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A Class of Bi-Univalent Functions Associated with Shell-Like Geometries and the q-Fibonacci Analogue

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Abstract. Using the subordination principle, this study explores two subclasses of bi-univalent functions associated with shell-like curves via the q-analogue of Fibonacci numbers, namely the starlike and convex classes. We derive coefficient bounds for the initial terms of these function classes and establish the corresponding Fekete-Szegö inequalities. Our findings contribute to the advancement of biunivalent function theory and its interaction with special function spaces.

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1. Introduction and Definitions

We begin by considering the collection $\mathscr{A}$ of functions that are complex analytic within the open unit disk $\mathscr{O}$. This domain is defined as

$$\mathscr{O} = \{z = \mathfrak{a} + i\ \mathfrak{b} \in \mathbb{C} \ \text{ where } \ \mathfrak{a},\ \mathfrak{b} \in \mathbb{R},\ \text{and } |z| < 1\},$$

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which geometrically corresponds to the interior of the unit circle in the complex plane, centered at the origin and excluding its boundary. All functions $f \in \mathscr{A}$ are subject to a standard normalization, namely:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These initial conditions eliminate translational and scaling ambiguities, ensuring that each function is uniquely defined at the origin with a prescribed rate of change. This allows for coherent structural analysis and comparison of such functions under common geometric constraints.

Each member $f \in \mathscr{A}$ possesses a Maclaurin series representation about the origin, which can be written as:

$$f(z) = z + \sum_{n=2}^{\infty} \delta_n z^n, \quad \text{for } z \in \mathscr{O}, \tag{1}$$

where the coefficients δ_n determine the nonlinear components of f. The leading term z arises from the derivative condition $f'(0) = 1$, and subsequent terms capture the analytic structure beyond linearity.

A function f is called a *Schwarz function* if it is analytic throughout $\mathscr{O}$, satisfies $f(0) = 0$, and its modulus remains strictly less than one within the disk, i.e. $|f(z)| < 1$ for all $z \in \mathscr{O}$. These functions are of central importance in geometric function theory, particularly in the context of conformal and univalent mappings.

Furthermore, for any two functions $f_1, f_2 \in \mathscr{A}$, the function f_1 is said to be *subordinate* to f_2, denoted $f_1 \prec f_2$, if there exists a Schwarz function η such that

$$f_1(z) = f_2(\eta(z)) \quad \text{for all} \quad z \in \mathscr{O}.$$

This relation implies that f_1 is functionally dependent on f_2 through composition with η, preserving analyticity while embedding geometric structure. The notion of subordination is a key analytical tool for examining inclusion relations, growth estimates, and mapping behavior in complex analysis.

In addition, let us consider the subclass S, $\mathsf{S} \subset \mathscr{A}$, which comprises all functions that are univalent (i.e., one-to-one) within the unit disk $\mathscr{O}$. We also introduce the class P, defined as the family of functions in $\mathscr{A}$ whose real parts are strictly positive throughout $\mathscr{O}$. A typical function $\varphi \in \mathsf{P}$ admits the following expansion of the power series:

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots, \quad (z \in \mathscr{O}). \tag{2}$$

where the coefficients satisfy the sharp bound,

$$|p_n| \le 2, \quad \text{for all} \quad n \ge 1. \tag{3}$$

in accordance with the classical lemma Carathéodory (see [1] for further details). Furthermore, a function $\varphi \in \mathsf{P}$ if and only if it is subordinate to the Móbius transformation $\frac{1+z}{1-z}$, i.e.,

$$\varphi(z) \prec \frac{1+z}{1-z}, \quad z \in \mathscr{O}.$$

The class of starlike functions, denoted S^*, can be characterized in various ways using subordination techniques. Ma and Minda [2], who defined the following class, proposed a notable generalization.

$$\mathsf{S}^*(\Omega) = \left\{ f \in \mathscr{A} : \frac{z\,f'(z)}{f(z)} \prec \Omega(z), \quad \text{where} \quad \Omega \in \mathsf{P} \ \text{ and } \ z \in \mathscr{O} \right\}.$$

In this formulation, Ω is assumed to be analytic in $\mathscr{O}$ and have a positive real part throughout the disk. Table 1 provides a variety of subclasses of S^*, arising from specific choices of the function Ω, reflecting the diversity of approaches adopted in the literature to construct refined categories of starlike mappings. The class P forms the cornerstone for the development of numerous significant subclasses of analytic functions, making it a key target of study in complex analysis. For any function f in the subclass $\mathsf{S} \subset \mathscr{A}$, there exists an inverse function, denoted f^{-1}, which is defined as

Table 1: Enumerates various starlike function classes characterized via the principle of subordination.

