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Zariski Topology of (Krasner) Hyperrings

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Abstract. In this paper, we investigate the Zariski topology on the prime spectrum of commutative Krasner hyperrings and explore its interplay with the underlying algebraic structure. We characterize the topological properties of the spectrum, such as connectedness, irreducibility, compactness, and separation axioms, and provide necessary and sufficient conditions for each. Notably, we show that the spectrum is irreducible if and only if the nilradical is a prime hyperideal, and it is connected precisely when the hyperring is not a nontrivial product. We also study functorial behavior of the Zariski topology in the category of hyperrings and analyze its correspondence with classical ring theory via the fundamental relation γ^*. Furthermore, we define a topology on the space of prime strongly regular relations and establish a homeomorphism with a subspace of the classical spectrum. These results contribute to the categorical and topological foundations necessary for developing a sheaf-theoretic framework in the context of hyperrings.

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1. Introduction

The theory of algebraic hyperstructures, initiated by F. Marty in 1934 [1] through the concept of hypergroups, extends classical algebraic systems by allowing multi-valued operations. This framework has found applications in group theory, rational and algebraic functions, and has since evolved into a rich area of research with implications in algebraic geometry, notably through the works of A. Connes and C. Consani [2–4], who revealed deep connections between hyperstructures and arithmetic geometry. The results established in this paper enable us, by defining sheaves of hyperrings, hyperringed spaces, and hyperschemes based on the topology induced by regular relations, to prove results analogous to those in [2–4] from the perspective of regular relations. One of the advantages of

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this approach is that it provides a more tangible understanding of the connection between the obtained results and the classical case.

In 1956 M. Krasner introduced the concept of hyperrings to use them as a technical tool on the approximation of valued fields [5]. A general hyperring is a hyperstructure $(R, +, \cdot)$ where $(R, +)$ is a hypergroup and $(R, \cdot)$ is semihypergroup such that "$\cdot$" is distributive with respect to "$+$". If $(R, +)$ is a canonical hypergroup and $(R, \cdot)$ is a semigroup such that zero element is absorbing, then $(R, +, \cdot)$ is the Krasner hyperring [6].

The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures can be derived from algebraic hyperstructures through them. The fundamental relation β^* on hypergroups was defined by Koskas [7], Corsini [6], Ferni [8], and Vogiouklis [9]. Then D. Ferni introduced the fundamental relation γ^* which is the transitive closure of γ and is the smallest relation such that H/γ^* is an abelian group. T. Vogiouklis generalized the fundamental relations in [9] to use on hyperrings and in [10], it has been demonstrated that relations β^* and γ^* are related together in the form of $\gamma^* = \delta * \beta^*$, where δ is the congruence relation with respect to the commutator subgroup.

In [11], a one-to-one correspondence was established between the lattice of strongly regular relations on a regular hypergroup H and the lattice of normal subhypergroups of H containing ω_H. References [12] and [13] explore the fundamental relations on H_v-modules and highlight significant applications of hyperstructure theory in chemistry, respectively. More recently, considerable attention has been devoted to studying the Zariski topology of algebraic hyperstructures. For a detailed examination of the Zariski topology in the context of multiplicative hyperrings, see [14].

Our paper is organized as follows. After this introduction, Section 2 presents the necessary background on Krasner hyperrings, hyperideals, and topological preliminaries. Section 3 develops the Zariski topology on the spectrum of prime hyperideals, focusing on its fundamental topological properties. In Section 4, we examine the categorical and functorial aspects of Zariski topology, establish its connection with classical ring theory, and define a topology on prime strongly regular relations. Section 5 offers further discussion and proposes directions for future work, while Section 6 concludes the study with a summary of our main findings.

This work lays the groundwork for future research on sheaf-theoretic constructions in hyperalgebraic geometry and provides a categorical bridge between hyperstructures and classical ring-theoretic frameworks, with promising applications in the study of hypermodules and beyond.

Also with the help of equivalence relation γ^* we will examine the relationship between this functor and its classical Zariski topology functor. Furthermore, a one-to-one correspondence between strongly regular relations and hyperideals containing $\gamma^*(0)$ will be introduced. Also, by introducing prime and primary strongly regular relations, a topology is defined on the set of all strongly regular relations, which in the next researches we intend to examine its connection with Zariski topology.

2. Preliminaries

This section reviews the fundamental concepts and definitions needed for the remainder of the paper. We recall basic properties of Krasner hyperrings, hyperideals, and regular relations [15], as well as essential topological notions relevant to the Zariski framework [16].

A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

- For every $x, y, z \in R$, $x + (y + z) = (x + y) + z$;
- For every $x, y \in R$, $x + y = y + x$;
- There exists $0 \in R$ such that $0 + x = \{x\}$, for every $x \in R$;
- For every $x \in R$ there is a unique $x' \in R$ that $0 \in x + x'$ (we use $-x$ for x');
- If $z \in x + y$ then $y \in -x + z$ and $x \in z - y$;
- $(R, \cdot)$ is a semigroup, and for every $x \in R$, $x \cdot 0 = 0 \cdot x = 0$;
- The operation "$\cdot$" is bilaterally distributive with respect to the hyperoperation "$+$".

A Krasner hyperring $(R, +, \cdot)$ is called commutative (with a unit element) if $(R, \cdot)$ is commutative (with a unit element) semigroup. If $(R - \{0\}, \cdot)$ is a group then $(R, +, \cdot)$ is called a Krasner hyperfield and if $(R, +, \cdot)$ is commutative Krasner hyperring with a unit element and $ab = 0$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$, then R is called hyperdomain. Throughout this paper, hyperring refers to commutative Krasner hyperring with a unit element.

Definition 1. *[15] Let R be a hyperring and let $I \subseteq R$ be a subhyperring.*

- *I is called a left hyperideal of R if $ra \in I$ for all $r \in R$, $a \in I$.*
- *I is called a right hyperideal of R if $ar \in I$ for all $r \in R$, $a \in I$.*
- *If I is both a left and a right hyperideal, it is called a (two-sided) hyperideal of R.*

A proper hyperideal $M \subsetneq R$ is called a maximal hyperideal if the only hyperideals of R that contain M are M and R itself.

A proper hyperideal $P \subsetneq R$ is called a prime hyperideal if for all hyperideals $A, B \subseteq R$, the condition $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Lemma 1. *[15] A nonempty subset I of a hyperring R is a left(right) hyperideal if and only if $a - b \subseteq I$ and $ra \in I (ra \in I)$, for all $a, b \in I$, $r \in R$.*

An equivalence relation θ on a Krasner hyperring R is called *regular* if the following implication holds:

$$a\ \theta\ b\ ,\ c\ \theta\ d\ \Rightarrow (a+c)\ \bar{\theta}\ (b+d)\ \ and\ \ ac\ \theta\ bd, \tag{1}$$

and is called *strongly regular* if

$$a\ \theta\ b\ ,\ c\ \theta\ d\ \Rightarrow (a+c)\ \bar{\bar{\theta}}\ (b+d)\ \ and\ \ ac\ \theta\ bd, \tag{2}$$

for every $a, b, c, d \in R$.

Definition 2. *[15] Let R and S be hyperrings. A mapping f from R to S is said to be a good homomorphism if for every $a, b \in R$: $f(a+b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$ and $f(0) = 0$.*

Let R be a commutative hyperring with a unit element and I be a proper hyperideal of R. Then there exists a maximal hyperideal of R containing I and each maximal hyperideal is a prime hyperideal. Additionally, I is prime if $ab \in P$ implies that $a \in P$ or $b \in P$, for every $a, b \in R$.

Proposition 1. *[15] Let R be a commutative hyperring with a unit element and I be a proper hyperideal of R. Then:*

(i) I is prime hyperideal if and only if R/I is a hyperdomain.

(ii) I is maximal hyperideal if and only if R/I is a hyperfield.

