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Hankel Determinant Estimates for Bi-Bazilevič-Type Functions Involving q-Fibonacci Numbers

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Abstract. This study focuses on a specific class of analytic and bi-univalent functions of Bazilevič-type, formulated within a geometric context shaped by shell-like curves and influenced by the q-analogue of Fibonacci numbers. By utilizing the subordination approach, we establish precise bounds for the initial coefficients in the Taylor–Maclaurin expansion of these functions. Moreover, the paper presents Fekete–Szegö-type inequalities and introduces novel bounds for the second Hankel determinant, thereby contributing to a deeper analytical insight into the behavior of this function class. These contributions not only broaden the scope of traditional coefficient problems related to bi-univalent functions but also highlight the intricate connections among geometric function theory, specialized function classes, and the principles of q-calculus. The outcomes pave the way for future studies aimed at deriving bounds for higher-order coefficients and examining determinant-related functionals under this theoretical model.

2020 Mathematics Subject Classifications: 30A36, 30C45, 81P68, 11B37

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Let $\mathcal{A}$ denote the class of functions that are analytic in the open unit disk $\mathbb{D}$, defined by

$$\mathbb{D} = \{z = a + ib \in \mathbb{C} : a, b \in \mathbb{R},\ |z| < 1\},$$

which represents the interior of the unit circle in the complex plane, centered at the origin and excluding the boundary.

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Each function $f \in \mathcal{A}$ is normalized such that

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These normalization conditions remove translational and dilational ambiguities, ensuring a standardized form that facilitates structural analysis and comparative study under shared geometric constraints.

Each member $f \in \mathcal{A}$ possesses a Maclaurin series representation about the origin, which can be written as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{for } z \in \mathbb{D}, \tag{1}$$

where the coefficients a_n determine the nonlinear components of f. The leading term z arises from the derivative condition $f'(0) = 1$, and subsequent terms capture the analytic structure beyond linearity.

A function f is called a *Schwarz function* if it is analytic throughout $\mathbb{D}$, satisfies $f(0) = 0$, and its modulus remains strictly less than one within the disk, i.e., $|f(z)| < 1$ for all $z \in \mathbb{D}$. These functions are of central importance in geometric function theory, particularly in the context of conformal and univalent mappings.

Furthermore, for any two functions $f_1, f_2 \in \mathcal{A}$, the function f_1 is said to be *subordinate* to f_2, denoted $f_1 \prec f_2$, if there exists a Schwarz function η such that

$$f_1(z) = f_2(\eta(z)) \quad \text{for all} \quad z \in \mathbb{D}.$$

This relation implies that f_1 is functionally dependent on f_2 through composition with η, preserving analyticity while embedding geometric structure. The notion of subordination is a key analytical tool for examining inclusion relations, growth estimates, and mapping behavior in complex analysis.

In addition, let us consider the subclass S, $\mathsf{S} \subset \mathcal{A}$, which comprises all functions that are univalent (i.e., one-to-one) within the unit disk $\mathbb{D}$. We also introduce the class P, defined as the family of functions in $\mathcal{A}$ whose real parts are strictly positive throughout $\mathbb{D}$. A typical function $\varphi \in \mathsf{P}$ admits the following power series expansion:

$$\mathsf{p}(z) = 1 + \sum_{n=1}^{\infty} \mathsf{p}_n z^n = 1 + \mathsf{p}_1 z + \mathsf{p}_2 z^2 + \mathsf{p}_3 z^3 + \ldots, \quad (z \in \mathbb{D}). \tag{2}$$

where the coefficients satisfy the sharp bound

$$|\mathsf{p}_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{3}$$

in accordance with the classical Carathéodory lemma (refer to [1] for further details). Furthermore, a function $\varphi \in \mathsf{P}$ if and only if it is subordinate to the Möbius transformation $\frac{1+z}{1-z}$, i.e.,

$$\varphi(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

The class of starlike functions, denoted S^*, can be characterized in various ways using subordination techniques. A notable generalization was proposed by Ma and Minda [2], who defined the following class:

$$\mathsf{S}^*(\Omega) = \left\{ f \in \mathcal{A} : \frac{z\, f'(z)}{f(z)} \prec \Omega(z), \quad \text{where} \quad \Omega \in \mathsf{P} \ \text{ and } \ z \in \mathbb{D} \right\}.$$

In this formulation, Ω is assumed to be analytic in $\mathbb{D}$ and possess a positive real part throughout the disk. Table 1 provides a variety of subclasses of S^*, arising from specific choices of the function Ω, reflecting the diversity of approaches adopted in the literature for constructing refined categories of starlike mappings.

Table 1: Enumerates various starlike function classes characterized via the principle of subordination.

	The subclasses of starlike functions	**Ref.**	**Author/s**
1	$\mathsf{S}^*\left(\frac{1+z}{1-z}\right) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$	[3]	Janowski
2	$\mathsf{S}^*(\vartheta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\vartheta)z}{1-z} \right\}$, where $0 \le \vartheta < 1$	[4]	Robertson
3	$\mathsf{SL}(\vartheta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2} \right\}$, where $\vartheta = \frac{1-\sqrt{5}}{2}$	[5]	Sokól
4	$\mathcal{SK}(\vartheta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{3}{3+(\vartheta-3)z-\vartheta^2 z^2} \right\}$, where $\vartheta \in (-3, 1]$	[6]	Sokól

The class P forms the cornerstone for the development of numerous significant subclasses of analytic functions, making it a key object of study in complex analysis. For any function f in the subclass $\mathsf{S} \subset \mathcal{A}$, there exists an inverse function, denoted f^{-1}, which is defined as

$$z = f^{-1}(f(z)) \text{ and } \xi = f(f^{-1}(\xi)), \quad (\mathsf{r}_0(f) \ge 0.25; \ \ |\xi| < \mathsf{r}_0(f); z \in \mathbb{D}). \tag{4}$$

where

$$\chi(\xi) = f^{-1}(\xi) = \xi - a_2\xi^2 + \left(2a_2^2 - a_3\right)\xi^3 - \left(5a_2^3 + a_4 - 5a_3a_2\right)\xi^4 + \cdots. \tag{5}$$

function $f \in \mathsf{S}$ is said to be bi-univalent if its inverse function $f^{-1} \in \mathsf{S}$. The subclass of S denoted by Σ contains all bi-univalent functions in $\mathbb{D}$. A table illustrating certain functions within the class Σ and their inverse functions is provided below.

Table 2: Representative examples of bi-univalent functions along with their corresponding inverse functions.

f	f^{-1}
$f_1(z) = \dfrac{z}{1+z}$	$f_1^{-1}(z) = \dfrac{z}{1-z}$
$f_2 = -\log(1-z)$	$f_1^{-1}(z) = \dfrac{e^{2z}-1}{e^{2z}+1}$
$f_3 = \frac{1}{2}\log\left(\dfrac{1+z}{1-z}\right)$	$f_1^{-1}(z) = \dfrac{e^{z}-1}{e^{z}}$

In contemporary mathematical and physical research, the study of quantum calculus—commonly referred to as q-calculus—has emerged as a vibrant and impactful area of inquiry. The foundational concepts of this theory were introduced in the late 19th century by Jackson [7, 8], who formulated the q-difference operator along with its integral counterpart. These contributions laid the groundwork for a novel approach to calculus, one that does not rely on traditional notions of limits. Expanding on Jackson's foundational work, Aral and Gupta [9] explored the q-extensions of classical mathematical tools and operators, particularly in the realm of geometric function theory.

