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Topological Approaches of Graphs Using j-Neighbourhoods and Their Applications

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Abstract. This paper investigates novel topological structures on graphs through the lens of j-neighbourhoods, specifically out, in, intersection, and union-based neighbourhoods. We develop a systematic framework for constructing subbases and topologies on directed graphs using these neighbourhoods and analyze their topological properties. Our work provides a rigorous comparative study of neighbourhood types, their interrelations, and their role in generating induced topologies. In addition, we explore potential applications in digital topology, spatial networks, and data structure. The theoretical results are supported by aircraft paths on an airline as an illustrative example and comparison tables that highlight structural differences and practical implications.

2020 Mathematics Subject Classifications: 05C10, 54A10, 05C20

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1. Introduction

The interplay between graph theory and topology has yielded powerful tools for modeling relationships in networks, decision systems, and digital structures. The foundation of topological graph theory, which seeks to apply topological concepts to graph structures, dates back to the foundational work of Kuratowski and others in the early twentieth century. This field has seen significant advances with the development of neighborhood systems and approximation spaces such as those initiated by Pawlak [1],[2] in rough set theory. Recent works have further expanded these concepts. Zhang et al. [3] and Lin [4] explored neighborhood operators in granular computing, while Yao [5] introduced rough sets based on covering with topological interpretations. Graph theory has recently established itself as an independent discipline [6, 7]. A graph $GR = (VE, ED)$ is an ordered pair of vertices $VE(GR)$ and edges $ED(GR)$. We say that the graph GR is finite (resp. infinite) if the set $VE(GR)$ is finite (resp. infinite). In medical decision-making systems, uncertain or imprecise concepts are often modeled using upper and lower approximations,

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which serve as rough estimates of these vague entities[8, 9]. Akram et al. have contributed significantly to this field by exploring intuitionistic fuzzy and fuzzy rough graph structures[10–12], and applying them effectively to various decision-making scenarios.
In [13], the solution of some problems in medicine and geography is deduced. A novel model for the blood circulation system of the human heart, structured around the characterization of blood flow pathways, is proposed by [14]. [15] Supplies a proof of the perfect graph theorem and features a revised chapter on the probabilistic method in graph theory with many results integrated throughout the text. Inspired by the diverse applications of general topology [16–27], our objective is to explore novel approaches to construct topological structures through generalized neighborhood systems. Experiments show that the algorithm proposed in [28] is more efficient than the traditional one when the graph is represented by an adjacency matrix. Graph theory serves as a fundamental mathematical framework that underpins a wide range of disciplines, including operational research, providing tools for modeling and solving complex relational structures [29, 30].Recent studies have extended topological approaches to graph structures, such as the work of Othman et al. [31] in L_2-directed topological spaces for directed graphs, and Alzubaidi et al. [32] who investigated topologies on simple graphs with applications in radar chart methods. Our work builds on these foundations by developing a systematic framework using j-neighbourhoods to construct topological spaces on directed graphs.

This paper begins with Section 2 which introduces the necessary preliminaries; Section 3 defines the j-neighborhood types; Section 4 discusses topological spaces on Air journies as an application; Section 5 includes conclusions and future work.

2. Preliminaries

In this section, some definitions and propositions on graphs that will be used throughout this paper are stated, and their topologies are generated.

Definition 1. *[1] Let (V, S) be an approximation space, where $V \neq \phi$ is a finite universe set, and S is an equivalence relation in V. Then:*
$LO_s(X) = \bigcup\limits_{x \subset V} \{S(x) : S(x) \subseteq X\}$,
$UP_s(X) = \bigcap\limits_{x \subset V} \{S(x) : S(x) \bigcap \neq \phi\}$.

Definition 2. *[13] Let (GR) be a graph with vertices $VE(GR)$ and S be a relation to GR. The open neighbourhoods of $(ve)_i$ are $(ve)_i S = \{(ve)_j : \overrightarrow{(ve)_i(ve)_j} \in E(GR)\}$. The subbase is $Sub_{GR} = \bigcup \{(ve)_i S : (ve)_i \in VE(GR)\}$*

Definition 3. *In [13], the topological space τ_{GR} in GR is induced by the base B_{GR} that is induced by the subbase Sub_{GR}.*

Definition 4. *[14] Let $GR(VE, ED)$ be a simple directed graph, and $ve \in VE(GR)$. The j-neighbourhoods of ve, say $N_j(ve), j \in \{t, n, int, un\}$, can be defined as:*

(i) *Out neighbourhood:* $N_t((ve)_i) = \bigcup_{i,k}\{(ve)_i, (ve)_k\} \bigcup \{(ed)_j : (ed)_j = \overrightarrow{(ve)_i(ve)_k}, (ed)_j \in ED, (ve)_i, (ve)_k \in VE\}$, *for each* $i, j, k \in I$.

(ii) *In neighbourhood:* $N_n((ve)_i) = \bigcup_{i,k}\{(ve)_i, (ve)_k\} \bigcup \{(ed)_j : (ed)_j = \overrightarrow{(ve)_k(ve)_i}, (ed)_j \in ED, (ve)_i, (ve)_k \in VE\}$, *for each* $i, j, k \in I$

(iii) *Intersection of neighbourhoods:* $N_{int}((ve)_i) = N_t \bigcap N_n$.

(iv) *Union of neighbourhoods:* $N_{un}((ve)_i) = N_t \bigcup N_n$.

Proposition 1. *[14] Let* $GR(VE, ED)$ *be a simple digraph,* N_j *be different kinds of neighbourhoods, where* $j \in \{t, n, int, un\}, K$ *and* M *are two subgraphs of* GR*. Then:*

(i) $LO_{N_j}((VE, ED)(K)) \subseteq (VE, ED)(K) \subseteq UP_{N_j}((VE, ED)(K))$;

(ii) $LO_{N_j}(GR) = UP_{N_j}(GR) = GR$;

(iii) $LO_{N_j}(\phi) = UP_{N_j}(\phi) = \phi$;

(iv) *If* $(VE, ED)(K) \subseteq (VE, ED)(M)$, *then* $LO_{N_j}((VE, ED)(K)) \subseteq LO_{N_j}((VE, ED)(M))$ *and* $UP_{N_j}((VE, ED)(K)) \subseteq UP_{N_j}((VE, ED)(M))$;

(v) $(UP_{N_j}((VE, ED)(K)))^c = LO_{N_j}((VE, ED)(K))^c$, *where* $((VE, ED)(K))^c$ *is a complement to* $(VE, ED)(K)$;

(vi) $UP_{N_j}((VE, ED)(K))^c = (LO_{N_j}((VE, ED)(K)))^c$.

Proposition 2. *[14] Let* $GR(VE, ED)$ *be a digraph,* $N_j(ve)$ *be different kinds neighbourhoods, where* $j \in \{t, n, int, un\}, K$ *and* M *be two subgraphs of* GR*. Then:*

(i) $LO_{N_j}((VE, ED)(K)) \bigcup LO_{N_j}((VE, ED)(M)) \subseteq LO_{N_j}((VE, ED)(K) \bigcup (VE, ED)(M))$;

(ii) $LO_{N_j}((VE, ED)(K) \bigcap (VE, ED)(M)) = LO_{N_j}((VE, ED)(K)) \bigcap LO_{N_j}((VE, ED)(M))$;

(iii) $UP_{N_j}((VE, ED)(K) \bigcup (VE, ED)(M)) = UP_{N_j}((VE, ED)(K)) \bigcup UP_{N_j}((VE, ED)(M))$;

(iv) $UP_{N_j}((VE, ED)(K) \bigcap (VE, ED)(M)) \subseteq UP_{N_j}((VE, ED)(K)) \bigcap UP_{N_j}((VE, ED)(M))$.

Proposition 3. *[14] Let* $GR(VE, ED)$ *be a simple digraph,* N_j *be different kinds of neighbourhoods, where* $j \in \{t, n, int, un\}, K$ *is a subgraph of* GR*. Then:*

(i) $LO_{N_j}((VE, ED)(GR) - (VE, ED)(K)) = (VE, ED)(GR) - UP_{N_j}((VE, ED)(K))$;

(ii) $UP_{N_j}(VE, ED)(GR) - (VE, ED)(K)) = (VE, ED)(GR) - LO_{N_j}((VE, ED)(K))$.

Proposition 4. *[14] Let* $GR(VE, ED)$ *be a simple digraph* N_j *be different kinds of neighbourhoods, where* $j \in \{t, n, int, un\}, K$ *and* M *are two subgraphs of* GR*. Then:*

(i) $LO_{N_j}((VE, ED)(K)) - (VE, ED)(M)) \subseteq LO_{N_j}((VE, ED)(K)) - LO_{N_j}((VE, ED)(M))$;

(ii) $Up_{N_j}((VE,ED)(K)) - UP_{N_j}((VE,ED)(M)) \subseteq UP_{N_j}((VE,ED)(K)) - (VE,ED)(M))$.

Proposition 5. *[14] Let $GR(VE,ED)$ be a graph and $((VE,ED)(GR),\tau_{N_j})$ its nanotopology K and M be induced subgraphs of GR. Then:*

(i) $(VE,ED)(K) \subseteq SU_{N_j}((VE,ED)(K))$;

(ii) *If $K \subseteq M$, then* $SU_{N_j}((VE,ED)(K)) \subseteq SU_{N_j}((VE,ED)(M))$;

(iii) $SU_{N_j}((VE,ED)(K) \bigcup (VE,ED)(M)) = SU_{N_j}((VE,ED)(K)) \bigcup SU_{N_j}((VE,ED)(M))$;

(iv) $SU_{N_j}((VE,ED)(K) \bigcap (VE,ED)(M)) \subseteq SU_{N_j}((VE,ED)(K)) \bigcap SU_{N_j}((VE,ED)(M))$;

(v) *If $K \subseteq M$ then* $RI_{N_j}((VE,ED)(K)) \subseteq RI_{N_j}((VE,ED)(M))$;

(vi) $RI_{N_j}((VE,ED)(K) \bigcup RI_{N_j}((VE,ED)(M)) \subseteq RI_{N_j}((VE,ED)(K) \bigcup (VE,ED)(M))$;

(vii) $RI_{N_j}((VE,ED)(K) \bigcap (VE,ED)(M)) = RI_{N_j}((VE,ED)(K) \bigcap RI_{N_j}((VE,ED)(M))$.

Proposition 6. *[14] Let $GR(VE,ED)$ be a graph and $((VE,ED)(GR),\tau_{N_j})$ its nanotopology, K be an induced subgraph of GR. Then:*

(i) $RI_{N_j}((VE,ED)(GR) - (VE,ED)(K)) = (VE,ED)(GR) - SU_{N_j}((VE,ED)(K))$;

(ii) $SU_{N_j}((VE,ED)(GR) - (VE,ED)(K)) = (VE,ED)(GR) - RI_{N_j}((VE,ED)(K))$.

3. New Types of Topological Spaces by Simple Directed Graphs

In this section, we introduce new kinds of j-neighbourhoods, j-lower(upper) approximations with illustrative tables, in addition to how to find a subbase, a base, and generating topologies. Moreover, some propositions with related remarks will be exposed.
The following definition defines and investigates four forms of neighbourhoods of vertices in simple directed graphs dependent on neighbouring vertices.

