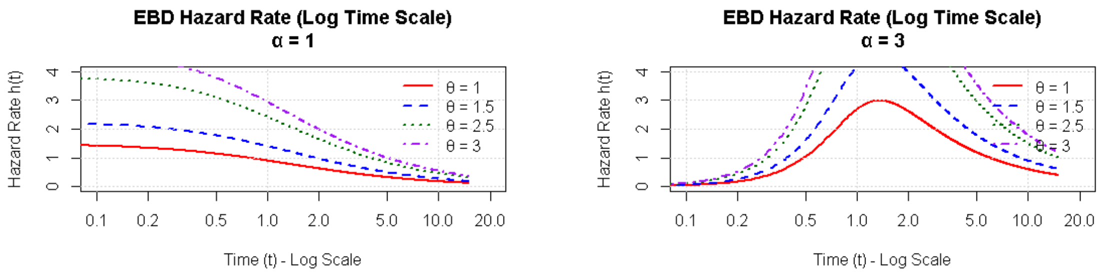
Then, the probability density function (PDF) of $X_{(i)}$ for $1 < i < n$ is given by

$$g_i(x) = \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[1 - G(x)]^{n-i},$$



(a) For $(\alpha, \theta) = (1, 1)$ (b) For $(\alpha, \theta) = (2, 1)$

Figure 3: Plots for the Reliability for different values of θ and α



(a) For $(\alpha, \theta) = (1, 1)$ (b) For $(\alpha, \theta) = (2, 1)$

Figure 4: Plots for the Hazard Rate for different values of θ and α

while the PDFs of the minimum $X_{(1)}$ and maximum $X_{(n)}$ are

$$g_1(x) = ng(x)[1 - G(x)]^{n-1}, \quad g_n(x) = ng(x)[G(x)]^{n-1}.$$

Substituting the EBD PDF and CDF, we get

$$g_1(x) = 3n\alpha\theta \left[x^{\alpha-1}(1 + x^\alpha)^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1 + x^\alpha)^{-3\theta-1}\right] \left[\frac{1}{2}(1 + x^\alpha)^{-3\theta} - \frac{3}{2}(1 + x^\alpha)^{-2\theta}\right],$$

$$g_n(x) = 3n\alpha\theta \left[x^{\alpha-1}(1 + x^\alpha)^{-2\theta-1} - \frac{1}{2}x^{\alpha-1}(1 + x^\alpha)^{-3\theta-1}\right] \left[1 - \frac{3}{2}(1 + x^\alpha)^{-2\theta} + \frac{1}{2}(1 + x^\alpha)^{-3\theta}\right].$$

7. Maximum Likelihood Estimation

One popular technique for estimating the distribution parameters is maximum likelihood estimation (MLE). The unknown parameters of Epanechnikov -Burr-XII were estimated using the MLE approach. Given the pdf described in equation 2, Let $x_1, x_2, \ldots, x_n$ be a random sample from EBD.

The joint PDF of $X_1, X_2, \ldots, X_n$ is called the likelihood function and is given by

$$L(\theta,\alpha)=\prod_{i=1}^{n}3\alpha\theta\left[x_i^{\alpha-1}(1+x_i^{\alpha})^{-2\theta-1}-\frac{1}{2}x_i^{\alpha-1}(1+x_i^{\alpha})^{-3\theta-1}\right],$$

which can also be expressed as

$$L(\theta,\alpha)=3^n\theta^n\alpha^n\prod_{i=1}^{n}\left[x_i^{\alpha-1}(1+x_i^{\alpha})^{-2\theta-1}\left(1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\right)\right].$$

The log-likelihood function is:

$$\ln L(\theta,\alpha)=n\ln 3+n\ln\theta+n\ln\alpha-(2\theta+1)\sum_{i=1}^{n}\ln(1+x_i^{\alpha})+(\alpha-1)\sum_{i=1}^{n}\ln(x_i)+\sum_{i=1}^{n}\ln\left(1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\right).$$

By deriving the log-likelihood function with respect to the parameters θ and α, we get