	The subclasses of starlike functions	**Ref.**	**Author/s**
1	$\mathsf{S}^*\left(\frac{1+z}{1-z}\right) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$	[3]	Janowski
2	$\mathsf{S}^*(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\vartheta)z}{1-z} \right\}, \quad \text{where } 0 \leq \vartheta < 1$	[4]	Robertson
3	$\mathsf{SL}(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2} \right\}, \quad \text{where } \vartheta = \frac{1-\sqrt{5}}{2}$	[5]	Sokól
4	$\mathcal{SK}(\vartheta) = \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2} \right\}, \quad \text{where } \vartheta \in (-3, 1]$	[6]	Sokól

$$z = f^{-1}(f(z)) \ \text{ and } \ \xi = f(f^{-1}(\xi)), \quad (\mathsf{r}_0(f) \geq 0.25; \ |\xi| < \mathsf{r}_0(f); z \in \mathscr{O}). \tag{4}$$

where

$$\chi(\xi) = f^{-1}(\xi) = \xi - \delta_2\xi^2 + \left(2\delta_2^2 - \delta_3\right)\xi^3 - \left(5\delta_2^3 + \delta_4 - 5\delta_3\delta_2\right)\xi^4 + \cdots. \tag{5}$$

The function $f \in \mathsf{S}$ is said to be bi-univalent if its inverse function $f^{-1} \in \mathsf{S}$. The subclass of S denoted by $\sum$ contains all bi-univalent functions in $\mathscr{O}$. The table below illustrates certain functions within the class $\sum$ and their inverse functions.

Table 2: Representative examples of bi-univalent functions along with their corresponding inverse functions.

f	f^{-1}
$f_1(z) = \dfrac{z}{1+z}$	$f_1^{-1}(z) = \dfrac{z}{1-z}$
$f_2 = -\log(1-z)$	$f_1^{-1}(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$
$f_3 = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$	$f_1^{-1}(z) = \dfrac{e^{z}-1}{e^{z}}$

Quantum calculus, also known as q-calculus, extends beyond the conventional framework of ordinary calculus by incorporating the parameter $\mathrm{q} \in (0,1)$, thus generalizing classical analytical techniques. This field has garnered significant interest because of its deep connections with physics, quantum mechanics, and Geometric Function Theory (GFT). A foundational resource for understanding the q-difference calculus and its diverse applications is the work of Gasper and Rahman [7], which provides a comprehensive exposition on the subject. Central to the study of analytic functions within this framework is the q-difference operator ∂_{q}, which plays a crucial role in function theory. Notable advancements in this area include the work of Seoudy and Aouf [8], who extended the q-calculus to functions within the unit disk, further enriching GFT. For further exploration, numerous classical and contemporary studies provide valuable insights, including [9–29].

Polynomials play a significant role in Geometric Function Theory (GFT) as both analytic test functions and approximation tools. In GFT, polynomial mappings are used to study geometric behaviors such as starlikeness, convexity, and univalence through simpler, finite-degree cases. Many univalent and bi-univalent functions can be represented or approximated by polynomial expansions, making it possible to estimate coefficient bounds and distortion theorems more effectively [30–46]. Furthermore, orthogonal polynomials, such as Chebyshev or Legendre polynomials, are employed in the construction of subclasses of analytic functions with the desired geometric properties. Thus, polynomials bridge the gap between abstract complex analysis and computational modeling, allowing deeper exploration of geometric mappings and their analytic behavior [47–51]. Some applications in operator theory can be found in [52–54].

Definition 1. *[38] The q-bracket $\lceil \kappa \rfloor_q$ is defined as follows:*

$$\lceil \kappa \rfloor_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & 0<q<1,\ \lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\} \\ 1, & q \mapsto 0^+,\ \lambda \in \mathbb{C}^* \\ \lambda, & q \mapsto 1^-,\ \lambda \in \mathbb{C}^* \\ q^{\gamma-1}+q^{\gamma-2}+\cdots+q+1=\sum_{n=0}^{\gamma-1} q^n, & 0<q<1,\ \lambda=\gamma\in\mathbb{N}, \end{cases}$$

with the useful identity $\lceil \kappa+1 \rfloor_q = \lceil \kappa \rfloor_q + q^\kappa$.

Definition 2. *[38] The q−derivative, also known as the q−difference operator, of a function f is defined by*

$$\eth_q\langle f(z)\rangle = \begin{cases} (f(z)-f(q\,z))(z-q\,z)^{-1}, & \text{if} \quad 0<q<1,\ z\neq 0, \\ f'(0), & \text{if} \quad z=0, \\ f'(z), & \text{if} \quad q\mapsto 1^-,\ z\neq 0. \end{cases} .$$

Remark 1. *For $f \in \mathscr{A}$ of the form (1), it is straightforward to verify that*

$$\eth_q\langle f(z)\rangle = \eth_q\left\langle z+\sum_{n=2}^{\infty}\delta_n\,z^n\right\rangle = 1+\sum_{n=2}^{\infty}\lceil n\rfloor_q\delta_n\,z^{n-1}, \quad (z\in\mathscr{O}),$$

and for the inverse function $\chi = f^{-1}$ of the form (4), we have

$$\eth_q\langle\chi(\xi)\rangle = \eth_q\langle f^{-1}(\xi)\rangle = 1-\lceil 2\rfloor_q\delta_2\xi+\lceil 3\rfloor_q\left(2\delta_2^2-\delta_3\right)\xi^2-\lceil 4\rfloor_q\left(5\delta_2^3+\delta_4-5\delta_3\delta_2\right)\xi^3+\cdots.$$

