EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6686
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Hierarchy Sets in Almost Distributive Lattices and Their Structural Properties

G. Chinnayya[1], Ramesh Sirisetti[2], G. Jogarao[3], Ravikumar Bandaru[4], Aiyared Iampan[5,*]

[1] *Department of Mathematics, GITAM School of Science, GITAM (Deemed to be a University), Visakhapatnam 530045, India*
[2] *Department of Mathematics, Aditya University, Surampalem, Kakinada, Andhra Pradesh-533437, India*
[3] *Department of BS & H, Aditya Institute of Technology and Management, Tekkali 532001, Srikakulam, India*
[4] *Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati 522237, Andhra Pradesh, India*
[5] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

Abstract. This paper studies hierarchy sets in almost distributive lattices, focusing on two key types: prime and maximal hierarchy sets. We show that every maximal hierarchy set is prime, but not vice versa, and use Zorn's Lemma to prove the existence of prime hierarchy sets extending a given one. We also introduce inverted-hierarchy sets, defined via join operations, and analyze their relation to filters. The results provide structural insights and extend ideal and filter theory within almost distributive lattices.

2020 Mathematics Subject Classifications: 06D99, 06D75

Key Words and Phrases: almost distributive lattices, ideals, filters, hierarchy sets, prime hierarchy sets, maximal hierarchy sets, inverted-hierarchy sets

---

1. Introduction

In lattice theory, the study of special subsets such as ideals, filters, and related constructions [1–6] plays a crucial role in understanding structural properties of the lattice. In particular, hierarchy sets [7], defined by their closure under the meet operation with a fixed generating set, offer a useful framework for analyzing elements in almost distributive

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6686

Email addresses: cgondu@gitam.in (G. Chinnayya),
ramesh.sirisetti@gmail.com (S. Ramesh), jogarao.gunda@gmail.com (G. Jogarao),
ravimaths83@gmail.com (R. Bandaru), aiyared.ia@up.ac.th (A. Iampan)

lattices (ADLs) [8]. This framework builds upon foundational work by Birkhoff [9], and relates to established studies in the structure of complemented and quasi-complemented lattices [10, 11], relatively complemented distributive lattices [12, 13], and the general theory of ideal-like constructs. Important extensions to the ADL setting have also been explored through weak relative complements, normal filters, and weakly complemented structures [14–17], offering alternative perspectives on distributivity and absorption in non-classical environments. Recently, Ramesh et al. introduced and developed the notion of hierarchy elements in ADLs [18], establishing foundational properties such as closure under meet, behavior with maximal elements, and conditions under which hierarchy elements generate ideals or sublattices. A natural extension of that work led to the formalization of hierarchy sets [7], shown to form a distributive lattice under union and a meet-based operation. In parallel, Khabyah's work on star filters and starlets enriched the lattice-theoretic landscape by introducing new substructures under relaxed closure properties. Additionally, Rafi et al. contributed to this growing body of work by introducing prime E-ideals [19], which refine the concept of absorption and primeness within lattice ideals, and by developing w-filters [20], a generalization of filters based on weakened closure under joins. Collectively, these developments underscore the need to investigate more nuanced classes of subsets in ADLs that extend classical notions like prime ideals and filters. Within this context, two important classes of hierarchy sets are introduced in this paper: prime hierarchy sets and maximal hierarchy sets. Prime hierarchy sets generalize the notion of prime ideals [21] by requiring that the meet of two elements lies in the set only if at least one of the elements does. Maximal hierarchy sets, on the other hand, represent the largest proper hierarchy sets under inclusion. We establish several structural properties, show that every maximal hierarchy set is necessarily prime, and use Zorn's Lemma to demonstrate the existence of prime hierarchy sets extending a given one while avoiding a particular subset. We provide examples to clarify the distinctions between these types of sets and highlight the fact that the converse of some implications does not hold in general. Also, we introduce inverted-hierarchy sets H^S, defined for a non-empty subset S of an almost distributive lattice L with maximal elements. Each H^S consists of elements in L that are idempotent under join with some $u \in S$. These sets exhibit notable algebraic properties, including closure under join and, in some cases, meet, forming filters or related substructures. We explore their characterizations, relationships to filters, and criteria under which H^S aligns with or differs from the filter generated by S.

2. Preliminaries

The study of lattice-theoretic structures has played a pivotal role in abstract algebra and its applications to computer science, logic, and information systems. In this aspect, the concept of an almost distributive lattice [8] was introduced by Swamy and Rao in 1981 as a common abstraction of both lattice-theoretic and ring-theoretic generalizations of a Boolean algebra (ring). It is an algebraic structure which satisfies all axioms of a distributive lattice $(L, \vee, \wedge, 0)$ with the zero element 0 except the commutativity of the binary operations $\vee, \wedge$, the right distributivity of $\vee$ over $\wedge$, and the associativity of $\vee$.

