EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6693
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Numerical Solutions of the SIR Mathematical Model of Computer Viruses Involving Non-linear Fractional Order Differential Equation

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Abstract. In this paper, the non-linear Fractional order susceptible-infected-Recovered(SIR) mathematical model of computer viruses is presented. For this Fractional differential transform method (FDTM) and the Laplace-Adomian decomposition method (LADM) are applied and compared the obtained results with Runge-Kutta Felhberg Method for $\gamma = 1$. Also, graphs of solutions are plotted up to five iterations from both the method. The fractional derivative used in the model is Caputo Fractional derivative. Using Lyapunov stability analysis,the stability verified of the given mathematical model. Solutions are obtained for three different fractional order values of γ. The validity of the result is confirmed by converting model into integer order. Errors from both the methods are compared.

2020 Mathematics Subject Classifications: 34A08
Key Words and Phrases: Caputo fractional derivative, fractional differential transform method, Laplace Adomian decomposition method, Lyapunov stability, Runge-Kutta Felhberg method

1. Introduction

Computer viruses are harmful programs for normal functioning of computers. There are various types of computer viruses that attack computers. Different types of mathematical models representing spread of computer viruses have been proposed in the literature[1] [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15],[16][17]. These models can be solved using mathematical techniques such as the Differential Transform Method, Collection Method [13], Homotopy Analysis Method [2],[15] Variational Iterational Method[18], and others[7],[16]. Various mathematical and engineering problems are solved using semi-analytical techniques [14],[19],[20],[21],[15],[22],[23],[24],[25]. The Differntial Transform Method and Laplace Adomian Decomposition Method are the tools to solve linear and non-linear preoblems in engineering, physics, mathematics, etc [25],[26],[27],[28],[29],[30],[31]

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6693

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,[32],[33],[34],[35],[36],[37],[38],[39],[40],[41],[42],[43],[44],[45],[46]
In this paper, we have designed and solved the non-linear fractional order system of the Susceptible-Infected-Recovered (SIR) mathematical model of computer viruses. For this purpose, the Fractional Differential Transform Method (FDTM) and Laplace Adomian Decomposition Method (LADM) is applied. The approximate solutions are estimated for different iterations (up to five iterations), and the results are compared with those obtained by 4/5 Runge-Kutta Frlhberg Algorithm (RKFA). The graphs of the approximate solutions up to five iterations and the absolute errors are plotted at different values of γ and t, illustrating the evolution of the system. Comparing the solutions and errors obtained by FDTM and LADM with RKFA, we have concluded that as compared to the LADM, the FDTM gives more accurate results under certain conditions. The methods and results presented in this chapter on the non-linear fractional order system for modeling computer virus propagation through the Susceptible-Infected-Recovered (SIR) framework have broader implications for mathematical modeling and systems analysis. The use of non-linear fractional order models is not limited to computer viruses; similar models can be applied to biological systems such as the spread of infectious diseases in populations. The methodologies outlined in this chapter can be adapted to various epidemiological contexts, potentially improving the understanding of disease dynamics and control measures. The methods employed, particularly the Fractional Differential Transform Method (FDTM) and Differential Transform Method (DTM), can be integrated with other mathematical techniques, such as Agent-based Modeling or Network Theory to provide a more comprehensive analysis of complex systems. This versatility allows the exploration of various dynamics and interactions in different contexts. The mathematical framework developed can be adapted for various applications. The most common and general model is the classical non-linear fractional order Susceptible-Infected-Recovered (SIR) model for computer virus propagation is given below.

2. The S-I-R Model and Parameters

The governing equations of the non-linear fractional-order SIR model[47] are given by:

$$\begin{aligned} \frac{d^\gamma s(t)}{dt^\gamma} &= f_1 - \lambda s(t)i(t) - ds(t), \\ \frac{d^\gamma i(t)}{dt^\gamma} &= f_2 + \lambda s(t)i(t) - \varepsilon i(t) - dr(t), \\ \frac{d^\gamma r(t)}{dt^\gamma} &= f_3 + \varepsilon i(t) - dr(t). \end{aligned} \tag{1}$$

with the initial conditions

$$s(0) = s_0, i(0) = i_0, r(0) = r_0 \tag{2}$$

The parameters and initial values of system 1 are given in Table 1.

Here the fractional derivative used is the Caputo fractional derivative and $\gamma \in (0, 1]$. The generalization of integer order differentiation is known as fractional order differentiation. As the reader can see, calculating derivatives of fractional order is far more difficult

Table 1: List of parameters and their values

Parameter	Meaning	Value
$s(t)$	Susceptible computers to the virus at time t	$s(0) = 20$
$i(t)$	Infected computers to the virus at time t	$i(0) = 15$
$r(t)$	Recovered computers from the virus at time t	$r(0) = 10$
f_1, f_2, f_3	Rate of other computers connecting to the network	0
λ	Rate of virus infection for susceptible computers	0.001
ε	Rate of recovery from virus for infected computers	0.1
d	Rate of removing from the network	0.1

than computing derivatives of traditional integer order. Nevertheless, it has been demonstrated that fractional order mathematical models many phenomena' attributes are better described by integrals and derivatives than by the previously employed integer order models.This is because systems are typically not flawless and can be influenced by outside factors, for instance. Consequently, it might not be possible to comprehend the trajectories of state variables using derivatives of integer order. With fractional derivatives, we may choose whatever fractional differential equation best captures the model's dynamics since we have an infinite number of derivative orders at our disposal. Among other places, [48]and [49] exhibit experimental data and techniques for certain real-world events demonstrating that fractional order derivatives offer more effective solution curve modelling.
In [50], Kamble, R., and Kukarni, P. have proved the existence and uniqueness of solutions for the following equation

$$D^{\alpha}D^{\beta}x(\tau) = f(t, x(\tau), \phi x(\tau), \psi x(\tau)), \ \tau \in [0,1], \ x(0) = x(1) = 0$$

where $0 < \alpha \leq 1, 0 < \beta \leq 1, D^{\alpha}, D^{\beta}$ are the Caputo fractional derivatives of order α , β respectively, $f : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function λ, $\delta : [0,1] \times [0, \ 1] \to [0, +\infty)$, and

$$\phi x(\tau) = \int_0^{\tau} \lambda(\tau, s)\, x(s)\, \mathrm{d}s, \quad \psi x(\tau) = \int_0^{\tau} \delta(\tau, s)\, x(s)\, \mathrm{d}s$$
$$\phi^* = \sup_{t\in[0,1]} \left| \int_0^t \lambda(\tau, s)\, \mathrm{d}s \right| < \infty, \quad \psi^* = \sup_{t\in[0,1]} \left| \int_0^t \delta(\tau, s)\, \mathrm{d}s \right| < \infty$$

In [51], Kamble, R., and Kukarni, P. have proved the existence and uniqueness of solutions for the conformable fractional order Volterra-Fredholm type integro-differentail equations of the form

$$\frac{d^{\alpha}x(t)}{dt} = f(t) + \int_0^t p(t, s, x(s))\, \mathrm{d}s + \int_0^b q(t, s, x(s))\, \mathrm{d}s$$

with the initial condition

$$x(0) = x_0$$

where the term $\frac{d^{\alpha}x(t)}{dt}$ represents the conformable fractional order derivative of fractional order $\alpha \in (0,1)$,$t \in I = [0,b]$, $f : I \to X$, $p, q : I \times I \times X \to X$ are continuous functions

and x_0 is an element of a real Banach space X, with the norm $\|.\|$.
In [52], Kamble, R., and Kukarni, P. have given new definition and Prove there properties

Definition 1. *Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function, $\alpha \in (0, 1], a > 0 \in \mathbb{R}, a \neq 1$. We define the (α, a)-Extended fractional derivative(EFD) of f at x, denoted $D_a^\alpha f(x)$, by*

$$D_a^\alpha f(x) = \lim_{h \to 0} \frac{f(xa^{hx^{(-\alpha)}}) - f(x)}{h}, \tag{3}$$

provided the limit exists.

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In [53], Kamble, R., and Kukarni, P. have proved, Existence and uniqueness of solutions for exponential fractional differential equations.
In [54], Kamble, R., and Kukarni, P. have developed method to solve, $(\alpha, 1)$ Fractional Differential Difference Equations With Conditions,Linear And Nonlinear with help of Laplace Transform And Laplace Decomposition Method.

3. Lyapunov Stability analysis of the given non Linear SIR model of Computer Viruses.

Using lyapunov stability theory we have investigated the stability of given SIR fractional order non linear mathematical model is stable.
The detail about lyapunov stability analysis of caputo fractional order nonliner system is given in [55]. Consider the following non linear system

$$D_{t_0}^\gamma \mathbf{Y}(t) = g(t, \mathbf{Y}(t)) = A\mathbf{Y}(t) + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0}$$

where $\gamma \in (0, 1)$,$\mathbf{Y} \in R^n$ called the state vector of the non linear system.
Here $A \in R^{n\times n}$ is the constant matrix of the given system, and $h : R_+ \times R^n \to R^n$ is the non linear part of the system, where $h(t, 0) = 0$ for all $t \geq 0$. The corresponding linear system is

$$D_{t_0}^\gamma \mathbf{Y}(t) = A\mathbf{Y}(t), \mathbf{Y}(t_0) = \mathbf{Y_0} \tag{4}$$

The detailed stability of the above linear system is given in [55].
Now consider the following non-linear system

$$D_{t_0}^\gamma \mathbf{Y}(t) = A\mathbf{Y}(t) + Bu + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0}$$

where $B \in R^{n\times p}$ is the constant matrix of the given system,$u \in R^p$ is the control input to be defined.
Writing the given SIR mathematical model of Computer Viruses in the form

$$D_{t_0}^\gamma \mathbf{Y}(t) = A\mathbf{Y}(t) + Bu + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0}$$

where $A = \begin{bmatrix} -d & 0 & 0 \\ 0 & -\epsilon & -d \\ 0 & \epsilon & -d \end{bmatrix}$, $B = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$, and $\mathbf{Y(t)} = \begin{bmatrix} s(t) & i(t) & r(t) \end{bmatrix}$ and $h(t, \mathbf{Y(t)}) = \begin{bmatrix} -\lambda s(t)i(t), & \lambda s(t)i(t), & 0 \end{bmatrix}$.

Since $B = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the system convert into

$$D_{t_0}^{\gamma} \mathbf{Y}(t) = A\mathbf{Y}(t) + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0} \tag{5}$$

Definition 2. *[55] The non linear function $h(t, \mathbf{Y})$ is said to be one sided Lipschitz if there exist $\rho \in R$ such that*
$\langle h(t, \mathbf{Y_1}) - h(t, \mathbf{Y_2}), \mathbf{Y_1} - \mathbf{Y_2} \rangle \leq \rho(\|\mathbf{Y_1} - \mathbf{Y_2}\|)^2$

Theorem 1. *[55] Let $\mathbf{Y} = 0$be an equilibrium point of the system 4. If the state matrix A is Hurwitz then the trivial solution of the fractional linear system 4 is fractional asymptotically stable.*

Theorem 2. *[55] Let $\mathbf{Y} = 0$ be an equilibrium point of the system 5. Let that the state matrix A is Hurwitz and the condition $\|h(t, \mathbf{Y}(t))\| < \rho$ holds.*
If there exist a positive definite matrix P such that the following inequality holds
$\rho < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$ where λ is eigen value of matrix.
where $A^T P + PA = -Q$, then the trivial solution of the fractional linear system 5 is fractional asymptotically stable.

Theorem 3. *[55] If the function $h(t, \mathbf{Y}(t))$ is Lipschitz continuous,L is Lipschitz Constant. Assume that the following assumption is hold : Then There exists a positive symmetric matrix P and positive constant ϵ such that the following inequalities hold $A^T P + PA + \epsilon I < 0$ and $L < \frac{\epsilon}{2\lambda_{max} P}$ where λ is eigen value of matrix, then the trivial solution of fractional system 5 is asymptotically stable.*

The given SIR mathematical model of Computer Viruses can be written in the form

$$D_{t_0}^{\gamma} \mathbf{Y}(t) = A\mathbf{Y}(t) + Bu + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0}$$

where $A = \begin{bmatrix} -d & 0 & 0 \\ 0 & -\epsilon & -d \\ 0 & \epsilon & -d \end{bmatrix}$, $B = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$, $\mathbf{Y(t)} = \begin{bmatrix} s(t) & i(t) & r(t) \end{bmatrix}$ and $h(t, \mathbf{Y(t)}) = \begin{bmatrix} -\lambda s(t)i(t), & \lambda s(t)i(t), & 0 \end{bmatrix}$ Since $B = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the system convert into

$$D_{t_0}^{\gamma} \mathbf{Y}(t) = A\mathbf{Y}(t) + h(t, \mathbf{Y}(t)), \mathbf{Y}(t_0) = \mathbf{Y_0}$$

For the given system $A = \begin{bmatrix} -d & 0 & 0 \\ 0 & -\epsilon & -d \\ 0 & \epsilon & -d \end{bmatrix} = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.1 & -0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$
A is Hurwitz matrix since the eigen values are $\lambda_1 = -0.1$,

$\lambda_2 = 0.1 + 0.447213595499996i, \lambda_3 = 0.1 - 0.447213595499996i$
We can find positive definite matrices P and Q as given below where
$P = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\rho = \frac{1}{10}$.
With these values, all the conditions of theorem 2 are satisfied. Also, with the P given as above and$\epsilon = 1$ $L = \frac{1}{20}$ all the condition of theorem 3 is satisfied. Therefore by theorem 3 the given fractional non linear system of SIR model is also asymptotically stable.