The essence of q-calculus lies in its ability to generalize conventional calculus through the use of q-differences, offering a robust analytical structure for examining complex classes of analytic functions. It has proven especially effective in characterizing and analyzing subclasses such as starlike, convex, and bi-univalent functions. Within this framework, the deformation parameter q, constrained to the interval $0 < q < 1$, plays a pivotal role. It ensures the convergence of q-series and the preservation of geometric and analytic properties necessary for the coherent definition of these subclasses. The q-derivative operator $\eth_q$, along with its associated constructs like q-numbers and q-factorials, provides a natural extension of classical operators. This facilitates refined estimates of coefficient bounds and enables the derivation of sharp inequalities within q-analytic function theory. Consequently, q-calculus offers new perspectives and methodologies for advancing both theoretical investigations and practical applications in complex analysis.

Definition 1. *[10] The q-bracket $\lceil \kappa \rfloor_q$ is defined as follows:*

$$\lceil \kappa \rfloor_q = \begin{cases} \frac{1-q^{\kappa}}{1-q}, & 0 < q < 1,\ \kappa \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ 1, & q \mapsto 0^+,\ \kappa \in \mathbb{C}^* \\ \kappa, & q \mapsto 1^-,\ \kappa \in \mathbb{C}^* \\ q^{\gamma-1} + q^{\gamma-2} + \cdots + q + 1 = \sum\limits_{n=0}^{\gamma-1} q^n, & 0 < q < 1,\ \kappa = \gamma \in \mathbb{N}, \end{cases}$$

with the useful identity $\lceil \kappa + 1 \rfloor_q = \lceil \kappa \rfloor_q + q^{\kappa}$.

Definition 2. *[10] The $q-$derivative, also known as the $q-$difference operator, of a function f is defined by*

$$\eth_q\langle f(z)\rangle = \begin{cases} (f(z)-f(q\,z))(z-q\,z)^{-1}, & \text{if} \quad 0<q<1,\ z\neq 0, \\ f'(0), & \text{if} \quad z=0, \\ f'(z), & \text{if} \quad q\mapsto 1^-,\ z\neq 0. \end{cases}.$$

Remark 1. *For $f\in\mathcal{A}$ of the form (1), it is straightforward to verify that*

$$\eth_q\langle f(z)\rangle = \eth_q\left\langle z+\sum_{n=2}^{\infty} a_n\, z^n \right\rangle = 1+\sum_{n=2}^{\infty}\lceil n\rfloor_q a_n\, z^{n-1}, \quad (z\in\mathbb{D}),$$

and for the inverse function $\chi=f^{-1}$ of the form (4), we have

$$\eth_q\langle\chi(\xi)\rangle = \eth_q\langle f^{-1}(\xi)\rangle = 1-\lceil 2\rfloor_q a_2\xi+\lceil 3\rfloor_q\left(2a_2^2-a_3\right)\xi^2-\lceil 4\rfloor_q\left(5a_2^3+a_4-5a_3a_2\right)\xi^3+\cdots.$$

In a more recent advancement, Alsoboh et al. [11] introduced a noteworthy class of functions known as q-starlike functions, denoted by SL_q, which were defined using the q-Jackson difference operators. The formal definition of this class is given by

$$\mathsf{SL}_q = \left\{ f\in\mathcal{A}: \quad \frac{z\,\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q), \quad z\in\mathbb{D}\right\}, \tag{6}$$

where the function $\Upsilon(z;q)$ is expressed explicitly as

$$\Upsilon(z;q)=\frac{1+q\vartheta_q^2z^2}{1-\vartheta_q z-q\vartheta_q^2z^2}, \tag{7}$$

and

$$\vartheta_q=\frac{1-\sqrt{4q+1}}{2q} \tag{8}$$

represents the q-analog of the Fibonacci numbers. Additionally, Alsoboh et al. [11] established a significant connection between these q-Fibonacci numbers, denoted as ϑ_q, and the related Fibonacci polynomials $\varphi_n(q)$. Specifically, they demonstrated that if

$$\Upsilon(z;q)=1+\sum_{n=1}^{\infty}\mathsf{p}_n z^n,$$

the coefficients p_n satisfy the following recurrence relation:

$$\mathsf{p}_n=\begin{cases} \vartheta_q, & \text{for } n=1, \\ (2q+1)\vartheta_q^2, & \text{for } n=2, \\ (3q+1)\vartheta_q^3, & \text{for } n=3, \\ \left(\varphi_{n+1}(q)+q\varphi_{n-1}(q)\right)\vartheta_q^n, & \text{for } n\geq 4. \end{cases} \tag{9}$$

Here, the q-Fibonacci polynomials $\varphi_{\mathsf{s}}(q)$ are defined as

$$\varphi_{\mathsf{s}}(q) = \frac{(1 - q\vartheta_q)^{\mathsf{s}} - (\vartheta_q)^{\mathsf{s}}}{\sqrt{4q+1}}, \quad \mathsf{s} \in \mathbb{N}. \tag{10}$$

This research presents a thorough framework for examining the relationship between the q-modified Fibonacci numbers and their corresponding polynomial representations.

The initial terms of the q-Fibonacci sequence, which constitutes a natural generalization of the classical Fibonacci numbers and converges to them as $q \to 1^-$, are enumerated in Table 3.

Table 3: Comparison of the classical Fibonacci numbers with their corresponding q-analogue terms from the q-Fibonacci sequence.

The classical Fibonacci numbers	The q-analogue of Fibonacci numbers
$\varphi_0 = 0$	$\varphi_0(q) = 0$
$\varphi_1 = 1$	$\varphi_1(q) = 1$
$\varphi_2 = 1$	$\varphi_2(q) = 1$
$\varphi_3 = 2$	$\varphi_3(q) = 1 + q$
$\varphi_4 = 3$	$\varphi_4(q) = 1 + 2q$

The function $\Upsilon(z;q)$, defined in (7), maps the unit circle to a curve Ω_q characterized by the parametric representation

$$\varkappa = \frac{\sqrt{4q+1}}{2(1 + 2q - 2q\cos\phi)}, \quad \mathsf{y} = \frac{\frac{\sin\phi}{2(1+\cos\phi)}(4q\cos\phi - 1)}{1 + 2q - 2q\cos\phi}, \quad \phi \in [0, 2\pi) \setminus \{\pi\}. \tag{11}$$

Notably, $\Upsilon(z;q)$ is not injective over the unit disk $\mathbb{D}$, as demonstrated by

$$\Upsilon(0;q) = \Upsilon\left(-\frac{1}{2q\vartheta_q}; q\right) = 1.$$

This non-injectivity results in a shell-like structure for the image curve Ω_q, which is symmetric with respect to the real axis, as illustrated in Fig. 1.

We distinguish two qualitative behaviors of the curve Ω_q depending on the value of the parameter q:

- **Case I:** $q \in \left(0, \frac{1}{4}\right)$
  In this regime, the curve Ω_q is smooth and resembles a conchoid without any self-intersections. The image under Υ remains injective on the unit circle, and the curve does not loop back on itself.