For any regular hypergroup H, if $\mathcal{SR}(H)$ is the set of all strongly regular relations on H and $N(S_\beta)$ is the set of all normal subhypergroups of H, containing S_β $(= \omega_H)$, then the map

$$\varphi : \mathcal{SR}(H) \underset{\rho\ \mapsto\ S_\rho}{\rightarrow} N(S_\beta) \tag{3}$$

where $S_\rho = \{x \in H;\ \rho(x) = e_{H/\rho}\}$, is an isomorphism of complete lattices [11].

Definition 3. *[16] Let T be a topological space.*

- *T is called disconnected if it can be written as the disjoint union of two nonempty closed subsets.*
- *T is called irreducible if every pair of nonempty open subsets of T has nonempty intersection.*
- *T is a T_0-space if for every pair of distinct points $a, b \in T$, there exists an open set that contains one of them but not the other.*
- *T is a T_1-space if for every pair of distinct points $a, b \in T$, there exist open sets $U, V \subseteq T$ such that $a \in U$, $b \notin U$, and $b \in V$, $a \notin V$.*
- *T is a T_2-space (or Hausdorff) if for every pair of distinct points $a, b \in T$, there exist open sets $U, V \subseteq T$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.*

Theorem 1. *[16] Let T be a topological space. Then:*

(i) If S is an irreducible subspace of T then $\bar{S}$ is irreducible.

(ii) Every irreducible subspace is contained in a maximal irreducible subspace.

(iii) The irreducible components of T are closed and cover T.

3. Zariski topology of Krasner hyperrings

In this section, we construct and analyze the Zariski topology on the spectrum of commutative Krasner hyperring. We investigate key topological properties such as irreducibility, connectedness, compactness, and separation, and relate them to the algebraic structure of the hyperring.

Everywhere in this section R is a commutative Krasner hyperring with a unit element. Let R be a hyperring and $Spec(R)$ be the set of all prime hyperideals of R and $mSpec(R)$ be the set of all maximal hyperideals of R. For all $x = P \in Spec(R)$ let $\mathbb{K}(x)$ be the quotient hyperfield of the hyperdomain R/P. For all $f \in R$ we have $R \to R/P \to \mathbb{K}(x)$ where $f \to f + P \to \frac{(f+P)(1+P)}{1+P}$.

Remark 1. *Since every maximal hyperideal is a prime hyperideal, it is always true that $mSpec(R) \subseteq Spec(R)$. Also, R is a hyperfield if and only if $0 \in mSpec(R)$, i.e. R has only trivial hyperideals.*

Definition 4. *For every nonempty subset $S \subseteq R$, let*

$$V(S) = \{x \in Spec(R)\ ;\ f(x) = 0,\ \forall f \in S\} = \{P \in Spec(R)\ ;\ S \subseteq P\}$$

,

$$V_m(S) = \{x \in mSpec(R)\ ;\ f(x) = 0,\ \forall f \in S\} = \{M \in mSpec(R)\ ;\ S \subseteq M\}.$$

Lemma 2. *The following properties hold:*

(i) If $I = I(S)$ is the hyperideal generated by S, then $V(S) = V(I)$.

(ii) If $S_1 \subseteq S_2$, then $V(S_2) \subseteq V(S_1)$.

(iii) $V(S) = \varnothing$ if and only if $1 \in I(S)$.

(iv) $V(M) = M$ if and only if $M \in mSpec(R)$.

Proof. The proof is straightforward and similar to the classical case.

Theorem 2. *The family of sets $\{V(I)\}_{I \lhd R}$, satisfy the axioms for closed sets in a topological space.*

Proof. We have $V(R) = \emptyset$ and $V(\{0\}) = Spec(R)$. Let $\{V(I_j)\}_{j\in J}$ be a family of closed sets and $P \in V(\sum_{j\in J} I_j)$. Then $I_i \subseteq \sum_{j\in J} I_j \subseteq P$, for every $i \in J$. Hence $V(\sum_{j\in J} I_j) \subseteq \bigcap_{j\in J} V(I_j)$. If $P \in \bigcap_{j\in J} V(I_j)$, then $I_i \subseteq P$, for every $i \in J$. So $I_i \subseteq \sum_{j\in J} I_j \subseteq P$, and $P \in V(\sum_{j\in J} I_j)$. Therefore, $V(\sum_{j\in J} I_j) \subseteq \bigcap_{j\in J} V(I_j)$.
Now let I and J be hyperideals of R and $P \in V(IJ)$. Since P is prime and $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. So $P \in V(I) \cup V(J)$. Let $P \in V(I) \cup V(J)$, then $I \subseteq P$ or $J \subseteq P$, so $IJ \subseteq P$ and $P \in V(IJ)$. Therefore, $V(IJ) = V(I) \cup V(J)$.

The resulting topology on $Spec(R)$ is called the Zariski topology of R, and the family $\{V_m(I)\}_{I\lhd R}$ forms a subspace for it.

Remark 2. *Let $\{S_j\}_{j\in J}$ be a family of subsets of hyperring R. Since*

$$I(\bigcup_{j\in J} S_j) = \sum_{j\in J} I(S_j), \tag{4}$$

then by Lemma 2, for every $i, j \in J$ we have

$$V(\textstyle\bigcup_{j\in J} S_j) = \bigcap_{j\in J} V(S_j) = \bigcap_{j\in J} V(I(S_j)) = V(\sum_{j\in J} I(S_j)),$$

$$V(S_i \cap S_j) = V(S_i) \cup V(S_j) = V(I(S_i)) \cup V(I(S_j)) = V(I(S_i) \cap I(S_j)).$$

Also, if I and J are hyperideals of R then by Lemma 2, $V(I) \cup V(J) \subseteq V(I \cap J)$. Let $P \in V(I \cap J)$ but $P \notin V(I) \cup V(J)$, then there are $x \in I - P$ and $y \in J - P$ such that $xy \in I \cap J$. So $xy \in P$, and it is a contradiction. Therefore, $V(IJ) = V(I \cap J)$.

Proposition 2. *If I and J are hyperideals of R, then:*

(i) $V(I) = V(\sqrt{I})$.

(ii) $V(I) \subseteq V(J) \Leftrightarrow \sqrt{J} \subseteq \sqrt{I}$.

Proof. $\sqrt{I} = \bigcap_{P\in V(I)} P = \{x \in R;\ x^n \in I, for\ some\ n \in \mathbb{N}\}$, *and the proof is similar to that of classic rings.*

Theorem 3. *If $f : R \to S$ is a good homomorphism of hyperrings, then $\bar{f} : Spec(S) \to Spec(R)$, defined by $\bar{f}(P) = f^{-1}(P)$, is continuous.*

Proof. Let $V(I)$ be a closed set of $Spec(R)$. Then:

$$\begin{aligned} \bar{f}^{-1}(V(I)) &= \{P \in Spec(S);\ \bar{f}(P) \in V(I)\} = \{P \in Spec(S);\ I \subseteq f^{-1}(P)\} \\ &= \{P \in Spec(S);\ f(I) \subseteq P\} = V(f(I)). \end{aligned}$$

If R is a hyperring and I is a hyperideal of R, then $\pi : R \to R/I$ is projection map and $\bar{\pi}$ is a continuous map from $Spec(R/I)$ to $Spec(R)$.

Theorem 4. *Let R be a hyperring and I be a hyperideal of R. Then:*

(i) $\bar{\pi}(Spec(R/I)) = V(I)$.

(ii) $\bar{\pi}$ *is injective.*

(iii) $Spec(R/I)$ *is homeomorphic to* $V(I)$ *by subspace topology.*

Proof. We have

$$\begin{aligned} \bar{\pi}(Spec(R/I)) &= \{P \in Spec(R);\ P/I \in Spec(R/I)\} \\ &= \{P \in Spec(R);\ I \subseteq P\} \\ &= V(I). \end{aligned}$$

Since there is a bijection between the hyperideals of R/I and the hyperideals of R containing I, then $\bar{\pi}$ is a bijection between $Spec(R/I)$ and $V(I)$.
By Theorem 3 $\bar{\pi}$ is continuous and also $\bar{\pi}^{-1} : V(I) \rightarrow Spec(R/I)$ defined by $\bar{\pi}^{-1}(P) = P/I$ is continuous, because every prime hyperideal of R/I is of the form P/I, where $P \in V(I)$.