Given an almost distributive lattice L and a non-empty subset $S \subseteq L$, an element $h \in L$ is said to be a hierarchy with respect to S [7] if it satisfies a specific absorption condition with at least one element of S. The set of all such elements, denoted by H_S, exhibits several interesting structural properties. In particular, this set is non-empty, contains S, and is closed under the meet operation. In this context, the notion of hierarchy elements and their associated hierarchy sets provides a useful framework for examining the internal organization of elements within an almost distributive lattice.

The study of hierarchy sets not only enhances our understanding of ideal-theoretic constructions in almost distributive lattices but also bridges concepts related to sub-almost distributive lattices, closure operators, and distributivity. Several results characterize the algebraic and order-theoretic behavior of hierarchy sets, including their stability under inclusion, intersection, and union. Additionally, under certain conditions, H_S forms an ideal and even a sub-almost distributive lattice.

Definition 1. *[7] Given a non-empty subset S of L, an element $h \in L$ is said to be* hierarchy with respect to S *if $s \wedge h = h$, for some $s \in S$. It is observed that the set H_S of hierarchy elements with respect to S is non-empty, containing S, and it is closed under $\wedge$.*

Lemma 1. *[7] For any non-empty subsets S_1, S_2 of L,*

(i) $S_1 \subseteq S_2$ *implies* $H_{S_1} \subseteq H_{S_2}$

(ii) $H_{S_1 \cup S_2} = H_{S_1} \cup H_{S_2}$

(iii) $H_{S_1 \cap S_2} \subseteq H_{S_1} \cap H_{S_2}$.

Lemma 2. *[7] For any non-empty subsets S of L,*

(i) If $m \in H_S$, then $H_S = L$, where m is a maximal element in L

(ii) $a \leq b$ implies $H_a \subseteq H_b$, where $H_a = \{h \in L \mid a \wedge h = h\}$

(iii) $a \leq b$ and $b \in H_S$ imply $a \in H_S$

(iv) $h \in H_S$ implies $(h] \subseteq H_S$, where $(h] = \{h \wedge a \mid a \in L\}$

(v) H_a is an ideal of L.

Theorem 1. *[7] For any non-empty subset S of L, the following are equivalent;*

(i) H_S is closed under $\vee$

(ii) H_S is a sub-almost distributive lattice of L

(iii) H_S is an ideal of L

(iv) H_S is the smallest ideal generated by S ($H_S = (S]$).

Theorem 2. *[7] The set $\mathcal{H}_S$ of hierarchy sets in L forms a distributive lattice with respect to the operations; $H_{S_1} \cup H_{S_2} = H_{S_1 \cup S_2}$ and $H_{S_1} \bigwedge H_{S_2} = \{s_1 \wedge s_2 \mid s_1 \in S_1, s_2 \in S_2\}$.*

3. Prime Hierarchy Sets in Almost Distributive Lattices

In this section, we investigate structural properties of hierarchy sets in an abstract distributive lattice. A hierarchy set H_S generated by a non-empty subset S of an almost distributive lattice L plays a central role in the algebraic and order-theoretic analysis of L. We begin by establishing foundational closure properties of hierarchy sets under meet operations. Building on this, we introduce and characterize two important classes of hierarchy sets: prime and maximal hierarchy sets. A prime hierarchy set satisfies the absorption condition that whenever a meet $a \wedge b$ belongs to the set, then at least one of the elements a or b must also belong to it. A maximal hierarchy set, on the other hand, is one that is not properly contained in any other proper hierarchy set. Through illustrative examples, we demonstrate that while every maximal hierarchy set is prime, the converse need not hold.

We further employ Zorn's Lemma to show the existence of prime hierarchy sets extending a given one, while avoiding a specific closed set under the meet operation. The section concludes by demonstrating that every hierarchy set can be represented as the intersection of all prime hierarchy sets that contain it.

Lemma 3. *If S is a non-empty subset and a is an element in L, then*

(i) $h \wedge a \in H_S$, *for all* $h \in H_S$

(ii) $a \wedge h \in H_S$, *for all* $h \in H_S$.

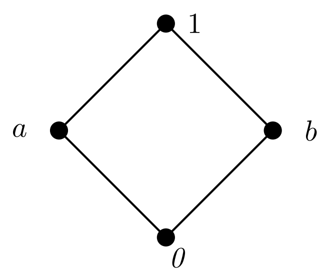
Proof. Let $h \in H_S$. Then there exists $s \in S$ such that $s \wedge h = h$. Given $a \in L$.
(i) $s \wedge (h \wedge a) = (s \wedge h) \wedge a = h \wedge a$. Therefore, $h \wedge a \in H_S$.
(ii) $s \wedge (a \wedge h) = (s \wedge a) \wedge h = (a \wedge s) \wedge h = a \wedge (s \wedge h) = a \wedge h$. Therefore, $a \wedge h \in H_S$.