Definition 5. *Let $SDG(L)$ be a simple directed graph, and $l \in L(SDG)$. The j-neighbourhood of l, say $N_j(l), j \in \{t, n, int, un\}$, abbreviations for $\{$out,in,intersection,union$\}$, can be defined as*

(i) *Outside neighbourhood (briefly, t-neighbourhood):* $N_t(l_i) = \{l_i \in L : i \neq j, l_i \Rightarrow l_j\}$, *for each $i, j \in I$;*

(ii) *Inside neighbourhood (briefly, n-neighbourhood):* $N_n(l_i) = \{l_j \in L : i \neq j, l_j \Rightarrow l_i\}$, *for each $i, j \in I$;*

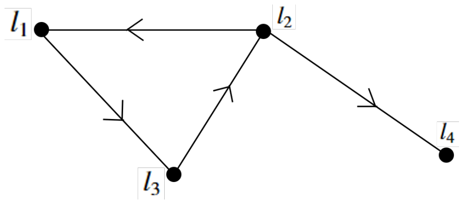
(iii) *Intersection of neighbourhoods (briefly, int-neighbourhood):* $N_{int}(l_i) = N_t(l_i) \bigcap N_n(l_i)$;

(iv) *Union of neighbourhoods (briefly, un-neighbourhood):* $N_{un}(l_i) = N_t(l_i) \bigcup N_n(l_i)$.

Example 1. *Consider Figure 1. The set of vertices of a simple digraph is* $L(G) = \{l_1, l_2, l_3, l_4\}$.

tableNeighbourhoods of simple digraph

$l \in L(SDG)$	l_1	l_2	l_3	l_4
$N_t(l_i)$	$\{l_3\}$	$\{l_1, l_4\}$	$\{l_2\}$	ϕ
$N_n(l_i)$	$\{l_2\}$	$\{l_3\}$	$\{l_1\}$	$\{l_2\}$
$N_{int}(l_i)$	ϕ	ϕ	ϕ	ϕ
$N_{un}(l_i)$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2\}$	$\{l_2\}$



figureA simple directed graph

Definition 6. *Let $SDG(L)$ be a simple directed graph, K be a subgraph of SDG and $N_j(l), j \in \{t, n, int, un\}$ be the j-neighbourhood of $l \in L(SDG)$. Then:*

(i) The j-lower approximation $LO_{N_j}((L)(K)) = \bigcup_{l \in (L)(SDG)} \{l : N_j(l) \subseteq (L)(K)\}$;

(ii) The j-upper approximation $UP_{N_j}((L)(K)) = \bigcup_{l \in (L)(SDG)} \{l : N_j(l) \bigcap (L)(K) \neq \phi\}$.

Example 2. *From Example 1. We get the following tables.*

Table 1: $LO_{N_j}((L)(K))$ w.r.t. Table 1

$L(K)$	$LO_{N_t}(L)(K)$	$LO_{N_n}(L)(K)$	$LO_{N_{int}}(L)(K)$	$LO_{N_{un}}(L)(K)$
ϕ	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$
$\{l_1\}$	$\{l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ
$\{l_2\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_3\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_4\}$	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$\{l_1, l_2\}$	$\{l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_1, l_3\}$	$\{l_1, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	ϕ
$\{l_1, l_4\}$	$\{l_2, l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ
$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$
$\{l_2, l_4\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_3, l_4\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_1, l_2, l_3\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$L(SDG)$	$\{l_1, l_3, l_4\}$
$\{l_1, l_2, l_4\}$	$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	$\{l_2\}$
$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$

Table 2: $UP_{N_j}((L)(K))$ w.r.t. Table 1

$L(K)$	$UP_{N_t}(L)(K)$	$UP_{N_n}(L)(K)$	$UP_{N_{int}}(L)(K)$	$UP_{N_{un}}(L)(K)$
ϕ	ϕ	ϕ	ϕ	ϕ
$L(SDG)$	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
$\{l_1\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2\}$	$\{l_3\}$	$\{l_1, l_4\}$	ϕ	$\{l_1, l_3, l_4\}$
$\{l_3\}$	$\{l_1\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_4\}$	$\{l_2\}$	ϕ	ϕ	$\{l_2\}$
$\{l_1, l_2\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_3\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_1, l_4\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2, l_3\}$	$\{l_1, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$
$\{l_2, l_4\}$	$\{l_2, l_3\}$	$\{l_1, l_4\}$	ϕ	$L(SDG)$
$\{l_3, l_4\}$	$\{l_1, l_2\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
$\{l_1, l_2, l_4\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_3, l_4\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_2, l_3, l_4\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$

Definition 7. *Let $SDG(L)$ be a simple directed graph $(L)(K)$ be a subgraph of $SDG, N_j(l)$ be different kinds of j-neighbourhoods, where $j \in \{t, n, int, un\}$ and $Sub_j(L)(K) = \{LO_{N_j}(L)(K), UP_{N_j}(L)(K)\}$ be a subbase where the base $B_j(L)(K)$ be the finite intersection of the subbase elements and $\tau_{N_j}((L)(K))$ be the topology on $(L)(SDG)$ with respect to $(L)(K)$ which can be find by an arbitrary unions of the base elements $((L)(SDG), \tau_{N_j}(L)(K))$ is a topology induced by different j-neighbourhoods. The topology $\tau = \bigcap\{\tau_{N_j}, j \in \{t, n, int, un\}\}$.*

Example 3. *From Example 2, we get the subbase $Sub_j(L)(K)$ in Table 4, the base $B_j(L)(K)$ in Table 5 and $\tau_{N_j}(L)(K)$ in Table 6. * Note that $\tau = \{\phi, L(SDG)\}$ for any $(L)(K)$.*

Table 3: $Sub_j(L)(K)$

$L(K)$	$Sub_t(L)(K)$	$Sub_n(L)(K)$	$Sub_{int}(L)(K)$	$Sub_{un}(L)(K)$
ϕ	$\{\phi, \{l_4\}\}$	$\{\phi\}$	$\{\phi, L(SDG)\}$	$\{\phi\}$
$L(SDG)$	$\{L(SDG), \{l_1, l_2, l_3\}\}$	$\{L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG)\}$
$\{l_1\}$	$\{\{l_2\}, \{l_4\}\}$	$\{\{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_2\}$	$\{\{l_3\}, \{l_3, l_4\}\}$	$\{\{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_4\}, \{l_1, l_3, l_4\}\}$
$\{l_3\}$	$\{\{l_1\}, \{l_1, l_4\}\}$	$\{\{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2\}\}$
$\{l_4\}$	$\{\{l_2\}, \{l_4\}\}$	$\{\phi\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2\}\}$
$\{l_1, l_2\}$	$\{\{l_2, l_3\}, \{l_3, l_4\}\}$	$\{\{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_3, l_4\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\{l_1, l_2\}, \{l_1, l_4\}\}$	$\{\{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3\}\}$
$\{l_1, l_4\}$	$\{\{l_2\}, \{l_2, l_4\}\}$	$\{\{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_2, l_3\}$	$\{\{l_1, l_3\}, \{l_1, l_3, l_4\}\}$	$\{\{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\{l_2, l_3\}, \{l_3, l_4\}\}$	$\{\{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_4\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\{l_1, l_2\}, \{l_1, l_4\}\}$	$\{\{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2\}\}$
$\{l_1, l_2, l_3\}$	$\{\{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_3, l_4\}\}$
$\{l_1, l_2, l_4\}$	$\{\{l_2, l_3\}, \{l_2, l_3, l_4\}\}$	$\{\{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_3, l_4\}\}$
$\{l_1, l_3, l_4\}$	$\{\{l_1, l_2\}, \{l_1, l_2, l_4\}\}$	$\{\{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_2\}, \{l_1, l_2, l_3\}\}$
$\{l_2, l_3, l_4\}$	$\{\{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{\{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4\}, L(SDG)\}$

Table 4: $B_j(L)(K)$

$L(K)$	$B_t(L)(K)$	$B_n(L)(K)$	$B_{int}(L)(K)$	$B_{un}(L)(K)$
ϕ	$\{\phi, L(SDG), \{l_4\}\}$	$\{L(SDG), \phi\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$
$L(SDG)$	$\{L(SDG), \{l_1, l_2, l_3\}\}$	$\{L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG)\}$
$\{l_1\}$	$\{\phi, L(SDG), \{l_2\}, \{l_4\}\}$	$\{\{L(SDG), \{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_2, l_3\}\}$
$\{l_2\}$	$\{L(SDG), \{l_3\}, \{l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_4\}, \{l_1, l_3, l_4\}\}$
$\{l_3\}$	$\{L(SDG), \{l_1\}, \{l_1, l_4\}\}$	$\{L(SDG), \{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2\}\}$
$\{l_4\}$	$\{L(SDG), \phi, \{l_2\}, \{l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_2\}\}$
$\{l_1, l_2\}$	$\{L(SDG), \{l_3\}, \{l_2, l_3\}, \{l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_3, l_4\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{L(SDG), \{l_1\}, \{l_1, l_2\}, \{l_1, l_4\}\}$	$\{L(SDG), \{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2, l_3\}\}$
$\{l_1, l_4\}$	$\{L(SDG), \{l_2\}, \{l_2, l_4\}\}$	$\{L(SDG), \{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_2, l_3\}\}$
$\{l_2, l_3\}$	$\{L(SDG), \{l_1, l_3\}, \{l_1, l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{L(SDG), \{l_3\}\{l_2, l_3\}, \{l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_4\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{L(SDG), \{l_1\}, \{l_1, l_2\}, \{l_1, l_4\}\}$	$\{L(SDG), \{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2\}\}$
$\{l_1, l_2, l_3\}$	$\{L(SDG), \{l_1, l_3\}, \{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_3, l_4\}\}$
$\{l_1, l_2, l_4\}$	$\{L(SDG), \{l_2, l_3\}, \{l_2, l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_3, l_4\}\}$
$\{l_1, l_3, l_4\}$	$\{L(SDG), \{l_1, l_2\}, \{l_1, l_2, l_4\}\}$	$\{L(SDG), \{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_2\}, \{l_1, l_2, l_3\}\}$
$\{l_2, l_3, l_4\}$	$\{L(SDG), \{l_1, l_3\}, \{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{L(SDG), \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4\}, L(SDG)\}$