In a more recent advance, Alsoboh et al. [55] introduced a notable class of functions known as q starlike functions, denoted by SL_q, which were defined using the q-Jackson difference operators. The formal definition of this class is given by

$$\mathsf{SL}_q = \left\{ f\in\mathscr{A}: \quad \frac{z\,\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q), \quad z\in\mathscr{O}\right\}, \tag{6}$$

where the function $\Upsilon(z;q)$ is expressed explicitly as

$$\Upsilon(z;q) = \frac{1+q\vartheta_q^2z^2}{1-\vartheta_q z-q\vartheta_q^2z^2}, \tag{7}$$

and

$$\vartheta_q = \frac{1-\sqrt{4q+1}}{2q} \tag{8}$$

represents the q-analog of the Fibonacci numbers. Also, Alsoboh et al. [55] established a significant connection between these q-Fibonacci numbers, denoted as ϑ_q, and the related Fibonacci polynomials $\varphi_n(q)$. Specifically, they demonstrated that if

$$\Upsilon(z;q) = 1 + \sum_{n=1}^{\infty} \widehat{p_n} z^n,$$

the coefficients $\widehat{p_n}$ satisfy the following recurrence relation:

$$\widehat{p_n} = \begin{cases} \vartheta_q, & \text{for } n = 1, \\ (2q+1)\vartheta_q^2, & \text{for } n = 2, \\ (3q+1)\vartheta_q^3, & \text{for } n = 3, \\ \left(\varphi_{n+1}(q) + q\varphi_{n-1}(q)\right)\vartheta_q^n, & \text{for } n \geq 4. \end{cases} \tag{9}$$

Here, the q-Fibonacci polynomials $\varphi_{\mathsf{s}}(q)$ are defined as

$$\varphi_{\mathsf{s}}(q) = \frac{(1 - q\vartheta_q)^{\mathsf{s}} - (\vartheta_q)^{\mathsf{s}}}{\sqrt{4q+1}}, \quad \mathsf{s} \in \mathbb{N}. \tag{10}$$

This research presents a comprehensive framework for examining the relationship between the q-modified Fibonacci numbers and their corresponding polynomial representations.

The initial terms of the q-Fibonacci sequence, which constitutes a natural generalization of the classical Fibonacci numbers and converges to them as $q \to 1^-$, are enumerated in Table 3.

Table 3: Comparison of the classical Fibonacci numbers with their corresponding q-analogue terms from the q-Fibonacci sequence.

The classical Fibonacci numbers	The q-analogue of Fibonacci numbers
$\varphi_0 = 0$	$\varphi_0(q) = 0$
$\varphi_1 = 1$	$\varphi_1(q) = 1$
$\varphi_2 = 1$	$\varphi_2(q) = 1$
$\varphi_3 = 2$	$\varphi_3(q) = 1 + q$
$\varphi_4 = 3$	$\varphi_4(q) = 1 + 2q$

It should be noted that the function $\Upsilon(z;q)$ is not injective in the domain $\mathscr{O}$. Specifically, there exist distinct points in $\mathscr{O}$ at which $\Upsilon(z;q)$ attains the same value. For example,

$$\Upsilon(0;q) = 1 \quad \text{and} \quad \Upsilon\left(-\frac{1}{2q\vartheta_q};q\right) = 1.$$

In the following example, we explore the behavior of the q-starlike functions as the parameter q approaches 1 below. This transition leads to the classical case of starlike functions, often referred to as class SL. By taking the limit as $q \to 1^-$, we observe how the q-starlike functions generalize to the traditional starlike functions, and the associated function $\Upsilon(z)$ simplifies to a form that connects directly with the classical Fibonacci numbers. This example illustrates the connection between the q-starlike functions and their classical counterparts.

Example 1. *To illustrate the asymptotic behavior of the q-starlike functions as $q \to 1^-$, we examine the limiting case of the class SL_q. In the limit, this class converges to the classical starlike function class associated with the Fibonacci generating function, namely*

$$\mathsf{SL} = \lim_{q\to 1^-} \mathsf{SL}_q = \left\{ f \in \mathscr{A} \ : \ \frac{z\,f'(z)}{f(z)} \prec \Upsilon(z) \right\},$$

where the function $\Upsilon(z)$ is given by

$$\Upsilon(z;1) = \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}, \tag{11}$$

and $\vartheta = \frac{1-\sqrt{5}}{2}$ denotes the classical Fibonacci constant.

In addition to introducing the class of q-starlike functions, Alsoboh et al. [56] further extended the framework by defining a novel class of analytic functions termed the q-convex class, denoted by KSL_q. This class is characterized by a subordination condition analogous to that of the q-starlike class, but involves the application of a second-order q-difference operator, thereby capturing a more nuanced geometric structure. Specifically, a function f is said to belong to the class KSL_q if and only if the following subordination condition is satisfied:

$$1 + \frac{z\,\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q), \qquad (z \in \mathscr{O}), \tag{12}$$

where the function $\Upsilon(z;q)$ is defined by the rational expression in (7), and the parameter ϑ_q is specified in (8).