4. Numerical Methods

The Fractional Differential Transform Method (FDTM) is used to solve the non-linear fractional order system of the Susceptible-Infected-Recovered mathematical model of computer viruses. In this section, we apply FDTM to solve the system of equations 1. The numerical solutions are compared with those obtained using the Differential Transform Method (DTM), verifying that our proposed results are valid when fractional order derivative $\gamma = 1$. Additionally, we have taken five iterations of the solutions ($k = 5$) and compute the values of $s(t)$, $i(t)$, and $r(t)$ for different time points.
The Laplace Adomian Decomposition method (LADM) is also used to solve the non-linear fractional order system of the Susceptible-Infected-Recovered mathematical model of computer viruses. In this section, we apply LADM to solve the system of equations 1. The numerical solutions are compared, verifying that our proposed results are valid when fractional order derivative $\gamma = 1$. Additionally, we have taken five iterations of the solutions ($k = 5$) and compute the values of $s(t)$, $i(t)$, and $r(t)$ for different time points.
The Runge-Kutta Felhberg Algorithm (RKFA) is also used to solve the non-linear fractional order system of the Susceptible-Infected-Recovered mathematical model of computer viruses for fractional order derivative $\gamma = 1$.
The results are visually presented through graphs that illustrate the evolution of the susceptible, infected, and recovered populations over time for both methods. A comparison of results obtained from FDTM and LADM is performed to analyze differences and evaluate the effectiveness of the fractional order model in capturing the dynamics of computer virus propagation. Future work could explore higher-order fractional derivatives, incorporate more complex virus spread scenarios, or apply the model to different types of computer security threats.

5. Fractional Differential Transform Method

The steps in the fractional differential transform method are given in [56]. The fractional differentiation in the Riemann-Liouville sense is defined as:

$$D_{t_0}^{\gamma} u(t) = \frac{1}{(n-\gamma)} \int_{t_0}^{t} \frac{d^m}{dx^m} (t-s)^{n-\gamma-1} u(s) ds \ , \tag{6}$$

where $n-1 \leq \gamma < m$, n is a positive integer, and $t > t_0$. The expansion of an analytic and continuous function $f(t)$ in terms of a fractional power series is given as:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^{k/\mu}, \tag{7}$$

where μ is the order of the fraction, and $U(k)$ is the fractional differential transform of $u(t)$. The relation between the Riemann-Liouville operator and the Caputo operator is given as follows:

$$^{C}D_{x_0}^{\gamma} u(t) = D_{x_0}^{\gamma}\left[u(t) - \sum_{k=0}^{n-1} \frac{u^k(t_0)}{k!}(t-t_0)^k\right]. \tag{8}$$

Here, the notation C is used for the Caputo operator to distinguish it from the Riemann-Liouville operator.

Since the initial conditions are implemented to the integer order differential equation, the transformed initial conditions can be obtained using the following relation:

$$U(k) = \begin{cases} \frac{1}{(k/\mu)!}\left[\frac{d^{(k/\mu)}u(t)}{dt^{(k/\mu)}}\right]_{x=x_0}, & \text{if } k/\mu \in \mathbb{Z}^+, \\ 0, & \text{if } k/\mu \notin \mathbb{Z}^+. \end{cases} \tag{9}$$

where $k = 0, 1, 2, \ldots, \gamma\mu - 1$, and γ is the order of the fractional differential equation. The parameter μ is chosen such that $\gamma\mu$ is a positive integer.

The following results[56] are applied in the proofs of the results in the next section.

Theorem 4. *[56] If $w(t) = u(t) \pm v(t)$, then its transformed form is $W(k) = U(k) \pm V(k)$.*

Theorem 5. *[56] If $w(t) = u(t)v(t)$, then $W(k) = \sum_{l=0}^{k} U(l)V(k-l)$.*

Theorem 6. *[56] If $w(t) = (t-t_0)^p$, then $W(k) = \delta(k - \lambda p)$, where:*

$$\delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases} \tag{10}$$

Theorem 7. *[56] If $w(t) = D_{t_0}^{\gamma} u(t)$, then:*

$$W(k) = \frac{\Gamma(\gamma + 1 + k/\mu)}{\Gamma(1 + k/\mu)} U(k + \gamma\mu). \tag{11}$$

6. Methodology of the Proposed Non-Linear Fractional Mathematical Model

Using the transformation rules from equations theorem4–theorem7, the transformed equations are

$$\begin{aligned} S(k+\gamma\mu) &= \frac{\Gamma(1+k/\mu)}{\Gamma(1+\gamma+k/\mu)}\left[f_1\delta(k) - \lambda\sum_{l=0}^{k} S(l)I(k-l) - dS(k)\right] \\ I(k+\gamma\mu) &= \frac{\Gamma(1+k/\mu)}{\Gamma(1+\gamma+k/\mu)}\left[f_2\delta(k) + \lambda\sum_{l=0}^{k} S(l)I(k-l) - \varepsilon I(k) - dR(k)\right] \\ R(k+\gamma\mu) &= \frac{\Gamma(1+k/\mu)}{\Gamma(1+\gamma+k/\mu)}\left[f_3\delta(k) + \varepsilon I(k) - dR(k)\right] \end{aligned} \tag{12}$$

The initial conditions transform into:

$$\begin{aligned} &S(k) = 0, k = 1, 2, ..., \gamma\mu - 1 \\ &I(k) = 0, k = 1, 2, ..., \gamma\mu - 1 \\ &R(k) = 0, k = 1, 2, ..., \gamma\mu - 1, \\ &S(k) = 20, \quad I(k) = 15, \quad R(k) = 10, \quad k = 0. \end{aligned} \tag{13}$$

The approximate series solution for system 1 is:

$$s(t) = \sum_{j=0}^{n} S(j)t^{j/\mu}, \quad i(t) = \sum_{j=0}^{n} I(j)t^{j/\mu}, \quad r(t) = \sum_{j=0}^{n} R(j)t^{j/\mu} \tag{14}$$

Case I: $\gamma = 1/10$
For $\gamma = 1/10$, setting $\mu = 10$, using 12, we obtain

$$\begin{aligned} S(k+10\gamma) &= \frac{\Gamma(1+k/10)}{\Gamma(1+\gamma+k/10)}\left[f_1\delta(k) - \lambda\sum_{l=0}^{k} S(l)I(k-l) - dS(k)\right] \\ I(k+10\gamma) &= \frac{\Gamma(1+k/10)}{\Gamma(1+\gamma+k/10)}\left[f_2\delta(k) + \lambda\sum_{l=0}^{k} S(l)I(k-l) - \varepsilon I(k) - dR(k)\right] \\ R(k+10\gamma) &= \frac{\Gamma(1+k/10)}{\Gamma(1+\gamma+k/10)}\left[f_3\delta(k) + \varepsilon I(k) - dR(k)\right] \end{aligned}$$

Substituting the initial conditions $S(0) = 20,\ I(0) = 15,\ R(0) = 10$, we obtain the values of $S(k),\ I(k),\ R(k),\ 0 \le k \le 4$ as follows. Thus, the approximate solution becomes

$$\begin{aligned} s(t) &= 20 - 2.4177t^{1/10} + 0.3350t^{2/10} - 0.04716t^{3/10} + 0.005841t^{4/10} \\ i(t) &= 15 - 2.3126t^{1/10} + 0.0996t^{2/10} + 0.03319t^{3/10} - 0.004836t^{4/10} \\ r(t) &= 10 + 0.5255t^{1/10} - 0.2981t^{2/10} + 0.04037t^{3/10} - 0.00070805t^{4/10} \end{aligned} \tag{15}$$

Table 2: Values of $S(k),\ I(k),\ R(k),\ \gamma = 1/10$.

Sr. No.	k	$S(k)$	$I(k)$	$R(k)$
1	0	20	15	10
2	1	-2.4177	-2.3126	0.5255
3	2	0.3350	0.0996	-0.2981
4	3	-0.04716	0.03319	0.04037
5	4	0.005841	-0.004836	-0.00070805

Case II: $\gamma = 1/2$
For $\gamma = 1/2$, setting $\mu = 2$, using (12), we obtain

$$S(k+2\gamma) = \frac{\Gamma(1+k/2)}{\Gamma(1+\gamma+k/2)}\left[f_1\delta(k) - \lambda\sum_{l=0}^{k} S(l)I(k-l) - dS(k)\right]$$
$$I(k+2\gamma) = \frac{\Gamma(1+k/2)}{\Gamma(1+\gamma+k/2)}\left[f_2\delta(k) + \lambda\sum_{l=0}^{k} S(l)I(k-l) - \varepsilon I(k) - dR(k)\right]$$
$$R(k+2\gamma) = \frac{\Gamma(1+k/2)}{\Gamma(1+\gamma+k/2)}\left[f_3\delta(k) + \varepsilon I(k) - dR(k)\right]$$

Using the initial conditions $S(0) = 20,\ I(0) = 15,\ R(0) = 10$, we obtain the values of $S(k),\ I(k),\ R(k),\ 0 \le k \le 4$ as follows.
Thus, the approximate solution takes the form

Table 3: Values of $S(k),\ I(k),\ R(k),\ \gamma = 1/2$.

Sr. No.	k	$S(k)$	$I(k)$	$R(k)$
1	0	20	15	10
2	1	-2.5953	-2.4825	0.5642
3	2	0.3084	0.0915	-0.2699
4	3	-0.02477	0.01479	0.02718
5	4	0.002359	-0.003505	-0.001411

$$\begin{aligned} s(t) &= 20 - 2.5953t^{1/2} + 0.3084t - 0.02477t^{3/2} + 0.002359t^2 \\ i(t) &= 15 - 2.4825t^{1/2} + 0.0915t + 0.01479t^{3/2} - 0.003505t^2 \\ r(t) &= 10 + 0.5642t^{1/2} - 0.2699t + 0.02718t^{3/2} - 0.001411t^2 \end{aligned} \tag{16}$$

Case III: $\gamma = 1$
For $\gamma = 1$, setting $\gamma\mu = 1$,i.e $\mu = 1$ we obtain (12) as follows

$$S(k+1) = \frac{\Gamma(1+k)}{\Gamma(2+k)}\left[f_1\delta(k) - \lambda\sum_{l=0}^{k} S(l)I(k-l) - dS(k)\right]$$

$$I(k+1) = \frac{\Gamma(1+k)}{\Gamma(2+k)}\left[f_2\delta(k) + \lambda\sum_{l=0}^{k} S(l)I(k-l) - \varepsilon I(k) - dR(k)\right]$$
$$R(k+1) = \frac{\Gamma(1+k)}{\Gamma(2+k)}\left[f_3\delta(k) + \varepsilon I(k) - dR(k)\right]$$

The table of values of $S(k),\ I(k),\ R(k),\ 0 \le k \le 4$ can be obtained as follows. The

Table 4: Values of $S(k),\ I(k),\ R(k),\ \gamma = 1$.