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta}-2\sum_{i=1}^{n}\ln(1+x_i^{\alpha})+\sum_{i=1}^{n}\frac{\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\ln(1+x_i^{\alpha})}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}} \quad (3)$$

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}-(2\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)}{1+x_i^{\alpha}}+\sum_{i=1}^{n}\ln(x_i)+\sum_{i=1}^{n}\frac{\frac{\theta}{2}x_i^{\alpha}\ln(x_i)(1+x_i^{\alpha})^{-\theta-1}}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}} \quad (4)$$

Since the likelihood equations do not admit closed-form solutions, the resulting system of nonlinear equations must be solved using numerical methods. We numerically verified that the score functions (Equations 3 and 4) evaluate to values close to zero at the reported maximum likelihood estimates, confirming that our optimization procedure successfully located the maximum of the likelihood function.

By Equating this equations 3 and 4 to 0, we get

$$\frac{n}{\theta}=2\sum_{i=1}^{n}\left(\ln(1+x_i^{\alpha})+\frac{\frac{1}{2}(1+x_i^{\alpha})^{-\theta}\ln(1+x_i^{\alpha})}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}\right),$$

$$\frac{n}{\alpha}=(2\theta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\ln(x_i)}{1+x_i^{\alpha}}-\sum_{i=1}^{n}\ln(x_i)-\sum_{i=1}^{n}\frac{\frac{\theta}{2}\cdot x_i^{\alpha}\ln(x_i)(1+x_i^{\alpha})^{-\theta-1}}{1-\frac{1}{2}(1+x_i^{\alpha})^{-\theta}}.$$

All summations are taken over $i=1$ to n, and the Newton-Raphson method is used for the numerical solution. In the numerical solution via the Newton-Raphson method, the choice of initial values is crucial for convergence. We recommend using the method of moments estimates from the sample data as initial guesses. Alternatively, initial values can be derived from the maximum likelihood estimates of the standard Burr-XII distribution, which often provide a robust starting point close to the EBD solution and ensure reliable convergence of the algorithm.

7.1. Theoretical Properties and Advantages

The Epanechnikov kernel provides several theoretical advantages over the standard Burr–XII distribution. First, the bounded quadratic transformation yields smoother tail behavior, moderating extreme values while maintaining flexibility. Second, the hazard function exhibits more nuanced shapes capable of capturing complex failure patterns. Notably, when $\alpha = 1$ and $\theta \to \infty$, the EBD converges to the Exponential(1) distribution, demonstrating its capacity to encompass simpler models. The moment existence condition $r < 2\alpha\theta$ reflects the distribution's heavy-tailed nature, which also explains the divergence of the moment generating function for $t > 0$. Practically, the bounded kernel ensures numerical stability during parameter estimation, as the likelihood function remains well-behaved and optimization algorithms converge reliably.

8. Simulation Study

We used the R (version 4.3.0) software to conduct a simulation study in which 1000 samples were generated from the EBD distribution using the inverse CDF method (based on Eq. 2.1) with varying sample sizes $n = 20, 50, 100, 150$, and 200, with $\theta = 0.5, 1, 1.5, 2$, and $\alpha = 1$, in order to examine the behavior of the MLE of the EBD parameter θ. The mean estimated value of $\hat{\theta}$ was calculated for each scenario. The results are displayed in Table 1. By examining these findings, we observe that as the sample size increases, the values of the Bias and MSE decrease, demonstrating that the MLEs preserve the consistency property for all parameter values. Additionally, both the Bias and MSE tend to increase as the values of θ increase.

The simulation results presented in Tables 1 and 2 decisively demonstrate the consistency and robustness of the Maximum Likelihood Estimators (MLEs) for the Epanechnikov-Burr XII distribution. As theoretically expected, for both parameters θ (Table 1) and α (Table 2), the Mean Squared Error (MSE) and Bias exhibit a noticeable and monotonic decline as the sample size increases from $n = 20$ to $n = 200$, across all true parameter values investigated.