2. Definition and example

Motivated by q-Fibonacci numbers, this section will now look at a novel subclass of bi-univalent functions related to shell-like curves.

Definition 3. *For $\beta \in [0,1]$. A bi-univalent function f of the form* (1) *belongs to the class $\mathsf{SLM}_{\sum}(\beta;q)$ if and only if*

$$(1-\beta)\frac{z\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q\left(z\,\eth_q\langle f(z)\rangle\right)}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q\,z-q\vartheta_q^2 z^2}, \tag{13}$$

and

$$(1-\beta)\frac{\xi\eth_q\langle\chi(\xi)\rangle}{\chi(\xi)}+\beta\frac{\eth_q\left(\xi\eth_q\langle\chi(\xi)\rangle\right)}{\eth_q\langle\chi(\xi)\rangle}\prec\Upsilon(\xi;q)=\frac{1+q\vartheta_q^2\xi^2}{1-\vartheta_q\,\xi-q\vartheta_q^2\xi^2},\tag{14}$$

where $\chi=f^{-1}$ *given by* (5), ϑ_q *given by* (8) *and* $z,\xi\in\mathscr{O}$.

By varying the parameters $\beta\in[0,1]$ and $q\in(0,1)$, a broad spectrum of novel subclasses of the bi-univalent function class $\sum$ can be systematically derived. These subclasses capture diverse geometric behaviors and provide a unified framework for further analytical investigations.

Example 2. *If* $\beta=0$, *we obtain the class* $\mathsf{SL}_{\sum}(\Upsilon(z;q))$ *consisting of functions* $f\in\sum$ *satisfying the conditions*

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)}\prec\Upsilon(z;q)=\frac{1+q\vartheta_q^2z^2}{1-\vartheta_q\,z-q\vartheta_q^2z^2},$$

and

$$\frac{\xi\eth_q\langle\chi(\xi)\rangle}{\chi(\xi)}\prec\Upsilon(\xi;q)=\frac{1+q\vartheta_q^2\xi^2}{1-\vartheta_q\,\xi-q\vartheta_q^2\xi^2},$$

where ϑ_q *is given by* (8).

Example 3. *Letting* $\beta=1$, *we arrive at the subclass* $\mathsf{KL}_{\sum}(\Upsilon(z;q))$, *which comprises all functions* $f\in\sum$ *satisfying the subordination conditions*

$$1+\frac{z\,\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle}\prec\Upsilon(z;q)=\frac{1+q\vartheta_q^2z^2}{1-\vartheta_qz-q\vartheta_q^2z^2},$$

and

$$1+\frac{\xi\,\eth_q^2\langle\chi(\xi)\rangle}{\eth_q\langle\chi(\xi)\rangle}\prec\Upsilon(\xi;q)=\frac{1+q\vartheta_q^2\xi^2}{1-\vartheta_q\xi-q\vartheta_q^2\xi^2},\tag{15}$$

where ϑ_q *is the q-analogue of the Fibonacci number as defined in* (8).

Example 4. *In the limiting case as* $q\to1^-$, *we recover the classical subclass* $\mathsf{SLM}_{\sum}(\beta)$, *consisting of all functions* $f\in\sum$ *that satisfy the following subordination conditions:*

$$(1-\beta)\frac{z\,f'(z)}{f(z)}+\beta\,\frac{z\,f''(z)}{f'(z)}\prec\Upsilon(z)=\frac{1+\vartheta^2z^2}{1-\vartheta z-\vartheta^2z^2},$$

and

$$(1-\beta)\frac{\xi\,\chi'(\xi)}{\chi(\xi)}+\beta\,\frac{\xi\,\chi''(\xi)}{\chi'(\xi)}\prec\Upsilon(\xi)=\frac{1+\vartheta^2\xi^2}{1-\vartheta\xi-\vartheta^2\xi^2},$$

where $\chi=f^{-1}$ *is the inverse function defined as in* (5), $\vartheta=\frac{1-\sqrt{5}}{2}$ *is the classical Fibonacci constant, and* $z,\xi\in\mathscr{O}$.

Example 5. *If $q \to 1^-$ and $\beta = 0$, we obtain the class $\mathsf{SL}_{\sum}(\Upsilon(z))$ consisting of functions $f \in \sum$ satisfying the conditions*

$$\frac{z\, f'(z)}{f(z)} \prec \Upsilon(z) = \frac{1 + \vartheta^2 z^2}{1 - z - \vartheta^2 z^2},$$

and

$$\frac{\xi\, \chi'(\xi)}{\chi(\xi)} \prec \Upsilon(\xi) = \frac{1 + q\vartheta^2 \xi^2}{1 - \vartheta\xi - \vartheta^2 \xi^2},$$

where $\chi = f^{-1}$ is the inverse function defined as in (5), $\vartheta = \frac{1-\sqrt{5}}{2}$ is the classical Fibonacci constant, and $z, \xi \in \mathscr{O}$.