Corollary 1. *If R is a hyperring, then $Spec(R) \cong Spec(R/nil(R))$.*

Proof. *By Theorem 4, $Spec(R/nil(R)) = V(nil(R))$ and since $nil(R) \subseteq P$, for every $P \in Spec(R)$ we have $V(nil(R)) = Spec(R)$.*

Theorem 5. *Let R_1 and R_2 be hyperrings, then:*

$$Spec(R_1 \times R_2) = Spec(R_1) \,\dot{\cup}\, Spec(R_2). \tag{5}$$

Where $\dot{\cup}$ means disjoint union.

Proof. *Let $P_1 \times P_2$ be a prime hyperideal of $R_1 \times R_2$. Then $R_1/P_1 \times R_2/P_2$, is hyperdomain. So $P_1 = R_1$ and $P_2 \in Spec(R_2)$ or $P_2 = R_2$ and $P_1 \in Spec(R_1)$.*

Definition 5. *[17] The hyperideals I and J of a hyperring R are said to be comaximal if $I + J = R$.*

Theorem 6 (Chinese Remainder Theorem)**.** *If $I_1, I_2, ..., I_k$ are hyperrings of R, then the map $R \rightarrow R/I_1 \times R/I_2 \times ... \times R/I_k$ defined by $r \longmapsto (r + I_1, r + I_2, ..., r + I_k)$ is a good homomorphism with kernel $I_1 \cap I_2 \cap ... \cap I_k$. If for each $i, j \in \{1, 2, ..., k\}$ with $i \neq j$ the hyperideals I_i and I_j are comaximal, then this map is surjective and $I_1 \cap I_2 \cap ... \cap I_k = I_1 I_2 ... I_k$, so:*

$$R/(I_1 I_2 ... I_k) = R/(I_1 \cap I_2 \cap ... \cap I_k) = R/I_1 \times R/I_2 \times ... \times R/I_k. \tag{6}$$

Proof. *Let $I = I_1$ and $J = I_2$. Consider the map $\varphi : R \rightarrow R/I \times R/J$, defined by $\varphi(r) = (r + I, r + J)$. Then φ is good homomorphism of hyperrings and $Ker\varphi = I \cap J$. Since R is a Krasner hyperring with a unit element and $I + J = R$, then there are elements $x \in I$ and $y \in J$ such that $1 \in x + y$. Hence $x \in 1 - y \subseteq 1 + J$ and $y \in 1 - x \subseteq 1 + I$. Also $1 - y + J = 1 + J$ and $1 - x + I = 1 + I$. So $\varphi(x) = (I, 1 + J) = (0_{R/I}, 1_{R/J})$ and $\varphi(y) = (1 + I, J) = (1_{R/I}, 0_{R/J})$. Let $(r_1 + I, r_2 + J) \in R/I \times R/J$, then*

$$\begin{aligned}(r_1 + I, r_2 + J) &= (r_1 + I, 0) + (0, r_2 + J)\\ &= (r_1 + I, r_1 + J)(1, 0) + (r_2 + I, r_2 + J)(0, 1)\\ &= \varphi(r_1)\varphi(y) + \varphi(r_2)\varphi(x)\\ &= \varphi(r_1 y + r_2 x).\end{aligned}$$

We know that $IJ \subseteq I \cap J$. Let $z \in I \cap J$, then $z = z.1 \in z(x + y) = zx + zy \in IJ$. Hence $I \cap J \subseteq IJ$. The general case follows by induction from the case of two hyperideals using $I = I_1$ and $J = I_2 I_3 ... I_k$.

Corollary 2. *The hyperring R is a product of hyperrings if and only if R has nontrivial idempotents.*

Proof. *Let x be a nontrivial idempotent of R. Hence (x) and $(1-x)$ are comaximal hyperideals of R and $(x)(1-x) = (0)$. Now by the Chinese Remainder Theorem: $R \cong R/(0) \cong R/(x) \times R/(1-x)$. Conversely suppose that $R = R_1 \times R_2$, then $(1,0)^2 = (1,0)$.*

Lemma 3. *If $R/nil(R)$ has nontrivial idempotent then R has nontrivial idempotent.*

Proof. *Let $x+nil(R) \in R/nil(R)$ be nontrivial idempotent. So $x^2+nil(R) = x+nil(R)$ and $x^2 - x \subseteq nil(R)$. Hence there is $n \in \mathbb{N}$ such that $(x^2-x)^n = x^n(x-1)^n = 0$. Also $(x^n) + ((x-1)^n) = R$, and $(x^n).((x-1)^n) = (0)$. Now by Chinese Remainder Theorem $\varphi : R \to R/(x^n) \times R/((x-1)^n)$, defined by $\varphi(x) = \Big(r + (x^n), r + ((x-1)^n)\Big)$ is an isomorphism and $(0,1) \in R/(x^n) \times R/((x-1)^n)$ is nontrivial idempotent. So $\varphi^{-1}((0,1)) \in R$ is nontrivial idempotent.*

Theorem 7. *If $Spec(R)$ is a disconnected space and $Spec(R) = C \,\dot{\cup}\, D$, then $R \cong R_1 \times R_2$. Here, $C = Spec(R_1)$ and $D = Spec(R_2)$.*

Proof. *Let I and J be hyperideals of R such that $C = V(I)$ and $D = V(J)$. Since $V(R) = \emptyset$ and $V(0) = Spec(R)$, then*

$$\emptyset = C \cap D = V(I) \cap V(J) = V(I+J) = V(R);$$

$$Spec(R) = C \cup D = V(I) \cup V(J) = V(IJ) = V(0).$$

By Lemma 2 we have $I+J = R$ and by Chinese Remainder Theorem $R/IJ \cong R/I \times R/J$. If R has no nilpotent elements, then $nil(R) = (0)$, and by Proposition 2 $IJ = 0$. So $R/(0) \cong R \cong R/I \times R/J$, and by Theorem 4 we have $V(I) = Spec(R/I)$ and $V(J) = Spec(R/J)$. Now since $Spec(R) = Spec(R/nil(R))$ so $Spec(R/nil(R))$ is disconnected and $R/nil(R)$ has no nilpotent elements. Hence $R/nil(R) \cong S \times T$, for some hyperrings S and T, and by Corollary 2, $R/nil(R)$ has nontrivial idempotents. Therefore, by Lemma 3, R has nontrivial idempotent and by Corollary 2, R is product of hyperrings.

Remark 3. *Consider $W(f) = \big\{P \in Spec(R); f \notin P\big\}$, for every $f \in R$. Then $W(0) = \emptyset$, $W(1) = X = Spec(R)$ and $X - V(E) = \bigcup_{f \in E} W(f)$, for $E \subseteq R$. Therefore, $\{W(f); f \in R\}$ is a basis for Zariski topology on $Spec(R)$, and it is clear that $W(f) \cap W(g) = W(fg)$. Also $W(f) = W(g)$ if and only if $\sqrt{(f)} = \sqrt{(g)}$, and $f \in R$ is nilpotent if and only if $W(f) = \emptyset$, and $f \in R$ is unit if and only if $W(f) = X$. Here too like the theory of classical rings, it can be proved that $W(f)$ is quasi-compact for every $f \in R$, and an open subset of X is quasi-compact if and only if it is a finite union of sets $W(f)$ [17]. On the other hand if we define Zariski topology based on closed subsets, then $\mathbb{B} = \{B(x); x \in R\}$ is a basis for Zariski topology on $Spec(R)$, where $B(x) = V((x)) = \{P \in Spec(R); x \in P\}$.*

Theorem 8. *The hyperideal $nil(R)$ of R is prime if and only if $Spec(R)$ is irreducible, (especially that if R is hyperdomain, then $Spec(R)$ is irreducible).*

Proof. *We know that* $\{W(f)\}_{f\in R}$*, is a basis for Zariski topology on* $Spec(R)$ *and any two non-empty sets will intersect if and only if any tow non-empty basis elements intersect. So* $Spec(R)$ *is irreducible if and only if any tow non-empty basis elements intersect, that is for every* $W(f) \neq \emptyset$ *and* $W(g) \neq \emptyset$ *we have* $W(fg) = W(f) \cap W(g) \neq \emptyset$*. Therefore,* $Spec(R)$ *is irreducible if and only if for every* $f, g \in R$*, if* $f, g \notin nil(R)$ *then* $fg \notin nil(R)$*, which means that* $nil(R)$ *is the prime hyperideal of* R*.*

The following result is obtained immediately from Theorem 8 regarding quotient hyperrings.