For instance, when estimating $\theta = 1.0$ (Table 1), the MSE decreases from 0.0875 to 0.0029, while the bias reduces from 0.2415 to 0.0065. A similar consistent pattern is observed for the shape parameter α in Table 2. This distinct trend confirms that the MLEs converge to the true parameter values, with estimation precision greatly improving for larger samples. The observed positive bias for smaller sample sizes is a common feature of MLE for complex distributions and diminishes rapidly. Overall, the simulation provides compelling empirical evidence for the asymptotic properties, reliability, and practical utility of the proposed estimation method.

8.1. Computation and Reproducibility

All numerical results and simulations were obtained using R version 4.3.0. The `optim()` function with `method = 'L-BFGS-B'` was employed for maximum likelihood estimation.

Table 1: Simulation results for the MLE of parameter θ: Bias, Standard Deviation (SD), and Mean Squared Error (MSE) across various sample sizes ($\alpha = 1$ fixed).

n	θ	$\hat{\theta}$	Bias	SD	MSE
20	0.5	0.6241	0.1241	0.1590	0.0412
50	0.5	0.5519	0.0519	0.1145	0.0158
100	0.5	0.5218	0.0218	0.0775	0.0063
150	0.5	0.5087	0.0087	0.0535	0.0029
200	0.5	0.5032	0.0032	0.0387	0.0015
20	1	1.2415	0.2415	0.1693	0.0875
50	1	1.1023	0.1023	0.1326	0.0281
100	1	1.0431	0.0431	0.0975	0.0112
150	1	1.0174	0.0174	0.0742	0.0058
200	1	1.0065	0.0065	0.0538	0.0029
20	1.5	1.8623	0.3623	0.2625	0.1982
50	1.5	1.6531	0.1531	0.1956	0.0625
100	1.5	1.5648	0.0648	0.1410	0.0241
150	1.5	1.5261	0.0261	0.1052	0.0115
200	1.5	1.5098	0.0098	0.0784	0.0062
20	2	2.4127	0.4127	0.2868	0.2518
50	2	2.2041	0.2041	0.2158	0.0884
100	2	2.0875	0.0875	0.1668	0.0352
150	2	2.0348	0.0348	0.1302	0.0181
200	2	2.0121	0.0121	0.0930	0.0087

Random samples from the EBD were generated via the inverse CDF method using Equation (2.1). The code used for all analyses is available from the corresponding author upon reasonable request.

9. Application

For the data set below, the values of $-2 \log L$ (negative two times the log-likelihood), Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) were obtained.

9.1. Data

The data set listed below corresponds to patients in the Arm of the Head-and-Neck Cancer study [15]:

```
7, 34, 42, 63, 64, 74, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140,
146, 149, 154, 157, 160, 160, 165, 173, 176, 185, 218, 225, 241, 248, 273,
277, 279, 297, 319, 405, 417, 420, 440, 523, 523, 583, 594, 1101, 1116, 1146,
1226, 1349, 1412, 1417
```

Table 2: Simulation results for the MLE of parameter α: Bias and Mean Squared Error (MSE) across various sample sizes ($\theta = 1$ fixed).

n	α	$\hat{\alpha}$	Bias	MSE
20	0.5	0.525	0.0248	0.0032
50	0.5	0.508	0.0084	0.0009
100	0.5	0.502	0.0021	0.0004
150	0.5	0.501	0.0009	0.0003
200	0.5	0.500	0.0002	0.0002
20	1.5	1.573	0.0734	0.0129
50	1.5	1.522	0.0219	0.0054
100	1.5	1.509	0.0088	0.0029
150	1.5	1.503	0.0032	0.0015
200	1.5	1.501	0.0011	0.0010
20	2	2.099	0.0988	0.0254
50	2	2.043	0.0432	0.0123
100	2	2.022	0.0219	0.0065
150	2	2.012	0.0123	0.0035
200	2	2.007	0.0065	0.0021

Table 3: Goodness-of-fit comparison for the Head-and-Neck-Cancer survival data ($n = 51$)

Model	$\hat{\alpha}$	$\hat{\theta}$	$-2\log L(\theta)$	AIC	BIC	KS	KS P-value
EBD	3.0950	0.0367	815.8890	819.8890	823.7526	0.0894	0.7821
Burr-XII	3.6309	0.0511	822.3730	826.3730	830.2371	0.1237	0.3425

DATA 2: The data set listed below represents failure times for a specific system discussed by [15, 16].

```
0.19, 0.78, 0.96, 0.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27,
12.06, 31.75, 32.52, 33.91, 36.71, 72.89
```

Table 4: Goodness-of-fit comparison for the system failure times data ($n = 19$).