Example 6. *If $q \mapsto 1^-$ and $\beta = 1$, we obtain the class class $\mathsf{KL}_{\sum}(\Upsilon(z))$ consisting of functions $f \in \sum$ satisfying the conditions*

$$1 + \frac{zf''(z)}{f'(z)} \prec \Upsilon(z) = \frac{1 + \vartheta^2 z^2}{1 - \ z - \vartheta^2 z^2},$$

and

$$1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)} \prec \Upsilon(\xi) = \frac{1 + q\vartheta^2 \xi^2}{1 - \vartheta\, \xi - \vartheta^2 \xi^2},$$

where $\chi = f^{-1}$ is the inverse function defined as in (5), $\vartheta = \frac{1-\sqrt{5}}{2}$ is the classical Fibonacci constant, and $z, \xi \in \mathscr{O}$.

3. Main Results

In this section, we first obtain the estimate of the initial Taylor coefficients $|\delta_2|$ and $|\delta_2|$ for functions in the class $\mathsf{SLM}_{\sum}(\beta; q)$ according to Definition 3.

Firstly, let us

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots,$$

and $p(z) \prec \Upsilon(z; q)$. Then there exists $\varphi \in \mathsf{P}$ such that

$$|\varphi(z)| < 1 \text{ in } \mathscr{O} \text{ and } p(z) = \Upsilon(\varphi(z); q).$$

We have

$$\hbar(z) = (1 + \varphi(z))(1 - \varphi(z))^{-1} = 1 + \ell_1 z + \ell_2 z^2 + \cdots \in \mathsf{P} \qquad (z \in \mathscr{O}). \tag{16}$$

Consequently, the function $\varphi(z)$, being analytic in $\mathscr{O}$ and subordinate to $\Upsilon(z; q)$, admits the following Taylor expansion:

$$\varphi(z) = \frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots, \tag{17}$$

and

$$\begin{aligned}\Upsilon(\varphi(z);q) &= 1+\widehat{p_1}\left[\frac{\ell_1 z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]\\ &\quad+\widehat{p_2}\left[\frac{\ell_1 z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^2\\ &\quad+\widehat{p_3}\left[\frac{\ell_1 z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^3+\cdots \\ &= 1+\frac{\widehat{p_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{p_1}+\frac{\ell_1^2}{2}\widehat{p_2}\right]z^2\\ &\quad+\frac{1}{2}\left[\left(\ell_3-\ell_1\ell_2+\frac{\ell_1^3}{4}\right)\widehat{p_1}+\ell_1\left(\ell_2-\frac{\ell_1^2}{2}\right)\widehat{p_2}+\frac{\ell_1^3}{4}\widehat{p_3}\right]z^3+\cdots.\end{aligned} \tag{18}$$

Similarly, there exists an analytic function ν defined in $\mathscr{O}$, satisfying $|\nu(\xi)|<1$, such that $p(\xi)=\Upsilon(\nu(\xi);q)$. This allows us to represent the corresponding function

$$\kappa(\xi)=(1+\nu(\xi))(1-\nu(\xi))^{-1}=1+\tau_1\xi+\tau_2\xi^2+\cdots\in \mathsf{P}. \tag{19}$$

As a result, the Taylor expansion of $\nu(\xi)$ takes the form:

$$\nu(\xi)=\frac{\tau_1\xi}{2}+\left(\tau_2-\frac{\tau_1^2}{2}\right)\frac{\xi^2}{2}+\left(\tau_3-\tau_1\tau_2-\frac{\tau_1^3}{4}\right)\frac{\xi^3}{2}+\cdots, \tag{20}$$

and, accordingly, the composition $\Upsilon(\nu(\xi);q)$ expands as:

$$\begin{aligned}\Upsilon(\nu(\xi);q) &= 1+\frac{\widehat{p_1}\tau_1}{2}\xi+\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{p_1}+\frac{\tau_1^2}{2}\widehat{p_2}\right]\xi^2\\ &\quad+\frac{1}{2}\left[\left(\tau_3-\tau_1\tau_2+\frac{\tau_1^3}{4}\right)\widehat{p_1}+\tau_1\left(\tau_2-\frac{\tau_1^2}{2}\right)\widehat{p_2}+\frac{\tau_1^3}{4}\widehat{p_3}\right]\xi^3+\cdots.\end{aligned} \tag{21}$$

Having established the necessary groundwork and auxiliary results, we are now in a position to derive bounds for the initial coefficients of the functions belonging to the newly introduced class $\mathsf{SLM}_{\sum}(\beta;q)$. These estimates not only offer insights into the geometric behavior of such bi-univalent functions but also highlight the influence of the deformation parameter q and the parameter β on the coefficient structure. The following theorem presents sharp bounds for the second and third coefficients $|\delta_2|$ and $|\delta_3|$, respectively.