Corollary 3. *Let* R *be a hyperring and* I *be a hyperideal of* R*. Then* $V(I)$ *is an irreducible subset of* $Spec(R/I)$ *if and only if* $\sqrt{I}$ *is prime.*

Proof. *By Theorem 8,* $Spec(R/I)$ *is irreducible if and only if* $\sqrt{I}$ *is prime. Also* $Spec(R/I) \cong V(I)$*.*

Remark 4. *The subset* $\{x\} = \{P\}$ *of* $Spec(R)$ *is closed (closed point), if and only if* $P \in mSpec(R)$*. So* $\overline{\{x\}} = V(P)$ *and* $y = Q \in \overline{\{x\}}$ *if and only if* $P \subseteq Q$*. Also* $Spec(R)$ *is a* T_0*-Space, because if* $x = P$ *and* $y = Q$ *are distinct points of* $Spec(R)$*, then* $P \nsubseteq Q$ *or* $Q \nsubseteq P$*. Without loss of generality assume that* $Q \nsubseteq P$*. Then* $x \notin \overline{\{y\}}$ *and so* $Spec(R) - \overline{\{y\}}$ *is an open set that contains* x*, and* $y \notin Spec(R) - \overline{\{y\}}$*.*

If $P \in Spec(R)$, the *height* of P denoted by $h(P)$ and defined to be the supremum of lengths of chains $P_0 \subsetneq P_1 \subsetneq ... \subsetneq P_n = P$, if this supremum exists, and ∞ otherwise. The dimension of R, denoted by $dim(R)$ and $dim(R) = sup\{h(P); P \in Spec(R)\}$. It is clear that if R is a hyperfield then $dim(R) = 0$.

Clearly, $Spec(R)$ is a T_1-Space if and only if $dim(R) = 0$. Because, if $dim(R) \neq 0$, then there are $P_1, P_2 \in Spec(R)$ such that $P_1 \subsetneq P_2$. So every neighborhood containing P_2 is contain P_1, and it is contradiction. Now suppose that $dim(R) = 0$. So there are $P_1, P_2 \in Spec(R)$ such that $x \in P_1 - P_2$ and $y \in P_2 - P_1$. Therefore, $(x) \subseteq P_1$, $(x) \nsubseteq P_2$ and $(y) \subseteq P_2$, $(y) \nsubseteq P_1$. So $P_1 \in Spec(R) - V((y))$, $P_1 \notin Spec(R) - V((x))$ and $P_2 \in Spec(R) - V((x))$, $P_2 \notin Spec(R) - V((y))$. In addition, we can also state the following theorem.

Theorem 9. $Spec(R)$ *is a Hausdorff space if and only if* $dim(R) = 0$*.*

Proof. *Every Hausdorff space (*T_2*-Space), is a* T_1*-Space. So assume that* $dimR = 0$*, and* $P, Q \in Spec(R)$ *and* $x \in P - Q$*. We know that* P_P *is maximal hyperideal of local hyperring* R_P*, and* $P_P = nil(R_P)$*. Let* $\frac{x}{1} \in P_P = nil(R_P)$*. Then there is* $n \in \mathbb{N}$ *such that* $\frac{x^n}{1} = 0_{R_P}$*. Hence for some* $s \in R - P$ *we have* $sx^n = 0$*.*
Since $x \notin Q$ *and* $s \notin P$ *then* $Q \in Spec(R) - V((x)) = W((x))$ *and* $P \in Spec(R) - V((s)) = W((s))$*. Also* $W((x)) \cap W((s)) = \{I \in Spec(R); s \notin I, x \notin I\} = \{I \in Spec(R); s \notin I, x^n \notin I\} = \{I \in Spec(R); sx^n \notin I\} = Spec(R) - V((sx^n)) = \emptyset$*, since* $V((sx^n)) = V((0)) = Spec(R)$*.*

Corollary 4. *$Spec(P)$ is T_1-space if and only if $mSpec(R) = Spec(R)$.*

Proposition 3. *For any hyperring R, $Spec(R)$ is a compact space under the Zariski topology.*

Proof. Let $\{I_j\}_{j\in J}$ be a family of hyperideals of R and $\{W(I_j)\}_{j\in J}$ be a family of open sets that $Spec(R) = \bigcup_{j\in J} W(I_j)$ and $W(I_j) = Spec(R) - V(I_j)$, for any $j \in J$. So $Spec(R) = \bigcup_{j\in J}(Spec(R) - V(I_j)) = Spec(R) - \bigcap_{j\in J} V(I_j) = Spec(R) - V(\sum_{j\in J} I_j)$. Hence $V(\sum_{j\in J} I_j) = \emptyset$ and $\sum_{j\in J} I_j = R$. So $1 \in \sum_{k\in K} a_k x_k$ where $a_k \in R$, $x_k \in I_k$ and K is a finite subset of J. Therefore, $\sum_{k\in K} I_k = R$ and $V(\sum_{k\in K} I_k) = \emptyset$. So $Spec(R) = \bigcup_{k\in K} W(I_k)$.

Proposition 4. *Let R be a hyperring and $x \in R$. Then $W(x)$ is a compact subset of $Spec(R)$. Also an open subset of $Spec(R)$ is compact if and only if it is a finite union of sets $W(x_i)$, where $x_i \in R$.*

Proof. Consider $W(x) = \bigcup_{i\in I} W(x_i)$. For every $P \in Spec(R)$, if $x \notin P$ then there is $i \in I$ such that $x_i \notin P$. Let $K = \sum_{i\in I}(x_i)$. Since $Spec(R) - V(x) = Spec(R) - V(K)$, then for every prime hyperideal P, that $K \subseteq P$, we have $x \in P$. So $x \in \sqrt{K}$ and $x^m \in K$, for some $m \in \mathbb{N}$. Hence for some $x_1, x_2, ..., x_n$ and $r_i \in R(1 \leq i \leq n)$, we have $x^m = \sum_{i=1}^{n} r_i x_i$. If $P \in Spec(R)$ and $\{x_1, x_2, ..., x_n\} \subseteq P$, then $x^m \in P$ and so $x \in P$. Also if $x \notin P$ then $x_i \notin P$, for some $i \in \{1, 2, ..., n\}$. Therefore, $W(x) = \bigcup_{i=1}^{n} W(x_i)$.
For the last part, let X be a compact and open subset of $Spec(R)$ and consider $X = \bigcup_{i\in I} W(x_i)$. Then there is a finite subcover $X = \bigcup_{i=1}^{n} W(x_i)$ for some $x_1, x_2, ..., x_n$. Also $X = \bigcup_{i=1}^{n} W(x_i)$ is compact, since the finite union of compact sets is compact.

Remark 5. *[17] For a hyperring R, similar to a ring it can be shown that if R is Artinian, then every prime hyperideal is maximal, and the number of maximal hyperideals is finite. Also, R is Artinian if and only if R is Noetherian and $dim(R) = 0$.*

By Proposition 2 , it conclude that if R is a Noetherian hyperring, then $Spec(R)$ is Noetherian space.

Proposition 5. *Let R be a Noetherian hyperring. The following assertions are equivalent:*

(i) R is Artinian.

(ii) $Spec(R)$ is a discrete and finite space.

(iii) $Spec(R)$ is a discrete space.