- **Case II:** $q \in \left(\frac{1}{4}, 1\right)$
  Here, the curve Ω_q develops a self-intersecting loop. Specifically, the relation

$$\Upsilon\left(e^{\pm i \arccos\left(\frac{1}{4q}\right)}; q\right) = \frac{\sqrt{4q+1}}{4q+1}$$

  implies that Ω_q intersects itself on the real axis at

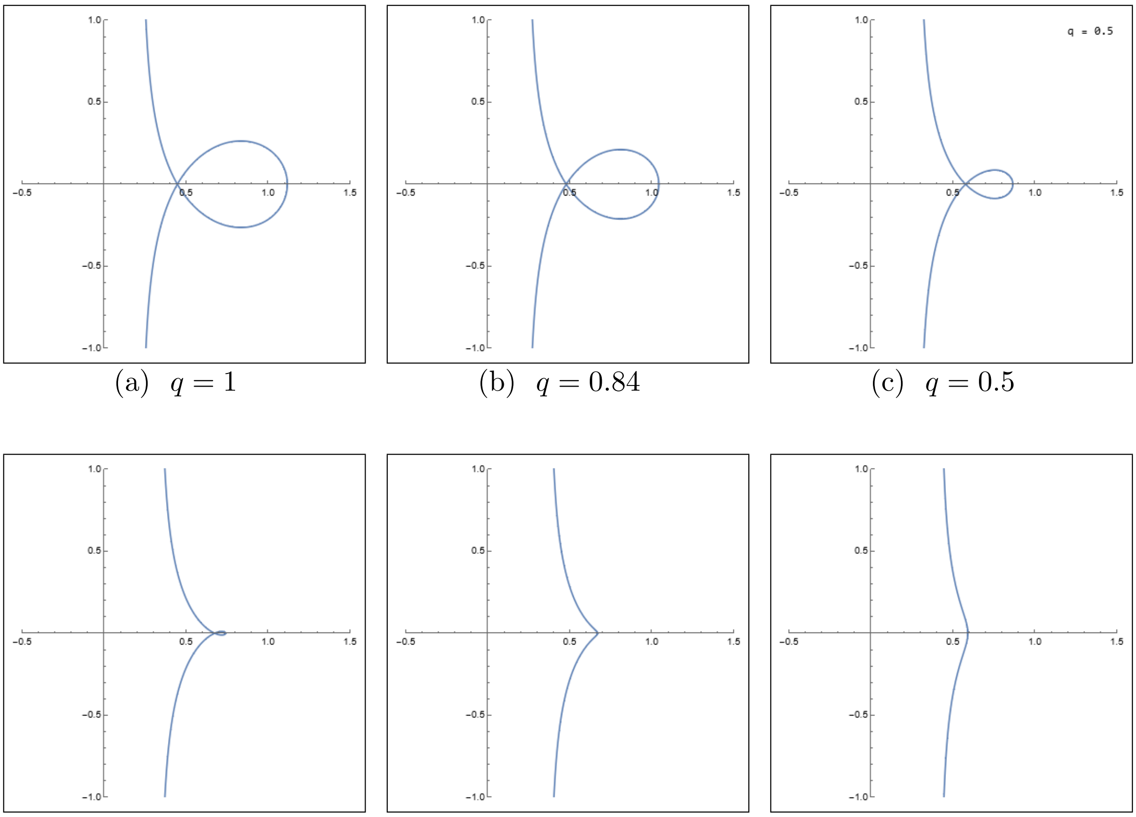
$$e_2 = \frac{\sqrt{4q+1}}{4q+1}.$$

  Consequently, the curve crosses the real axis at two distinct points:

$$e_1 = \frac{\sqrt{4q+1}}{2}, \qquad e_2 = \frac{\sqrt{4q+1}}{4q+1}.$$

  This results in a closed loop in the geometry of Ω_q.

Figure 1: The curve Ω_q for various values of q.



(a) $q = 1$ (b) $q = 0.84$ (c) $q = 0.5$

(d) $q = 0.3$ (e) $q = 0.2$ (f) $q = 0.1$

In the following example, we explore the behavior of the q-starlike functions as the parameter q approaches 1 from below. This transition leads to the classical case of starlike functions, often referred to as the class SL. By taking the limit as $q \to 1^-$, we observe how the q-starlike functions generalize to the traditional starlike functions, and the associated function $\Upsilon(z)$ simplifies to a form that connects directly with the classical Fibonacci numbers. This example illustrates the connection between the q-starlike functions and their classical counterparts.

Example 1. *To illustrate the asymptotic behavior of the q-starlike functions as $q \to 1^-$, we examine the limiting case of the class SL_q. In the limit, this class converges to the classical starlike function class associated with the Fibonacci generating function, namely*

$$\mathsf{SL} = \lim_{q\to 1^-} \mathsf{SL}_q = \left\{ f \in \mathcal{A} \ : \ \frac{z\, f'(z)}{f(z)} \prec \Upsilon(z) \right\},$$

where the function $\Upsilon(z)$ is given by

$$\Upsilon(z;1) = \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z - \vartheta^2 z^2}, \tag{12}$$

and $\vartheta = \frac{1-\sqrt{5}}{2}$ denotes the classical Fibonacci constant.

The development of q-calculus has profoundly advanced the field of analytic function theory by facilitating the construction and investigation of new subclasses characterized by rich geometric structures and intricate algebraic behavior. This analytical framework highlights the intrinsic adaptability of q-calculus in generalizing classical results and revealing previously unexplored mathematical phenomena. Its influence extends beyond mere theoretical curiosity, providing a unified platform for significant insights and promising applications in diverse areas of mathematical analysis. As such, q-calculus serves as a powerful bridge between traditional methodologies and contemporary innovations, laying a solid groundwork for sustained research and future breakthroughs in the discipline [12–30].

1. Definition and example

Motivated by q-Fibonacci numbers, this section will now look at a novel subclass of bi-univalent functions related to shell-like curves.

Definition 3. *A bi-univalent function f of the form* (1) *belongs to the class $\mathsf{B}_\Sigma(\mu;q)$ if and only if*

$$\frac{z^{1-\mu}\ \eth_q\langle f(z)\rangle}{\big(f(z)\big)^{1-\mu}} \prec \Upsilon(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q\, z - q\vartheta_q^2 z^2}, \tag{13}$$

and

$$\frac{\xi^{1-\mu}\ \eth_q\langle \chi(\xi)\rangle}{\big(\chi(\xi)\big)^{1-\mu}} \prec \Upsilon(\xi;q) = \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q\, \xi - q\vartheta_q^2 \xi^2}, \tag{14}$$

where $\mu \in \mathbb{R}^+ \cup \{0\}$, $\chi = f^{-1}$ given by (5), *ϑ_q is given by* (8) *and $z, \xi \in \mathbb{D}$.*

By varying the parameters $\mu \in \mathbb{R}^{+} \cup \{0\}$ and $q \in (0,1)$, a broad spectrum of novel subclasses of the bi-univalent function class $\sum$ can be systematically derived. These subclasses capture diverse geometric behaviors and provide a unified framework for further analytical investigations.

Example 2. *If $\mu = 0$, we obtain the class $\mathsf{SL}_{\Sigma}(\Upsilon(z;q))$, defined as consisting of functions $f \in \Sigma$ satisfying the conditions*

$$\frac{z\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q) = \frac{1+q\vartheta_q^2 z^2}{1-\vartheta_q\, z - q\vartheta_q^2 z^2},$$

and

$$\frac{\xi\eth_q\langle \chi(\xi)\rangle}{\chi(\xi)} \prec \Upsilon(\xi;q) = \frac{1+q\vartheta_q^2 \xi^2}{1-\vartheta_q\, \xi - q\vartheta_q^2 \xi^2},$$

where ϑ_q is given by (8), $\chi = f^{-1}$ defined as in (5), $\vartheta = \frac{1-\sqrt{5}}{2}$ is the classical Fibonacci constant, and $z, \xi \in \mathbb{D}$. This class was studied by Alsoboh et al. [11].