Table 5: $\tau_j(L)(K)$

$L(K)$	$\tau_{N_t}(L)(K)$	$\tau_{N_n}(L)(K)$	$\tau_{N_{int}}(L)(K)$	$\tau_{N_{un}}(L)(K)$
ϕ	$\{\phi, L(SDG), \{l_4\}\}$	$\{L(SDG), \phi\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$
$L(SDG)$	$\{\phi, L(SDG), \{l_1, l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$
$\{l_1\}$	$\{\phi, L(SDG), \{l_2\}, \{l_4\}, \{l_2, l_4\}\}$	$\{\phi, \{L(SDG), \{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_2, l_3\}\}$
$\{l_2\}$	$\{\phi, L(SDG), \{l_3\}, \{l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_4\}, \{l_1, l_3, l_4\}\}$
$\{l_3\}$	$\{\phi, L(SDG), \{l_1\}, \{l_1, l_4\}\}$	$\{\phi, L(SDG), \{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2\}\}$
$\{l_4\}$	$\{L(SDG), \phi, \{l_2\}, \{l_4\}, \{l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_2\}\}$
$\{l_1, l_2\}$	$\{\phi, L(SDG), \{l_3\}, \{l_2, l_3\}, \{l_3, l_4\}, \{l_2, l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_3, l_4\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\phi, L(SDG), \{l_1\}, \{l_1, l_2\}, \{l_1, l_4\}, \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG), \{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2, l_3\}\}$
$\{l_1, l_4\}$	$\{\phi, L(SDG), \{l_2\}, \{l_2, l_4\}\}$	$\{\phi, L(SDG), \{l_3\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_2, l_3\}\}$
$\{l_2, l_3\}$	$\{\phi, L(SDG), \{l_1, l_3\}, \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\phi, L(SDG), \{l_3\}\{l_2, l_3\}, \{l_3, l_4\}, \{l_2, l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_4\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\phi, L(SDG), \{l_1\}, \{l_1, l_2\}, \{l_1, l_4\}, \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG), \{l_2\}\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \phi, \{l_1, l_2\}\}$
$\{l_1, l_2, l_3\}$	$\{\phi, L(SDG), \{l_1, l_3\}, \{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_1, l_3, l_4\}\}$
$\{l_1, l_2, l_4\}$	$\{\phi, L(SDG), \{l_2, l_3\}, \{l_2, l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_3, l_4\}\}$
$\{l_1, l_3, l_4\}$	$\{\phi, L(SDG), \{l_1, l_2\}, \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG), \{l_2, l_3\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_2\}, \{l_1, l_2, l_3\}\}$
$\{l_2, l_3, l_4\}$	$\{\phi, L(SDG), \{l_1, l_3\}, \{l_1, l_2, l_3\}, \{l_1, l_3, l_4\}\}$	$\{\phi, L(SDG), \{l_1, l_2, l_4\}\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4\}, L(SDG)\}$

Table 6: $(LO_{N_j}((L)(K)))^c$ w.r.t. Table 2

$L(K)$	$(LO_{N_t}(L)(K))^c$	$(LO_{N_n}(L)(K))^c$	$(LO_{N_{int}}(L)(K))^c$	$(LO_{N_{un}}(L)(K))^c$
ϕ	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
$L(SDG)$	ϕ	ϕ	ϕ	ϕ
$\{l_1\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$
$\{l_2\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_3\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_4\}$	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
$\{l_1, l_2\}$	$\{l_1, l_2\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_1, l_3\}$	$\{l_2, l_3\}$	$\{l_1, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_4\}$	$\{l_1, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$
$\{l_2, l_3\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2, l_4\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_3, l_4\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_3\}$	$\{l_2\}$	ϕ	ϕ	$\{l_2\}$
$\{l_1, l_2, l_4\}$	$\{l_1\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_1, l_3, l_4\}$	$\{l_3\}$	$\{l_1, l_4\}$	ϕ	$\{l_1, l_3, l_4\}$
$\{l_2, l_3, l_4\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$

Table 7: $(UP_{N_j}((L)(K)))^c$ w.r.t. Table 3

$L(K)$	$(UP_{N_t}(L)(K))^c$	$(UP_{N_n}(L)(K))^c$	$(UP_{N_{int}}(L)(K))^c$	$(UP_{N_{un}}(L)(K))^c$
ϕ	$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$
$L(SDG)$	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$\{l_1\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$
$\{l_2\}$	$\{l_1, l_2, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	$\{l_2\}$
$\{l_3\}$	$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$L(SDG)$	$\{l_1, l_3, l_4\}$
$\{l_1, l_2\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_1, l_3\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_1, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$
$\{l_2, l_3\}$	$\{l_2, l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ
$\{l_2, l_4\}$	$\{l_1, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	ϕ
$\{l_3, l_4\}$	$\{l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_1, l_2, l_3\}$	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$\{l_1, l_2, l_4\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_1, l_3, l_4\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_2, l_3, l_4\}$	$\{l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ

Table 8: $LO_{N_j}((L)(K))^c$

$(L(K))^c$	$LO_{N_t}((L)(K))^c$	$LO_{N_n}((L)(K))^c$	$LO_{N_{int}}((L)(K))^c$	$LO_{N_{un}}((L)(K))^c$
$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$
ϕ	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$
$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	$\{l_2\}$
$\{l_1, l_2, l_4\}$	$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_1, l_2, l_3\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$L(SDG)$	$\{l_1, l_3, l_4\}$
$\{l_3, l_4\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_2, l_4\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	$\{l_1, l_2, l_4\}$	$L(SDG)$	$\{l_1, l_4\}$
$\{l_1, l_4\}$	$\{l_2, l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ
$\{l_1, l_3\}$	$\{l_1, l_4\}$	$\{l_2, l_3\}$	$L(SDG)$	ϕ
$\{l_1, l_2\}$	$\{l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	$\{l_3, l_4\}$
$\{l_4\}$	$\{l_4\}$	ϕ	$L(SDG)$	ϕ
$\{l_3\}$	$\{l_1, l_4\}$	$\{l_2\}$	$L(SDG)$	ϕ
$\{l_2\}$	$\{l_3, l_4\}$	$\{l_1, l_4\}$	$L(SDG)$	$\{l_4\}$
$\{l_1\}$	$\{l_4\}$	$\{l_3\}$	$L(SDG)$	ϕ

Table 9: $UP_{N_j}((L)(K))^c$

$(L(K))^c$	$UP_{N_t}((L)(K))^c$	$UP_{N_n}((L)(K))^c$	$UP_{N_{int}}((L)(K))^c$	$UP_{N_{un}}((L)(K))^c$
$L(SDG)$	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{l_2, l_3, l_4\}$	$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_3, l_4\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_1, l_2, l_4\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_3\}$	$\{l_1, l_2, l_3\}$	$L(SDG)$	ϕ	$L(SDG)$
$\{l_3, l_4\}$	$\{l_1, l_2\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_2, l_4\}$	$\{l_2, l_3\}$	$\{l_1, l_4\}$	ϕ	$L(SDG)$
$\{l_2, l_3\}$	$\{l_1, l_3\}$	$\{l_1, l_2, l_4\}$	ϕ	$L(SDG)$
$\{l_1, l_4\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_1, l_3\}$	$\{l_1, l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_1, l_2, l_3\}$
$\{l_1, l_2\}$	$\{l_2, l_3\}$	$\{l_1, l_3, l_4\}$	ϕ	$L(SDG)$
$\{l_4\}$	$\{l_2\}$	ϕ	ϕ	$\{l_2\}$
$\{l_3\}$	$\{l_1\}$	$\{l_2\}$	ϕ	$\{l_1, l_2\}$
$\{l_2\}\}$	$\{l_3\}$	$\{l_1, l_4\}$	ϕ	$\{l_1, l_3, l_4\}$
$\{l_1\}$	$\{l_2\}$	$\{l_3\}$	ϕ	$\{l_2, l_3\}$

Definition 8. *Let $SDG(L)$ be a simple directed graph, and let $(L)(K)$ be a subgraph of SDG. Then the j-closure(j-interior) of K is the j-upper(j-lower) approximation of K and can be defined by $SU_{N_j}(L)(K) = UP_{N_j}(L)(K)(RI_{N_j}(L)(K) = LO_{N_j}(L)(K))$, where $j \in \{t, n, int, un\}$.*

Proposition 7. *Let $SDG(L)$ be a simple directed graph, N_j be different kinds of j-neighbourhoods, where $j \in \{t, n, int, un\}, K$ and M are two subgraphs of SDG. Then:*

(i) $LO_{N_j}(SDG) = SDG$;
(ii) $UP_{N_j}(\phi) = \phi$;
(iii) If $(L)(K) \subseteq (L)(M)$, then $LO_{N_j}(L)(K) \subseteq LO_{N_j}(L)(M)$ and $UP_{N_j}(L)(K) \subseteq UP_{N_j}(L)(M)$;
(iv) $(UP_{N_j}(L)(K))^c = LO_{N_j}((L)(K))^c$, where $((L)(K))^c$ is a complement of $(L)(K)$;
(v) $UP_{N_j}((L)(K))^c = (LO_{N_j}((l)(K)))^c$.

Proof. (i) and (ii) are obvious from Definition 6. (iii) Let $l \in LO_{N_j}((L)(K))$. Then $N_j(l) \subseteq (L)(K)$ by Definition 6, but $(L)(K) \subseteq (L)(M)$. Then $N_j(l) \subseteq (L)(M)$ and so $l \in LO_{N_j}(L)(M)$. Therefore, $LO_{N_j}(L)(K) \subseteq LO_{N_j}(L)(M)$. The proof of $UP_{N_j}(L)(K) \subseteq UP_{N_j}(L)(M)$ is similar. (iv) $(UP_{N_j}(L)(K))^c = (\bigcup_{l \in (L)(SDG)} \{l : N_j(l) \bigcap (L)(K) \neq \phi\})^c = \{l \in (L)(SDG) : N_j(l) \bigcap (L)(K) = \phi\} = \{l \in (L)(SDG) : N_j(l) \subseteq ((L)(K))^c\} = LO_{N_j}((L)(K))^c$. (v) Similar to (iv).

Remark 1. *Let $SDG(L)$ be a simple directed graph. N_j are different kinds of j-neighbourhoods, where $j \in \{t, n, int, un\}, K$ is a subgraph of SDG. Then the following:*

(i) $LO_{N_j}(L)(K) \subseteq (L)(K) \subseteq UP_{N_j}(L)(K)$;

(ii) $UP_{N_j}(SDG) = SDG, LO_{N_j}(\phi) = \phi$*; are not always true.*

Example 4. *According to Tables 2 and 3,*

(i) $LO_{N_t}(\{l_1\}) = \{l_4\} \nsubseteq \{l_1\} \nsubseteq \{l_2\} = UP_{N_t}(\{l_1\})$*;*

(ii) $UP_{N_t}(SDG) = \{l_1, l_2, l_3\} \neq SDG, LO_{N_t}(\phi) = \{l_4\} \neq \phi$*.*

Proposition 8. *Let* $SDG(L)$ *be a simple directed graph,* $N_j(l)$ *be different kinds of neighbourhoods, where* $j \in \{t, n, int, un\}, K$ *and* M *be two subgraphs of* SDG*. Then:*

(i) $LO_{N_j}(L)(K) \bigcup LO_{N_j}(L)(M) \subseteq LO_{N_j}((L)(K) \bigcup (L)(M))$*;*

(ii) $LO_{N_j}((L)(K) \bigcap (L)(M)) = LO_{N_j}(L)(K) \bigcap LO_{N_j}(L)(M)$*;*

(iii) $UP_{N_j}((L)(K) \bigcup (L)(M)) = UP_{N_j}(L)(K) \bigcup UP_{N_j}(L)(M)$*;*

(iv) $UP_{N_j}((L)(K) \bigcap (L)(M)) \subseteq UP_{N_j}(L)(K) \bigcap UP_{N_j}(L)(M)$*.*

Proof.