Sr. No.	k	$S(k)$	$I(k)$	$R(k)$
1	0	20	15	10
2	1	-2.3	-2.2	0.5
3	2	0.15425	0.04575	-0.135
4	3	-0.007904	0.005737	0.006025
5	4	0.0003097	-0.0004061	-0.0000216

approximate solution up to five iterations in this case is

$$\begin{aligned} s(t) &= 20 - 2.3t + 0.15425t^2 - 0.007904t^3 + 0.0003097t^4 \\ i(t) &= 15 - 2.2t + 0.04575t^2 + 0.005737t^3 - 0.0004061t^4 \\ r(t) &= 10 + 0.5t - 0.135t^2 + 0.006025t^3 - 0.0000216t^4 \end{aligned} \tag{17}$$

7. Laplace-Adomian Decomposition Method (LADM)

Now we solve the system of equations (1) using the Laplace-Adomian Decomposition Method. Denoting the Laplace transform $\mathcal{L}[\phi(t)] = \Phi(z)$ and the Laplace transform of the Caputo fractional derivative of the function ϕ by

$$\mathcal{L}\Big[\frac{d^\gamma \phi(t)}{dt}\Big] = z^\gamma \mathcal{L}[\phi(t)] - \sum_{k=0}^{n-1} z^{\gamma-k-1}\phi^{(k)}(0)$$

Since $\gamma \in (0,1]$ i.e. $n = 1$, the above equation become

$$\mathcal{L}\Big[\frac{d^\gamma \phi(t)}{dt}\Big] = z^\gamma \mathcal{L}[\phi(t)] - z^{\gamma-1}\phi(0)$$

Using this relation and the system of equations (1), we get

$$\mathcal{L}[s(t)] = \frac{s(0)}{z} - \lambda\frac{\mathcal{L}[s(t)i(t)]}{z^\gamma} - d\frac{\mathcal{L}[s(t)]}{z^\gamma}$$
$$\mathcal{L}[i(t)] = \frac{i(0)}{z} + \lambda\frac{\mathcal{L}[s(t)i(t)]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i(t)]}{z^\gamma} - d\frac{\mathcal{L}[r(t)]}{z^\gamma}$$

$$\mathcal{L}[r(t)] = \frac{r(0)}{z} + \varepsilon\frac{\mathcal{L}[i(t)]}{z^\gamma} - d\frac{\mathcal{L}[r(t)]}{z^\gamma}$$

Setting $A(t) = s(t)i(t)$, we have

$$\begin{aligned}
\mathcal{L}[s(t)] &= \frac{s(0)}{z} - \lambda\frac{\mathcal{L}[A(t)]}{z^\gamma} - d\frac{\mathcal{L}[s(t)]}{z^\gamma} \\
\mathcal{L}[i(t)] &= \frac{i(0)}{z} + \lambda\frac{\mathcal{L}[A(t)]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i(t)]}{z^\gamma} - d\frac{\mathcal{L}[r(t)]}{z^\gamma} \\
\mathcal{L}[r(t)] &= \frac{r(0)}{z} + \varepsilon\frac{\mathcal{L}[i(t)]}{z^\gamma} - d\frac{\mathcal{L}[r(t)]}{z^\gamma}
\end{aligned}$$

The non-linear term A is called Adomian polynomial and it is given by

$$A = \sum_{j=0}^{\infty} A_j,\ A_n = \sum_{j=0}^{n} s(j).i(n-j)$$

and

$$s = \sum_{j=0}^{\infty} s_j, \quad i = \sum_{j=0}^{\infty} i_j, \quad r = \sum_{j=0}^{\infty} r_j$$

The terms $s_j,\ i_j,\ r_j$ are obtained as follows.

$$\begin{aligned}
\mathcal{L}[s_0] &= \frac{s(0)}{z} \implies s_0 = s(0) = 20 \\
\mathcal{L}[i_0] &= \frac{i(0)}{z} \implies s_0 = i(0) = 15 \\
\mathcal{L}[r_0] &= \frac{r(0)}{z} \implies r_0 = r(0) = 10 \\
\mathcal{L}[s_1] &= -\lambda\frac{\mathcal{L}[s_0 i_0]}{z^\gamma} - d\frac{\mathcal{L}[s_0]}{z^\gamma} \implies s_1 = -2.3\frac{t^\gamma}{\Gamma(\gamma+1)} \\
\mathcal{L}[i_1] &= \lambda\frac{\mathcal{L}[s_0 i_0]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i_0]}{z^\gamma} - d\frac{\mathcal{L}[r_0]}{z^\gamma} \implies i_1 = -2.2\frac{t^\gamma}{\Gamma(\gamma+1)} \\
\mathcal{L}[r_1] &= \varepsilon\frac{\mathcal{L}[i_0]}{z^\gamma} - d\frac{\mathcal{L}[r_0]}{z^\gamma} \implies r_1 = 0.5\frac{t^\gamma}{\Gamma(\gamma+1)} \\
\mathcal{L}[s_2] &= -\lambda\frac{\mathcal{L}[A_1]}{z^\gamma} - d\frac{\mathcal{L}[s_1]}{z^\gamma} \implies s_2 = 0.3085\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \\
\mathcal{L}[i_2] &= \lambda\frac{\mathcal{L}[A_1]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i_1]}{z^\gamma} - d\frac{\mathcal{L}[r_1]}{z^\gamma} \implies i_2 = 0.0915\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \\
\mathcal{L}[r_2] &= \varepsilon\frac{\mathcal{L}[i_1]}{z^\gamma} - d\frac{\mathcal{L}[r_1]}{z^\gamma} \implies r_2 = -0.27\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \\
\mathcal{L}[s_3] &= -\lambda\frac{\mathcal{L}[A_2]}{z^\gamma} - d\frac{\mathcal{L}[s_2]}{z^\gamma} \implies s_3 = -0.0193325\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}
\end{aligned}$$

$$\mathcal{L}[i_3] = \lambda\frac{\mathcal{L}[A_2]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i_2]}{z^\gamma} - d\frac{\mathcal{L}[r_2]}{z^\gamma} \implies i_3 = 0.00293675\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}$$
$$\mathcal{L}[r_3] = \varepsilon\frac{\mathcal{L}[i_2]}{z^\gamma} - d\frac{\mathcal{L}[r_2]}{z^\gamma} \implies r_3 = 0.03615\frac{t^{3\gamma}}{\Gamma(3\gamma+1)}$$
$$\mathcal{L}[s_4] = -\lambda\frac{\mathcal{L}[A_3]}{z^\gamma} - d\frac{\mathcal{L}[s_3]}{z^\gamma} \implies s_4 = 0.0025250375\frac{t^{4\gamma}}{\Gamma(4\gamma+1)}$$
$$\mathcal{L}[i_4] = \lambda\frac{\mathcal{L}[A_3]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i_3]}{z^\gamma} - d\frac{\mathcal{L}[r_3]}{z^\gamma} \implies i_4 = -0.0045004625\frac{t^{4\gamma}}{\Gamma(4\gamma+1)}$$
$$\mathcal{L}[r_4] = \varepsilon\frac{\mathcal{L}[i_3]}{z^\gamma} - d\frac{\mathcal{L}[r_3]}{z^\gamma} \implies r_4 = -0.0033321325\frac{t^{4\gamma}}{\Gamma(4\gamma+1)}$$

It can be verified that for $j \geq 1$, we have

$$\mathcal{L}[s_j] = -\lambda\frac{\mathcal{L}[A_{j-1}]}{z^\gamma} - d\frac{\mathcal{L}[s_{j-1}]}{z^\gamma}$$
$$\mathcal{L}[i_j] = \lambda\frac{\mathcal{L}[A_{j-1}]}{z^\gamma} - \varepsilon\frac{\mathcal{L}[i_{j-1}]}{z^\gamma} - d\frac{\mathcal{L}[r_{j-1}]}{z^\gamma}$$
$$\mathcal{L}[r_j] = \varepsilon\frac{\mathcal{L}[i_{j-1}]}{z^\gamma} - d\frac{\mathcal{L}[r_{j-1}]}{z^\gamma}$$

The solution of the SIR model of computer viruses by Laplace Adomian decomposition method up to five iteration is given by

$$\begin{aligned} s(t) = \sum_{j=0}^{4} s_j &= 20 - 2.3\frac{t^\gamma}{\Gamma(\gamma+1)} + 0.3085\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - 0.0193325\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \\ &+ 0.0025250375\frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \\ i(t) = \sum_{j=0}^{4} i_j &= 15 - 2.2\frac{t^\gamma}{\Gamma(\gamma+1)} + 0.0915\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + 0.00293675\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \\ &- 0.0045004625\frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \qquad (18) \\ r(t) = \sum_{j=0}^{4} r_j &= 10 + 0.5\frac{t^\gamma}{\Gamma(\gamma+1)} - 0.27\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + 0.03615\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \\ &- 0.0033321325\frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \end{aligned}$$

As in the earlier case, the solutions for different values of γ are obtained as follows.
CaseI: $\gamma = 1/10$

$$\begin{aligned} s(t) =& 20 - 2.4176145292t^{1/10} + 0.3359947896t^{2/10} - 0.021541986t^{3/10} \\ &+ 0.0028458694t^{4/10} \end{aligned}$$

$$\begin{aligned} i(t) =& 15 - 2.312500854t^{1/10} + 0.0996548566t^{2/10} + 0.00322722506t^{3/10} \\ & - 0.0050722925t^{4/10} \\ r(t) =& 10 + 0.5255683759t^{1/10} - 0.2940635112t^{2/10} + 0.0402798531t^{3/10} \\ & - 0.0037555141t^{4/10} \end{aligned} \tag{19}$$

Case II: $\gamma = 1/2$

$$\begin{aligned} s(t) &= 20 - 2.595271866t^{1/2} + 0.3084t - 0.0145429311t^{3/2} + 0.0012625188t^2 \\ i(t) &= 15 - 2.4824339588t^{1/2} + 0.0915t + 0.002209179t^{3/2} - 0.0022502313t^2 \\ r(t) &= 10 + 0.5641895361t^{1/2} - 0.27t + 0.0271939459t^{3/2} - 0.0016660663t^2 \end{aligned} \tag{20}$$

Case III: $\gamma = 1$

$$\begin{aligned} s(t) &= 20 - 2.3t + 0.15425t^2 - 0.0032220833t^3 + 0.0001052099t^4 \\ i(t) &= 15 - 2.2t + 0.04575t^2 + 0.0004894583t^3 - 0.0001875193t^4 \\ r(t) &= 10 + 0.5t - 0.135t^2 + 0.006025t^3 - 0.0001388389t^4 \end{aligned} \tag{21}$$

8. Numerical Approximations and Estimate of Errors

Using the equations (15), (16), which are obtained using the Fractional Differential Transform Method and the equations (19), (20), obtained using the Laplace Adomian Decomposition Method, the approximate values of the solutions $s(t), i(t)$ and $r(t)$ for the cases $\gamma = 0.1$ and $\gamma = 0.5$ are as shown by the tables 5 to 10. In these tables, the values are obtained by using the MATLAB software correct up to fourteen digits after the decimal. The notations $s(t)FDTM, i(t)FDTM$ and $r(t)FDTM$ are used to represent the solutions $s(t), i(t)$ and $r(t)$ obtained using the Fractional Differential Transform Method respectively. Similarly, the notations $s(t)LADM, i(t)LADM$ and $r(t)LADM$ are used to represent the solutions $s(t), i(t)$ and $r(t)$ obtained using the Laplace Adomian Decomposition Method respectively. The approximate values of the absolute errors in $s(t), i(t)$ and $r(t)$ are represented by $Er[s(t)], Er[i(t)]$ and $Er[r(t)]$ and are obtained by the formulae

$$\begin{aligned} Er[s(t)] &= |s(t)FDTM - s(t)LADM| \\ Er[i(t)] &= |i(t)FDTM - i(t)LADM| \\ Er[r(t)] &= |r(t)FDTM - r(t)LADM| \end{aligned}$$

Table 5: Estimate of errors in $s(t)$ for $\gamma = 0.1$.

t	s(t)FDTM	s(t)LADM	Er[s(t)]
0	20.00000000000000	20.00000000000000	0.00000000000000
5	17.55702229544161	17.59431154300001	0.03728924755840
10	17.40781060925223	17.45308607588309	0.04527546663086
15	17.31712461031913	17.36782451716560	0.05069990684647
20	17.25124112568540	17.30616983119194	0.05492870550654
25	17.19925180577707	17.25769584483516	0.05844403905810
30	17.15619659362801	17.21767482183604	0.06147822820803

Table 6: Estimate of errors in $i(t)$ for $\gamma = 0.1$.

t	i(t)FDTM	i(t)LADM	Er[i(t)]
0	15.00000000000000	15.00000000000000	0.00000000000000
5	12.46558082849279	12.41676378653062	0.04881704196217
10	12.30053972810497	12.24037435176146	0.06016537634351
15	12.19982823339885	12.13183803127759	0.06799020212126
20	12.12647775783722	12.05232598332779	0.07415177450943
25	12.06849040276133	11.98917726395060	0.07931313881073
30	12.02039883376837	11.93660290522253	0.08379592854584

Table 7: Estimate of errors in $r(t)$ for $\gamma = 0.1$.

t	r(t)FDTM	r(t)LADM	Er[r(t)]
0	10.00000000000000	10.00000000000000	0.00000000000000
5	10.27004286714835	10.26974491443414	0.00029795271420
10	10.26787884169264	10.26652744914304	0.00135139254960
15	10.26545056249648	10.26327199707570	0.00217856542077
20	10.26315827919200	10.26027696818093	0.00288131101106
25	10.26103572422611	10.25753353094654	0.00350219327957
30	10.25906856101875	10.25500465803592	0.00406390298284

Table 8: Estimate of errors in $s(t)$ for $\gamma = 0.5$.

t	s(t)FDTM	s(t)LADM	Er[s(t)]
0	20.00000000000000	20.00000000000000	0.00000000000000
5	15.52077075898145	15.61380112841204	0.09303036943060
10	14.32954461214130	14.55193192445817	0.22238731231687
15	13.66621435930453	14.02421111980599	0.35799676050146
20	13.28956940388280	13.77791436087322	0.48834495699041
25	13.11162500000000	13.71834853250000	0.60672353250000
30	13.08993014024754	13.79851856567137	0.70858842542383

Table 9: Estimate of errors in $i(t)$ for $\gamma = 0.5$.

t	i(t)FDTM	i(t)LADM	Er[i(t)]
0	15.00000000000000	15.00000000000000	0.00000000000000
5	9.98419347279288	9.87505250806161	0.10914096473127
10	8.18184657457090	7.90969179323354	0.27215478133736
15	6.82841519839620	6.38011427914472	0.44830091925149
20	5.64878030720242	5.02572029367717	0.62306001352524
25	4.53312500000000	3.74508301850000	0.78804198150000
30	3.42353249758467	2.48594621404391	0.93758628354076

Table 10: Estimate of errors in $r(t)$ for $\gamma = 0.5$.

t	r(t)FDTM	r(t)LADM	Er[r(t)]
0	10.00000000000000	10.00000000000000	0.00000000000000
5	10.18069619104760	10.17395205545793	0.00674413558967
10	9.80356412390077	9.77746541222377	0.02609871167700
15	9.39817751417899	9.34005725374527	0.05812026043372
20	8.99183221094853	8.88900625418141	0.10282595676712
25	8.58912500000000	8.42889948050000	0.16022551950000
30	8.18948040334127	7.95915496540499	0.23032543793628

9. Runge-Kutta Felhberg Algorithm

An important family of predictor-corrector methods to solve linear and non-linear ordinary differential equations is known to the Runge-Kutta methods, named after the German mathematicians C. Runge (1856–1927) and M. W. Kutta (1867–1944).