Example 3. *If $q \to 1^{-}$ then $\mathsf{B}_{\Sigma}(\mu;q)$ is reduce to the subclass $\mathsf{B}_{\Sigma}(\mu)$ studied by Pulala [31].*

$$\frac{z^{1-\mu}\ f'(z)}{\big(f(z)\big)^{1-\mu}} \prec \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z - \vartheta^2 z^2},$$

and

$$\frac{\xi^{1-\mu}\ \chi'(\xi)}{\big(\chi(\xi)\big)^{1-\mu}} \prec \Upsilon(\xi) = \frac{1+\vartheta^2 \xi^2}{1-\vartheta \xi - \vartheta^2 \xi^2},$$

As demonstrated in [32], the most precise agreement is attained in the case where

$$\mathsf{B}_{\Sigma}(\mu;q) := \mathcal{J}_{\Sigma}^{(1,\mu)}\big(Y(z;1^{-})\big),$$

Example 4. *In the limiting case as $q \to 1^{-}$, and $\mu = 0$ we recover the classical subclass SLM_{Σ}, consisting of all functions $f \in \Sigma$ that satisfy the following subordination conditions:*

$$\frac{z\, f'(z)}{f(z)} \prec \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z - \vartheta^2 z^2},$$

and

$$\frac{\xi\, \chi'(\xi)}{\chi(\xi)} \prec \Upsilon(\xi) = \frac{1+\vartheta^2 \xi^2}{1-\vartheta \xi - \vartheta^2 \xi^2},$$

where $\chi = f^{-1}$ is the inverse function defined as in (5), $\vartheta = \frac{1-\sqrt{5}}{2}$ is the classical Fibonacci constant, and $z, \xi \in \mathbb{D}$. This class was initially investigated by Sokół [5, 6], and subsequently studied in greater depth by Özgür and Sokół [33].

2. Coefficient bounds of $\mathsf{B}_{\Sigma}(\mu; q)$

In this section, we aim to estimate the initial Taylor coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathsf{B}_{\Sigma}(\mu; q)$, as defined in Definition 3.

Consider the analytic function

$$\mathsf{p}(z) = 1 + \mathsf{p}_1 z + \mathsf{p}_2 z^2 + \mathsf{p}_3 z^3 + \ldots,$$

which satisfies the subordination condition $\mathsf{p}(z) \prec \Upsilon(z; q)$. By the principle of subordination, there exists a Schwarz function $\varphi \in \mathsf{P}$ such that

$$|\varphi(z)| < 1 \quad \text{for all } z \in \mathbb{D}, \quad \text{and} \quad \mathsf{p}(z) = \Upsilon(\varphi(z); q).$$

Define the function

$$\hbar(z) = \frac{1 + \varphi(z)}{1 - \varphi(z)} = 1 + \ell_1 z + \ell_2 z^2 + \cdots \in \mathsf{P}, \qquad (z \in \mathbb{D}). \tag{15}$$

Since $\varphi(z)$ is analytic in $\mathbb{D}$ and subordinate to $\Upsilon(z; q)$, it admits the following Taylor expansion:

$$\varphi(z) = \frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots. \tag{16}$$

Substituting this into the function $\Upsilon(\varphi(z); q)$, we obtain:

$$\begin{aligned}
\Upsilon(\varphi(z); q) &= 1 + \mathsf{p}_1\left[\frac{\ell_1 z}{2} + \left(\ell_2 - \frac{\ell_1^2}{2}\right)\frac{z^2}{2} + \left(\ell_3 - \ell_1\ell_2 - \frac{\ell_1^3}{4}\right)\frac{z^3}{2} + \cdots\right] \\
&\quad + \mathsf{p}_2\left[\frac{\ell_1 z}{2} + \cdots\right]^2 + \mathsf{p}_3\left[\frac{\ell_1 z}{2} + \cdots\right]^3 + \cdots \\
&= 1 + \frac{\mathsf{p}_1\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\mathsf{p}_1 + \frac{\ell_1^2}{2}\mathsf{p}_2\right]z^2 \\
&\quad + \frac{1}{2}\left[\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right)\mathsf{p}_1 + \ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right)\mathsf{p}_2 + \frac{\ell_1^3}{4}\mathsf{p}_3\right]z^3 + \cdots \\
&= 1 + \frac{\vartheta_q\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\vartheta_q + \frac{(1+2q)\vartheta_q^2\ell_1^2}{2}\right]z^2 \\
&\quad + \frac{1}{2}\left[\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right)\vartheta_q + \ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right)(1+2q)\vartheta_q^2 + \frac{(1+3q)\vartheta_q^3\ell_1^3}{4}\right]z^3 + \cdots
\end{aligned} \tag{17}$$

Similarly, there exists an analytic function ν, defined in $\mathbb{D}$, such that $|\nu(\xi)| < 1$ and $\mathsf{p}(\xi) = \Upsilon(\nu(\xi); q)$. Accordingly, we define

$$\kappa(\xi) = \frac{1 + \nu(\xi)}{1 - \nu(\xi)} = 1 + \tau_1\xi + \tau_2\xi^2 + \cdots \in \mathsf{P}. \tag{18}$$

The corresponding expansion for $\nu(\xi)$ is then given by:

$$\nu(\xi) = \frac{\tau_1 \xi}{2} + \left(\tau_2 - \frac{\tau_1^2}{2}\right)\frac{\xi^2}{2} + \left(\tau_3 - \tau_1\tau_2 - \frac{\tau_1^3}{4}\right)\frac{\xi^3}{2} + \cdots . \tag{19}$$

Therefore, the expansion of $\Upsilon(\nu(\xi); q)$ becomes:

$$\Upsilon(\nu(\xi); q) = 1 + \frac{\mathsf{p}_1\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\mathsf{p}_1 + \frac{\tau_1^2}{2}\mathsf{p}_2\right]\xi^2 + \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\vartheta_q + \tau_1\left(\tau_2 - \frac{(1+2q)\vartheta_q^2\tau_1^2}{2}\right) + \frac{(1+3q)\vartheta_q^3\tau_1^3}{4}\right]\xi^3 + \cdots . \tag{20}$$

Having established the necessary groundwork and auxiliary results, we are now in a position to derive bounds for the initial coefficients of functions belonging to the newly introduced class $\mathsf{B}_\Sigma(\mu; q)$. These estimates not only offer insights into the geometric behavior of such bi-univalent functions, but also highlight the influence of the deformation parameter q and the parameter β on the coefficient structure. The following theorem presents sharp bounds for the second and third coefficients, $|a_2|$ and $|a_3|$, respectively.

To proceed, we first introduce the following lemma, which plays a fundamental role in the theoretical analysis that follows.

Lemma 1. *[34] If the function $\mathfrak{r} \in \mathsf{P}$ is given by the series*

$$\mathfrak{r}(z) = 1 + \ell_1 z + \ell_2 z^2 + \ell_3 z^3 + \ldots,$$

then the following coefficient estimates hold:

$$2\ell_2 = \ell_1^2 + x(4 - \ell_1^2),$$
$$4\ell_3 = \ell_1^3 + 2\ell_1(4 - \ell_1^2)x - \ell_1(4 - \ell_1^2)x^2 + 2(4 - \ell_1^2)(1 - |x|^2)z,$$

for some $x, z \in \mathbb{C}$ with $\max\{|x|, |z|\} \leq 1$.

Theorem 1. *For $\mu \in \mathbb{R}^+ \cup \{0\}$, let $f \in \mathsf{B}_\Sigma(\mu; q)$. Then*

$$|a_2| \leq \min\left\{\frac{|\vartheta_q|}{\mu + q}, \sqrt{\frac{2\vartheta_q^2}{\psi(q, \mu)}}\right\} \tag{21}$$

$$|a_3| \leq \min\left\{\frac{\vartheta_q^2}{(\mu + q)^2} + \frac{|\vartheta_q|}{\mu + q^2 + q}, \frac{2\vartheta_q^2}{\psi(q, \mu)} + \frac{|\vartheta_q|}{\mu + q^2 + q}\right\}, \tag{22}$$

where

$$\psi(q, \mu) = 2(\mu + q)^2\left((1 + 2q)\vartheta_q - 1\right) + \vartheta_q\left(2q^2 + (2q + 1)\mu + \mu^2\right). \tag{23}$$