(i) As $(L)(K) \subseteq (L)(K) \bigcup (L)(M)$ and $(L)(M) \subseteq (L)(K) \bigcup (L)(M)$, then $LO_{N_j}(L)(K) \subseteq LO_{N_j}((L)(K) \bigcup (L)(M))$ and $LO_{N_j}(L)(M) \subseteq LO_{N_j}((L)(K) \bigcup (L)(M))$. Hence, $LO_{N_j}(L)(K) \bigcup LO_{N_j}(L)(M) \subseteq LO_{N_j}((L)(K) \bigcup (L)(M))$.

(ii) As $(L)(K) \bigcap (L)(M) \subseteq (L)(K)$ and $(L)(K) \bigcap (L)(M) \subseteq (L)(M)$, then $LO_{N_j}((L)(K) \bigcap (L)(M)) \subseteq LO_{N_j}(L)(K)$ and $LO_{N_j}((L)(K) \bigcap (L)(M)) \subseteq LO_{N_j}(L)(M)$. So, $LO_{N_j}((L)(K) \bigcap (L)(M)) \subseteq LO_{N_j}(L)(K) \bigcap LO_{N_j}(L)(M)$. Let $l \in LO_{N_j}(L)(K) \bigcap LO_{N_j}(L)(M)$, then $l \in LO_{N_j}(L)(K)$ and $l \in LO_{N_j}(L)(M)$. By Definition 6(i), $N_j(l) \subseteq (L)(K)$ and $N_j(l) \subseteq (L)(M)$, and thus $N_j(l) \subseteq (L)(K) \bigcap (L)(M)$. So, $l \in LO_{N_j}((L)(K) \bigcap (L)(M))$. Therefore, $LO_{N_j}(L)(K) \bigcap LO_{N_j}(L)(M) \subseteq LO_{N_j}((L)(K) \bigcap (L)(M))$. Hence, the result.

(iii) As $(L)(K) \subseteq (L)(K) \bigcup (L)(M)$ and $(L)(M) \subseteq (L)(K) \bigcup (L)(M)$, then $UP_{N_j}(L)(K) \subseteq UP_{N_j}((L)(K) \bigcup (L)(M))$ and $UP_{N_j}(L)(M) \subseteq UP_{N_j}((L)(K) \bigcup (L)(M))$. So, $UP_{N_j}(L)(K) \bigcup UP_{N_j}(L)(M) \subseteq UP_{N_j}((L)(K) \bigcup (L)(M))$. Let $l \in UP_{N_j}((L)(K) \bigcup (L)(M))$. Then by Definition 6 (ii), $l \in \bigcup_{l \in L(SDG)} \{l : N_j(l) \bigcap ((L)(K) \bigcup (L)(M)) \neq \phi\}$, then $l \in \bigcup_{l \in L(SDG)} \{l : N_j(l) \bigcap (L)(K) \neq \phi\}$ or $l \in \bigcup_{l \in L(SDG)} \{l : N_j(l) \bigcap (L)(M)) \neq \phi\}$, that is $l \in UP_{N_j}(L)(K)$ or $l \in UP_{N_j}(L)(M)$ so $l \in UP_{N_j}(L)(K) \bigcup UP_{N_j}(L)(M)$. Therefore, $UP_{N_j}((L)(K) \bigcup (L)(M)) \subseteq UP_{N_j}(L)(K) \bigcup UP_{N_j}(L)(M)$. Hence the result.

(iv) As $(L)(K) \bigcap (L)(M) \subseteq (L)(K)$ and $(L)(K) \bigcap (L)(M) \subseteq (L)(M)$, then $UP_{N_j}((L)(K) \bigcap (L)(M)) \subseteq UP_{N_j}(L)(K)$ and $UP_{N_j}((L)(K) \bigcap (L)(M)) \subseteq UP_{N_j}(L)(M)$. Hence, the result.

Proposition 9. *Let $(SDG)(L)$ be a simple directed graph, N_j be different kinds of neighbourhoods where $j \in \{t, n, int, un\}, K$ be a subgraph of SDG. Then:*

(i) $LO_{N_j}((L)(SDG) - (L)(K)) = (L)(SDG) - UP_{N_j}(L)(K)$;

(ii) $UP_{N_j}((L)(SDG) - (L)(K)) = (L)(SDG) - LO_{N_j}(L)(K)$.

Proof.

(i) As $LO_{N_j}((L)(SDG) - (L)(K)) = LO_{N_j}((L)(K))^c = (UP_{N_j}(L)(K))^c = (L)(SDG) - UP_{N_j}(L)(K)$, by using Proposition 7(iv).

(ii) As $UP_{N_j}((L)(SDG) - (L)(K)) = UP_{N_j}((L)(K))^c = (LO_{N_j}(L)(K))^c = (L)(SDG) - LO_{N_j}(L)(K)$, using Proposition 7(v).

Proposition 10. *Let $SDG(L)$ be a simple directed graph, N_j be different kinds of neighbourhoods where $j \in \{t, n, int, un\}, K$ and M are two subgraphs of SDG. Then: $UP_{N_j}(L)(K) - UP_j(L)(M) \subseteq UP_{N_j}((L)(K) - (L)(M))$.*

Proof. Let $l \in (UP_{N_j}(L)(K) - UP_{N_j}(L)(M))$, then by Definition 6, $l \in \bigcup\{l : N_j(l) \bigcap (L)(K)$
$\neq \phi\}$ and $l \notin \bigcup\{l : N_j(l) \bigcap (L)(M) \neq \phi\}$. That is $l \in \bigcup\{l : N_j(l) \bigcap ((L)(M))^c \neq \phi\}$. Therefore, $l \in \bigcup\{l : N_j(l) \bigcap [(L)(K) \bigcap ((L)(M))^c] \neq \phi\}$. So, $l \in \bigcup\{l : N_j(l) \bigcap [(L)(K) - (L)(M)] \neq \phi\}$. Hence, $l \in UP_{N_j}((L)(K) - (L)(M))$ Then the result.

Remark 2. *Let $SDG(L)$ be a simple directed graph, N_j be different kinds of neighbourhoods where $j \in \{t, n, int, un\}, K$ and M are two subgraphs of SDG. Then $LO_{N_j}((L)(K) - (L)(M)) \nsubseteq LO_{N_j}(L)(K) - LO_{N_j}(L)(M)$.*

Example 5. *From Table 2, $LO_{N_t}(\{l_1, l_2\} - \{l_1, l_3\}) = LO_{N_t}(\{l_2\}) = \{l_3, l_4\} \nsubseteq \{l_3\} = \{l_3, l_4\} - \{l_1, l_4\} = LO_{N_t}(\{l_1, l_2\}) - LO_{N_t}(l_1, l_3\})$.*

From Definition 8 and Propositions 7, 8 and 9,we obtain the following.

Proposition 11. *Let $SDG(L)$ be a simple directed graph, $(L)(K)$ and $(L)(M)$ be subgraphs of SDG. Then:*

(i) If $(L)(K) \subseteq (L)(M)$, *then* $SU_{N_j}(L)(K) \subseteq SU_{N_j}(L)(M)$ *and* $RI_{N_j}(L)(K) \subseteq RI_{N_j}(L)(M)$;

(ii) $SU_{N_j}((L)(K) \bigcup (L)(M)) = SU_{N_j}(L)(K) \bigcup SU_{N_j}(L)(M)$;

(iii) $SU_{N_j}((L)(K) \bigcap (L)(M)) \subseteq SU_{N_j}(L)(K) \bigcap SU_{N_j}(L)(M)$;

(iv) $RI_{N_j}(L)(K) \bigcup RI_{N_j}(L)(M) \subseteq RI_{N_j}((L)(K) \bigcup (L)(M))$;

(v) $RI_{N_j}((L)(K) \bigcap (L)(M)) = RI_{N_j}(L)(K) \bigcap RI_{N_j}(L)(M)$;

(vi) $RI_{N_j}((L)(SDG) - (L)(K)) = (L)(SDG) - SU_{N_j}(L)(K)$;

(vii) $SU_{N_j}((L)(SDG) - (L)) = (L)(SDG) - RI_{N_j}(L)(K)$.

4. An Application to Air Traffic Networks

In this section, depending on new kinds of j-neighbourhoods and their topological spaces generated by them, we present various air journies ranking via these. The illustration illustrates one of the air line companies' routing plan for aircraft.

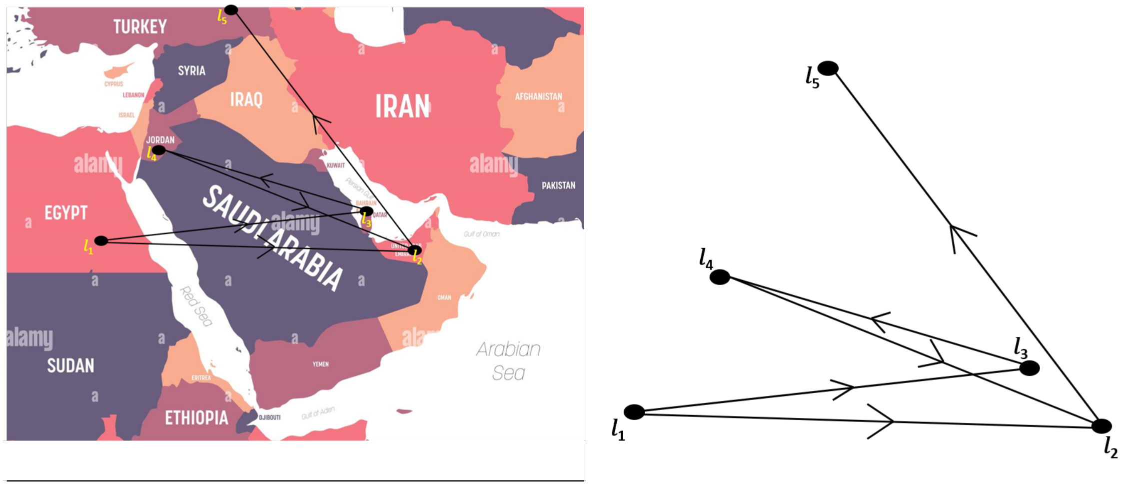


Figure 1: A simple digraph of air journies

Table 10: Neighbourhoods of simple digraph of the air journeys

$l \in L(SDG)$	l_1	l_2	l_3	l_4	l_5
$N_t(l_i)$	$\{l_2, l_3\}$	$\{l_5\}$	$\{l_4\}$	$\{l_2\}$	ϕ
$N_n(l_i)$	ϕ	$\{l_1, l_4\}$	$\{l_1\}$	$\{l_3\}$	$\{l_2\}$
$N_{int}(l_i)$	ϕ	ϕ	ϕ	ϕ	ϕ
$N_{un}(l_i)$	$\{l_2, l_3\}$	$\{l_1, l_4, l_5\}$	$\{l_1, l_4\}$	$\{l_2, l_3\}$	$\{l_2\}$