Proof. $1 \Rightarrow 2$: If R is Artinian hyperring by Remark 5, $mSpec(R) = Spec(R)$ and each point of $Spec(R)$ is closed and the number of maximal hyperideals of R is finite. Hence $Spec(R)$ is a discrete and finite space.
$3 \Rightarrow 1$: Since $Spec(R)$ is a discrete space, then each point is closed. By Lemma 2, any closed point is maximal hyperideal. So $mSpec(R) = Spec(R)$ and $dim(R) = 0$. Now by Remark 5, R is Artinian.

Proposition 6. *Let R be a hyprring. Then the irreducible components of $Spec(R)$ are closed sets $V(I)$, where I is a minimal prime hyperideal of R.*

Proof. If X is a maximal irreducible subset of $Spec(R)$, then by Theorem 1, X is closed and so $X = V(I)$, for some hyperideal I of R. By Corollary 3, $\sqrt{I}$ is prime hyperideal and if $P \in Spec(R)$ such that $V(P)$ is irreducible and $P \subseteq I$, then $X = V(I) \subseteq V(P)$. So $V(I) = V(P)$ and therefore, $\sqrt{P} = \sqrt{I}$.

Theorem 10. *If R is a local hyprring, then $Spec(R)$ is connected.*

Proof. Let $x \in R$ be a nontrivial idempotent and M be a maximal hyperideal of R. Then $x \notin M = nil(R)$. Since x is not unit, then X is contained in some maximal hyperideal. So $x \in M$ and it is contradiction.

Remark 6. *[6] Let S be a multiplicative subset of hyperring R, and I be a hyperideal of R. Then $S \cap I \neq \emptyset$ if and only if $S^{-1}I = S^{-1}R$. Also there is a one to one correspondence between the sets $\{P; P \in Spec(R), P \cap S = \emptyset\}$ and $Spec(S^{-1}R)$, under the mapping $P \mapsto S^{-1}P$.*

Proposition 7. *Let S be a multiplicative subset of hyperring R. Then $Spec(R)$ is irreducible and $nil(R) \cap S = \emptyset$ if and only if $Spec(S^{-1}R)$ is irreducible.*

Proof. If $Spec(R)$ is irreducible, then by Theorem 8, $nil(R) \in Spec(R)$ and since $nil(R) \cap S = \emptyset$ and $nil(S^{-1}R) = S^{-1}nil(R)$, then $nil(S^{-1}R) \in Spec(S^{-1}R)$ and so $Spec(S^{-1}R)$ is irreducible.
If $Spec(S^{-1}R)$ is irreducible, then $nil(S^{-1}R) = S^{-1}nil(R) \in Spec(S^{-1}R)$. Hence $nil(R) \cap S = \emptyset$ and $nil(R) \in Spec(R)$.

Proposition 8. *$Spec(S^{-1}R)$ is disconnected if and only if $Spec(R)$ is disconnected.*

Proof. Since $S^{-1}(R_1 \times R_2) \cong S^{-1}R_1 \times S^{-1}R_2$, then by Theorem 5, and Theorem 7, the result immediately holds.

Example 1. *Let $(R, +, .)$ be a commutative ring with a unit element and G be a subgroup of monoid $(R - \{0\}, .)$ and I be an ideal of R.*
(1) Consider $\bar{R} = R/G = \{rG; r \in R\}$ and $rG \oplus sG = \{tG;\ t \in rG + sG\}$, $rG \odot sG = rsG$. Then $(\bar{R}, \oplus, \odot)$ is hyperring and $0_{\bar{R}} = \{0\}$, $1_{\bar{R}} = G$. Moreover if R is a field, then $(\bar{R}, \oplus, \odot)$ is a hyperfield,(For more details see [7]).
Then it is easy to verify R are of the form I/G such that I is ideal of R. Let $P \in Spec(R)$ and $rG \odot sG = rsG \in P/G$, then $rs \in P$. So $r \in P$ or $s \in P$ and therefore, $rG \in P/G$ or $sG \in P/G$. and $P/G \in Spec(\bar{R})$. Also for any $A \in Spec(\bar{R})$, there is a prime ideal $P \in Spec(R)$ such that $A = P/G$. So $Spec(\bar{R}) = \{P/G; P \in Spec(R)\}$.
(2) Consider $\bar{R} = R/I = \{r + I; r \in R\}$ and define $r + I \oplus s + I = (r + s) + I$ and $r + I \odot s + I = \{t + I; t \in rs + I\}$. Then $(\bar{R}, \oplus, \odot)$ is multiplicative hyperring with $0_{\bar{R}} = I$, $1_{\bar{R}} = 1 + I$. It is clear that $Spec(\bar{R}) = \{P/I; P \in Spec(R)\} \cong V(I)$.
(3) If $\bar{R} = \{r + I; r \in R\}$, $r + I \oplus s + I = \{t + I; t \in r + s + I\}$ and $r + I \odot s + I = \{t + I; t \in rs + I\}$, then $(\bar{R}, \oplus, \odot)$ is general hyperring and $Spec(\bar{R}) = \{P/I; P \in Spec(R)\} \cong V(I)$.

Corollary 5. *$Spec(\bar{R})$ is irreducible (disconnected) if and only if $Spec(R)$ is irreducible (disconnected).*

Proof. *Clearly, $P \in Spec(R)$ if and only if $P/G \in Spec(\bar{R})$. Since $1_R \in G$, then $nil(\bar{R}) = \{rG;\ r^m G = 0,\ for\ some\ m \in \mathbb{N}\} = \{rG;\ r^m = 0,\ for\ some\ m \in \mathbb{N}\} = \{rG;\ r \in nil(R)\} = nil(R)/G$. So $nil(R) \in Spec(R)$ if and only if $nil(\bar{R}) = nil(R)/G \in Spec(\bar{R})$. By Theorem 8, $Spec(R)$ is irreducible if and only if $Spec(\bar{R})$ is irreducible. Since $(R_1 \times R_2)/G \cong R_1/G \times R_2/G$, it is clear that $Spec(R)$ is disconnected if and only if $Spec(\bar{R})$ is disconnected.*

Example 2. *Let $R = \mathbb{Z}$ and $G = \{\pm 1\}$. Then $\bar{R} = \{\{\pm a\}; a \in \mathbb{Z}\}$ and $\{\pm a\} \oplus \{\pm b\} = \{\{\pm(a+b)\}, \{\pm(a-b)\}\}$, $\{\pm a\} \odot \{\pm b\} = \{\pm ab\}$. For every prime $p \in \mathbb{Z}$; $(p)/G = \{\{\pm kp\}; k \in \mathbb{N} \cup \{0\}\}$. Also $\mathbb{Z}$ is integral domain therefore, $\bar{R}$ is hyperdomain and $nil(\bar{R})$ is a prime hyperideal of $\bar{R}$. So $Spec(\bar{R})$ is irreducible and connected.*

Example 3. *In Example 2, if F is a field and $R = F[x]$ and $G = F - \{0\}$, then $Spec(\bar{R}) = \big\{(f(x))/G\ ;\ f(x) \in F[x] \text{ is irreducible}\big\}$.*

4. From Zariski topology of hyperrings to Zariski topology of rings

This section develops a categorical and functorial approach to the Zariski topology of hyperrings. We explore the role of the fundamental relation γ^* in connecting hyperring spectra to their classical counterparts and introduce a topology on prime strongly regular relations.

Definition 6. *[18] Let $\mathcal{U}$ be the set of all finite sums of finite products of elements of a general hyperring R. Then $(a, b) \in \gamma^*$ if and only if there exists $(a = z_1, z_2, ..., b = z_{n+1}) \in R^{n+1}$ and $U_1, U_2, ..., U_n \in \mathcal{U}$ such that $\{z_i, z_{i+1}\} \subseteq U_i$, for any $i \in \{1, 2, ..., n\}$.*

The relation γ^* is the smallest equivalence relation on the general hyperring R such that the quotient R/γ^* is a ring. R/γ^* is called the fundamental ring [15].