Proof. Let $f \in \mathsf{B}_\Sigma(\mu;q)$ and $\xi = f^{-1}$. Considering (13) and (14) we have

$$\frac{z^{1-\mu}\ \eth_q\langle f(z)\rangle}{\big(f(z)\big)^{1-\mu}} = \Upsilon(\varphi(z);q), \qquad (z \in \mathbb{D}), \tag{24}$$

and

$$\frac{\xi^{1-\mu}\ \eth_q\langle \chi(\xi)\rangle}{\big(\chi(\xi)\big)^{1-\mu}} = \Upsilon(\nu(\xi);q), \qquad (\xi \in \mathbb{D}). \tag{25}$$

Since

$$\begin{aligned}\frac{z^{1-\mu}\ \eth_q\langle f(z)\rangle}{\big(f(z)\big)^{1-\mu}} &= 1 + (\mu+q)\, a_2 z + \left[\left(\mu + q\lceil 2\rfloor_q\right)a_3 + \frac{1}{2}\Big(\left(-2q\lceil 2\rfloor_q - 3\right)\mu + \mu^2\Big)a_2^2\right] z^2 \cdots \\ &+ \Big[\left(\mu + q\lceil 3\rfloor_q\right)a_4 + \left(-(\lceil 3\rfloor_q + \lceil 2\rfloor_q - 2) + (\lceil 3\rfloor_q + \lceil 2\rfloor_q - 3)\mu + \mu^2\right) a_2 a_3 \\ &\quad + \frac{1}{6}\left(6q + (11 - 9\lceil 2\rfloor_q)\mu + 3(\lceil 2\rfloor_q - 2)\mu^2 + \mu^3\right) a_2^3\Big] z^3 + \cdots,\end{aligned} \tag{26}$$

and

$$\begin{aligned}\frac{\xi^{1-\mu}\ \eth_q\langle \chi(\xi)\rangle}{\big(\chi(\xi)\big)^{1-\mu}} &= 1 - (\mu+q)a_2\xi \\ &+ \left[\frac{1}{2}\left(-2(\lceil 2\rfloor_q - 2\lceil 3\rfloor_q + 1) + (2\lceil 2\rfloor_q + 1)\mu + \mu^2\right) a_2^2 - (\mu + \lceil 3\rfloor_q - 1)a_3\right]\xi^2 \\ &+ \Big[-(\mu + \lceil 4\rfloor_q - 1)a_4 + \left(5\lceil 4\rfloor_q - \lceil 2\rfloor_q - \lceil 3\rfloor_q - 3 + (\lceil 3\rfloor_q + \lceil 2\rfloor_q + 2)\mu + \mu^2\right) a_2 a_3 \\ &\quad + \frac{1}{6}\big(12 - 30\lceil 4\rfloor_q + 6\lceil 2\rfloor_q + 12\lceil 3\rfloor_q + (-5 - 3\lceil 2\rfloor_q - 12\lceil 3\rfloor_q)\mu \\ &\quad + (-3\lceil 2\rfloor_q - 6)\mu^2 - \mu^3\big) a_2^3\Big]\xi^3 + \cdots.\end{aligned} \tag{27}$$

It follows from the equations (24), (17), and (26) that

$$(\mu+q)a_2 = \frac{\vartheta_q \ell_1}{2} \tag{28}$$

$$(\mu + q\lceil 2\rfloor_q)a_3 + \frac{1}{2}\left(-2q + (2\lceil 2\rfloor_q - 3)\mu + \mu^2\right)a_2^2 = \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\vartheta_q + \frac{(1+2q\vartheta_q^2)\ell_1^2}{2}\right] \tag{29}$$

$$\begin{aligned}&(\mu + q\lceil 3\rfloor_q)a_4 + \left(-(\lceil 3\rfloor_q + \lceil 2\rfloor_q - 2) + (\lceil 3\rfloor_q + \lceil 2\rfloor_q - 3)\mu + \mu^2\right) a_2 a_3 \\ &\quad + \frac{1}{6}\left(6(\lceil 2\rfloor_q - 1) + (11 - 9\lceil 2\rfloor_q)\mu + 3(\lceil 2\rfloor_q - 2)\mu^2 + \mu^3\right) a_2^3 \\ &= \frac{\vartheta_q}{2}\left(\ell_3 - \ell_1\ell_2 + \frac{\ell_1^3}{4}\right) + \frac{(1+2q)\vartheta_q^2}{2}\ell_1\left(\ell_2 - \frac{\ell_1^2}{2}\right) + \frac{(1+3q)\vartheta_q^3}{8}\ell_1^3.\end{aligned} \tag{30}$$

Similarly, from the equations (25), (20), and (27) we obtain:

$$-(\mu+q)a_2 = \frac{\vartheta_q}{2}\tau_1, \tag{31}$$

$$\begin{aligned}\frac{1}{2}\left(-2(\lceil 2\rfloor_q - 2\lceil 3\rfloor_q + 1) + (2\lceil 2\rfloor_q + 1)\mu + \mu^2\right)a_2^2 - (\mu + q\lceil 2\rfloor_q)a_3 \\ = \frac{\vartheta_q}{2}\left(\tau_2 - \frac{\tau_1^2}{2}\right) + \frac{(1+2q)\vartheta_q^2}{4}\tau_1^2,\end{aligned} \tag{32}$$

$$\begin{aligned}&-(\mu + q\lceil 3\rfloor_q)a_4 + \left(5\lceil 4\rfloor_q - \lceil 2\rfloor_q - \lceil 3\rfloor_q - 3 + (\lceil 3\rfloor_q + \lceil 2\rfloor_q + 2)\mu + \mu^2\right)a_2a_3 \\ &+ \frac{1}{6}\left(12 - 30\lceil 4\rfloor_q + 6\lceil 2\rfloor_q + 12\lceil 3\rfloor_q - (5 + 3\lceil 2\rfloor_q + 12\lceil 3\rfloor_q)\mu - (3\lceil 2\rfloor_q + 6)\mu^2 - \mu^3\right)a_2^3 \\ &= \frac{\vartheta_q}{2}\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right) + \frac{(1+2q)\vartheta_q^2}{2}\tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right) + \frac{(1+3q)\vartheta_q^3}{8}\tau_1^3.\end{aligned} \tag{33}$$

From (28) and (31) we have:

$$a_2 = \frac{\vartheta_q}{2(\mu+q)}\ell_1 = -\frac{\vartheta_q}{2(\mu+q)}\tau_1, \tag{34}$$

and this indicates that

$$\ell_1 = -\tau_1. \tag{35}$$

Using (3) to equation (34), we obtain that

$$a_2 \leq \left|\frac{\vartheta_q}{\mu+q}\right| = \frac{|\vartheta_q|}{\mu+q}. \tag{36}$$

The squaring and addition of (28) and (31) lead to

$$a_2^2 = \frac{\vartheta_q^2(\ell_1^2 + \tau_1^2)}{8(\mu+q)^2}. \tag{37}$$

In addition, the sum of (29) and (32) gives

$$a_2^2\left(2(\lceil 3\rfloor_q - \lceil 2\rfloor_q) + (2\lceil 2\rfloor_q - 1)\mu + \mu^2\right) = \frac{\vartheta_q}{2}(\ell_2 + \tau_2) + \frac{(1+2q)\vartheta_q^2 - \vartheta_q}{2}\cdot\frac{\ell_1^2 + \tau_1^2}{2}.$$

From (37), we get

$$a_2^2\left(2(\lceil 3\rfloor_q - \lceil 2\rfloor_q) + (2\lceil 2\rfloor_q - 1)\mu + \mu^2\right) = \frac{\vartheta_q}{2}(\ell_2 + \tau_2) + \frac{(1+2q)\vartheta_q^2 - \vartheta_q}{2}\cdot\frac{4(\mu+q)^2a_2^2}{\vartheta_q^2},$$

which means

$$a_2^2 = \frac{(\ell_2 + \tau_2)\,\vartheta_q^2}{-4(\mu+q)^2\left(-1 + (1+2q)\vartheta_q\right) + 2\vartheta_q\Big(2(\lceil 3\rfloor_q - \lceil 2\rfloor_q) + (2\lceil 2\rfloor_q - 1)\mu + \mu^2\Big)}.$$