A hierarchy set H_S in L is said to be proper if $H_S \neq L$.

Definition 2. *A proper hierarchy set H_S of L is said to be* prime*, if given a, $b \in L$, $a \wedge b \in H_S$ implies $a \in H_S$ or $b \in H_S$.*

Remark 1. *Every hierarchy set does not need to be prime. For example, see the following example:*

Example 1. *Let $L = \{0, a, b, 1\}$ be an almost distributive lattice whose Hasse diagram is given below:*



For $S = \{0\}$, $H_S = \{0\}$ is not prime.

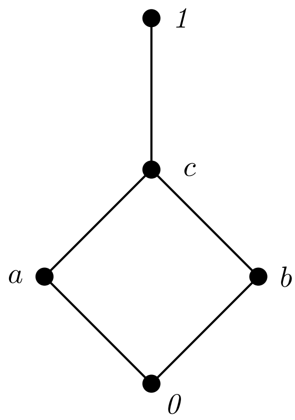
Definition 3. *A proper hierarchy set H_S of L is said to be* maximal*, if, given a proper hierarchy set H_T in L, $H_S \subseteq H_T$ implies $H_S = H_T$.*

Theorem 3. *Every maximal hierarchy set is prime.*

Proof. Let H_S be a maximal hierarchy set in L and $a, b \in L$ such that $a \wedge b \in H_S$. If $a \notin H_S$, then $H_S \subseteq H_S \cup H_a = H_{S\cup\{a\}} = L$ (since H_S is maximal). Now, for this $b \in L = H_S \cup H_a$. we have $b \in H_S$, or $b \in H_{\{a\}}$, or $b \in H_S \cap H_{\{a\}}$. If $b \in H_S$, or $b \in H_S \cap H_{\{a\}}$, then nothing else to do. If $b \in H_{\{a\}}$, then $b = a \wedge b \in H_S$. Therefore, H_S is prime.

Remark 2. *The converse of Theorem* (3) *need not be true. That is, every prime hierarchy set need not be maximal. For example, see the following example:*

Example 2. *Let $L = \{0, a, b, c, 1\}$ be an almost distributive lattice whose Hasse diagram is given below:*



Let $S_1 = \{a, b\}$ and $S_2 = \{b, c\}$. Then $H_{S_1} = \{0, a, b\}$ and $H_{S_2} = \{0, a, b, c\}$. Therefore, H_{S_1} and H_{S_2} are two prime hierarchy sets and $H_{S_1} \subsetneqq H_{S_2}$. Hence, H_{S_1} is prime but not maximal.

Theorem 4. *Let H_S be a hierarchy set and K is a non-empty subset of L which is closed under $\wedge$ such that $H_S \cap K = \emptyset$. Then there exists a prime hierarchy set H_P of L such that $H_S \subseteq H_P$ and $H_P \cap K = \emptyset$.*

Proof. Let H_S be a hierarchy set and K be a non-empty subset of L and closed under $\wedge$ such that $H_S \cap K = \emptyset$. Consider $\mathcal{Q} = \{H_T \mid H_S \subseteq H_T \text{ and } H_T \cap K = \emptyset\}$. Then $\mathcal{Q} \neq \emptyset$ (since $H_S \cap K = \emptyset$) and $(\mathcal{Q}, \subseteq)$ is a partially ordered set with respect to the inclusion order. Let $\{0\} = H_{S_1} \subseteq H_{S_2} \subseteq H_{S_3} \subseteq \ldots$ be an increasing chain in $\mathcal{Q}$. Then $\bigcup\limits_{i\in I} H_{S_i} = H_{S_1} \cup H_{S_2} \ldots = H_{(\bigcup\limits_{i\in I} S_i)}$, $H_{(\bigcup\limits_{i\in I} S_i)} \cap K = \bigcup\limits_{i\in I}(H_{S_i} \cap K) = \bigcup\limits_{i\in I} \emptyset = \emptyset$, and

$H_S \subseteq H_{S_i}$, for all $i \in I$. Therefore, $H_{(\bigcup_{i\in I} S_i)}$ is an upper bound of the chain in $\mathcal{Q}$. By Zorn's lemma, $\mathcal{Q}$ has maximal element, say H_P. Let $a, b \in L$ such that $a \notin H_P$ and $b \notin H_P$. Then $a \notin P$ and $b \notin P$. Now, $H_P \cup H_a = H_{P\cup\{a\}}$ and $H_P \cup H_b = H_{P\cup\{b\}}$. Since H_P is maximal, $H_{P\cup\{a\}} \cap K \neq \emptyset$ and $H_{P\cap\{b\}} \cap K \neq \emptyset$. Let $x \in H_{P\cup\{a\}} \cap K$ and $y \in H_{P\cup\{b\}} \cap K$. By Lemma (3), $x \wedge y \in H_{P\cup\{a\}} \cap H_{P\cup\{b\}} \cap K$. Then $x \wedge y \in (H_P \cup H_a) \cap (H_P \cup H_b) \cap K = [H_P \cup (H_a \wedge H_b)] \cap K = (H_P \cup H_{\ a\wedge b}) \cap K = H_{P\cup\{a\wedge b\}} \cap K$. If $a \wedge b \in H_P$, then $H_{P\cup\{a\wedge b\}} = H_P$. So that $x \wedge y \in H_P \cap K$ and hence $H_P \cap K \neq \emptyset$. Which is a contradiction. Therefore, $a \wedge b \notin H_P$ and hence H_P is a prime hierarchy set of L.