Consider the ordinary differential equation

$$Y'(t) = F[t, Y(t)],\ Y(0) = Y_0$$

where $Y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T,\ F : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $a, b \in \mathbb{R}$. In case the function F is non-linear and the analytical solutions do not exist, we need to obtain an approximate solution by dividing the interval $[a, b]$ into n equal parts, each of length $h = \frac{b-a}{n},\ n \in \mathbb{N}$,$h$ is also called as the step size. Considering the mesh points $t_i = a + ih,\ i = 0, 1, \cdots, n$, the $n + 1$th approximate solution at the mth stage to the initial value problem is given by:

$$Y(t_{n+1}) = Y_{n+1} = Y_n + \sum_{i=1}^{m} a_i k_i$$

where

$$k_m = F(t_n + c_m h + Y_n + h \sum_{i=1}^{m-1} d_{mi} k_i)$$

The coefficients $c_i,\ 2 \leq i \leq m,\ d_{ij},\ 1 \leq j < i \leq m$, and $a_i,\ i \leq i \leq m$ are to be determined in the form of a table, known as *Butcher Tableau.* In order to improve the rate of convergence, the algorithm based on step-size adjustment was developed by Felhberg and is known as Runge-Kutta Felhberg Algorithm (RKFA). This algorithm is an adaptive algorithm where at each step, two different approximate solutions are obtained and are compared. If the two approximate solutions agree up to a desired level of accuracy, the approximate solution is accepted. In case the two approximate solutions did not agree, the step-size is reduced and the same procedure is repeated.

This algorithm evaluates $F[t, Y(t)]$ six times per step using the embedded 4th order and 5th order Runge-Kutta method. Moreover, the algorithm also evaluates the amount of errors. The approximate solution to the initial value problem by Runge-Kutta method of order 4 is given by:

$$Y_{k+1} = Y_k + \frac{25k_1}{216} + \frac{1408k_2}{2565} + \frac{2197k_4}{4101} - \frac{k_5}{5}$$

where at each step, the following six values are calculated.

$$\begin{aligned}
k_1 &= hF(t_k, Y_k) \\
k_2 &= hF\left(t_k + \frac{h}{4}, Y_k + \frac{k_1}{4}\right) \\
k_3 &= hF\left(t_k + \frac{3h}{8}, Y_k + \frac{3k_1}{32} + \frac{9k_2}{32}\right) \\
k_4 &= hF\left(t_k + \frac{12h}{13}, Y_k + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right) \\
k_5 &= hF\left(t_k + h, Y_k + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}\right) \\
k_6 &= hF\left(t_k + \frac{h}{2}, Y_k - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40}\right)
\end{aligned}$$

A better approximate solution is given by the 5th order Runge-Kutta Felhberg Algorithm, which is given by

$$Y_{k+1} = Y_k + \frac{16k_1}{135} + \frac{6656k_3}{12825} + \frac{28561k_4}{56430} - \frac{9k_5}{50} + \frac{2k_6}{55}$$

It has been proved that RKFA provides more accurate solutions as compared to the Runge-Kutta method of order 4. Moreover, RKFA requires less number of iterations as compared to the R-K method to arrive at the desired level of accuracy. Using the RKFA, we have obtained the numerical values of $s(t), i(t)$ and $r(t)$ and compared them with the values obtained by FDTM and LADM for the case $\gamma = 1$. Moreover, we have obtained the estimates of the errors in these solutions with respect to the RKFA. The approximate values of the absolute errors in $s(t), i(t)$ and $r(t)$ are represented by $ErFDTM[s(t)], ErFDTM[i(t)], ErFDTM[r(t)]$, etc. and are obtained by the formulae

$$\begin{aligned} ErFDTM[s(t)] &= |s(t)FDTM - s(t)RKFA| \\ ErFDTM[i(t)] &= |i(t)FDTM - i(t)RKFA| \\ ErFDTM[r(t)] &= |r(t)FDTM - r(t)RKFA| \\ ErLADM[s(t)] &= |s(t)LADM - s(t)RKFA|, etc. \end{aligned}$$

The estimate of errors in $s(t), i(t), r(t)$ for $\gamma = 1$ are as shown in the tables 11, 12, 13.

Table 11: Estimate of errors in $s(t)$ for $\gamma = 1$.

t	s(t)FDTM	s(t)LADM	s(t)RKFA	ErFDTM[s(t)]	ErLADM[s(t)]
0.0	20.00000000000000	20.00000000000000	20.00000000000000	0.00000000000000	0.00000000000000
0.2	19.54610726352000	19.54614439166944	19.546107256391014	0.00000000712900	0.00003713527840
0.4	19.10418207232000	19.10447648004224	19.104181955888297	0.00000011643180	0.00029452415400
0.6	18.67386287312000	18.67484766521024	18.673862145750277	0.00000072736980	0.00098551946000
0.8	18.25480000512000	18.25711338732544	18.254797171108123	0.00000283401190	0.00231621621730
1.0	17.84665570000000	17.85113312660000	17.846647378609305	0.00000832139070	0.00448574799070
1.2	17.44910408192000	17.45677040330624	17.449083821756879	0.00002026016320	0.00768658154940
1.4	17.06183116752001	17.07389277777664	17.061787968569263	0.00004319895080	0.01210480920740
1.6	16.68453486592000	16.70237185040384	16.684451412115802	0.00008345380420	0.01792043828800
1.8	16.31692497872000	16.34208326164064	16.316775584419219	0.00014939430080	0.02530767722140
2.0	15.95872320000000	15.99290669200000	15.958471474156529	0.00025172584350	0.03443521784350

Table 12: Estimate of errors in $i(t)$ for $\gamma = 1$.

t	i(t)FDTM	i(t)LADM	i(t)RKFA	ErFDTM[i(t)]	ErLADM[i(t)]
0.0	15.00000000000000	15.00000000000000	15.00000000000000	0.00000000000000	0.00000000000000
0.2	14.56187524624000	14.56183361563552	14.561875256612106	0.00000001037210	0.00004164097660
0.4	14.12767677184000	14.12734652483712	14.127676929730777	0.00000015789070	0.00033040489360
0.6	13.69765656144000	13.69655142049152	13.697657518861158	0.00000095742110	0.00110609836960
0.8	13.27205100544000	13.26945379474432	13.272054690053119	0.00000368461310	0.00260089530880
1.0	12.85108090000000	12.84605193900000	12.851091662011378	0.00001076201130	0.00503972301130
1.2	12.43495144704000	12.42633694392192	12.538540600282305	0.10358915324230	0.11220365636040
1.4	12.02385225424000	12.01029269943232	12.023907974631463	0.00005572039140	0.01361527519910
1.6	11.61795733504000	11.59789589471232	11.618065010548600	0.00010767550860	0.02016911583630
1.8	11.21742510864000	11.18911601820192	11.217618014408080	0.00019290576800	0.02850199620610
2.0	10.82239840000000	10.78391535760000	10.822723791238749	0.00032539123870	0.03880843363870

Table 13: Estimate of errors in $r(t)$ for $\gamma = 1$.

t	r(t)FDTM	r(t)LADM	r(t)RKFA	ErFDTM[r(t)]	ErLADM[r(t)]
0.0	10.00000000000000	10.00000000000000	10.00000000000000	0.00000000000000	0.00000000000000
0.2	10.09464816544000	10.09464817778578	10.094648185984170	0.00000002054410	0.00000000819840
0.4	10.17878504704000	10.17878524457242	10.178785335767941	0.00000028872790	0.00000009119550
0.6	10.25269860064000	10.25269960064786	10.252699863108273	0.00000126246820	0.00000026246040
0.8	10.31667595264000	10.31667911315866	10.316679322626605	0.00000336998660	0.00000020946800
1.0	10.37100340000000	10.37101111611000	10.371010136930472	0.00000673693040	0.00000097917960
1.2	10.41596641024000	10.41598241036570	10.415977336803497	0.00001092656340	0.00000507356230
1.4	10.45184962144000	10.45187926364818	10.451864314231031	0.00001469279100	0.00001494941710
1.6	10.47893684224000	10.47898741053850	10.478952588015369	0.00001574577530	0.00003482252320
1.8	10.49751105184000	10.49759205247634	10.497521581722806	0.00001052988280	0.00007047075350
2.0	10.50785440000000	10.50797785776000	10.507848413694944	0.00000598630510	0.00012944406510

10. Results and Discussion

In the second section of this paper, we have formulated the non-linear fractional order SIR model of computer viruses with the specific initial conditions as described by the table 1. In the third section, we concluded that the system of equations 1 is asymptotically stable. In the sixth section, we have obtained the numerical values of the solutions which are given by the equations (15) and (16) for the cases $\gamma = 0.1,\ 0.5$. Whereas using the FDTM the solutions are obtained for the cases $\gamma = 0.1,\ 0.5$ as given by equations (19) and (20). To have a better understanding of the solution curves and errors, we have compared the numerical values obtained by FDTM and LADM with those obtained by RKFA. As can be noticed, we don't get a clear picture of the solutions and the errors just from the tabulated values. For a better understanding of the system, we plot the graphs of the approximate solutions and errors for different values of γ.