Proposition 9. *If ρ is a strongly regular relation on Krasner hyperring R, then $\rho(0) = \{x \in R; \rho(x) = 0_{R/\rho}\}$ is a normal hyperideal of R.*

Proof. *For every $x, y \in \rho(0)$; $\rho(x-y) = \rho(x) - \rho(y) = 0_{R/\rho}$. So $x - y \subseteq \rho(0)$. Let $r \in R$, then $\rho(rx) = \rho(r)\rho(x) = \rho(r)0_{R/\rho} = 0_{R/\rho}$. So $rx \in \rho(0)$, and similarly $xr \in \rho(0)$. Also, since $\gamma^* \subseteq \rho$ and $r - r \subseteq \gamma^*(0)$, then $r - r \subseteq \rho(0)$ and $r + \rho(0) - r \subseteq \rho(0)$. Thus $\rho(0)$ is normal.*

Clearly every hyperideal I of R containing $\gamma^*(0)$ is normal, and R/I is Krasner hyperring. Denote the set of all strongly regular relations on R, by $\mathcal{SR}(R)$ and the set of all hyperideals containing $\gamma^*(0)$, by $\mathcal{I}(\gamma^*(0))$.

Lemma 4. *If $\rho \in \mathcal{SR}(R)$, then $\rho(x) = x + \rho(0)$ for every $x \in R$.*

Proof. *If $z \in \rho(x)$, then $-x + z \subseteq \rho(0)$ and $z \in x + (-x + z) \subseteq x + \rho(0)$. So $z \in x + \rho(0)$ and $\rho(x) \subseteq x + \rho(0)$. Conversely, if $z \in x + \rho(0)$, then $z \in x + y$, for some $y \in \rho(0)$. So $\rho(z) = \rho(x)$ and $z \in \rho(x)$. Therefore, $x + \rho(0) \subseteq \rho(x)$.*

Lemma 5. *If $I \in \mathcal{I}(\gamma^*(0))$, then the congruence relation modulo I is strongly regular.*

Proof. Let $x, y, z \in R$ and $x + I = y + I$. Then $(z + x) + I = (z + y) + I$ and since $S_\beta \subseteq I$, then for every $r \in z + x$ and $s \in z + y$; $(z+x)+I = r+I$ and $(z+y)+I = s+I$. Hence the congruence relation modulo I is strongly regular.

Let $\mathcal{SR}(R)$ be the set of all strongly regular relations on canonical hypergroup $(R, +)$. If $\rho, \sigma \in \mathcal{SR}(R)$, then $\rho \vee \sigma \in \mathcal{SR}(R)$, and the lattices $N(S_\beta) = \{H;\ S_\beta \subseteq H \lhd R\}$ and $\mathcal{SR}(R)$ are isomorph, [11].

Theorem 11. *For every hyperring R, $\mathcal{SR}(R) \cong \mathcal{I}(\gamma^*(0))$ is an isomorphism of complete lattices.*

Proof. Since R is a Krasner hyperring, then for every $\rho, \sigma \in \mathcal{SR}(R)$, $A \subseteq \mathcal{SR}(R)$, $I, J \in \mathcal{I}(\gamma^*(0))$ and $B \subseteq \mathcal{I}(\gamma^*(0))$, we have $\rho \vee \sigma, \rho \cap \sigma \in \mathcal{SR}(R)$ and $I \vee J = I + J, I \cap J \in \mathcal{I}(\gamma^*(0))$. Also

$$\bigvee_{\rho \in A} \rho \in \mathcal{SR}(R), \quad \bigcap_{\rho \in \mathcal{SR}(R)} \rho = \gamma^*, \quad \bigvee_{I \in B} I = \sum_{I \in B} I \in \mathcal{I}(\gamma^*(0)), \quad \bigcap_{I \in \mathcal{I}(\gamma^*(0))} I = \gamma^*(0).$$

Let $f : \mathcal{SR}(R) \to \mathcal{I}(\gamma^*(0))$ by $\rho \mapsto \rho(0)$ and $g : \mathcal{I}(\gamma^*(0)) \to \mathcal{SR}(R)$ by $I \mapsto \rho_I := \{(x, y) \in R^2;\ x + I = y + I\}$. By Lemmas 4 and 5, f and g are well defined, and

$$\begin{aligned} f \circ g(I) &= f(\rho_I) \\ &= \{r \in R;\ (x + I) + (r + I) = x + I, \forall x \in R\} \\ &= \{r \in R;\ (x + r) + I = x + I, \forall x \in R\} = I, \end{aligned}$$

because, for every $x \in R$, $0 \in -x + x \subseteq I$. Also

$$\begin{aligned} g \circ f(\rho) &= g(\rho(0)) \\ &= \{(x, y) \in R^2;\ x + \rho(0) = y + \rho(0)\} \\ &= \{(x, y) \in R^2;\ \rho(x) = \rho(y)\} = \rho. \end{aligned}$$

Now let $\rho, \sigma \in \mathcal{SR}(R)$ and $I, J \in \mathcal{I}(\gamma^*(0))$ such that $\rho \subseteq \sigma$ and $I \subseteq J$. Then $f(\rho) = \rho(0) \subseteq \sigma(0) = f(\sigma)$ and if $(x, y) \in R^2$ such that $x + I = y + I$, then $x + I + J = y + I + J$. Therefore, $x + J = y + J$ and $f^{-1}(I) \subseteq f^{-1}(J)$.

Corollary 6. *If $\rho \in \mathcal{SR}(R)$, then $R/\rho \cong R/\rho(0)$ is a ring isomorphism.*

Proof. Since for every $x, y \in R$ and $z \in x + y$; $(x + \rho(0)) + (y + \rho(0)) = (x + y) + \rho(0) = z + \rho(0)$, then $R/\rho(0)$ is a ring. Consider the bijection map $\phi : R/\rho \to R/\rho(0)$ by $\rho(x) \mapsto x + \rho(0)$, then: $\phi(\rho(x) + \rho(y)) = \phi(\rho(x + y)) = (x + y) + \rho(0) = (x + \rho(0)) + (y + \rho(0)) = \phi(\rho(x)) + \phi(\rho(y))$ and $\phi(\rho(x)\rho(y)) = \phi(\rho(xy)) = (xy) + \rho(0) = (x + \rho(0))(y + \rho(0)) = \phi(\rho(x))\phi(\rho(y))$.

Therefore, if $\rho, \sigma \in \mathcal{SR}(R)$ and $I, J \in \mathcal{I}(\gamma^*(0))$, then $\rho(0) + \sigma(0) = (\rho \vee \sigma)(0)$, $\rho(0) \cap \sigma(0) = (\rho \cap \sigma)(0)$ and $\rho_I \vee \rho_J = \rho_{I+J}$, $\rho_I \cap \rho_J = \rho_{I \cap J}$.
The hyperideal I of a Krasner hyperring R is a normal hyperideal if and only if $x + I - x \subseteq I$, for all $x \in R$, [18]. Also it is proved that R/I is a ring if and only if I is a normal hyperideal of R, [18].

Corollary 7. *Let I be a hyperideal of Krasner hyperring R. Then, I is normal if and only if $\gamma^*(0) \subseteq I$.*

Proof. *Since $0 \in I$, then the hyperideal I is normal if and only if $x - x \subseteq I$, for all $x \in R$. If $y \in \gamma^*(0)$, then there are $n \in \mathbb{N}$ and $(x_1, ..., x_n) \in R^n$ such that $0, y \in \sum_{i=1}^n x_i$. Therefore, $y \in \sum_{i=1}^n x_i - \sum_{i=1}^n x_i \subseteq I$. Let $\gamma^*(0) \subseteq I$. Since $x - x \subseteq \gamma^*(0)$, for all $x \in R$, then I is normal.*

So we can say that if R is a Krasner hyperring, then R/I is ring if and only if $I \in \mathcal{I}(\gamma^*(0))$.