Corollary 1. *Let H_S be a hierarchy set in L and $a \in L$ such that $a \notin H_S$. Then there exists a prime hierarchy set H_P of L such that $H_S \subseteq H_P$ and $a \notin H_P$.*

Proof. Let H_S be a hierarchy set in L and $a \in L$ such that $a \notin H_S$. If $K = \{a\}$, then $H_S \cap K = \emptyset$ and K is closed under $\wedge$. By Theorem 4, there exists a prime hierarchy set H_P in L such that $H_S \subseteq H_P$, and $H_P \cap K = \emptyset$. Hence, $a \notin H_P$ and $H_S \subseteq H_P$. Since $P \subseteq H_P$, $a \notin P$.

Theorem 5. *If H_S is a hierarchy set in L, then H_S is the intersection of all prime hierarchy sets containing H_S in L.*

Proof. Let S be a non-empty subset of L and $a \in L$ such that $a \notin H_S$. Consider a set $\mathcal{Q} = \{H_T \mid a \notin H_T \text{ and } H_S \subseteq H_T\}$. Then $\mathcal{Q} \neq \emptyset$ (since $a \notin H_S$) and it is a poset with the inclusion order. Let $H_{S_1} \subseteq H_{S_2} \subseteq \dots$ be an increasing chain in $\mathcal{Q}$. Then $\bigcup_{i\in I} H_{S_i} = H_{(\bigcup_{i\in I} S_i)}$. If $a \in H_{(\bigcup_{i\in I} S_i)}$, then $a \in \bigcup_{i\in I} H_{S_i}$. Therefore, $a \in H_{S_i}$, for some $i \in I$. Which is a contradiction to $a \notin H_{S_i}$. So that $a \notin H_{(\bigcup_{i\in I} S_i)}$. Since $H_S \subseteq H_{S_i}$, for all $i \in I$, $H_S \subseteq H_{(\bigcup_{i\in I} S_i)}$. Therefore, $H_{(\bigcup_{i\in I} S_i)} \in \mathcal{Q}$ and it is an upper bound for the chain. By Zorn's lemma, $\mathcal{Q}$ has a maximal element, say H_P. That is $a \notin H_P$ and $H_S \subseteq H_P$. Let $a, b \in L$ such that $a \notin H_P$ and $b \notin H_P$. Then $H_P \cup H_a = H_{P\cup\{a\}}$ and $H_P \cup H_b = H_{P\cup\{b\}}$. Since H_P is maximal in $\mathcal{Q}$, $a \in H_P \cup H_a$ and $a \in H_P \cup H_b$, and then $a \in (H_P \cup H_a) \cap (H_P \cup H_b) = H_P \cup (H_a \cap H_b) = H_P \cup H_{a\wedge b}$. If $a \wedge b \in H_P$, then $\{a \wedge b\} \subseteq H_P$, and $H_{\{a\wedge b\}} \subseteq H_{H_P} = H_P$. Therefore, $a \in H_P$. Which is a contradiction. Hence, H_P is a prime hierarchy set in L. Thus, $H_S = \bigcap\{H_P \mid H_P$ is a hierarchy set containing H_S in L.

A non-empty subset F of L is said to be a filter [8] if it is closed under $\wedge$ and given $a \in L, b \in F, a \vee b \in F$. A proper filter F of L is called prime [8] if given $a, b \in L$, $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Theorem 6. *If H_P is a prime hierarchy set of L, where P is a non-empty subset of L, then $L \setminus H_P$ is a filter of L.*

Proof. Let H_P be a prime hierarchy set, where P is a non-empty subset of L. Since H_P is proper, we can choose $a, b \in L \setminus H_P$. If $a \wedge b \in H_P$, then $a \in H_P$ or $b \in H_P$ (since

H_P is prime). Which is not possible. Therefore, $a \wedge b \in L \setminus H_P$ and hence $L \setminus H_P$ is closed under $\wedge$. Let $c \in L$ and $a \in L \setminus H_P$. If $c \vee a \in H_P$, then $p \wedge (c \vee a) = c \vee a$, for some $p \in H_P$. So that $p \wedge (c \vee a) \wedge a = (c \vee a) \wedge a$ and then $p \wedge a = a$. It means that $a \in H_P$. Which is not true. Therefore, $c \vee a \in L \setminus H_P$ and hence $L \setminus H_P$ is a filter.