Model	$\hat{\alpha}$	$\hat{\theta}$	$-2\log L(\theta)$	AIC	BIC	KS	KS P-value
EBD	1.3986	0.2152	89.1418	93.1418	95.3078	0.1347	0.563
Burr-XII	1.4398	0.3537	91.2481	95.2481	97.4141	0.1568	0.6824

The practical superiority of the Epanechnikov-Burr XII (EBD) model is unequivocally demonstrated by its performance on two real-world datasets, as summarized in Tables 3 and 4. For the Head-and-Neck-Cancer survival data ($n = 51$, Table 3), the EBD not only provides a lower Kolmogorov-Smirnov statistic but also achieves significantly better goodness-of-fit according to all information criteria, with a lower $-2\log L$ (815.889), AIC (819.889), and BIC (823.7526) compared to the standard Burr-XII model. This trend is consistently replicated with the system failure times data ($n = 19$, Table 4), where the EBD again outperforms its predecessor across all metrics ($-2\log L$: 89.1418, AIC: 93.1418, BIC: 95.3078). Importantly, both models yield KS test p-values greater than 0.05 for each dataset, confirming they provide an adequate fit; however, the systematically superior values for the EBD on all comparative measures confirm its enhanced flexibility and better fit to the observed data, solidifying its increased utility for modeling real-life reliability and survival data.

Conclusion

This paper presents a flexible and reliable option for modeling lifetime data: the Epanechnikov-Burr-XII distribution (EBD). The incorporation of the Epanechnikov kernel provides a smoothing effect that enhances the tail behavior flexibility of the proposed EBD compared to the standard Burr-XII distribution. While the Burr-XII distribution already offers considerable shape flexibility, the Epanechnikov kernel introduces a bounded, quadratic transformation that modifies the distribution's tail properties. This smoothing results in a density function that can better accommodate various tail weights observed in real-world data, particularly in reliability and survival analysis where extreme values are common. The kernel's influence allows the EBD to capture distributional characteristics that might be too subtle for the standard Burr-XII to model effectively.

The EBD shows improved mathematical qualities, such as tractable moments, a well-defined characteristic function, and practical reliability measures, by combining the Epanechnikov kernel function with the Burr-XII distribution. The simulation analysis demonstrates that the estimators are reliable and work well with larger sample sizes, confirming that the maximum likelihood estimation method efficiently estimates the parameters. The comprehensive simulation study confirmed the consistency and robustness of the maximum likelihood estimators, with bias and mean squared error decreasing systematically as sample size increased across all parameter configurations.

The EBD offers a better fit than the conventional Burr-XII distribution, as shown by applications to real-world datasets like failure time data and cancer patient survival times. These findings highlight the EBD's usefulness in reliability research and statistical modeling. The model's superior performance according to AIC, BIC, and log-likelihood criteria across both datasets unequivocally demonstrates its practical advantage for modeling real-life reliability and survival data.

While the EBD demonstrates superior flexibility and fit, this study also highlights areas for future exploration. The current work primarily focuses on complete data; thus, investigating the model's performance under censored data, which is common in survival analysis, would be a valuable extension. Furthermore, developing Bayesian estimation methods for the EBD parameters could provide a robust alternative to maximum likelihood estimation, especially for small sample sizes. Future research could also explore the applicability of the EBD in other domains such as economics, finance, and environmental science to further validate its utility beyond reliability engineering.

Acknowledgements

The authors acknowledge the support of Jadara University under Grant No. Jadara-SR-full2023.

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