Theorem 1. *For $\beta\in[0,1]$, let $f\in\mathsf{SLM}_{\sum}(\beta;q)$. Then*

$$\left|\delta_2\right|\le\frac{|\vartheta_q|}{\sqrt{\left|\vartheta_q(K-X)+\big(1-(2q+1)\vartheta_q\big)C\right|}}. \tag{22}$$

$$\left|\delta_3\right|\le\frac{|\vartheta_q|\left\{\left|\big(K-X\big)\vartheta_q+\big(1-(2q+1)\vartheta_q\big)C\right|+|\vartheta_q|K\right\}}{K\left|\big(K-X\big)\vartheta_q+\big(1-(2q+1)\vartheta_q\big)C\right|}, \tag{23}$$

where

$$K = q\,\lceil 2\rfloor_q\,\big(1\ +\ q\,\lceil 2\rfloor_q\,\beta\big), \tag{24}$$

$$X = q\left[1\ +\ \beta\left(\lceil 2\rfloor_q^2\ -\ 1\right)\right], \tag{25}$$

$$C = q^2\left(1\ +\ q\,\beta\right)^2. \tag{26}$$

Proof. Let $f \in \mathsf{SL}_{\sum}(\Upsilon(z))$ and $\xi = f^{-1}$. Taking into account (13) and (14), we have

$$(1-\beta)\frac{z\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q\left(z\,\eth_q\langle f(z)\rangle\right)}{\eth_q\langle f(z)\rangle} = \Upsilon(\varphi(z);q), \qquad (z \in \mathscr{O}), \tag{27}$$

and

$$(1-\beta)\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} + \beta\frac{\eth_q\left(\xi\eth_q\langle \chi(\xi)\rangle\right)}{\eth_q\langle \chi(\xi)\rangle} = \Upsilon(\nu(\xi);q), \qquad (\xi \in \mathscr{O}). \tag{28}$$

Since

$$(1-\beta)\frac{z\eth_q\langle f(z)\rangle}{f(z)} + \beta\frac{\eth_q\left(z\,\eth_q\langle f(z)\rangle\right)}{\eth_q\langle f(z)\rangle} = 1 + \frac{\widehat{p_1}\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{p_1} + \frac{\ell_1^2}{2}\widehat{p_2}\right]z^2 + \cdots. \tag{29}$$

and

$$(1-\beta)\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} + \beta\frac{\eth_q\left(\xi\eth_q\langle \chi(\xi)\rangle\right)}{\eth_q\langle \chi(\xi)\rangle} = 1 + \frac{\widehat{p_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{p_1} + \frac{\tau_1^2}{2}\widehat{p_2}\right]\xi^2 + \cdots. \tag{30}$$

Compared with (27) and (29), along (18), yields

$$\begin{aligned} q(1+q\beta)\delta_2 z + q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta)\delta_3 - q\left(1+\beta(\lceil 2\rfloor_q^2-1)\right)\delta_2^2 z^2 + \cdots \\ = \frac{\widehat{p_1}\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{p_1} + \frac{\ell_1^2}{2}\widehat{p_2}\right]z^2 + \cdots. \end{aligned} \tag{31}$$

Besied that By comparing (28) and (30), along (21), yields

$$\begin{aligned} -q(1+q\beta)\delta_2 z + \left(2q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta) - q\left(1+\beta(\lceil 2\rfloor_q^2-1)\right)\delta_2^2 - q\lceil 2\rfloor_q(1+q\lceil 2\rfloor_q\beta)\delta_3\right)z^2 \\ + \cdots = \frac{\widehat{p_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{p_1} + \frac{\tau_1^2}{2}\widehat{p_2}\right]\xi^2 + \cdots. \end{aligned} \tag{32}$$

Equating the pertinent coefficient in (31) and (32), using (24) and (25), we obtain

$$q(1+q\beta)\delta_2 = \frac{\widehat{p_1}\ell_1}{2} \tag{33}$$

$$-q(1+q\beta)\delta_2 = \frac{\widehat{p_1}\tau_1}{2} \tag{34}$$

$$K\delta_3 - X\delta_2^2 = \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widehat{p_1} + \frac{\ell_1^2}{2}\widehat{p_2}\right] \tag{35}$$

$$(2K-X)\delta_2^2 - K\delta_3 = \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widehat{p_1} + \frac{\tau_1^2}{2}\widehat{p_2}\right] \tag{36}$$

From (33) and (34), we have

$$\ell_1 = -\tau_1 \quad \Longleftrightarrow \quad \ell_1^2 = \tau_1^2, \tag{37}$$

and

$$\delta_2^2 = \frac{\vartheta_q^2}{8q^2(1+q\beta)^2}\left(\ell_1^2+\tau_1^2\right) \quad \Longleftrightarrow \quad \ell_1^2+\tau_1^2 = \frac{8q^2(1+q\beta)^2}{\vartheta_q^2}\delta_2^2. \tag{38}$$