Remark 3. *If H_P is a prime hierarchy set in L, then $L \setminus H_P$ need not be prime. For example, see the following example:*

Example 3. *Let $L = \{0, a, b, c, m_1, m_2\}$ be an almost distributive lattice with maximal elements m_1, m_2, where the operations $\wedge$ and $\vee$ are defined below:*

$\wedge$	0	a	b	c	m_1	m_2
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
m_1	0	a	b	c	m_1	m_2
m_2	0	a	b	c	m_1	m_2

$\vee$	0	a	b	c	m_1	m_2
0	0	a	b	c	m_1	m_2
a	a	a	c	c	m_1	m_2
b	b	c	b	c	m_1	m_2
c	c	c	c	c	m_1	m_2
m_1	m_1	m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2	m_2	m_2

Take $P = \{a, b\}$. Then $H_P = \{0, a, b\}$ is a prime hierarchy set in L and $L \setminus H_P = \{c, m_1, m_2\}$ is a filter of L, but not prime.

4. Inverted-Hierarchy Sets in an Almost Distributive Lattice with Maximal Elements

This section introduces and develops the theory of inverted-hierarchy sets, denoted by H^S, where S is a non-empty subset of an almost distributive lattice L with maximal elements. These sets consist of elements in L that are idempotent over at least one element of S under the join operation. Such sets exhibit rich algebraic properties, including closure under join and, under suitable conditions, closure under meet, forming filters and substructures of the lattice. We explore various characterizations and properties of H^S, its relationship to filters, and conditions under which it coincides with or differs from the filter generated by S.

Definition 4. *An element h in L with maximal elements is said to be inverted-hierarchy with respect to a non-empty set S in L, if $h \vee s = h$, for some $s \in S$. Let us denote H^S as the set of inverted-hierarchy elements with respect to a non-empty set S in L. Then it is easy to observe that $H^S \neq \emptyset$ (because $m \vee s = m$, for all $s \in S$, where m is a maximal element in L) and $S \subseteq H^S$.*

Theorem 7. *For any non-empty subset S of L, we have*

(i) S is closed under $\vee$

(ii) For any $h \in H^S$, $[h) \subseteq H^S$

(iii) For any $h \in H^S$ *and* $a \in L$, $a \vee h$, $h \vee a \in H^S$

(iv) If $a \in L$ *and* $s \in S$ *such that* $s \leq a$, *then* $a \in H^S$.

Proof. (i) Let $h_1, h_2 \in H^S$. Then $h_1 \vee s_1 = h_1$ and $h_2 \vee s_2 = h_2$, for some $s_1, s_2 \in S$. Now, $(h_1 \vee h_2) \vee s_2 = h_1 \vee (h_2 \vee s_2) = h_1 \vee h_2$. Then $h_1 \vee h_2 \in H^S$. Therefore, H^S is closed under $\vee$.

(ii) Let $h \in H^S$. Then $h \vee s = h$, for some $s \in S$. Let $a \in [h)$. Then $a = b \vee h$, for some $b \in L$. Now, $a \vee s = (b \vee h) \vee s = b \vee (h \vee s)$. Then $a \in H^S$. Therefore, $[h) \subseteq H^S$.

(iii) Let $h \in H^S$. Then $h \vee s = s$, for some $s \in S$. Given $a \in L$, $(a \vee h) \vee s = a \vee (h \vee s) = a \vee h$. Therefore, $a \vee h \in H^S$. Similarly, $(h \vee a) \wedge s = (a \vee h) \wedge s = (a \wedge s) \vee (h \wedge s) = (a \wedge s) \vee s = s$. Therefore, $(h \vee a) \vee s = h \vee a$ and hence $h \vee a \in H^S$.

(iv) Let $a \in L$ and $s \in S$ such that $s \leq a$. Then $a \vee s = a$. Therefore, $a \in H^S$.

Lemma 4. *If* S_1, S_2 *are any two non-empty subsets of* L, *then*

(i) $S_1 \subseteq S_2$ *implies* $H^{S_1} \subseteq H^{S_2}$

(ii) $H^{S_1} \cup H^{S_2} = H^{S_1 \cup S_2}$

(iii) $H^{S_1 \cap S_2} \subseteq H^{S_1} \cap H^{S_2}$.

Proof. Let S_1, S_2 be two non-empty subsets in L.

(i) Let $h \in H^{S_1}$. Then $h \vee s_1 = h$, for some $s_1 \in S_1 \subseteq S_2$. Therefore, $h \in H^{S_2}$ and hence $H^{S_1} \subseteq H^{S_2}$.