Figure 1: Comparison of graphs of $s(t)$ by FDTM and LADM for $\gamma = 0.1$

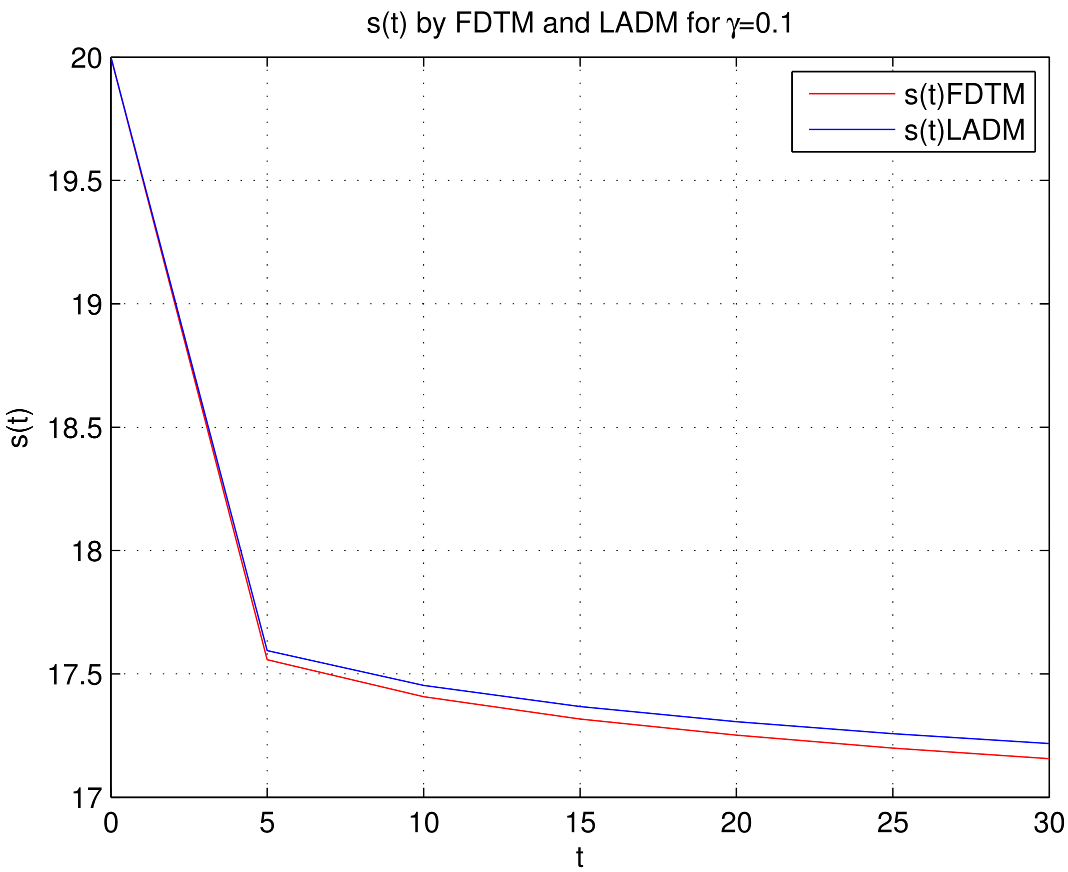


Figure 2: Comparison of graphs of $i(t)$ by FDTM and LADM for $\gamma = 0.1$

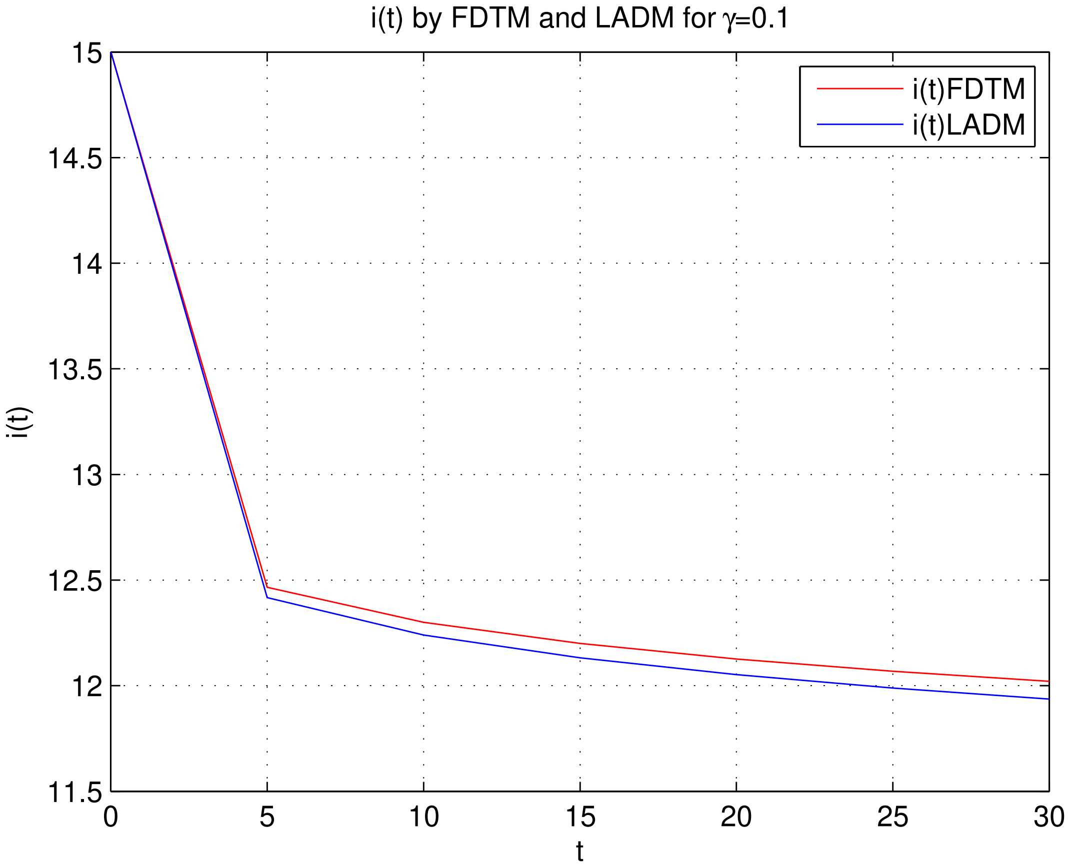
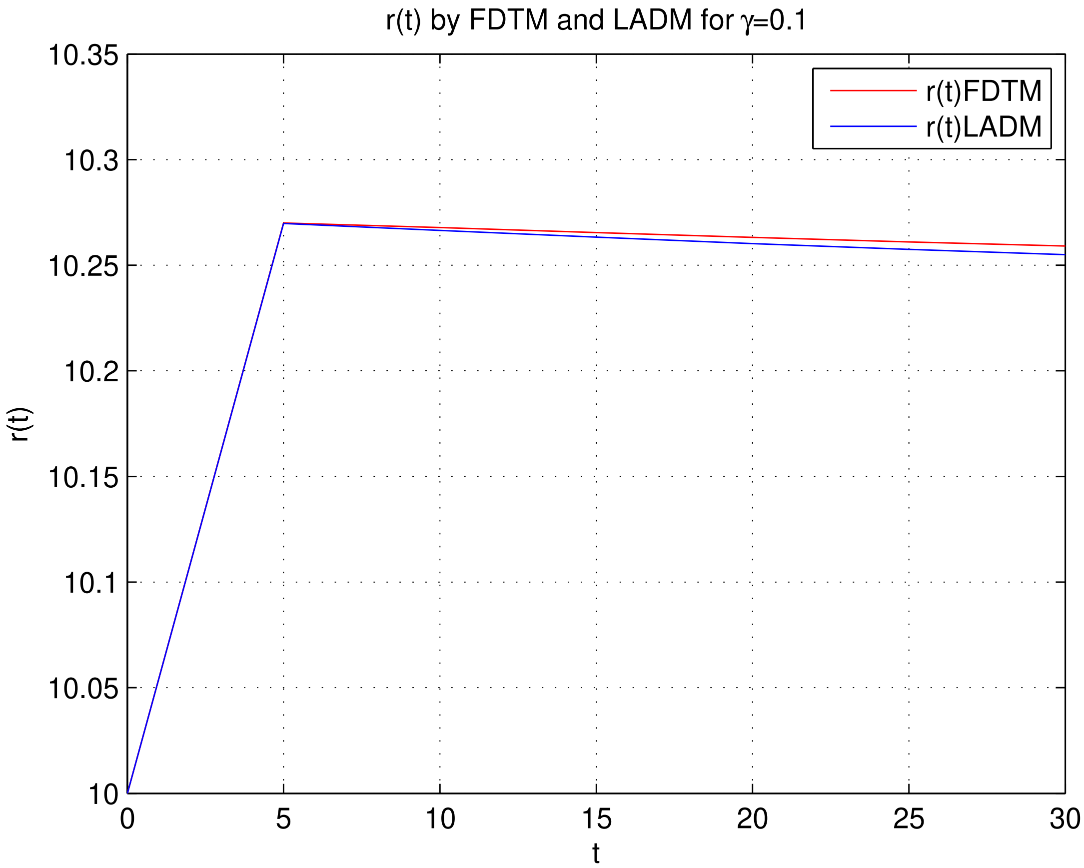
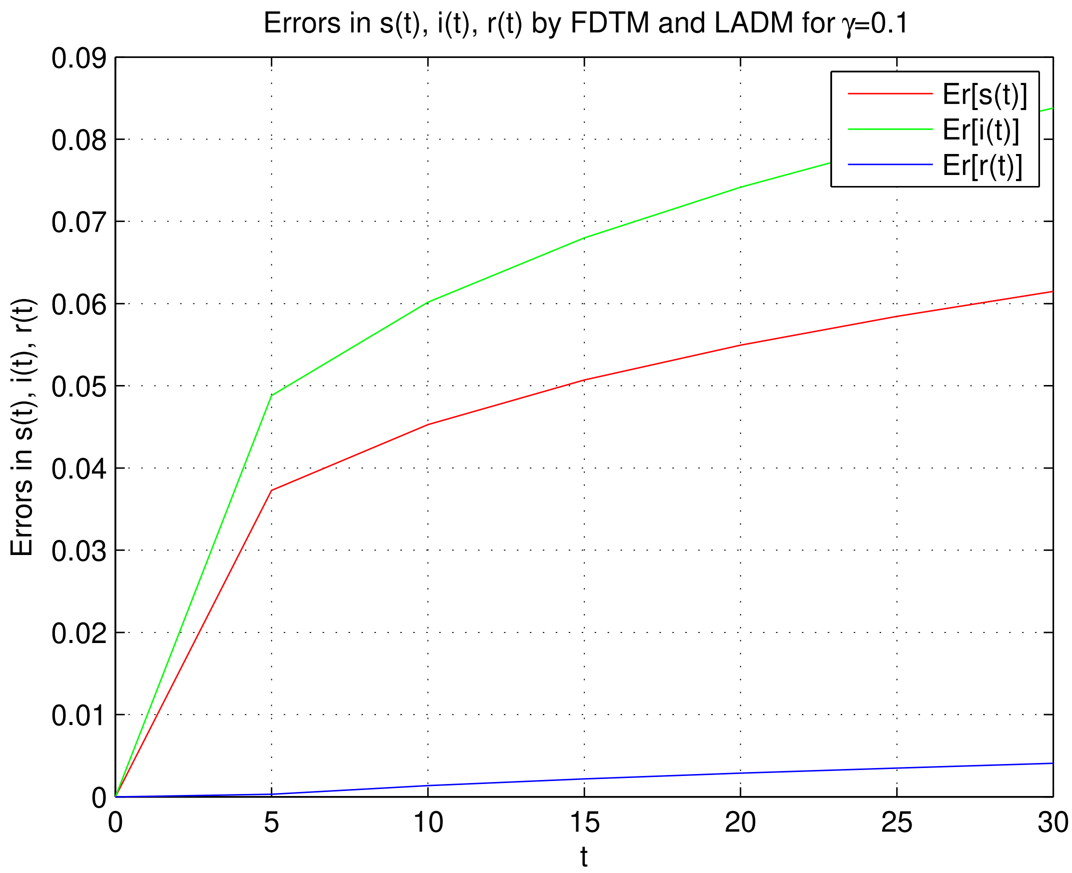


Figure 3: Comparison of graphs of $r(t)$ by FDTM and LADM for $\gamma = 0.1$



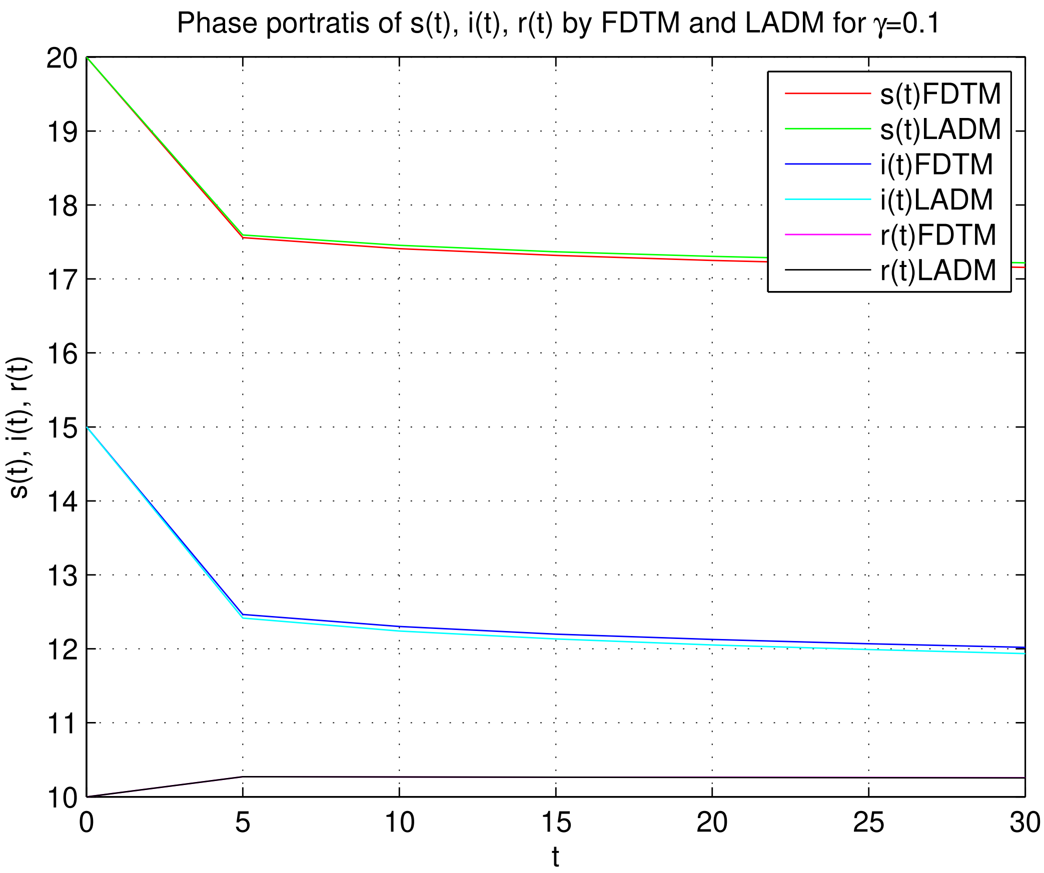
The graphs shown by figures 1, 2 and 3 represent the solution curves for $s(t), i(t)$ and $r(t)$ by FDTM and LADM for $\gamma = 0.1$. From these figures, it can be observed that the solutions obtained by FDTM and LADM almost agree for first few values of the variable t, to be specific, for $t \in [0, 5]$. But as t increases, slight differences in the solution curves are observed. The absolute errors in the solutions by both the methods can be observed in figure 4.

Figure 4: Comparison of errors in $s(t), i(t), r(t)$ by FDTM and LADM for $\gamma = 0.1$



The phase-plane portraits of $s(t), i(t), r(t)$ by both the methods are as plotted in figure 5. From this figure we note that the solution curves for $s(t), i(t), r(t)$ progress almost hand-in-hand for the initial values of the variable t, but differ later on.