Definition 7. *A strongly regular relation ρ on the commutative and of unity Krasner hyperring R is called prime, if for every x and y in R, we have:*

$$(xy, 0) \in \rho \ \Rightarrow \ (x, 0) \in \rho \ \text{ or } \ (y, 0) \in \rho. \tag{7}$$

Also ρ is primitive, if for every x and y in R, we have:

$$(xy, 0) \in \rho \ , \ (x, 0) \notin \rho \ \Rightarrow \ \exists n \in \mathbb{N} \ \ s.t. \ \ (y^n, 0) \in \rho. \tag{8}$$

Since, $(x, 0) \in \rho$ if and only if $x \in \rho(0)$, then ρ is prime if and only if $xy \in \rho(0)$ results in $x \in \rho(0)$ or $y \in \rho(0)$, if and only if $\rho(x)\rho(y) = \rho(0)$ results in $\rho(x) = \rho(0)$ or $\rho(y) = \rho(0)$ if and only if R/ρ is an integral domain. Therefore, ρ is prime if and only if $\rho(0)$ is a prime hyperideal of R.
Additionally, since the lattices $\mathcal{SR}(R)$ and $\mathcal{I}(\gamma^*(0))$ are isomorph, then ρ is maximal if and only if $\rho(0)$ is a maximal hyperideal of R.

Example 4. *Consider the Krasner hyperring R as follows:*

$+$	0	1	a	b	c	d	e	f
0	0	1	a	b	c	d	e	f
1	1	a,d	b,e	c	$1,f$	e	$0,c$	a
a	a	b,e	$0,c$	1	a,d	c	$1,f$	e
b	b	c	1	d	e	f	a	0
c	c	$1,f$	a,d	e	$0,c$	a	b,e	1
d	d	e	c	f	a	0	1	b
e	e	$0,c$	$1,f$	a	b,e	1	a,d	c
f	f	a	e	0	1	b	c	d

.	0	1	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0
1	0	1	a	b	c	d	e	f
a	0	a	c	d	c	0	a	d
b	0	b	d	f	0	d	f	b
c	0	c	c	0	c	0	c	0
d	0	d	0	d	0	0	d	d
e	0	e	a	f	c	d	1	b
f	0	f	d	b	0	d	b	f

Then $\gamma^* = \{(a,d),(d,a),(b,e),(e,b),(1,f),(f,1),(0,c),(c,0)\} \cup \Delta_R$ is a primitive strongly regular relation on R that is not prime because $(ad,0) \in \gamma^*$ but $(a,0),(d,0) \notin \gamma^*$. Equivalently, $\gamma^*(0) = \{0,c\}$ is a primitive hyperideal that is not prime.
Also, $R/\gamma^* = \{\{0,c\},\{1,f\},\{a,d\},\{b,e\}\} \cong \mathbb{Z}_4$, which is not an integral domain, and $\mathcal{I}(\gamma^*(0)) = \{\{0,c\},\{0,c,a,d\}\}$, where $\{0,c,a,d\} = V(\gamma^*(0)) = \sqrt{\gamma^*(0)}$ is a maximal hyperideal of R and

$$R/\sqrt{\gamma^*(0)} = \{\{0,c,a,d\},\{1,f,b,e\}\} \cong \mathbb{Z}_2$$

is a field. Let $I = \{0,d\}$ and $M = \{0,b,d,f\}$. Because $a,f \notin I$ while $af = d \in I$ and $\gamma^*(0) \nsubseteq I$, then I is neither prime nor normal. Also, M is a maximal hyperideal which is not normal. So, $Spec(R) = \{M, \sqrt{\gamma^*(0)}\}$.

Theorem 12. *Let R be a hyperring. Then $Spec(R/\gamma^*) = \{P/\gamma^*; P \in Spec(R)\}$.*

Proof. Let I be a prime ideal of R/γ^* and $P = \{s \in R; \gamma^*(r) \in I\}$. So $I = P/\gamma^*$ and if $s,t \in P$ and $r,k \in R$, then $\gamma^*(s) - \gamma^*(t) = \gamma^*(s-t) \in I$ and $\gamma^*(r)\gamma^*(s) = \gamma^*(rs) \in I$. Hence $s - t \subseteq P$ and $rs \in P$. If $\gamma^*(r)\gamma^*(k) = \gamma^*(rk) \in I$, then $\gamma^*(r) \in I$ or $\gamma^*(k) \in I$. So if $rk \in P$, then $r \in P$ or $k \in P$.

Lemma 6. *Let $S \subset Spec(R)$. Then $\bar{S} = V(\bigcap_{P\in S} P)$.*

Proof. Let I be the radical hyperideal such that $V(I) = \bar{S}$, then $I \subset \bigcap_{P\in S} P$. Since $V(\bigcap_{P\in S} P)$ is closed, then $S \subset V(\bigcap_{P\in S} P)$ and so $\bar{S} \subset V(\bigcap_{P\in S} P)$. Hence $V(I) \subset V(\bigcap_{P\in S} P)$ and by Proposition 2, $\bigcap_{P\in S} P = I$.

Theorem 13. *Let $f : R \to S$ be a good homomorphism of hyperrings. Then:*

(i) If I is a hyperideal of R, then $\bar{f}^{-1}(V(I)) = V(I^e)$.

(ii) For every $x \in R$, $\bar{f}^{-1}(W(x)) = W(f(x))$.

(iii) If J is a hyperideal of S, then $\overline{\bar{f}(V(J))} = V(J^c)$.

(iv) If f is surjective, then $\bar{f}$ is a homiomorphism of $Spec(S)$ on to the closed subset $V(Ker(f))$ of $Spec(R)$.

(v) If f is injective, then $\bar{f}(Spec(S))$ is dense in $Spec(R)$. In fact, the image $\bar{f}(Spec(S))$ is dense in $Spec(R)$ if and only if $Ker(f) \subset \sqrt{0}$.

(vi) If $g : S \to T$ is another homomorphism of hyperrings, then $\overline{g \circ f} = \bar{f} \circ \bar{g}$.

Proof. (i) By Theorem 3, we have $\bar{f}^{-1}(V(I)) = V(f(I))$ and since $(f(I)) = I^e$, by Lemma 2, $V(f(I)) = V(I^e)$.
(ii) For every $x \in R$ we have:

$$\bar{f}^{-1}(W(x)) = \{P \subset S; x \notin f^{-1}(P)\} = \{P \subset S; f(x) \notin P\} = W(f(x)).$$

(iii) By Lemma 6, $\overline{\bar{f}(V(J))} = V(\bigcap_{P\in\bar{f}(V(J))} P)$ but

$$\textstyle\bigcap_{P\in\bar{f}(V(J))} P = \bigcap_{J\subset Q} f^{-1}(Q) = f^{-1}(\bigcap_{J\subset Q} Q) = f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}.$$

Now by Proposition 2, we have $V(\bigcap_{P\in\bar{f}(V(J))} P) = V(f^{-1}(J))$.
(iv) Since f *is surjective, consider* $S = R/Ker(f)$. *We know that there is an inclusion perserving one to one correspondence between prime hyperideals of* $R/Ker(f)$ *and prime hyperideals of* R *containing* $Ker(f)$. *So* $\bar{f}$ *is a continuous bijection onto the closed subset* $V(Ker(f))$, *and* $\bar{f}^{-1}$ *is continuous because:*

$$\bar{f}(I/Ker(f)) = \{P \in Spec(R); I/Ker(f) \subset P/Ker(f) \in Spec(R/Ker(f))\} = V(I).$$

(v) Since $\bar{f}(Spec(S)) = \bar{f}(V(0))$, *then by (3) we have* $\overline{\bar{f}(Spec(S))} = V(Ker(f))$. *So* $\bar{f}(Spec(S))$ *is dense in* $Spec(R)$ *if and only if* $V(Ker(f)) = Spec(R)$ *if and only if* $Ker(f) \subset \sqrt{0}$.
(vi) It is clear that $\overline{g\circ f}(P) = (g\circ f)^{-1}(P) = (f^{-1}\circ g^{-1})(P) = \bar{f}(\bar{g}(P))$.

Let R be a hyperring. By *ZTop.HRg*, we mean the category of Zariski topology of (Krasner) hyperrings, which objects are $X = Spec(R)$ and for $X = Spec(R)$ and $Y = Spec(S)$, $Hom(X,Y)$ is the set of all continuous maps induced by hyperring homomorphisms, with usual combinations of functions. Also, *ZTop.Rg* denotes the category of Zariski topology of all rings.