Since $\lceil 2\rfloor_q = q+1$ and $\lceil 3\rfloor_q = q^2+q+1$, then

$$a_2^2 = \frac{(\ell_2+\tau_2)\,\vartheta_q^2}{2\Big[2(\mu+q)^2\left((1+2q)\vartheta_q-1\right)+\vartheta_q\left(2q^2+(2q+1)\mu+\mu^2\right)\Big]}. \tag{38}$$

Thus, (3) implies that

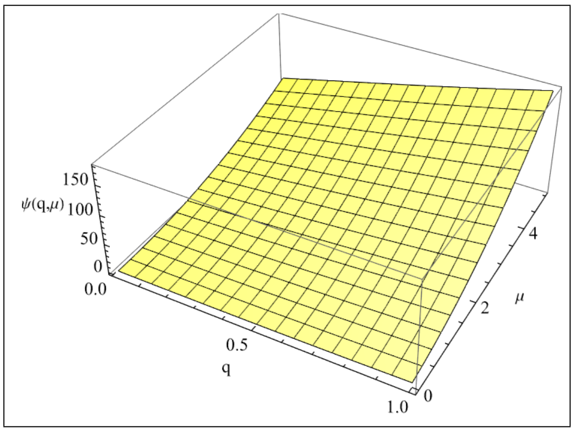
$$|a_2| \leq \sqrt{\frac{2\vartheta_q^2}{2(\mu+q)^2\left((1+2q)\vartheta_q-1\right)+\vartheta_q\left(2q^2+(2q+1)\mu+\mu^2\right)}}.$$

and hence,

$$|a_2| \leq \sqrt{\frac{2\vartheta_q^2}{\psi(q,\mu)}}, \tag{39}$$

is satisfied for all $\mu \geq 0$ (see Figure 2), where $\psi(q,\mu)$ is given by (23).

Figure 2: The plot of the function $\psi(q,\mu)$.



Subsequently, by subtracting (32) from (29), we obtain

$$a_3 = a_2^2 + \frac{\vartheta_q(\ell_2-\tau_2)}{4(\mu+q\lceil 2\rfloor_q)} = a_2^2 + \frac{\vartheta_q(\ell_2-\tau_2)}{4(\mu+q^2+q)}. \tag{40}$$

Hence,

$$|a_3| \leq |a_2|^2 + \frac{|\vartheta_q|}{\mu+q^2+q}. \tag{41}$$

Substituting equations (36) and (39) into (41), respectively, we obtain

$$|a_3| \leq \frac{\vartheta_q^2}{(\mu+q)^2} + \frac{|\vartheta_q|}{\mu+q^2+q},$$

and

$$|a_3| \leq \frac{2\vartheta_q^2}{\psi(q,\mu)} + \frac{|\vartheta_q|}{\mu+q^2+q}.$$

Hence, the desired result follows, completing the proof with elegance and clarity.

In the next result, we derive the sharp bound for the functional $\left|a_3 - \eta a_2^2\right|$ for functions $f \in \mathcal{B}_\Sigma(\mu, q)$, where $\eta \in \mathbb{R}$.

Theorem 2. *For $\eta \in \mathbb{R}^+ \cup \{0\}$, let $f \in \mathcal{B}_\Sigma(\mu, q)$. Then*

$$\left|a_3 - \eta\, a_2^2\right| \leq \begin{cases} \dfrac{|\vartheta_q|}{\mu + q^2 + q}, & \textit{if} \quad |1-\eta| \leq \dfrac{\psi(q,\mu)}{2(\mu + q^2 + q)|\vartheta_q|}, \\ \dfrac{2(1-\eta)\,\vartheta_q^2}{\psi(q,\mu)}, & \textit{if} \quad |1-\eta| \geq \dfrac{\psi(q,\mu)}{2(\mu + q^2 + q)|\vartheta_q|}, \end{cases} \tag{42}$$

where $\psi(q, \mu)$ is defined in (23).

Proof. Assuming that $f \in \mathcal{B}_\Sigma(\mu, q)$, it follows from equations (38) and (40) that

$$\begin{aligned} a_3 - \eta\, a_2^2 &= \frac{(1-\eta)\,\vartheta_q^2(\ell_2 + \tau_2)}{2\left[-2(\mu+q)^2(-1 + (1+2q)\vartheta_q) + \vartheta_q(2q^2 + (2q+1)\mu + \mu^2)\right]} + \frac{\vartheta_q(\ell_2 - \tau_2)}{4(\mu + q^2 + q)} \\ &= \left(\Xi(q,\mu,\eta) + \frac{\vartheta_q}{4(\mu + q^2 + q)}\right)\ell_2 + \left(\Xi(q,\mu,\eta) - \frac{\vartheta_q}{4(\mu + q^2 + q)}\right)\tau_2, \end{aligned} \tag{43}$$

where

$$\Xi(q,\mu,\eta) = \frac{(1-\eta)\,\vartheta_q^2}{2\left[-2(\mu+q)^2(-1 + (1+2q)\vartheta_q) + \vartheta_q(2q^2 + (2q+1)\mu + \mu^2)\right]}. \tag{44}$$

Accordingly, taking the modulus of (43), we obtain:

$$\left|a_3 - \eta\, a_2^2\right| \leq \begin{cases} \dfrac{|\vartheta_q|}{\mu + q^2 + q}, & \text{if} \quad 0 \leq \Xi(q,\mu,\eta) \leq \dfrac{\vartheta_q}{4(\mu + q^2 + q)}, \\ 4\Xi(q,\mu,\eta), & \text{if} \quad \Xi(q,\mu,\eta) \geq \dfrac{\vartheta_q}{4(\mu + q^2 + q)}. \end{cases}$$

After straightforward computations, we obtain the result in (42).

3. On coefficient bounds of the second Hankel determinant for the class $\mathcal{B}_\Sigma(\mu, q)$

This section is devoted to deriving a coefficient inequality for the second Hankel determinant associated with the class $\mathcal{B}_\Sigma(\mu, q)$, as presented in the following theorem:

Theorem 3. *Let $f \in \mathcal{B}_\Sigma(\mu, q)$. Then*

$$\left|H_{2,2}(f)\right| \leq \begin{cases} \mathcal{Y}(2^-) & \text{if } \mathcal{X}_1 \geq 0 \text{ and } \mathcal{X}_2 \geq 0, \\ \max\left\{\dfrac{\vartheta_q^2}{(q+q^2+\mu)^2}, \mathcal{Y}(2^-)\right\} & \text{if } \mathcal{X}_1 > 0 \text{ and } \mathcal{X}_2 < 0, \\ \dfrac{\vartheta_q^2}{(q+q^2+\mu)^2} & \text{if } \mathcal{X}_1 \leq 0 \text{ and } \mathcal{X}_2 \leq 0, \\ \max\left\{\mathcal{Y}(2^-), \mathcal{Y}\left(\sqrt{\dfrac{-12\mathcal{X}_2}{\mathcal{X}_1}}\right)\right\} & \text{if } \mathcal{X}_1 < 0 \text{ and } \mathcal{X}_2 > 0, \end{cases} \tag{45}$$