(ii) By *(i)*, we have $H^{S_1}, H^{S_2} \subseteq H^{S_1 \cup S_2}$. Therefore, $H^{S_1} \cup H^{S_2} \subseteq H^{S_1 \cup S_2}$. Let $h \in H^{S_1 \cup S_2}$. Then $h \vee s = h$ for some $s \in S_1 \cup S_2 \subseteq H^{S_1} \cup H^{S_2}$. By Theorem 7 (iii), $h = h \vee s H^{S_1} \cup H^{S_2}$ and hence $H^{S_1 \cup S_2} \subseteq H^{S_1} \cup H^{S_2}$. Thus, $H^{S_1} \cup H^{S_2} = H^{S_1 \cup S_2}$.

(iii) We have $S_1 \cap S_2 \subseteq S_1, S_2$. Then $H^{S_1 \cap S_2}, \subseteq H^{S_1}, H^{S_2}$ (by (ii)). Therefore, $H^{S_1 \cap S_2} \subseteq H^{S_1} \cap H^{S_2}$.

Remark 4. $H^{S_1 \cap S_2}$ *need not be equal to* $H^{S_1} \cap H^{S_2}$. *For, in Example* (1); *Let* $S_1 = \{a, b\}$ *and* $S_2 = \{0, a\}$. *Then* $S_1 \cap S_2 = \{a\}$, $H^{S_1} = \{a, b, 1\}$, $H^{S_2} = \{0, a, b, 1\}$, $H^{S_1 \cap S_2} = \{a, 1\}$ *and* $H^{S_1} \cap H^{S_2} = \{a, b, 1\}$. *Therefore,* $H^{S_1} \cap H^{S_2} \neq H^{S_1 \cap S_2}$.

Lemma 5. *If* $s_1 \in S_1$, $s_2 \in S_2$ *and* $s_1 \wedge s_2 \in S_1 \cap S_2$, *then* $H^{S_1 \cap S_2} = H^{S_1} \cap H^{S_2}$.

Proof. Let $h \in H^{S_1} \cap H^{S_2}$. Then $h \vee s_1 = h$ and $h \vee s_2 = h$, for some $s_1 \in S_1$ and $s_2 \in S_2$. Since $s_1 \wedge s_2 \in S_1 \cap S_2$, $h \vee (s_1 \wedge s_2) = (h \vee s_1) \wedge (h \vee s_2) = h \wedge h = h$. Therefore, $h \in H^{S_1 \cap S_2}$. Hence, $H^{S_1} \cap H^{S_2} \subseteq H^{S_1 \cap S_2}$. Thus, $H^{S_1} \cap H^{S_2} = H^{S_1 \cap S_2}$ (by Lemma 7 (iv)).

Theorem 8. *Let* S *be a non-empty subset of* L *and* S *be closed under* $\wedge$. *Then*

(i) H^S *is closed under* $\wedge$

(ii) H^S *is a sub-almost distributive lattice of* L

(iii) H^S *is a filter of* L

(iv) H^S *is the smallest filter containing* S.

Proof. Let S be a non-empty subset of L and S is closed under $\wedge$.

(i) Let $h_1, h_2 \in H^S$. Then $h_1 \vee s_1 = h_1$ and $h_2 \vee s_2 = h_2$, for some $s_1, s_2 \in S$. Now, $h_1 \wedge h_2 \wedge (s_1 \wedge s_2) = h_1 \wedge s_1 \wedge h_2 \wedge s_2 = s_1 \wedge s_2$. Then $(h_1 \wedge h_2) \vee (s_1 \wedge s_2) = h_1 \wedge h_2$. Since S is closed under $\wedge$, $h_1 \wedge h_2 \in H^S$. Therefore, H^S is closed under $\wedge$.

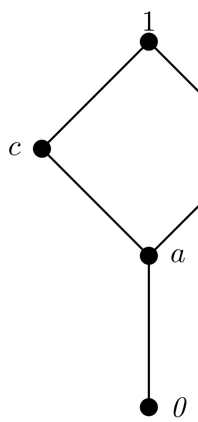
(ii) By (i) and Theorem 7 (iii), H^S is a sub-almost distributive lattice of L.

(iii) By (i) and Theorem 7 (iii), H^S is a filter of L.

(iv) Let F be a filter of L such that $S \subseteq F$. By Lemma 4 (i), $H^S \subseteq H^F$ and $F \subseteq H^F$. Let $h \in H^F$. Then $h \vee s = h$, for some $s \in F$. For $s \in F$, $h \vee s = h \in F$ (since F is a filter). Therefore, $h \in F$. So that $H^F \subseteq F$. Hence, $H^S \subseteq F = H^F$. Thus, H^S is the smallest filter containing S.