Table 11: $LO_{N_j}((L)(K))$ with respect to Table 11

$L(K)$	$LO_{N_t}(L)(K)$	$LO_{N_n}(L)(K)$	$LO_{N_{int}}(L)(K)$	$LO_{N_{un}}(L)(K)$
ϕ	$\{l_5\}$	$\{l_1\}$	$L(SDG)$	ϕ
$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$	$L(SDG)$
$\{l_1\}$	$\{l_5\}$	$\{l_1, l_3\}$	$L(SDG)$	ϕ
$\{l_2\}$	$\{l_4, l_5\}$	$\{l_1, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_3\}$	$\{l_5\}$	$\{l_1, l_4\}$	$L(SDG)$	ϕ
$\{l_4\}$	$\{l_3, l_5\}$	$\{l_1\}$	$L(SDG)$	ϕ
$\{l_5\}$	$\{l_2, l_5\}$	$\{l_1\}$	$L(SDG)$	ϕ
$\{l_1, l_2\}$	$\{l_4, l_5\}$	$\{l_1, l_3, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_1, l_3\}$	$\{l_5\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	ϕ

Continued on next page

Table 11 – Continued from previous page

$L(K)$	$LO_{N_t}(L)(K)$	$LO_{N_n}(L)(K)$	$LO_{N_{int}}(L)(K)$	$LO_{N_{un}}(L)(K)$
$\{l_1, l_4\}$	$\{l_3, l_5\}$	$\{l_1, l_2, l_3\}$	$L(SDG)$	$\{l_3\}$
$\{l_1, l_5\}$	$\{l_2, l_5\}$	$\{l_1, l_3\}$	$L(SDG)$	ϕ
$\{l_2, l_3\}$	$\{l_1, l_4, l_5\}$	$\{l_1, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$
$\{l_2, l_4\}$	$\{l_3, l_4, l_5\}$	$\{l_1, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_2, l_5\}$	$\{l_2, l_4, l_5\}$	$\{l_1, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_3, l_4\}$	$\{l_3, l_5\}$	$\{l_1, l_4\}$	$L(SDG)$	ϕ
$\{l_3, l_5\}$	$\{l_2, l_5\}$	$\{l_1, l_4\}$	$L(SDG)$	ϕ
$\{l_4, l_5\}$	$\{l_2, l_3, l_5\}$	$\{l_1\}$	$L(SDG)$	ϕ
$\{l_1, l_2, l_3\}$	$\{l_1, l_4, l_5\}$	$\{l_1, l_3, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$
$\{l_1, l_2, l_4\}$	$\{l_3, l_4, l_5\}$	$\{l_1, l_2, l_3, l_5\}$	$L(SDG)$	$\{l_3, l_5\}$
$\{l_1, l_2, l_5\}$	$\{l_2, l_4, l_5\}$	$\{l_1, l_3, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_1, l_3, l_4\}$	$\{l_3, l_5\}$	$\{l_1, l_2, l_3, l_4\}$	$L(SDG)$	$\{l_3\}$
$\{l_1, l_3, l_5\}$	$\{l_2, l_5\}$	$\{l_1, l_3, l_4\}$	$L(SDG)$	ϕ
$\{l_1, l_4, l_5\}$	$\{l_2, l_3, l_5\}$	$\{l_1, l_2, l_3\}$	$L(SDG)$	$\{l_2, l_3\}$
$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4, l_5\}$	$\{l_1, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$
$\{l_2, l_3, l_5\}$	$\{l_1, l_2, l_4, l_5\}$	$\{l_1, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$
$\{l_2, l_4, l_5\}$	$\{l_2, l_3, l_4, l_5\}$	$\{l_1, l_5\}$	$L(SDG)$	$\{l_5\}$
$\{l_3, l_4, l_5\}$	$\{l_2, l_3, l_5\}$	$\{l_1, l_4\}$	$L(SDG)$	ϕ
$\{l_1, l_2, l_3, l_4\}$	$\{l_1, l_3, l_4, l_5\}$	$L(SDG)$	$L(SDG)$	$\{l_1, l_3, l_4, l_5\}$
$\{l_1, l_2, l_3, l_5\}$	$\{l_1, l_2, l_4, l_5\}$	$\{l_1, l_3, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$
$\{l_1, l_2, l_4, l_5\}$	$\{l_2, l_3, l_4, l_5\}$	$\{l_1, l_2, l_3, l_5\}$	$L(SDG)$	$\{l_2, l_3, l_5\}$
$\{l_1, l_3, l_4, l_5\}$	$\{l_2, l_3, l_5\}$	$\{l_1, l_2, l_3, l_4\}$	$L(SDG)$	$\{l_3\}$
$\{l_2, l_3, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$	$L(SDG)$	$\{l_1, l_4, l_5\}$

Table 12: $UP_{N_j}((L)(K))$ with respect to Table 11

$L(K)$	$UP_{N_t}(L)(K)$	$UP_{N_n}(L)(K)$	$UP_{N_{int}}(L)(K)$	$UP_{N_{un}}(L)(K)$
ϕ	ϕ	ϕ	ϕ	ϕ
$L(SDG)$	$\{l_1, l_2, l_3, l_4\}$	$\{l_2, l_3, l_4, l_5\}$	ϕ	$L(SDG)$
$\{l_1\}$	ϕ	$\{l_2, l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2\}$	$\{l_1, l_4\}$	$\{l_5\}$	ϕ	$\{l_1, l_4, l_5\}$
$\{l_3\}$	$\{l_1\}$	$\{l_4\}$	ϕ	$\{l_1, l_4\}$
$\{l_4\}$	$\{l_3\}$	$\{l_2\}$	ϕ	$\{l_2, l_3\}$
$\{l_5\}$	$\{l_2\}$	ϕ	ϕ	$\{l_2\}$
$\{l_1, l_2\}$	$\{l_1, l_4\}$	$\{l_2, l_3, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_3\}$	$\{l_1\}$	$\{l_2, l_3, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_1, l_4\}$	$\{l_3\}$	$\{l_2, l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_1, l_5\}$	$\{l_2\}$	$\{l_2, l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2, l_3\}$	$\{l_1, l_4\}$	$\{l_4, l_5\}$	ϕ	$\{l_1, l_4, l_5\}$

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Table 12 – Continued from previous page

$L(K)$	$UP_{N_t}(L)(K)$	$UP_{N_n}(L)(K)$	$UP_{N_{int}}(L)(K)$	$UP_{N_{un}}(L)(K)$
$\{l_2, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_2, l_5\}$	ϕ	$L(SDG)$
$\{l_2, l_5\}$	$\{l_1, l_2, l_4\}$	$\{l_5\}$	ϕ	$\{l_1, l_2, l_4, l_5\}$
$\{l_3, l_4\}$	$\{l_1, l_3\}$	$\{l_2, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_3, l_5\}$	$\{l_1, l_2\}$	$\{l_4\}$	ϕ	$\{l_1, l_2, l_4\}$
$\{l_4, l_5\}$	$\{l_2, l_3\}$	$\{l_2\}$	ϕ	$\{l_2, l_3\}$
$\{l_1, l_2, l_3\}$	$\{l_1, l_4\}$	$\{l_2, l_3, l_4, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_2, l_3, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_5\}$	$\{l_1, l_2, l_4\}$	$\{l_2, l_3, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_3, l_4\}$	$\{l_1, l_3\}$	$\{l_2, l_3, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_1, l_3, l_5\}$	$\{l_1, l_5\}$	$\{l_2, l_3, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_1, l_4, l_5\}$	$\{l_2, l_3\}$	$\{l_2, l_3\}$	ϕ	$\{l_2, l_3\}$
$\{l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_2, l_4, l_5\}$	ϕ	$L(SDG)$
$\{l_2, l_3, l_5\}$	$\{l_1, l_2, l_4\}$	$\{l_4, l_5\}$	ϕ	$\{l_1, l_2, l_4, l_5\}$
$\{l_2, l_4, l_5\}$	$\{l_1, l_2, l_3, l_4\}$	$\{l_2, l_5\}$	ϕ	$L(SDG)$
$\{l_3, l_4, l_5\}$	$\{l_1, l_2, l_3\}$	$\{l_2, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_1, l_2, l_3, l_4\}$	$\{l_1, l_3, l_4\}$	$\{l_2, l_3, l_4, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_3, l_5\}$	$\{l_1, l_2, l_4\}$	$\{l_2, l_3, l_4, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_2, l_4, l_5\}$	$\{l_1, l_2, l_3, l_4\}$	$\{l_2, l_3, l_5\}$	ϕ	$L(SDG)$
$\{l_1, l_3, l_4, l_5\}$	$\{l_1, l_2, l_3\}$	$\{l_2, l_3, l_4\}$	ϕ	$\{l_1, l_2, l_3, l_4\}$
$\{l_2, l_3, l_4, l_5\}$	$\{l_1, l_2, l_3, l_4\}$	$\{l_2, l_4, l_5\}$	ϕ	$L(SDG)$

Table 13: $Sub_t(L)(K)$ and $Sub_n(L)(K)$ with respect to Tables 12 and 13

$L(K)$	$Sub_t(L)(K)$	$Sub_n(L)(K)$
ϕ	$\{\phi, \{l_5\}\}$	$\{\phi, \{l_1\}\}$
$L(SDG)$	$\{\{l_1, l_2, l_3, l_4\}, L(SDG)\}$	$\{\{l_2, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1\}$	$\{\phi, \{l_5\}\}$	$\{\{l_1, l_3\}, \{l_2, l_3\}\}$
$\{l_2\}$	$\{\{l_1, l_4\}, \{l_4, l_5\}\}$	$\{\{l_5\}, \{l_1, l_5\}\}$
$\{l_3\}$	$\{\{l_1\}, \{l_5\}\}$	$\{\{l_4\}, \{l_1, l_4\}\}$
$\{l_4\}$	$\{\{l_3\}, \{l_3, l_5\}\}$	$\{\{l_1\}, \{l_2\}\}$
$\{l_5\}$	$\{\{l_2\}, \{l_2, l_5\}\}$	$\{\phi, \{l_1\}\}$
$\{l_1, l_2\}$	$\{\{l_1, l_4\}, \{l_4, l_5\}\}$	$\{\{l_1, l_3, l_5\}, \{l_2, l_3, l_5\}\}$
$\{l_1, l_3\}$	$\{\{l_1\}, \{l_5\}\}$	$\{\{l_1, l_3, l_4\}, \{l_2, l_3, l_4\}\}$
$\{l_1, l_4\}$	$\{\{l_3\}, \{l_3, l_5\}\}$	$\{\{l_2, l_3\}, \{l_1, l_2, l_3\}\}$
$\{l_1, l_5\}$	$\{\{l_2\}, \{l_2, l_5\}\}$	$\{\{l_1, l_3\}, \{l_2, l_3\}\}$
$\{l_2, l_3\}$	$\{\{l_1, l_4\}, \{l_1, l_4, l_5\}\}$	$\{\{l_4, l_5\}, \{l_1, l_4, l_5\}\}$
$\{l_2, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}\}$	$\{\{l_2, l_5\}, \{l_1, l_5\}\}$
$\{l_2, l_5\}$	$\{\{l_1, l_2, l_4\}, \{l_2, l_4, l_5\}\}$	$\{\{l_5\}, \{l_1, l_5\}\}$

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Table 13 – Continued from previous page