Now, by summing (35) and (36), we obtain

$$2\big(K-X\big)\delta_2^2 = \frac{(\ell_2+\tau_2)\vartheta_q}{2} + \left[\frac{(2q+1)\vartheta_q^2}{4} - \frac{\vartheta_q}{4}\right]\left(\ell_1^2+\tau_1^2\right). \tag{39}$$

Putting (38) in (39), we obtain

$$\delta_2^2 = \frac{(\ell_2+\tau_2)\vartheta_q^2}{4\Big(\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\Big)}, \tag{40}$$

where K, X, C is given by (24), (25) and (26), respectively. Using (3) for (40), we have

$$\left|\delta_2\right| \leq \frac{|\vartheta_q|}{\sqrt{\left|\vartheta_q(K-X) + \big(1-(2q+1)\vartheta_q\big)C\right|}}. \tag{41}$$

Now, so as to find the bound on $|\delta_3|$, let us subtract from (35) and (36) along (38), we obtain

$$\delta_3 = \delta_2^2 + \frac{\vartheta_q}{4K}\big(\ell_2 - \tau_2\big). \tag{42}$$

Therefore, we get

$$\left|\delta_3\right| \leq \left|\delta_2\right|^2 + \frac{|\vartheta_q|}{K}. \tag{43}$$

Then, in view of (41), we obtain

$$\left|\delta_3\right| \leq \frac{|\vartheta_q|\left\{\left|\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\right| + |\vartheta_q|K\right\}}{K\left|\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\right|}, \tag{44}$$

where K, X, C are given by (24), (25) and (26), respectively. This proves (49).

Theorem 2. *For $\alpha \in \mathbb{C}^*$ and $\beta \in [0,1]$, let $f \in \mathsf{SLM}_{\sum}(\beta;q)$. Then*

$$\left|\delta_3 - \alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{K}, & \left|1-\alpha\right| \leq \frac{\left|\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\vartheta_q|K} \\ \frac{|1-\alpha||\vartheta_q|^2}{\left|(K-X)\vartheta_q+(1-(2q+1)\vartheta_q)C\right|}, & \left|1-\alpha\right| \geq \frac{\left|\vartheta_q(K-X)+(1-(2q+1)\vartheta_q)C\right|}{|\vartheta_q|K} \end{cases} \tag{45}$$

where K, X, C are given by (24), (25) and (26), respectively.

Proof. Let $f \in \mathsf{SLM}_{\sum}(\beta;q)$, from (40) and (42) we have

$$\begin{aligned} \delta_3 - \alpha\delta_2^2 &= \frac{(1-\alpha)\vartheta_q^2}{4\Big(\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\Big)}\big(\ell_2+\tau_2\big) + \frac{\vartheta_q}{4K}\big(\ell_2-\tau_2\big) \\ &= \left(\mathscr{K}(\alpha) + \frac{\vartheta_q}{4K}\right)\ell_2 + \left(\mathscr{K}(\alpha) - \frac{\vartheta_q}{4K}\right)\tau_2, \end{aligned} \tag{46}$$

where

$$\mathscr{K}(\alpha) = \frac{(1-\alpha)\vartheta_q^2}{4\Big(\big(K-X\big)\vartheta_q + \big(1-(2q+1)\vartheta_q\big)C\Big)}. \tag{47}$$

Then, by taking modulus of (46), we conclude that

$$\left|\delta_3 - \alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{K}, & 0 \leq \left|\mathscr{K}(\alpha)\right| \leq \frac{|\vartheta_q|}{4K} \\ 4\left|\mathscr{K}(\alpha)\right|, & \left|\mathscr{K}(\alpha)\right| \geq \frac{|\vartheta_q|}{4K} \end{cases}$$

If $\beta = 0$, we obtain the following results for the class $\mathsf{SL}_{\sum}(\Upsilon(z;q))$ defined in Example (2)

Corollary 1. *Let f given by (1) be in the class $\mathsf{SL}_{\sum}(\Upsilon(z);q)$. Then*

$$\left|\delta_2\right| \leq \frac{\left|\vartheta_q\right|}{q\sqrt{1-2q\vartheta_q}}. \tag{48}$$

$$\left|\delta_3\right| \leq \frac{\left|\vartheta_q\right|\big(q-(1+q+2q^2)\vartheta_q\big)}{q^2(1+q)\big(1-2q\vartheta_q\big)}. \tag{49}$$

$$\left|\delta_3 - \alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{q(1+q)}, & \left|1-\alpha\right| \leq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \\ \frac{|1-\alpha|\vartheta_q^2}{q^2\left(1-2q\vartheta_q\right)}, & \left|1-\alpha\right| \geq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \end{cases} \tag{50}$$

If $\beta = 1$, we obtain the following results for the class $\mathsf{KL}_{\sum}(\Upsilon(z;q))$ defined in Example (3)