where

$$\begin{aligned}\mathcal{X}_1 :=\ & \left|\vartheta_q^4\big(6q^3(1+6q) + 6q^2(5+18q)\mu + 3q\mu(-1+(11+36q)\mu)\right. \\ & \left.+\mu\big(-2+\mu(-3+(11+36q)\mu)\big)\big)\right| (q+q^2+\mu)^2 \\ & - 6(q+\mu)^2\Big[q^3(1+q(3+q))\vartheta_q^2 - q(1+q)\big(1+2q(1+q(1+q)(3+4q))\big)\vartheta_q^3 \\ & - \Big(-3q^2(1+2q)\vartheta_q^2 + \vartheta_q^3 + q\big(3+q(15+18q+4q^2+8(1+q)(3+q)q)\big)\vartheta_q^3\Big)\mu \\ & - \Big(q(-3+(-3+q)q)\vartheta_q^2 + \vartheta_q^3 + 4q(3+2q)(1+2q)\vartheta_q^3\Big)\mu^2 \\ & + \Big(\vartheta_q^2 - 4(\vartheta_q^3 + 2q\vartheta_q^3)\Big)\mu^3\Big],\end{aligned}$$

$$\begin{aligned}\mathcal{X}_2 :=\ & -(q+\mu)^2\Big[2q^5(3+4q)\vartheta_q^3 + q\vartheta_q^3\big(1+3\mu+12(1+2q)\mu^2\big) \\ & + \vartheta_q^3\mu\big(1+\mu+(4+8q)\mu^2\big) + 2q^3\big(\vartheta_q^2(-2+\mu)\mu + \vartheta_q^3(4+4q+9\mu+16q\mu)\big) \\ & + q^2\big(-2\vartheta_q^2\mu^2 + \vartheta_q^3\big(3+3(5+8q)\mu+8(1+2q)\mu^2\big)\big) \\ & + 2q^4\big(\vartheta_q^2(-1+\mu) + 2\vartheta_q^3(3+\mu+2q(2+\mu))\big)\Big],\end{aligned}$$

$$\mathcal{Y}(2^-) = \frac{\vartheta_q^2}{(q+q^2+\mu)^2} + \frac{\mathcal{X}_1 + 6\mathcal{X}_2c^2}{6(q+\mu)^4(q+q^2+\mu)^2(q+q^2+q^3+\mu)},$$

and

$$\mathcal{Y}\left(\sqrt{\frac{-12\mathcal{X}_2}{\mathcal{X}_1}}\right) = \frac{\vartheta_q^2}{(q+q^2+\mu)^2} - \frac{3\mathcal{X}_2^2}{2\mathcal{X}_1(q+\mu)^4(q+q^2+\mu)^2(q+q^2+q^3+\mu)}.$$

Proof. Let $f \in \mathcal{B}_\Sigma(\mu, q)$. By employing an argument analogous to that used in the proof of Theorem 1, subtracting equation (33) from (30) and utilizing (34) and (40) yields

$$a_4 = \frac{1}{-1+\lceil 4\rfloor_q+\mu}\Bigg(-\frac{1}{8}\vartheta_q\Big(-4\ell_3 - \ell_1^3 + 4\ell_1\tau_2 + 4\tau_3 + 2\ell_1^3\vartheta_q - 4\ell_1\tau_2\vartheta_q + 4\ell_1^3q\vartheta_q - 8\ell_1\tau_2q\vartheta_q$$

$$- \ell_1^3\vartheta_q^2 - 3\ell_1^3 q\vartheta_q^2 - 4\ell_1\ell_2(-1+\vartheta_q+2q\vartheta_q) - \ell_1^3\big(1-2(1+2q)\vartheta_q+(1+3q)\vartheta_q^2\big)\Big)$$
$$- \frac{1}{48(-1+\lceil 2\rfloor_q+\mu)^3(-1+\lceil 3\rfloor_q+\mu)}\ell_1\vartheta_q^2(-1+\mu)\Big(6\ell_2(-1+\lceil 2\rfloor_q+\mu)^2(-2+\lceil 2\rfloor_q+\lceil 3\rfloor_q+\mu)$$
$$- 6\tau_2(-1+\lceil 2\rfloor_q+\mu)^2(-2+\lceil 2\rfloor_q+\lceil 3\rfloor_q+\mu) + \ell_1^2\vartheta_q(-1+\lceil 3\rfloor_q+\mu)(-6+6\lceil 3\rfloor_q+\mu+3\lceil 2\rfloor_q\mu$$
$$+\mu^2)\Big)\Bigg).$$

Since $\lceil 2\rfloor_q = q+1$, $\lceil 3\rfloor_q = q^2+q+1$, and $\lceil 4\rfloor_q = q^3+q^2+q+1$, then, after some simplifications, we obtain

$$\begin{aligned} a_4 = {}& \frac{\vartheta_q(\ell_3-\tau_3)}{2(q^3+q^2+q+\mu)} - \frac{\vartheta_q(1-\vartheta_q-2q\vartheta_q)\ell_1(\ell_2+\tau_2)}{2(q^3+q^2+q+\mu)} \\ & + \frac{\vartheta_q^2(\mu-1)(q^2+2q+\mu)\ell_1(\ell_2-\tau_2)}{8(q+\mu)(q^2+q+\mu)(q^3+q^2+q+\mu)} \\ & + \left(-\frac{\vartheta_q(-2+4\vartheta_q+8q\vartheta_q-2\vartheta_q^2-6q\vartheta_q^2)}{8(q^3+q^2+q+\mu)} - \frac{\vartheta_q^3(-1+\mu)(6q^2+6q+3q\mu+4\mu+\mu^2)}{48(q+\mu)^3(q^3+q^2+q+\mu)} \right)\ell_1^3. \end{aligned} \tag{46}$$

Therefore, by using (34), (40), and (46), we deduce that

$$\begin{aligned} a_2a_4 - a_3^2 = {}& \frac{\begin{array}{c}\vartheta_q^4(1-\mu)\left(6q^2+3q(2+\mu)+\mu(4+\mu)\right) \\ +12(q+\mu)^3\vartheta_q^2\left[1-2(1+2q)\vartheta_q+(1+3q)\vartheta_q^2\right] - 6\vartheta_q^4(q+q^2+q^3+\mu)\end{array}}{96(q+\mu)^4(q+q^2+q^3+\mu)}\ell_1^4 \\ & - \frac{\vartheta_q^2(\ell_2-\tau_2)^2}{16(q+q^2+\mu)^2} - \frac{\vartheta_q^3(1+2q+2q^2+2q^3+\mu)\ell_1^2(\ell_2-\tau_2)}{16(q+\mu)^2(q+q^2+\mu)(q+q^2+q^3+\mu)} \\ & - \frac{(\vartheta_q^2-\vartheta_q^3-2q\vartheta_q^3)\ell_1^2(\ell_2+\tau_2)}{4(q+\mu)(q+q^2+q^3+\mu)} + \frac{\vartheta_q^2\ell_1(\ell_3-\tau_3)}{4(q+\mu)(q+q^2+q^3+\mu)}. \end{aligned} \tag{47}$$

From Lemma 1, it can be concluded that

$$2\ell_2 = \ell_1^2 + (4-\ell_1^2)x, \quad \text{and} \quad 2\tau_2 = \tau_1^2 + (4-\tau_1^2)y.$$

Therefore, in view of (35), we obtain

$$\ell_2 - \tau_2 = \frac{4-\ell_1^2}{2}(x-y), \tag{48}$$

and

$$\ell_2 + \tau_2 = \ell_1^2 + \frac{4-\ell_1^2}{2}(x+y). \tag{49}$$

Moreover, we have

$$4\ell_3 = \ell_1^3 + 2(4-\ell_1^2)\ell_1 x - \ell_1(4-\ell_1^2)x^2 + 2(4-\ell_1^2)(1-|x|^2)z,$$

and

$$4\tau_3 = \tau_1^3 + 2(4-\tau_1^2)\tau_1 y - \tau_1(4-\tau_1^2)y^2 + 2(4-\tau_1^2)(1-|y|^2)w.$$

for some x, y, z and w, with $\max\{|x|,|y|,|z|,|w|\} \leq 1$, and $\ell_1, \tau_1 \in [0,2]$. Thus, we have

$$\ell_3 - \tau_3 = \frac{\ell_1^3}{2} + \frac{\ell_1(4-\ell_1^2)}{2}(x+y) - \frac{\ell_1(4-\ell_1^2)}{4}(x^2+y^2) + \frac{4-\ell_1^2}{2}\left[(1-|x|^2)z - (1-|y|^2)w\right]. \tag{50}$$