Theorem 14. *The mapping* $F : ZTop.HRg \to ZTop.Rg$ *defined by* $Spec(R) \mapsto Spec(R/\gamma^*)$ *and*

$$\big(f : Spec(R) \to Spec(S)\big) \mapsto \big(f^* : Spec(R/\gamma^*) \to Spec(S/\gamma^*)\big).$$

Then the following statements are satisfied:

(i) F *is functor.*

(ii) The following diagram is commutative (π_R *and* π_S *are canonical projections):*

$$\begin{array}{ccc} Spec(R) & \xrightarrow{\ f\ } & Spec(S) \\ \downarrow{\scriptstyle \pi_R} & & \downarrow{\scriptstyle \pi_S} \\ Spec(R/\gamma^*) & \xrightarrow[f^*]{} & Spec(S/\gamma^*) \end{array}$$

Proof. *(i) Closed subsets of* $Spec(S/\gamma^*)$ *are of the form* $V(J/\gamma^*)$ *where* J *is hyperideal of* S, *and* $V(J/\gamma^*) = \{P/\gamma^* \in Spec(S/\gamma^*); p \in V(J)\}$. *So* ${g^*}^{-1}(V(J/\gamma^*)) = \{Q/\gamma^* \in Spec(R/\gamma^*); Q \in g^{-1}(V(J))\}$. *Since* g *is continuous, then* $g^{-1}(V(J))$ *is closed subset of* $Spec(R)$ *and hence there is hyperideal* I *of* R *such that* $g^{-1}(V(J)) = V(I)$. *Thus* ${g^*}^{-1}(V(J/\gamma^*)) = V(I/\gamma^*)$. *It is clear that* $F(1_{Spec(R)}) = 1_{Spec(R/\gamma^*)}$ *and* $(h\circ g)^* = h^*\circ g^*$.
(ii) let $P \in Spec(R)$, *then* $f^*\pi_R(P) = f^*(P/\gamma^*) = f(P)/\gamma^* = \pi_S f(P)$.

Theorem 15. *The mapping* $Spec(-) : H.Rg \to ZTop.HRg$, *by* $R \mapsto Spec(R)$ *and* $f \mapsto \bar{f}$ *is a contravariant functor.*

Proof. *By Theorem 3,* $\bar{f}$ *is continuous and by Theorem 13,* $\overline{g\circ f} = \bar{f}\circ\bar{g}$.

Theorem 16. *The following diagram is commutative:*

$$\begin{array}{ccc} H.Rg & \xrightarrow{Spec(-)} & ZTop.HRg \\ \gamma^{*}\downarrow & & \downarrow F \\ Rg & \xrightarrow[Spec(-)]{} & ZTop.Rg \end{array}$$

Proof. *It is clear that* $F(Spec(R)) = Spec(\gamma^*(R))$. *If* $f : R \to S$ *is a good homomorphism of hyperrings then we have* $\bar{f} : Spec(R) \to Spec(S)$ *by* $P \mapsto f^{-1}(P)$ *and* $f^* : R/\gamma^* \to S/\gamma^*$ *by* $\gamma^*(r) \mapsto \gamma^*(f(r))$. *So* $\bar{f}^{*} = \bar{f^*}$, *because* $\bar{f}^{*}(P/\gamma^*) = f^{-1}(P)/\gamma^* = {f^*}^{-1}(P/\gamma^*) = \bar{f^*}(P/\gamma^*)$.

Example 5. *Consider hyperring* $(\bar{\mathbb{Z}}, \oplus, \odot)$, *in which* $\bar{\mathbb{Z}} = \{\bar{n}; n \in \mathbb{Z}\}$, $\bar{n} = \{-n, n\}$, $\bar{n} \oplus \bar{m} = \{\overline{n+m}, \overline{n-m}\}$, *and* $\bar{n} \odot \bar{n} = \overline{nm}$.
Analogous to the classical case, $\bar{\mathbb{Z}}$ *is a principal hyperideal domain; which is, its hyperideals are of the form* $(\bar{a}) = \{\overline{na}; n \in \mathbb{Z}\}$, *for every* $a \in \mathbb{Z}$. *Moreover, the prime hyperideals of* $\bar{\mathbb{Z}}$ *are of the form* $(\bar{p})$, *where* p *is a prime number. So* $Spec(\bar{\mathbb{Z}}) = \{(\bar{p});\ p \text{ is a prime number}\}$.
Since $\gamma^*(\bar{0}) = \bar{\mathbb{Z}}$ *(or, equivalently,* $\gamma^* = \bar{\mathbb{Z}} \times \bar{\mathbb{Z}}$*), we have:*

$$Spec\big(\gamma^*(\bar{\mathbb{Z}})\big) = F\big(Spec(\bar{\mathbb{Z}})\big) \cong \mathbb{Z}_1.$$

5. Discussion

The framework developed in this paper establishes a deep connection between the algebraic properties of commutative Krasner hyperrings and the topological structure of their spectra under the Zariski topology. By characterizing connectedness, irreducibility, compactness, and separation in terms of properties like the presence of nontrivial idempotents or the nature of the nilradical, we provide a bridge between hyperstructure theory and topological algebra.

A central contribution of this work is the introduction of a topological structure on the set of prime strongly regular relations, which reflects the hyperring's internal congruence geometry. The demonstration of a homeomorphism between this space and a subspace of the classical spectrum via the fundamental relation γ^* offers a novel perspective, enabling a reinterpretation of classical results through hyperalgebraic lenses. This correspondence not only enriches the conceptual understanding of hyperrings but also sets the stage for potential generalizations of geometric tools, such as schemes and sheaves, to hyperring contexts.

Additionally, our categorical treatment of Zariski topology via the $Spec(-)$ functor reveals the functorial behavior of spectra under good homomorphisms. We demonstrated how the interplay between strongly regular relations and the fundamental relation γ^* enables the transfer of structural information from hyperrings to their associated fundamental rings. This opens the door to developing new categorical frameworks for hyperrings that mirror the structural richness found in classical algebraic geometry.

An open question addressed in this study concerns the existence of functors from hyperrings to integral domains via families of strongly regular relations. Although such sequences are not functorial in general, we propose sufficient conditions under which they may define a contravariant functor, potentially leading to a functorial theory of hyperring localization.

This exploratory work highlights the untapped potential of combining topological, algebraic, and categorical methods in the study of hyperstructures, and invites further developments in hyperalgebraic geometry and its applications.

6. Conclusion

In this paper, we presented a detailed study of the Zariski topology on the spectrum of prime hyperideals in commutative Krasner hyperrings. We established precise algebraic conditions under which the spectrum exhibits topological properties such as connectedness, irreducibility, compactness, and various separation axioms. Our results reveal that these properties are intrinsically tied to the structure of the hyperring—specifically the presence of nontrivial idempotents, the nature of the nilradical, and the behavior of hyperideals under multiplication.

Beyond the classical spectrum, we introduced a topology on the set of prime strongly regular relations and demonstrated its homeomorphism with a subspace of the spectrum determined by the fundamental relation γ^*. This dual topological viewpoint offers a new route for developing geometric theories over hyperstructures, particularly in connection with sheaves and categorical constructions.

By incorporating functorial methods and exploring the behavior of spectra under good homomorphisms, we laid a foundational platform for extending algebraic geometry tools to hyperrings. The categorical insights and topological frameworks developed here are expected to contribute to ongoing research in the theory of hypermodules, spectral constructions, and algebraic geometry beyond the confines of traditional ring theory.

The primary motivation of this paper was to establish a framework for the study of sheaves on the Zariski topology of Krasner hyperrings and hypermodules, and to examine the properties of the associated hyperringed spaces and hyperschemes, with the aim of conducting investigations in hyperalgebraic geometry. In addition, extending the obtained results to other types of hyperrings and hypermodules, as well as exploring the role of regular relations in this context, will be among our future works.

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