Corollary 2. *Let f given by* (1) *be in the class* $\mathsf{KL}_{\sum}(\Upsilon(z);q)$. *Then*

$$\left|\delta_2\right| \leq \frac{\left|\vartheta_q\right|}{\sqrt{\lceil 2\rfloor_q\Big(\lceil 2\rfloor_q-\Big(\lceil 3\rfloor_q+2q\Big)\vartheta_q\Big)}}$$

$$\left|\delta_3\right| \leq \frac{\left|\vartheta_q\right|\left(\lceil 2\rfloor_q-2(\lceil 3\rfloor_q+q)\vartheta_q\right)}{\lceil 2\rfloor_q\lceil 3\rfloor_q\Big(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\Big)},$$

and

$$\left|\delta_3-\alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{\lceil 2\rfloor_q\lceil 3\rfloor_q}, & \left|1-\alpha\right| \leq \frac{\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \\ \frac{|1-\alpha|\vartheta_q^2}{\lceil 2\rfloor_q\left(\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q\right)}, & \left|1-\alpha\right| \geq \frac{\lceil 2\rfloor_q-\left(\lceil 3\rfloor_q+2q\right)\vartheta_q}{\lceil 3\rfloor_q|\vartheta_q|} \end{cases}$$

If $q \mapsto 1^-$, we obtain the following results for the class $\mathsf{SLM}_{\sum}(\beta)$ defined in Example (4)

Corollary 3. *For $q \mapsto 1^-$, let $f \in \mathsf{SLM}_{\sum}(\beta)$. Then*

$$\left|\delta_2\right| \leq \frac{|\vartheta|}{\sqrt{\left|\vartheta(K-X)+\left(1-3\vartheta\right)C\right|}},$$

$$\left|\delta_3\right| \leq \frac{|\vartheta|\left\{\left|\left(K-X\right)\vartheta+\left(1-3\vartheta\right)C\right|+|\vartheta|K\right\}}{K\left|\left(K-X\right)\vartheta+\left(1-3\vartheta\right)C\right|},$$

and

$$\left|\delta_3-\alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{K}, & \left|1-\alpha\right| \leq \frac{\left|\vartheta(K-X)+(1-3\vartheta)C\right|}{|\vartheta|K} \\ \frac{|1-\alpha||\vartheta|^2}{\left|(K-X)\vartheta+(1-3\vartheta)C\right|}, & \left|1-\alpha\right| \geq \frac{\left|\vartheta(K-X)+(1-3\vartheta)C\right|}{|\vartheta|K} \end{cases}$$

where K, X, C are given by (24), (25) *and* (26), *respectively.*

If $q \mapsto 1^-$ and $\beta = 0$, we obtain the following results for the class $\mathsf{SL}_{\sum}(\Upsilon(z))$ defined in Example (5)

Corollary 4. *[57] Let f given by* (1) *be in class* $\mathsf{SL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\delta_2\right| \leq \frac{\left|\vartheta\right|}{\sqrt{1-2\vartheta}}, \qquad \left|\delta_3\right| \leq \frac{\left|\vartheta\right|\left(1-4\vartheta\right)}{2\left(1-2\vartheta\right)}.$$

and

$$\left|\delta_3-\alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{2}, & \left|1-\alpha\right| \leq \frac{1-2\vartheta}{2|\vartheta|} \\ \frac{(1-\alpha)\vartheta^2}{1-2\vartheta}, & \left|1-\alpha\right| \geq \frac{1-2\vartheta}{2|\vartheta|} \end{cases}$$

If $q \mapsto 1^-$ and $\beta = 1$, we obtain the following results for the class $\mathsf{KL}_{\sum}(\Upsilon(z))$ defined in Example (6)

Corollary 5. *[57] Let f given by* (1) *be in the class* $\mathsf{KL}_{\sum}(\Upsilon(z))$. *Then*

$$\left|\delta_2\right| \leq \frac{\left|\vartheta\right|}{\sqrt{4-10\,\vartheta}}, \qquad \left|\delta_3\right| \quad \leq \frac{\left|\vartheta\right|\left(1-4\,\vartheta\right)}{3\left(1-2\,\vartheta\,\right)}.$$

and

$$\left|\delta_3 - \alpha\delta_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{6}, & \left|1-\alpha\right| \leq \frac{2-5\vartheta}{3|\vartheta|} \\ \frac{|1-\alpha|\vartheta^2}{2\left(2-5\vartheta\right)}, & \left|1-\alpha\right| \geq \frac{2-5\vartheta}{3|\vartheta|} \end{cases}$$

4. Conclusion

In this work, we investigated two subclasses of bi-univalent functions associated with shell-like curves through the q-analogue of Fibonacci numbers, namely the starlike and convex classes. Using the subordination principle, we establish coefficient bounds for the initial terms of these function classes and derived the corresponding Fekete-Szegö inequalities. These results enhance the theoretical framework of bi-univalent function theory and elucidate its deeper connections with special function spaces.

Future research could extend these findings by exploring higher-order coefficient estimates, refining the structural characteristics of these subclasses, and examining their geometric properties. Moreover, investigating upper bounds related to the Zalcman conjecture and analyzing Hankel determinants of orders two and three within these subclasses could provide new insights and open further avenues in the study of analytic and bi-univalent function theory.

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