Figure 5: Phase plane portraits of $s(t), i(t), r(t)$ by FDTM and LADM for $\gamma = 0.1$



The solution curves, errors and the phase-plain portrait for $\gamma = 0.5$ are as shown by figures 6 to figure 10. In this case also we have a similar type of conclusion as in the case for $\gamma = 0.1$. This type of pattern can be observed for the both the methods for $\gamma \in (0, 1)$.

Figure 6: Comparison of graphs of $s(t)$ by FDTM and LADM for $\gamma = 0.5$

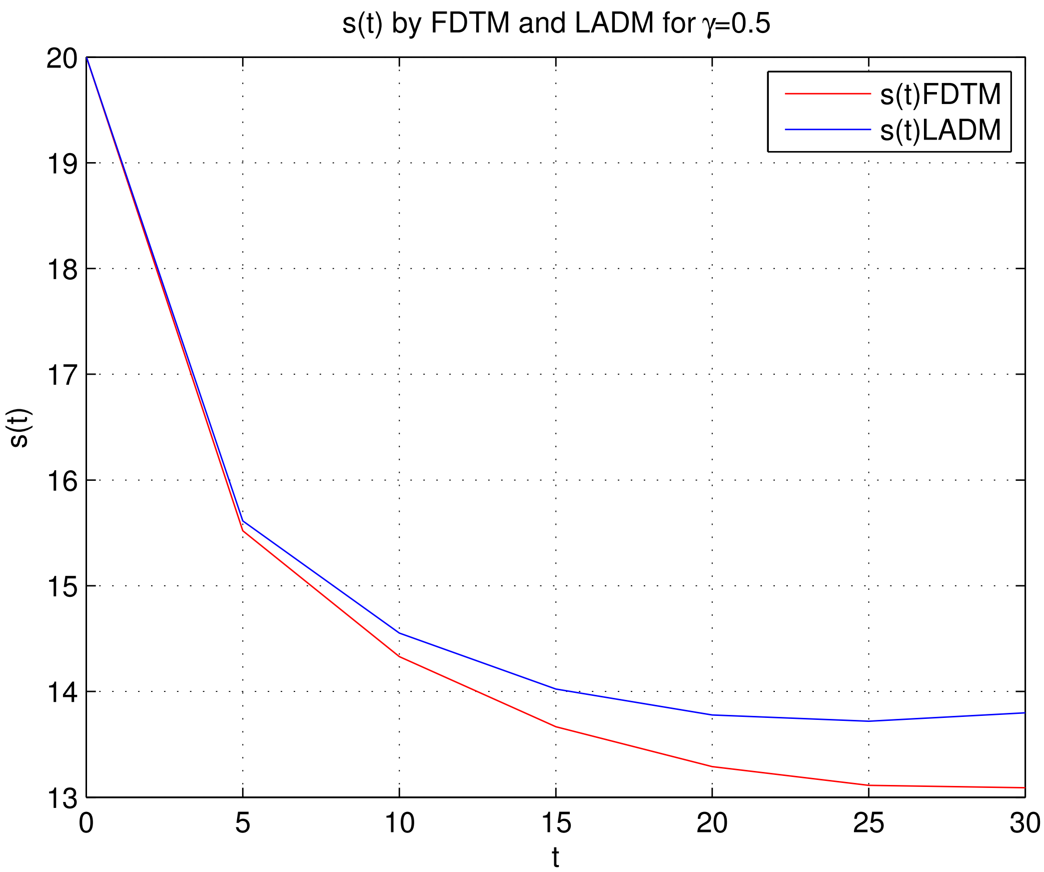


Figure 7: Comparison of graphs of $i(t)$ by FDTM and LADM for $\gamma = 0.5$

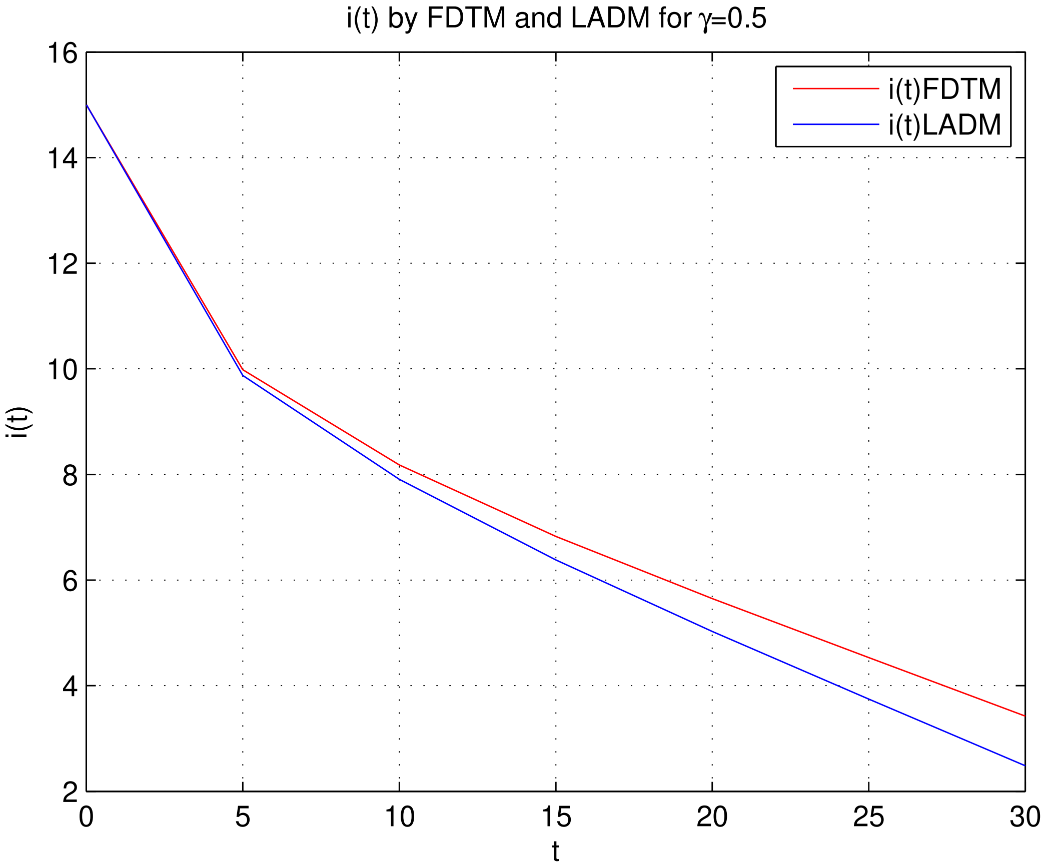


Figure 8: Comparison of graphs of $r(t)$ by FDTM and LADM for $\gamma = 0.5$

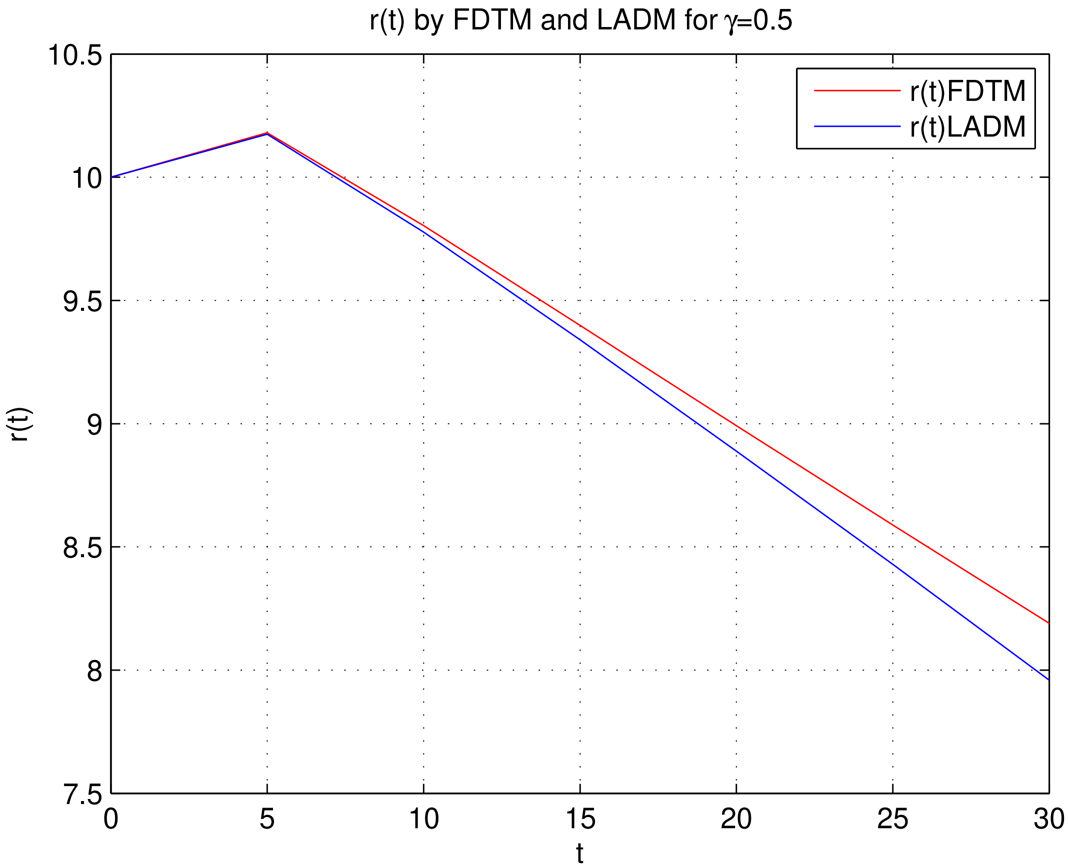


Figure 9: Comparison of errors in $s(t), i(t), r(t)$ by FDTM and LADM for $\gamma = 0.5$

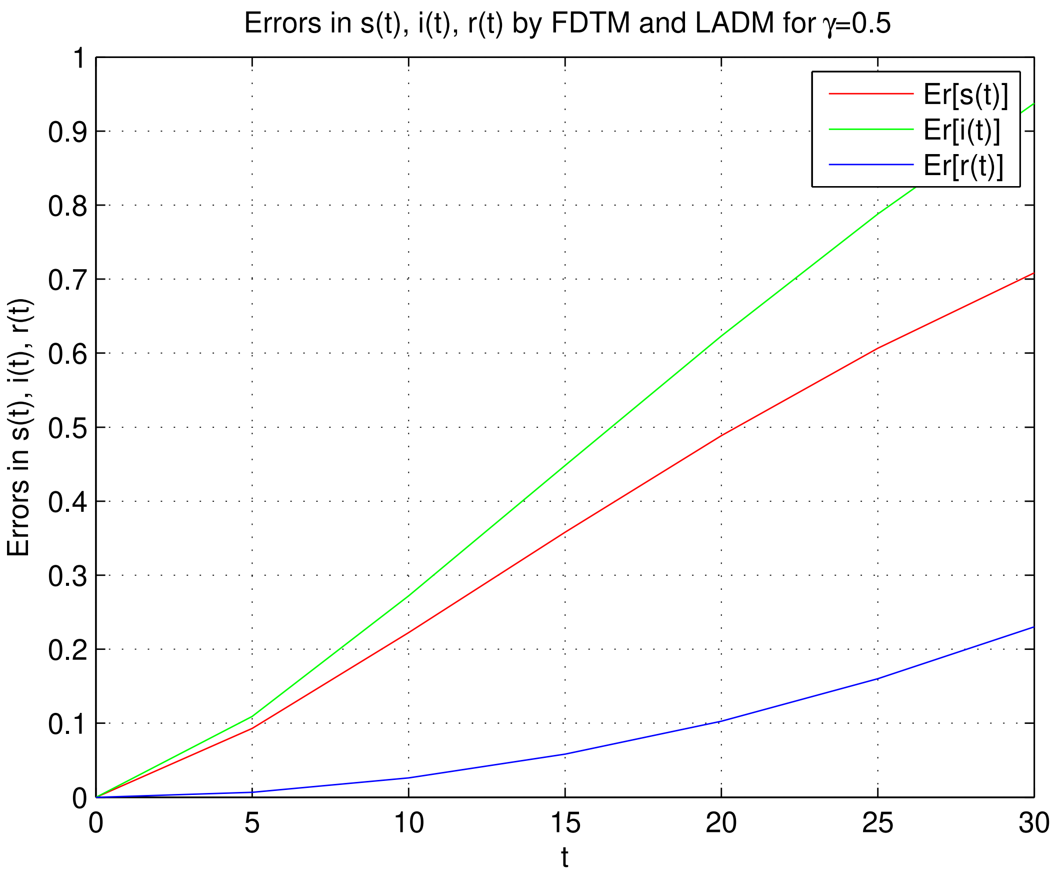
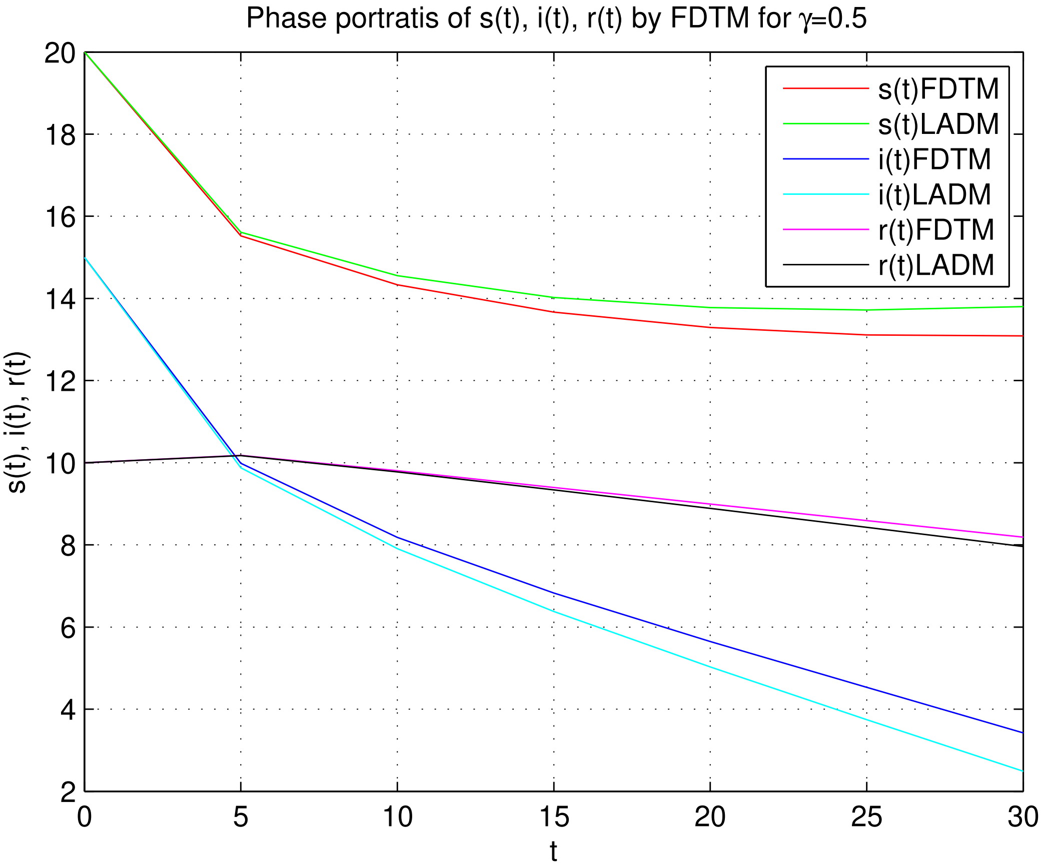