$L(K)$	$Sub_t(L)(K)$	$Sub_n(L)(K)$
$\{l_3, l_4\}$	$\{\{l_1, l_3\}, \{l_3, l_5\}\}$	$\{\{l_1, l_4\}, \{l_2, l_4\}\}$
$\{l_3, l_5\}$	$\{\{l_1, l_2\}, \{l_2, l_5\}\}$	$\{\{l_4\}, \{l_1, l_4\}\}$
$\{l_4, l_5\}$	$\{\{l_2, l_3\}, \{l_2, l_3, l_5\}\}$	$\{\{l_1\}, \{l_2\}\}$
$\{l_1, l_2, l_3\}$	$\{\{l_1, l_4\}, \{l_1, l_4, l_5\}\}$	$\{\{l_1, l_3, l_4, l_5\}, \{l_2, l_3, l_4, l_5\}\}$
$\{l_1, l_2, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}\}$	$\{\{l_1, l_2, l_3, l_5\}, \{l_2, l_3, l_5\}\}$
$\{l_1, l_2, l_5\}$	$\{\{l_1, l_2, l_4\}, \{l_2, l_4, l_5\}\}$	$\{\{l_2, l_3, l_5\}, \{l_1, l_3, l_5\}\}$
$\{l_1, l_3, l_4\}$	$\{\{l_3, l_5\}, \{l_1, l_3\}\}$	$\{\{l_2, l_3, l_4\}, \{l_1, l_2, l_3, l_4\}\}$
$\{l_1, l_3, l_5\}$	$\{\{l_1, l_5\}, \{l_2, l_5\}\}$	$\{\{l_2, l_3, l_4\}, \{l_1, l_3, l_4\}\}$
$\{l_1, l_4, l_5\}$	$\{\{l_2, l_3\}, \{l_2, l_3, l_5\}\}$	$\{\{l_2, l_3\}, \{l_1, l_2, l_3\}\}$
$\{l_2, l_3, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_1, l_3, l_4, l_5\}\}$	$\{\{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}\}$
$\{l_2, l_3, l_5\}$	$\{\{l_1, l_2, l_4\}, \{l_1, l_2, l_4, l_5\}\}$	$\{\{l_4, l_5\}, \{l_1, l_4, l_5\}\}$
$\{l_2, l_4, l_5\}$	$\{\{l_1, l_2, l_3, l_4\}, \{l_2, l_3, l_4, l_5\}\}$	$\{\{l_2, l_5\}, \{l_1, l_5\}\}$
$\{l_3, l_4, l_5\}$	$\{\{l_1, l_2, l_3\}, \{l_2, l_3, l_5\}\}$	$\{\{l_2, l_4\}, \{l_1, l_4\}\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_1, l_3, l_4, l_5\}\}$	$\{\{l_2, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\{l_1, l_2, l_4\}, \{l_1, l_2, l_4, l_5\}\}$	$\{\{l_2, l_3, l_4, l_5\}, \{l_1, l_3, l_4, l_5\}\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\{l_1, l_2, l_3, l_4\}, \{l_2, l_3, l_4, l_5\}\}$	$\{\{l_2, l_3, l_5\}, \{l_1, l_2, l_3, l_5\}\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\{l_1, l_2, l_3\}, \{l_2, l_3, l_5\}\}$	$\{\{l_2, l_3, l_4\}, \{l_1, l_2, l_3, l_4\}\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\{l_1, l_2, l_3, l_4\}, L(SDG)\}$	$\{\{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}\}$

Table 14: $Sub_{int}(L)(K)$ and $Sub_{un}(L)(K)$ with respect to Tables 12 and 13

$L(K)$	$Sub_{int}(L)(K)$	$Sub_{un}(L)(K)$
ϕ	$\{\phi, L(SDG)\}$	$\{\phi\}$
$L(SDG)$	$\{\phi, L(SDG)\}$	$\{L(SDG)\}$
$\{l_1\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_2\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, \{l_1, l_4, l_5\}\}$
$\{l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4\}\}$
$\{l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2\}\}$
$\{l_1, l_2\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}\}$
$\{l_1, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_3\}, \{l_2, l_3\}\}$
$\{l_1, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4, l_5\}\}$
$\{l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_5\}\}$
$\{l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4, l_5\}, \{l_5\}\}$
$\{l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}\}$
$\{l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4\}, \phi\}$

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Table 14 – Continued from previous page

$L(K)$	$Sub_{int}(L)(K)$	$Sub_{un}(L)(K)$
$\{l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}\}$
$\{l_1, l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_3, l_4\}, \{l_3\}\}$
$\{l_1, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}\}$
$\{l_1, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_2, l_3\}\}$
$\{l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4, l_5\}, \{l_1, l_4, l_5\}\}$
$\{l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_5\}\}$
$\{l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_3, l_4, l_5\}\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_2, l_3, l_5\}\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_3\}, \{l_1, l_2, l_3, l_4\}\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG)\{l_1, l_4, l_5\}\}$

Table 15: $B_t(L)(K)$ with respect to Table 14

$L(K)$	$B_t(L)(K)$
ϕ	$\{\phi, \{l_5\}, L(SDG)\}$
$L(SDG)$	$\{\{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1\}$	$\{\phi, \{l_5\}, L(SDG)\}$
$\{l_2\}$	$\{\{l_1, l_4\}, \{l_4, l_5\}, \{l_4\}, L(SDG)\}$
$\{l_3\}$	$\{\{l_1\}, \{l_5\}, \phi, L(SDG)\}$
$\{l_4\}$	$\{\{l_3\}, \{l_3, l_5\}, L(SDG)\}$
$\{l_5\}$	$\{\{l_2\}, \{l_2, l_5\}, L(SDG)\}$
$\{l_1, l_2\}$	$\{\{l_1, l_4\}, \{l_4, l_5\}, \{l_4\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\{l_1\}, \{l_5\}, \phi, L(SDG)\}$
$\{l_1, l_4\}$	$\{\{l_3\}, \{l_3, l_5\}, L(SDG)\}$
$\{l_1, l_5\}$	$\{\{l_2\}, \{l_2, l_5\}, L(SDG)\}$
$\{l_2, l_3\}$	$\{\{l_1, l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}, \{l_3, l_4\}, L(SDG)\}$
$\{l_2, l_5\}$	$\{\{l_1, l_2, l_4\}, \{l_2, l_4, l_5\}, \{l_2, l_4\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\{l_1, l_3\}, \{l_3, l_5\}, \{l_3\}, L(SDG)\}$
$\{l_3, l_5\}$	$\{\{l_1, l_2\}, \{l_2, l_5\}, \{l_2\}, L(SDG)\}$
$\{l_4, l_5\}$	$\{\{l_2, l_3\}, \{l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3\}$	$\{\{l_1, l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4\}$	$\{\{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}, \{l_3, l_4\}, L(SDG)\}$

Continued on next page

Table 15 – Continued from previous page

$L(K)$	$B_t(L)(K)$
$\{l_1,l_2,l_5\}$	$\{\{l_1,l_2,l_4\},\{l_2,l_4,l_5\},\{l_2,l_4\},L(SDG)\}$
$\{l_1,l_3,l_4\}$	$\{\{l_3,l_5\},\{l_1,l_3\},\{l_3\},L(SDG)\}$
$\{l_1,l_3,l_5\}$	$\{\{l_1,l_5\},\{l_2,l_5\},\{l_5\},L(SDG)\}$
$\{l_1,l_4,l_5\}$	$\{\{l_2,l_3\},\{l_2,l_3,l_5\},L(SDG)\}$
$\{l_2,l_3,l_4\}$	$\{\{l_1,l_3,l_4\},\{l_1,l_3,l_4,l_5\},L(SDG)\}$
$\{l_2,l_3,l_5\}$	$\{\{l_1,l_2,l_4\},\{l_1,l_2,l_4,l_5\},L(SDG)\}$
$\{l_2,l_4,l_5\}$	$\{\{l_1,l_2,l_3,l_4\},\{l_2,l_3,l_4,l_5\},\{l_2,l_3,l_4\},L(SDG)\}$
$\{l_3,l_4,l_5\}$	$\{\{l_1,l_2,l_3\},\{l_2,l_3,l_5\},\{l_2,l_3\},L(SDG)\}$
$\{l_1,l_2,l_3,l_4\}$	$\{\{l_1,l_3,l_4\},\{l_1,l_3,l_4,l_5\},L(SDG)\}$
$\{l_1,l_2,l_3,l_5\}$	$\{\{l_1,l_2,l_4\},\{l_1,l_2,l_4,l_5\},L(SDG)\}$
$\{l_1,l_2,l_4,l_5\}$	$\{\{l_1,l_2,l_3,l_4\},\{l_2,l_3,l_4,l_5\},\{l_2,l_3,l_4\},L(SDG)\}$
$\{l_1,l_3,l_4,l_5\}$	$\{\{l_1,l_2,l_3\},\{l_2,l_3,l_5\},\{l_2,l_3\},L(SDG)\}$
$\{l_2,l_3,l_4,l_5\}$	$\{\{l_1,l_2,l_3,l_4\},L(SDG)\}$

Table 16: $B_n(L)(K)$ with respect to Table 14

$L(K)$	$B_n(L)(K)$
ϕ	$\{\phi,\{l_1\},L(SDG)\}$
$L(SDG)$	$\{\{l_2,l_3,l_4,l_5\},L(SDG)\}$
$\{l_1\}$	$\{\{l_1,l_3\},\{l_2,l_3\},\{l_3\},L(SDG)\}$
$\{l_2\}$	$\{\{l_5\},\{l_1,l_5\},L(SDG)\}$
$\{l_3\}$	$\{\{l_4\},\{l_1,l_4\},L(SDG)\}$
$\{l_4\}$	$\{\{l_1\},\{l_2\},\phi,L(SDG)\}$
$\{l_5\}$	$\{\phi,\{l_1\},L(SDG)\}$
$\{l_1,l_2\}$	$\{\{l_1,l_3,l_5\},\{l_2,l_3,l_5\},\{l_3,l_5\},L(SDG)\}$
$\{l_1,l_3\}$	$\{\{l_1,l_3,l_4\},\{l_2,l_3,l_4\}\{l_3,l_4\},L(SDG)\}$
$\{l_1,l_4\}$	$\{\{l_2,l_3\},\{l_1,l_2,l_3\},L(SDG)\}$
$\{l_1,l_5\}$	$\{\{l_1,l_3\},\{l_2,l_3\},\{l_3\},L(SDG)\}$
$\{l_2,l_3\}$	$\{\{l_4,l_5\},\{l_1,l_4,l_5\},L(SDG)\}$
$\{l_2,l_4\}$	$\{\{l_2,l_5\},\{l_1,l_5\},\{l_5\},L(SDG)\}$
$\{l_2,l_5\}$	$\{\{l_5\},\{l_1,l_5\},L(SDG)\}$
$\{l_3,l_4\}$	$\{\{l_1,l_4\},\{l_2,l_4\},\{l_4\},L(SDG)\}$
$\{l_3,l_5\}$	$\{\{l_4\},\{l_1,l_4\},L(SDG)\}$
$\{l_4,l_5\}$	$\{\{l_1\},\{l_2\},\phi,L(SDG)\}$
$\{l_1,l_2,l_3\}$	$\{\{l_1,l_3,l_4,l_5\},\{l_2,l_3,l_4,l_5\},\{l_3,l_4,l_5\},L(SDG)\}$
$\{l_1,l_2,l_4\}$	$\{\{l_1,l_2,l_3,l_5\},\{l_2,l_3,l_5\},L(SDG)\}$
$\{l_1,l_2,l_5\}$	$\{\{l_2,l_3,l_5\},\{l_1,l_3,l_5\},\{l_3,l_5\},L(SDG)\}$
$\{l_1,l_3,l_4\}$	$\{\{l_2,l_3,l_4\},\{l_1,l_2,l_3,l_4\},L(SDG)\}$
$\{l_1,l_3,l_5\}$	$\{\{l_2,l_3,l_4\},\{l_1,l_3,l_4\},\{l_3,l_4\},L(SDG)\}$