Remark 5. H^S *need not be closed under* $\wedge$. *See the following example:*

Example 4. *Let* $L = \{0, a, b, c, 1\}$ *be an almost distributive lattice with maximal element* 1*, whose Hasse diagram is given below:*



Let $S_1 = \{b, c\}$. *Then* $H^S = \{b, c, 1\}$. *Let* $b, c \in H^S$. *Then* $b \wedge c = a \notin H^S$. *Therefore,* H^S *is not closed under* $\wedge$.

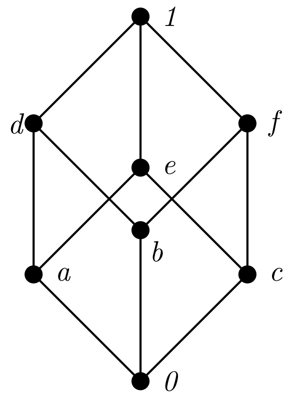
Given a non-empty set S of L, it is known that $[S) = \{a \vee (\bigwedge_{i=1}^{n} s_i) \mid a \in L \text{ and } s_i \in S\}$ is the smallest filter containing S.

Lemma 6. *For any non-empty subset* S *of* L, $H^S \subseteq [S)$.

Proof. Let $h \in H^S$. Then $h \vee s = h$ and $h \wedge s = s$, for some $s \in S$. Since $[S)$ is a filter generated by S, $h \vee s = h \in [S)$. Therefore, $H^S \subseteq [S)$.

Remark 6. H^S *need not be equal to* $[S)$. *See the following example:*

Example 5. *Let $L = \{0, a, b, c, d, e, f, 1\}$ be an almost distributive lattice whose Hasse diagram is given below:*



Let $S = \{e, f\}$. Then $[S) = \{c, e, f, 1\}$ is a filter of L and $H^S = \{e, f, 1\}$. Therefore, $H^S \neq [S)$.

Lemma 7. *Let S be a non-empty subset of L and $H^S = H^F$, for some filter F of L. Then $[S) = F$.*

Proof. Suppose $H^S = H^F$, for some filter F of L. Since $F \subseteq H^F = H^S$ and F is closed under $\wedge$, $F \subseteq H^F = H^S \subseteq [S)$ and H^F is a filter of L. Therefore, $F = H^S \subseteq [S)$. Now, $S \subseteq H^S = H^F = F$. Then $[S) \subseteq H^F = F$. Hence, $F = [S)$.

Remark 7. *The converse of Lemma 7 need not be true. For, in Example (5); let $L = \{0, a, b, c, d, e, f, 1\}$ be an almost distributive lattice whose Hasse-diagram is given in Example 5. Let $S = \{d, e\}$. Then $[S) = F = \{a, d, e, 1\}$ is a filter of L and $H^S = \{d, e, 1\}$, but $H^S \neq H^F = F$.*

Theorem 9. *For any non-empty subsets S_1, S_2 of L, we have*

(i) $H^{S_1} \cup H^{S_1} = H^{S_1}$ *(Idempotent Law)*

(ii) $H^{S_1} \cup H^{S_2} = H^{S_1 \cup S_2} = H^{S_2} \cup H^{S_1} = H^{S_2 \cup S_1}$ *(Commutative Law)*

(iii) $(H^{S_1} \cup H^{S_2}) \cup H^{S_3} = H^{(S_1 \cup S_2) \cup S_3} = H^{S_1} \cup (H^{S_2} \cup H^{S_3})$ *(Associative Law).*

Proof. (i) It is easy to observe that $H^{S_1} \cup H^{S_1} = H^{S_1}$, by Lemma 4 (i).

(ii) By Lemma 4 (ii), we can prove $H^{S_1} \cup H^{S_2} = H^{S_1 \cup S_2} = H^{S_2 \cup S_1} = H^{S_2} \cup H^{S_1}$.

(iii) Since the set union satisfies associative law and by (ii), we have $(H^{S_1} \cup H^{S_2}) \cup H^{S_3} = H^{(S_1 \cup S_2) \cup S_3} = H^{S_1} \cup (H^{S_2} \cup H^{S_3})$.

Theorem 10. *For any non-empty subset S of L and S_M is the set of maximal elements in L, we have* *(i)* $\bigcup_{S\subseteq L} H^S = L$

(ii) $S_M \subseteq H^S$

(iii) $H^{S_M} = S_M$

(iv) $\bigcap_{S\subseteq L} H^S = S_M$

(v) $H^{H^S} = H^S$.

Proof. (i) Since $L \subseteq H^L$, $H^L = L$ and $\bigcup_{S\subseteq L} H^S = L$.

(ii) Let $m \in S_M$. Then $m \vee s = m$, for all $s \in S$. Therefore, $m \in H^S$ and $S_M \subseteq H^S$.

(iii) Since S_M is a filter L, $H^{S_M} = S_M$.