Figure 10: Phase plane portraits of $s(t), i(t), r(t)$ by FDTM and LADM for $\gamma = 0.5$



For $\gamma = 1$, figure 11 shows solution curves for $s(t)$ by FDTM, LADM and RKFA. In the graph, the curves are respectively named as $s(t)FDTM, s(t)LADM$ and $s(t)RKFA$. In the figure, we can not find any differences in the curves just by naked eyes and the curves appear to be coincident and one can not have better understanding of which method among FDTM and LADM is close enough to RKFD. To overcome this difficulty, we have obtained a zoom-in figure of figure 11, which is shown in figure 12. From this figure, we observe that the curve obtained by FDTM is comparatively colse to the one obtained by RKFA. A similar conclusion for the curves $i(t)$ and $r(t)$ can be drawn form the figure 13 to figure 16.

Figure 11: Comparison of graphs of $s(t)$ by FDTM, LADM and RKFA for $\gamma = 1$

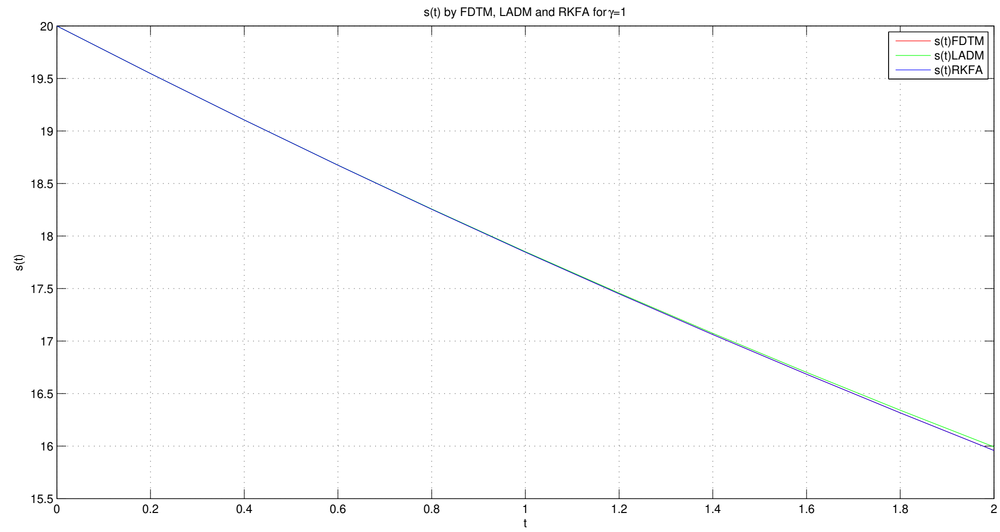


Figure 12: Zoom-in $s(t)$ by FDTM and LADM for $\gamma = 1$

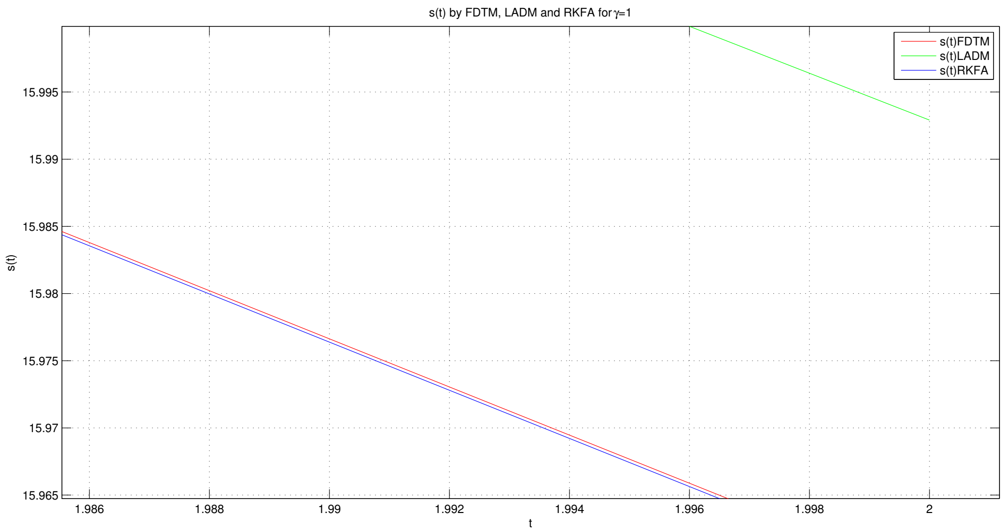


Figure 13: Comparison of graphs of $i(t)$ by FDTM and LADM for $\gamma = 1$

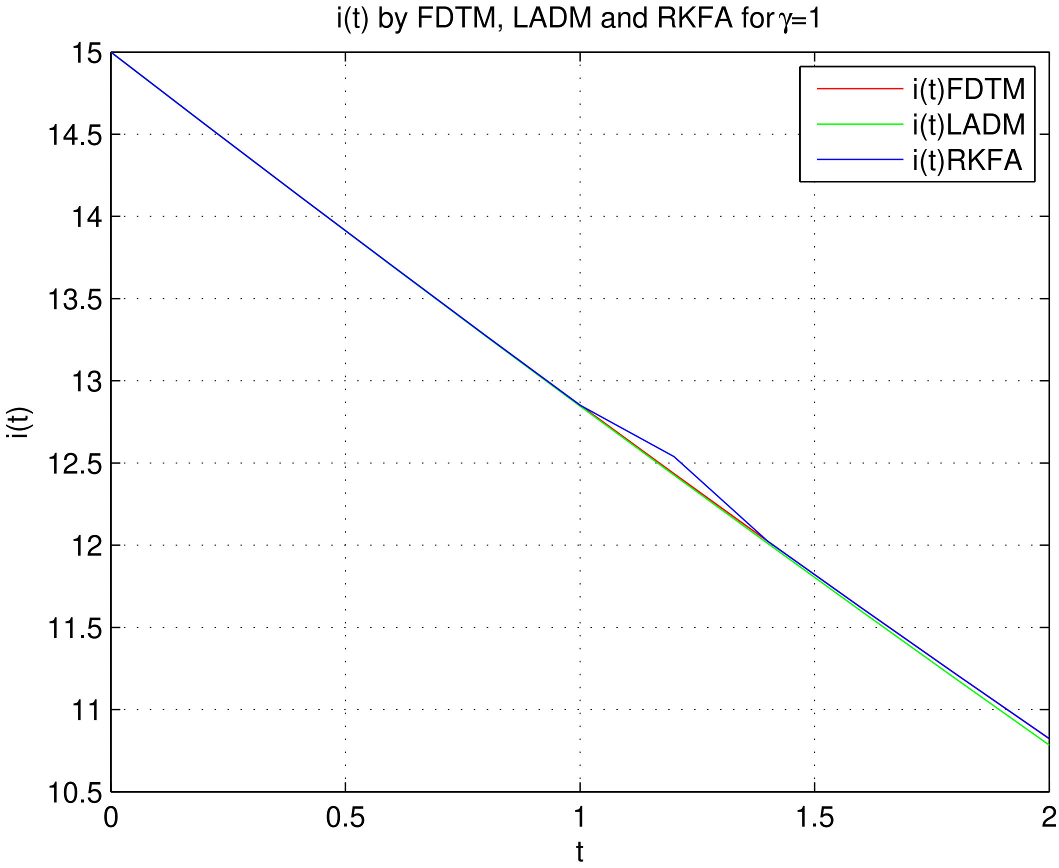


Figure 14: Zoom-in $i(t)$ by FDTM and LADM for $\gamma = 1$

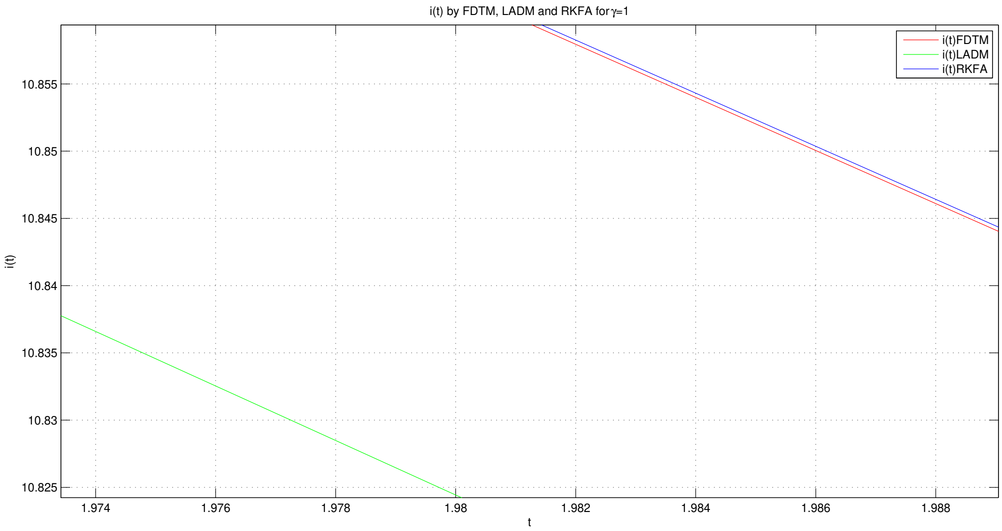


Figure 15: $r(t)$ by FDTM and LADM for $\gamma = 1$

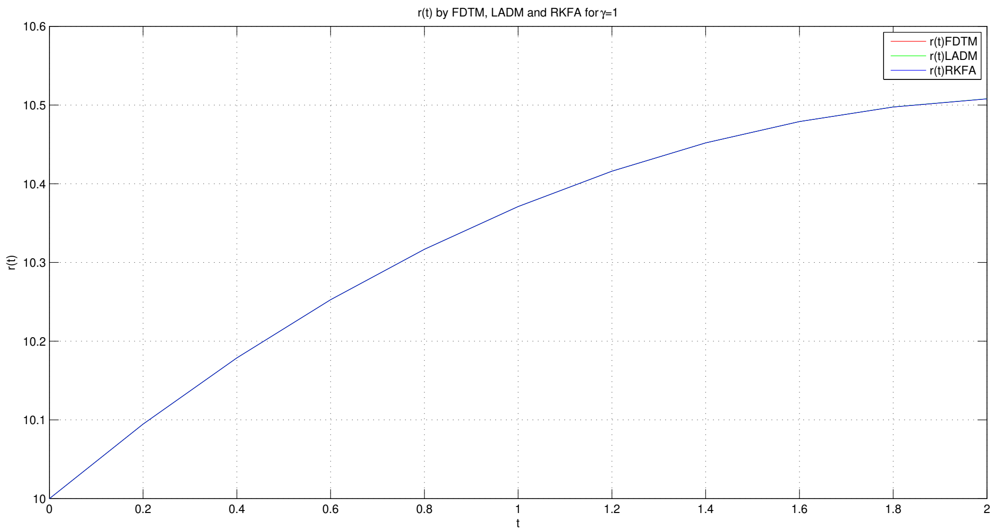
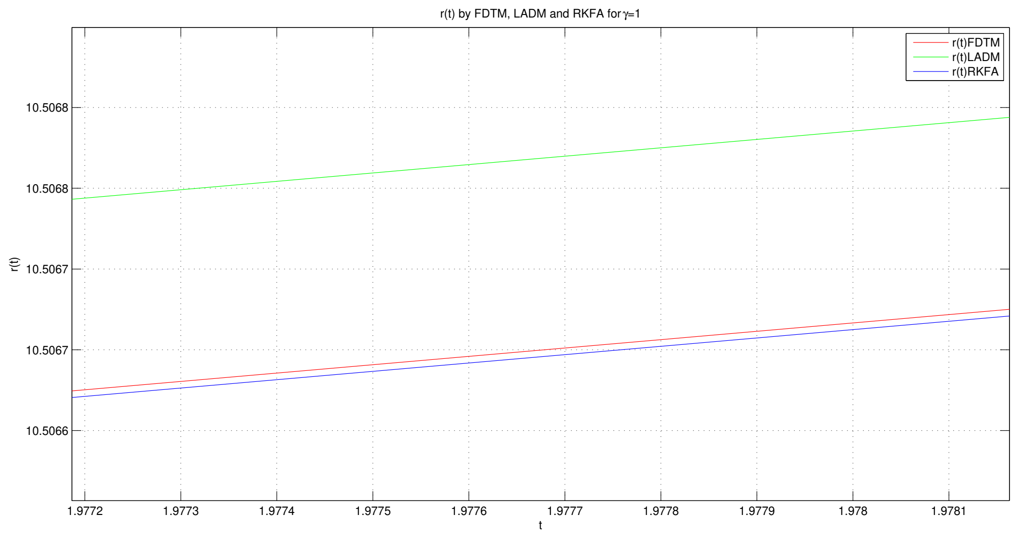


Figure 16: Zoom-in $r(t)$ by FDTM and LADM for $\gamma = 1$



The figure 17 shows the estimate of absolute errors in $s(t)$ by FDTM and LADM with respect to RKFA. In the figure, we observe that the absolute error by FDTM is almost close to zero and the curve is going parallel to the x-axis while the error by LADM is though in between 10^{-2} and 10^{-4}, but still growing exponentially. The figure 18 shows the error in $i(t)$ and the figure 19 shows the error in $r(t)$.

Figure 17: Error in $s(t)$ by FDTM and LADM for $\gamma = 1$

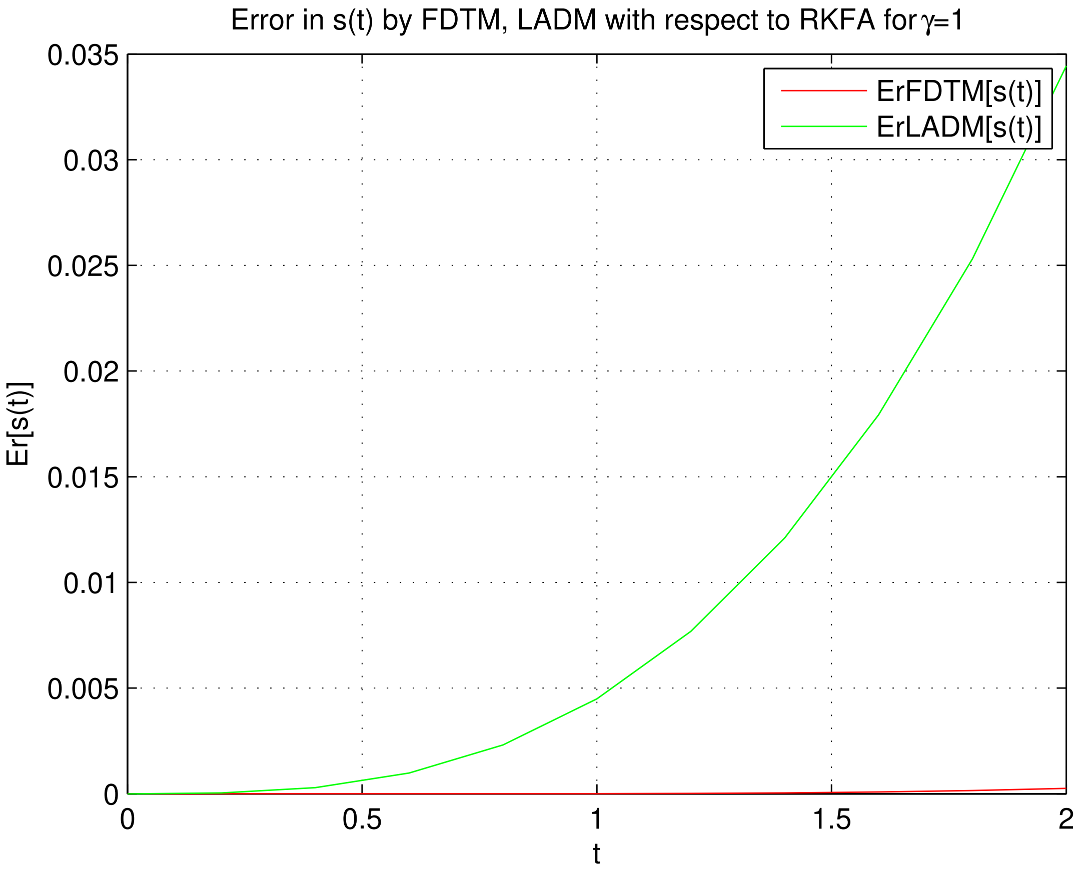


Figure 18: Error in $i(t)$ by FDTM and LADM for $\gamma = 1$