Continued on next page

Table 16 – Continued from previous page

$L(K)$	$B_n(L)(K)$
$\{l_1, l_4, l_5\}$	$\{\{l_2, l_3\}, \{l_1, l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3, l_4\}$	$\{\{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, \{l_4, l_5\}, L(SDG)\}$
$\{l_2, l_3, l_5\}$	$\{\{l_4, l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4, l_5\}$	$\{\{l_2, l_5\}, \{l_1, l_5\}, \{l_5\}, L(SDG)\}$
$\{l_3, l_4, l_5\}$	$\{\{l_2, l_4\}, \{l_1, l_4\}, \{l_4\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\{l_2, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\{l_2, l_3, l_4, l_5\}, \{l_1, l_3, l_4, l_5\}, \{l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\{l_2, l_3, l_5\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\{l_2, l_3, l_4\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, \{l_4, l_5\}, L(SDG)\}$

Table 17: $B_{int}(L)(K)$ and $B_{un}(L)(K)$ with respect to Table 15

$L(K)$	$B_{int}(L)(K)$	$B_{un}(L)(K)$
ϕ	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$
$L(SDG)$	$\{\phi, L(SDG)\}$	$\{L(SDG)\}$
$\{l_1\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_2\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4\}, L(SDG)\}$
$\{l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2\}, L(SDG)\}$
$\{l_1, l_2\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_3\}, \{l_2, l_3\}, L(SDG)\}$
$\{l_1, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_5\}\}$
$\{l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4, l_5\}, \{l_5\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4\}, \phi, L(SDG)\}$
$\{l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_1, l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_3, l_4\}, \{l_3\}, L(SDG)\}$
$\{l_1, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$

Continued on next page

Table 17 – Continued from previous page

$L(K)$	$B_{int}(L)(K)$	$B_{un}(L)(K)$
$\{l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_5\}\}$
$\{l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_3, l_4, l_5\}\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG), \{l_2, l_3, l_5\}\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_3\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{L(SDG)\{l_1, l_4, l_5\}\}$

Table 18: $\tau_{N_t}(L)(K)$ with respect to Table 16

$L(K)$	$\tau_{N_t}(L)(K)$
ϕ	$\{\phi, \{l_5\}, L(SDG)\}$
$L(SDG)$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1\}$	$\{\phi, \{l_5\}, L(SDG)\}$
$\{l_2\}$	$\{\phi, \{l_1, l_4\}, \{l_4, l_5\}, \{l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_3\}$	$\{\{l_1\}, \{l_5\}, \{l_1, l_5\}, \phi, L(SDG)\}$
$\{l_4\}$	$\{\phi, \{l_3\}, \{l_3, l_5\}, L(SDG)\}$
$\{l_5\}$	$\{\phi, \{l_2\}, \{l_2, l_5\}, L(SDG)\}$
$\{l_1, l_2\}$	$\{\phi, \{l_1, l_4\}, \{l_4, l_5\}, \{l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\{l_1\}, \{l_5\}, \{l_1, l_5\}, \phi, L(SDG)\}$
$\{l_1, l_4\}$	$\{\phi, \{l_3\}, \{l_3, l_5\}, L(SDG)\}$
$\{l_1, l_5\}$	$\{\phi, \{l_2\}, \{l_2, l_5\}, L(SDG)\}$
$\{l_2, l_3\}$	$\{\phi, \{l_1, l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\phi, \{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}, \{l_3, l_4\}, \{l_1, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_5\}$	$\{\phi, \{l_1, l_2, l_4\}, \{l_2, l_4, l_5\}, \{l_2, l_4\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\phi, \{l_1, l_3\}, \{l_3, l_5\}, \{l_3\}, \{l_1, l_3, l_5\}, L(SDG)\}$
$\{l_3, l_5\}$	$\{\phi, \{l_1, l_2\}, \{l_2, l_5\}, \{l_2\}, \{l_1, l_2, l_5\}, L(SDG)\}$
$\{l_4, l_5\}$	$\{\phi, \{l_2, l_3\}, \{l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3\}$	$\{\phi, \{l_1, l_4\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4\}$	$\{\phi, \{l_1, l_3, l_4\}, \{l_3, l_4, l_5\}, \{l_3, l_4\}, \{l_1, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_5\}$	$\{\phi, \{l_1, l_2, l_4\}, \{l_2, l_4, l_5\}, \{l_2, l_4\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4\}$	$\{\phi, \{l_3, l_5\}, \{l_1, l_3\}, \{l_3\}, \{l_1, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_3, l_5\}$	$\{\phi, \{l_1, l_5\}, \{l_2, l_5\}, \{l_5\}, \{l_1, l_2, l_5\}, L(SDG)\}$
$\{l_1, l_4, l_5\}$	$\{\phi, \{l_2, l_3\}, \{l_2, l_3, l_5\}, L(SDG)\}$
$\{l_2, l_3, l_4\}$	$\{\phi, \{l_1, l_3, l_4\}, \{l_1, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_3, l_5\}$	$\{\phi, \{l_1, l_2, l_4\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4, l_5\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, \{l_2, l_3, l_4, l_5\}, \{l_2, l_3, l_4\}, L(SDG)\}$
$\{l_3, l_4, l_5\}$	$\{\phi, \{l_1, l_2, l_3\}, \{l_2, l_3, l_5\}, \{l_2, l_3\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$

Continued on next page

Table 18 – Continued from previous page

$L(K)$	$\tau_{N_t}(L)(K)$
$\{l_1, l_2, l_3, l_4\}$	$\{\phi, \{l_1, l_3, l_4\}, \{l_1, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\phi, \{l_1, l_2, l_4\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, \{l_2, l_3, l_4, l_5\}, \{l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\phi, \{l_1, l_2, l_3\}, \{l_2, l_3, l_5\}, \{l_2, l_3\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$

Table 19: $\tau_{N_n}(L)(K)$ with respect to Table 17

$L(K)$	$\tau_{N_n}(L)(K)$
ϕ	$\{\phi, \{l_1\}, L(SDG)\}$
$L(SDG)$	$\{\phi, \{l_2, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1\}$	$\{\phi, \{l_1, l_3\}, \{l_2, l_3\}, \{l_3\}, \{l_1, l_2, l_3\}, L(SDG)\}$
$\{l_2\}$	$\{\phi, \{l_5\}, \{l_1, l_5\}, L(SDG)\}$
$\{l_3\}$	$\{\phi, \{l_4\}, \{l_1, l_4\}, L(SDG)\}$
$\{l_4\}$	$\{\phi, \{l_1\}, \{l_2\}, \{l_1, l_2\}, \phi, L(SDG)\}$
$\{l_5\}$	$\{\phi, \{l_1\}, L(SDG)\}$
$\{l_1, l_2\}$	$\{\phi, \{l_1, l_3, l_5\}, \{l_2, l_3, l_5\}, \{l_3, l_5\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\phi, \{l_1, l_3, l_4\}, \{l_2, l_3, l_4\}\{l_3, l_4\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4\}$	$\{\phi, \{l_2, l_3\}, \{l_1, l_2, l_3\}, L(SDG)\}$
$\{l_1, l_5\}$	$\{\phi, \{l_1, l_3\}, \{l_2, l_3\}, \{l_3\}, \{l_1, l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3\}$	$\{\phi, \{l_4, l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\phi, \{l_2, l_5\}, \{l_1, l_5\}, \{l_5\}, \{l_1, l_2, l_5\}, L(SDG)\}$
$\{l_2, l_5\}$	$\{\phi, \{l_5\}, \{l_1, l_5\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\phi, \{l_1, l_4\}, \{l_2, l_4\}, \{l_4\}, \{l_1, l_2, l_4\}, L(SDG)\}$
$\{l_3, l_5\}$	$\{\phi, \{l_4\}, \{l_1, l_4\}, L(SDG)\}$
$\{l_4, l_5\}$	$\{\{l_1\}, \{l_2\}, \{l_1, l_2\}, \phi, L(SDG)\}$
$\{l_1, l_2, l_3\}$	$\{\phi, \{l_1, l_3, l_4, l_5\}, \{l_2, l_3, l_4, l_5\}, \{l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4\}$	$\{\phi, \{l_1, l_2, l_3, l_5\}, \{l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_5\}$	$\{\phi, \{l_2, l_3, l_5\}, \{l_1, l_3, l_5\}, \{l_3, l_5\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4\}$	$\{\phi, \{l_2, l_3, l_4\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_3, l_5\}$	$\{\phi, \{l_2, l_3, l_4\}, \{l_1, l_3, l_4\}, \{l_3, l_4\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4, l_5\}$	$\{\phi, \{l_2, l_3\}, \{l_1, l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3, l_4\}$	$\{\phi, \{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, \{l_4, l_5\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_3, l_5\}$	$\{\phi, \{l_4, l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4, l_5\}$	$\{\phi, \{l_2, l_5\}, \{l_1, l_5\}, \{l_5\}, \{l_1, l_2, l_5\}, L(SDG)\}$
$\{l_3, l_4, l_5\}$	$\{\phi, \{l_2, l_4\}, \{l_1, l_4\}, \{l_4\}, \{l_1, l_2, l_4\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\phi, \{l_2, l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\phi, \{l_2, l_3, l_4, l_5\}, \{l_1, l_3, l_4, l_5\}, \{l_3, l_4, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\phi, \{l_2, l_3, l_5\}, \{l_1, l_2, l_3, l_5\}, L(SDG)\}$

Continued on next page

Table 19 – Continued from previous page

$L(K)$	$\tau_{N_n}(L)(K)$
$\{l_1, l_3, l_4, l_5\}$	$\{\phi, \{l_2, l_3, l_4\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\phi, \{l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, \{l_4, l_5\}, \{l_1, l_2, l_4, l_5\}, L(SDG)\}$

Table 20: $\tau_{int}(L)(K)$ and $\tau_{un}(L)(K)$ with respect to Table 18

$L(K)$	$\tau_{int}(L)(K)$	$\tau_{un}(L)(K)$
ϕ	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\}$
$L(SDG)$	$\{\phi, L(SDG)\}$	$\phi, \{L(SDG)\}$
$\{l_1\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_2\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4\}, L(SDG)\}$
$\{l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2\}, L(SDG)\}$
$\{l_1, l_2\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_5\}, L(SDG)\}$
$\{l_1, l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_3\}, \{l_2, l_3\}, L(SDG)\}$
$\{l_1, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_5\}\}$
$\{l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_4, l_5\}, \{l_5\}, L(SDG)\}$
$\{l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\{l_1, l_2, l_4\}, \phi, L(SDG)\}$
$\{l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_1, l_2, l_3\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_3, l_5\}, L(SDG)\}$
$\{l_1, l_2, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_5\}, L(SDG)\}$
$\{l_1, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, \{l_3\}, L(SDG)\}$
$\{l_1, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_2, l_3\}, L(SDG)\}$
$\{l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_4, l_5\}, \{l_1, l_4, l_5\}, L(SDG)\}$
$\{l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_5\}\}$
$\{l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_1, l_2, l_3, l_4\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_1, l_3, l_4, l_5\}\}$
$\{l_1, l_2, l_3, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_1, l_4, l_5\}\}$
$\{l_1, l_2, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG), \{l_2, l_3, l_5\}\}$
$\{l_1, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, \{l_3\}, \{l_1, l_2, l_3, l_4\}, L(SDG)\}$
$\{l_2, l_3, l_4, l_5\}$	$\{\phi, L(SDG)\}$	$\{\phi, L(SDG)\{l_1, l_4, l_5\}\}$

Remark 3. *Figure 2 shows that the reversal of stocks is not achieved and this has been*

clearly made clear through examples 4 and 5.