(iv) From (ii), $S_M \subseteq H^S$, for all $S \subseteq L$, so that $S_M \subseteq \bigcap_{S\subseteq L} H^S$. Let $h \in \bigcap_{S\subseteq L} H^S$. Then $h \vee s = h$, for some $s \in S$ and for all $S \subseteq L$. Let $m \in S_M$. Since S_M is a filter of L, $h = h \vee m \in S_M$. Therefore, $\bigcap_{S\subseteq L} H^S \subseteq S_M$ and hence $\bigcap_{S\subseteq L} H^S = S_M$.

(v) Since $S \subseteq H^S$, $H^S \subseteq H^{H^S}$. Let $h \in H^{H^S}$. Then $h \vee t = h$, for some $t \in H^S$. For this $t \in H^S$, $t \vee s = t$, for some $s \in S$. Now, $h \wedge s = h \wedge (t \wedge s) = (h \wedge t) \wedge s = t \wedge s = s$. Then $h \vee s = h$, for some $s \in S$. Therefore, $h \in H^S$ and $H^{H^S} \subseteq H^S$. Thus, $H^{H^S} = H^S$.

5. Conclusion

These findings not only clarify the distinctions and relationships between prime, maximal, and inverted-hierarchy sets but also enrich the broader theory of lattices by highlighting how such structures interact with classical notions, such as ideals and filters, in almost distributive lattices.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2026, Grant No. 2252/2568).

References

[1] N. H. McCoy and D. Mantgomery. A representation of generalized boolean rings. *Duke. Math. J.*, 3:455–459, 1937.

[2] A. A. Meuborn. Regular rings and Baer rings. *Math. Z.*, 121:211–219, 1971.

[3] M. H. Stone. The theory of representation for Boolean algebras. *Trans. Am. Math. Soc.*, 40:37–111, 1936.

[4] M. H. Stone. Topological representation of distributive lattices and Brouwerian logics. *Čas. Mat. Fys.*, 1:1–25, 1937.

[5] I. Sussman. A generalization of Boolean rings. *Math. Ann.*, 136:326–338, 1981.

[6] J. Von Neuman. On regular rings. *Proc. Nat. Acad. Sci., U.S.A.*, 22:707–713, 1963.

[7] S. Ramesh, G. Chinnayya, G. Jogarao, and A. Iampan. Hierarchy sets in almost distributive lattice. *Eur. J. Pure Appl. Math.*, 17(3):1691–1704, 2024.

[8] U. M. Swamy and G. C. Rao. Almost distributive lattices. *J. Aust. Math. Soc., Ser. A*, 31:77–91, 1981.

[9] G. Birkhoff. *Lattice theory.* Amer. Math. Soc. Colloq. XXV, Providence, U.S.A., 1967.

[10] W. H. Cornish. Quasi complemented lattices. *Comment. Math. Univ. Carolin.*, 15:501–511, 1974.

[11] W. H. Cornish. Normal lattices. *J. Aust. Math. Soc.*, 14:200–215, 1972.

[12] R.-S. Dong. Relatively complemented distributive lattice. *Sci. China, Ser. A*, 34(12):1427–1437, 1991.

[13] G. Grätzer and E. T. Schmidt. Characterizations of relatively complemented distributive lattices. *Publ. Math. Debr.*, 5(3-4):257–287, 1958.

[14] S. Ramesh and G. Jogarao. Characterizations of weak relatively complemented almost distributive lattices. *Int. J. Open Probl. Compt. Math.*, 10(1):11–23, 2017.

[15] S. Ramesh and G. Jogarao. Normal filters in almost distributive lattices. *J. Int. Math. Virtual Inst.*, 7:37–51, 2017.

[16] S. Ramesh and G. Jogarao. Weak relative complements in almost distributive lattices. *Discuss. Math., Gen. Algebra Appl.*, 38:5–18, 2018.

[17] S. Ramesh and G. Jogarao. Weak relatively complemented almost distributive lattices. *Palest. J. Math.*, 6(2):448–457, 2017.

[18] S. Ramesh, G. Chinnayya, G. Jogarao, R. Bandaru, and A. Iampan. Hierarchy elements in an almost distributive lattice. *Eur. J. Pure Appl. Math.*, 17(3):1691–1704, 2024.

[19] N. Rafi, Y. Monikarchana, R. Bandaru, and A. Iampan. On prime E-ideals of almost distributive lattices. *Int. J. Anal. Appl.*, 11:85, 2023.

[20] N. Rafi, P. Vijaya Saradhi, and M. Balaiah. w-Filters of almost distributive lattices. *J. Algebr. Syst.*, 13(2):37–52, 2025.

[21] Y. S. Pawar and I. A. Shaikh. On prime, minimal prime and annihilator ideals in an almost distributive lattice. *Eur. J. Pure Appl. Math.*, 6(1):107–118, 2013.