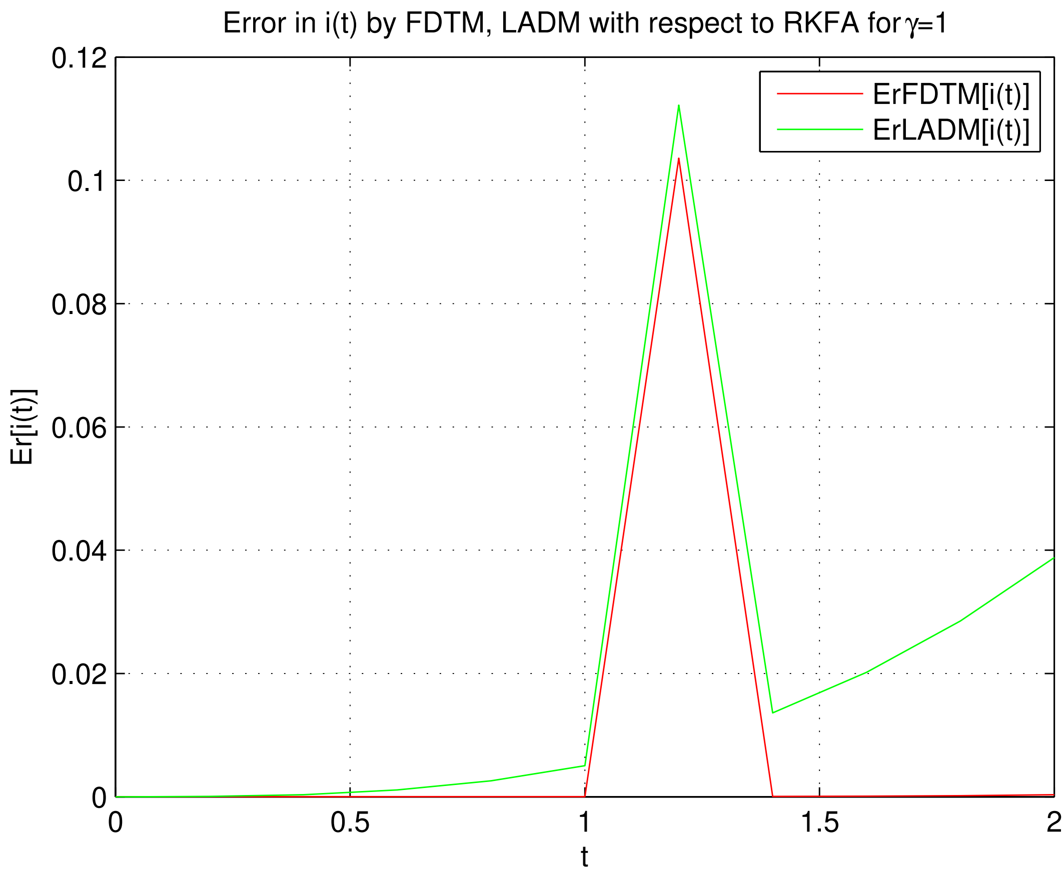
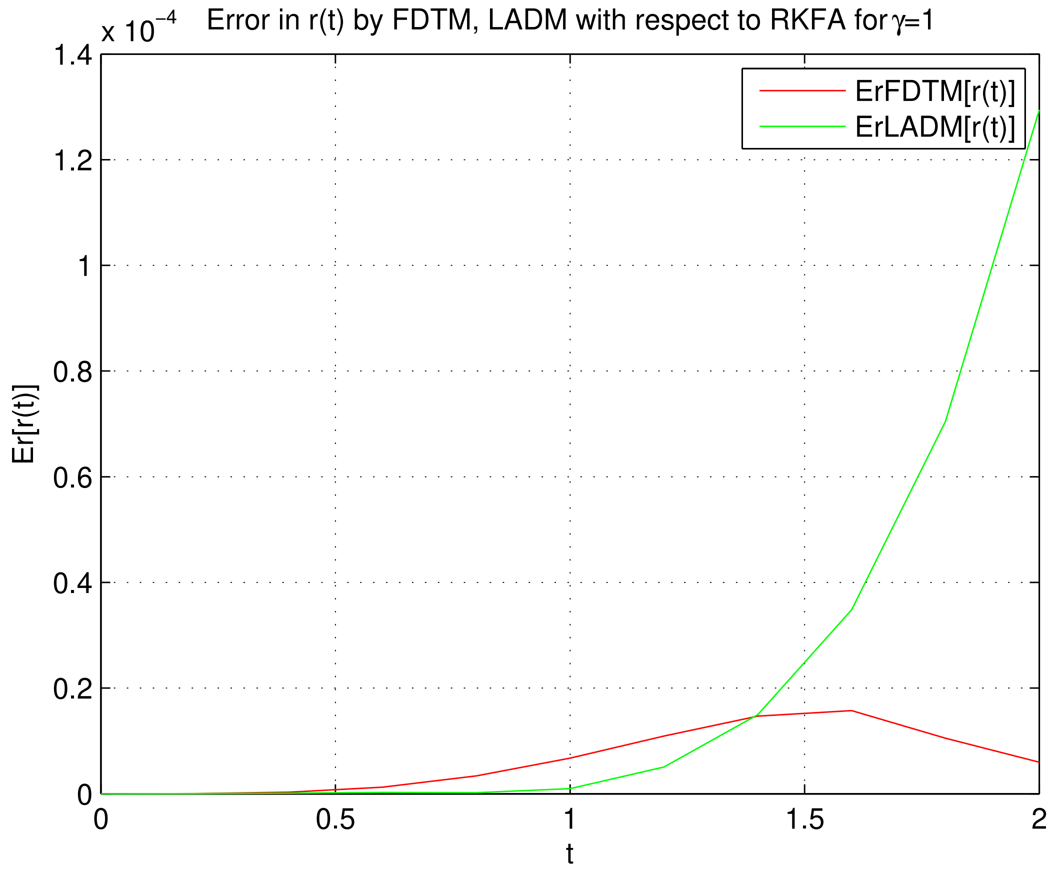


Figure 19: Error in $r(t)$ by FDTM and LADM for $\gamma = 1$



11. Conclusion

In this paper, the non-linear Fractional order susceptible-infected-Recovered(SIR) mathematical model of computer viruses is presented.The fractional derivative used in the model is Caputo Fractional derivative. Using Lyapunov stability analysis,the stability verified of the given mathematical model. The FDTM and the LADM are two reliable and useful techniques that were used in this work to solve the non-linear epidemiological model of computer viruses Involving Non-linear Fractional Order Differential Equation. These models are among the useful computer engineering models that we may use to monitor, assess, and forecast computer viruses within a network. To demonstrate the effectiveness and precision of the approach that was described, the residual errors for five iterations were displayed using the LADM, FDTM, and RKFA. Additionally, residual error graphs were shown to illustrate the capabilities of the methodologies that were provided. These findings suggest that the FDTM is more easy and applicable than the LADM without adomian polynomial. From the graphical analysis of the solution curves and errors by by FDTM and LADM, we conclude that the FDTM for solving the non-linear system of equations 1 presents better solutions as compared to the LADM. However, one should note that due to the complexity of calculations, while finds the solutions by FDTM and LADM, we have considered only five iterations and a small interval for the values of the variable t. The conclusions drawn here may vary if one considers a large number of iterations and find out the long term behaviour of the SIR model. Hence further investigation in this matter is needed. By including a few more parameters, we may improve the model. In many other fields, this model may also be used to provide alternative models.

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