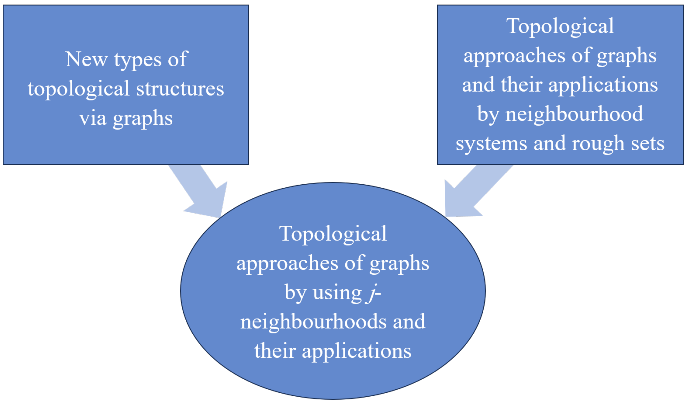

Figure 2: Comparison between results.

Remark 4. *The relation between $\tau_t, \tau_n, \tau_{int}, \tau_{un}$ and T_0 and T_1*

(i) Necessary and sufficient conditions for T_o.
A topological space is T_0 if for every pair of distinct vertices $x \neq y$, there exists an open set containing one but not the other. Firstly, for τ_t: The subbase is $\{LO_{N_t}(K), UP_{N_t}(K)\}$, and the topology is generated from out-neighbourhoods $N_t(v)$. The condition for $T_0 : \tau_t$ is T_0 iff for all $u \neq v, N_t(u) \setminus \{v\} \neq N_t(v) \setminus \{u\}$ is necessary but not sufficient. Sufficient condition: τ_t is T_0 iff for all $u \neq v$, there exists w such that $w \in N_t(u) \triangle N_t(v)$ and this difference yields an open set containing one but not the other. In the airline example (Fig. 2), Table 19 shows τ_t is not discrete, but likely fails T_0 for some vertices, e.g. if $N_t(u) = N_t(v)$ then they are topologically indistinguishable. Secondly, for τ_n: the condition for $T_0 : \tau_n$ is T_0 iff for all $u \neq v, N_n(u) \neq N_n(v)$, or the topology distinguishes them through the in-links. Third, for $\tau_{int} : N_{int}(v) = N_t(v) \cap N_n(v)$. If $N_{int}(v) = \phi$ for all v, then $LO_{N_{int}}(K)$ is either ϕ or $L(SDG)$ depending on K, and $UP_{N_{int}}(K)$ is ϕ or $L(SDG)$ similarly. Then $\tau_{int} = \{\phi, L\}$ is trivial. The trivial topology is not T_o (cannot separate any points). So, τ_{int} is T_0 only if N_{int} is nonempty enough to distinguish vertices, which is rare. Finally, for $\tau_{un} : N_{un}(v) = N_t(v) \cup N_n(v)$. τ_{un} is T_0 iff $N_{un}(u) \neq N_{un}(v)$ for all $u \neq v$.

(ii) Can any of $\tau_t, \tau_n, \tau_{int}, \tau_{un}$ be T_1?
A space is T_1 if for every $u \neq v$, there is an open set containing u but not v and another containing v but not u. τ_t is T_1 iff the digraph has no edges (discrete topology). τ_n is T_1 iff there are no edges (discrete). τ_{int} is almost never T_1. τ_{un} is T_1 iff $N_{un}(v) = \{v\}$ for all v iff the graph has no edges (discrete). In general, T_0 is achievable for $\tau_t, \tau_n, \tau_{un}$ under certain distinguishing neighbourhood conditions, but τ_{int} is almost never T_0. T_1 is impossible for any nontrivial digraph for all four topologies, they are T_1 only in the discrete case (not edges).

Remark 5. *Analysis exploring whether continuous self maps on the constructed spaces satisfy any known fixed-point theorems.*

(i) ***Defining a "j-Neighbourhood Preserving" Mapping***
Definition (j-Continuous Map): *Let (V, τ_j) be a topological space constructed from a simple directed graph $SDG(V)$ using the j-neighbourhood system $j \in \{t, n, int, un\}$). A function $f : V \to V$ is called j-continuous if it is continuous with respect to the topology τ_j, i.e., for every open set $U \in \tau_j$, its pre-image $f^{-1}(U)$ is also open in τ_j.*

Definition (j-Neighbourhood Preserving Map): *A function $f : V \to V$ is j-neighbourhood Preserving if for every vertex $v \in V$, the image of the j-neighbourhood is contained in the j-neighbourhood of the image: $f(N_j(v)) \subseteq N_j(f(v))$. This is a direct analogue of a continuous map in a neighbourhood space: points "close" to v (in N_j)) are mapped to points "close" to $f(v)$.*

(ii) ***Connecting to known fixed-point Theorems***
Classical fixed-point theorems (Brouwer, Schauder) require topological structure like compactness and convexity, which our finite discrete spaces possess trivially (they are compact), but they lack the convex structure. The most relevant theorem for finite topological spaces is a direct consequence of the Lefschetz Fixed-Point Theorem.

(iii) ***Fixed- Point Theorem for j-Continuous Maps***
Theorem 1 (Fixed point in acyclic graphs): *Let $SDG(V)$ be a finite, acyclic directed graph. Let $f : V \to V$ be a j-continuous map for $j \in \{t, n\}$. Then f has a fixed point.*
Proof: *Acyclicity implies a "source" or "sink": In a finite acyclic digraph, there exists at least one source (vertex with $N_n(v) = \phi$) and one sink (vertex with $N_t(v) = \phi$). Topology of τ_t in an acyclic graph: Consider the out-neighbourhood topology τ_t. For a sink vertex $s, N_t(s) = \phi$. Therefore, $LO_{N_t}(\{S\}) = \{x \in V : N_t(x) \subseteq \{s\}\}$. Since s is a sink, $N_t(s) = \phi \subseteq \{s\}$, so $s \in LO_{N_t}(\{s\})$. In fact, for many acyclic graphs, the set of sinks form minimal open sets. The fixed-point argument: Let S be the set of sink vertices. This set is open in τ_t because for each sink s, the singleton $\{s\}$ might be open, or S is a union of such minimal open sets. A τ_t-continuous map f must map sinks to sinks. Suppose s is a sink and $f(s)$ is not a sink. Then there is an edge $f(s) \to w$. The set $U = \{v \in V : w \notin N_t(v)\}$ is an open neighbourhood of s (since $N_t(s) = \phi, w \notin N_t(s)$). By continuity, $f^{-1}(U)$ is open and contains s. But $f(s) \notin U$, which is a contradiction regarding the pre-image. Thus, $f(s)$ must be a sink. Since f maps the finite set S to itself, by the pigeonhole principle (or the finite fixed point property), f has a fixed point within S.*

Theorem 2 (Fixed point in strongly connected graphs with j-contractive property): *Let $SDG(V)$ be a finite, strongly connected directed graph. Let $f : V \to V$ be a j-continuous map that is j-contracting for $j = un$, meaning: $f(N_{un}(v)) \subseteq N_{un}(f(v))$ for all $v \in V$ where $N_{un}(v) \neq \{v\}$. Then f has a fixed point.*

proof: Strong connectivity and τ_{un} : *In a strongly connected graph, the union neighbourhood topology* τ_{un} *is highly non-trivial. The space* (V, τ_{un}) *is connected. Furthermore, because every very vertex is reachable from every other, the minimal open sets (given by* $LO_{N_{un}}(\{v\})$*) are not singletons (unless the graph is a single vertex). The contracting property implies a unique "center": A contracting map on a finite, connected topological space that is "sphere-like" (in the sense of the neighbourhood structure) must have a unique fixed point. The argument is analogous to the Branch fixed-point theorem but for finite metric spaces or ultra-metric spaces. Here, the "metric" is the graph distance in the underlying undirected graph of* N_{un}*. Fixed point via minimal closed set: Consider the family* $\mathcal{F}$ *of non-empty, closed subsets* C *of* (V, τ_{un}) *such that* $f(C) \subseteq C$*. This family is non-empty (as* V *itself is in it). Since* V *is finite, there is a minimal such set* C_0*. If* C_0 *has more than one point, the contracting property and the connectivity of* C_0 *(in the subspace topology) would force* f *to map* C_0 *to a proper subset of itself that is also closed and* f*-invariant, contradicting minimality. Therefore,* C_0 *must be a singleton* $\{v_0\}$*, and hence* $f(v_0) = v_0$*.*

5. Conclusions and future work

This study successfully introduced and investigated new types of topological spaces in simple directed graphs, leveraging the concept of j-neighbourhoods. We defined four distinct j-neighbourhoods (outside, inside, intersection, and union) and subsequently used them to establish j-lower and j-upper approximations for subgraphs. The detailed examples and tables provided throughout the document illustrate the practical application of these definitions and approximations. We also rigorously proved several propositions that highlight the fundamental properties of these approximations concerning set operations such as inclusion, union, intersection, and complementation. This theoretical framework offers a more refined approach to analyzing graph structures and their inherent relationships.

For future work, this research can be extended in several directions: Firstly, further exploration of topological properties: Investigate additional topological properties induced by j-neighbourhoods, such as connectedness, compactness, and separation axioms, to gain a deeper understanding of these new topological spaces. Secondly, applications in diverse fields: Explore the applicability of these new topological spaces in other complex networks beyond air travel, such as social networks, biological networks, and communication networks, to model and analyze their structural characteristics and dynamics. Thirdly, weighted and fuzzy graphs: Extend the definitions of j-neighbourhoods and approximations to weighted graphs, where edges have associated values, and fuzzy graphs, where relationships are not sharply defined, to enhance the model's capacity to represent real-world complexities. Fourthly, algorithmic development: Develop efficient algorithms for computing J-neighbourhoods, lower and upper approximations, and the induced topological spaces for large-scale graphs. Fifthly, comparison with existing topological graph theories: Conduct a comparative analysis of this new topological approach with existing methods in topological graph theory to identify its strengths, limitations, and potential for integration.

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