In addition, the substitution of (48)-(50) into (47) yields
$a_2a_4 - a_3^2 =$

$$\begin{aligned}
&\frac{\ell_1^4\vartheta_q^4\big(6q^3(1+6q) + 6q^2(5+18q)\mu + 3q\mu(-1+(11+36q)\mu) + \mu(-2+\mu(-3+(11+36q)\mu))\big)}{96(q+\mu)^4(q+q^2+q^3+\mu)} \\
&+ \left(\frac{-\vartheta_q^3(1+2q+2q^2+2q^3+\mu)(x-y)}{32(q+\mu)^2(q+q^2+\mu)(q+q^2+q^3+\mu)} - \frac{(-\vartheta_q^3 - 2q\vartheta_q^3)(x+y)}{8(q+\mu)(q+q^2+q^3+\mu)}\right)\ell_1^2(4-\ell_1^2) \\
&- \frac{\vartheta_q^2\ell_1^2(4-\ell_1^2)}{16(q+\mu)(q+q^2+q^3+\mu)}(x^2+y^2) - \frac{\vartheta_q^2(4-\ell_1^2)^2(x-y)^2}{64(q+q^2+\mu)^2} \\
&+ \frac{\vartheta_q^2(4-\ell_1^2)\ell_1}{8(q+\mu)(q+q^2+q^3+\mu)}\left[(1-|x|^2)z - (1-|y|^2)w\right].
\end{aligned}$$

Hence, $|a_2a_4 - a_3^2| \leq$

$$\begin{aligned}
&\left|\frac{\vartheta_q^4\left(6q^3(1+6q) + 6q^2(5+18q)\mu + 3q\mu(-1+(11+36q)\mu) + \mu(-2+\mu(-3+(11+36q)\mu))\right)}{96(q+\mu)^4(q+q^2+q^3+\mu)}\right|\ell_1^4 \\
&+ \frac{\left(-4(1+2q)(q+\mu)(q+q^2+\mu)\vartheta_q^3 - \vartheta_q^3[1+2q(1+q+q^2)+\mu]\right)}{32(q+\mu)^2(q+q^2+\mu)(q+q^2+q^3+\mu)}\ell_1^2(4-\ell_1^2)(|x|+|y|) \\
&+ \frac{\vartheta_q^2\ell_1^2(4-\ell_1^2)}{16(q+\mu)(q+q^2+q^3+\mu)}(|x|^2+|y|^2) + \frac{\vartheta_q^2(4-\ell_1^2)^2}{64(q+q^2+\mu)^2}(|x|+|y|)^2 \\
&+ \frac{\vartheta_q^2(4-\ell_1^2)\ell_1}{8(q+\mu)(q+q^2+q^3+\mu)}\left(2-(|x|^2+|y|^2)\right) \\
&= \left|\frac{\vartheta_q^4\left(6q^3(1+6q) + 6q^2(5+18q)\mu + 3q\mu(-1+(11+36q)\mu) + \mu(-2+\mu(-3+(11+36q)\mu))\right)}{96(q+\mu)^4(q+q^2+q^3+\mu)}\right|\ell_1^4 \\
&+ \frac{2\vartheta_q^2(4-\ell_1^2)\ell_1}{8(q+\mu)(q+q^2+q^3+\mu)} \\
&+ \frac{\left(-4(1+2q)(q+\mu)(q+q^2+\mu)\vartheta_q^3 - \vartheta_q^3[1+2q(1+q+q^2)+\mu]\right)}{32(q+\mu)^2(q+q^2+\mu)(q+q^2+q^3+\mu)}\ell_1^2(4-\ell_1^2)(|x|+|y|) \\
&+ \frac{\vartheta_q^2\ell_1(\ell_1-2)(4-\ell_1^2)}{16(q+\mu)(q+q^2+q^3+\mu)}(|x|^2+|y|^2) + \frac{\vartheta_q^2(4-\ell_1^2)^2}{64(q+q^2+\mu)^2}(|x|+|y|)^2.
\end{aligned}$$

Letting $|x| = \epsilon_1$, $|y| = \epsilon_2$, and $\ell_1 = c$, the following result can be derived straightforwardly:

$$|a_2a_4 - a_3^2| \leq \Lambda_1 + \Lambda_2(\epsilon_1+\epsilon_2) + \Lambda_3(\epsilon_1^2+\epsilon_2^2) + \Lambda_4(\epsilon_1+\epsilon_2)^2 =: \mathcal{F}(\epsilon_1, \epsilon_2),$$

where

$$\Lambda_1(c) = \left| \frac{\vartheta_q^4\left(6q^3(1+6q)+6q^2(5+18q)\mu+3q\mu(-1+(11+36q)\mu)+\mu(-2+\mu(-3+(11+36q)\mu))\right)}{96(q+\mu)^4(q+q^2+q^3+\mu)} \right| c^4$$
$$+\frac{2\vartheta_q^2(4-c^2)c}{8(q+\mu)(q+q^2+q^3+\mu)} \geq 0,$$
$$\Lambda_2(c) = \left(\frac{-4(1+2q)(q+\mu)(q+q^2+\mu)\vartheta_q^3 - \vartheta_q^3[1+2q(1+q+q^2)+\mu]}{32(q+\mu)^2(q+q^2+\mu)(q+q^2+q^3+\mu)} \right) c^2(4-c^2) \geq 0,$$
$$\Lambda_3(c) = \frac{\vartheta_q^2 c(c-2)(4-c^2)}{16(q+\mu)(q+q^2+q^3+\mu)} \leq 0,$$
$$\Lambda_4(c) = \frac{\vartheta_q^2(4-c^2)^2}{64(q+q^2+\mu)^2} \geq 0.$$

We aim to determine the maximum value of the function $\mathcal{F}(\epsilon_1, \epsilon_2)$ over the closed square

$$\mathbf{\Lambda} := \{(\epsilon_1, \epsilon_2) : \epsilon_1, \epsilon_2 \in [0,1]\}.$$

Given that $\Lambda_3 < 0$ and $\Lambda_3 + 2\Lambda_4 > 0$ for all $c \in (0,1)$, it follows that

$$\mathcal{F}_{\epsilon_1\epsilon_1}\mathcal{F}_{\epsilon_2\epsilon_2} - (\mathcal{F}_{\epsilon_1\epsilon_2})^2 < 0$$

throughout the square $\mathbf{\Lambda}$. This inequality implies that $\mathcal{F}$ cannot attain a local maximum in the interior of the square $\mathbf{\Lambda}$. Accordingly, we proceed to examine the boundary of $\mathbf{\Lambda}$ in search of the maximum value. When setting $\epsilon_1 = 0$ and $\epsilon_2 \in [0,1]$ (similarly for $\epsilon_2 = 0$ and $\epsilon_1 \in [0,1]$), the function reduces to

$$\mathcal{F}(0, \epsilon_2) = \mathcal{G}(\epsilon_2) = \Lambda_1 + \Lambda_2\epsilon_2 + (\Lambda_3 + \Lambda_4)\epsilon_2^2.$$

Case (i): When $\Lambda_3 + \Lambda_4 \geq 0$. In this situation, for any fixed $c \in [0,2)$, the derivative $\mathcal{G}'(\epsilon_2) = 2(\Lambda_3 + \Lambda_4)\epsilon_2 + \Lambda_2$ remains strictly positive for $0 < \epsilon_2 < 1$. This implies that $\mathcal{G}(\epsilon_2)$ is monotonically increasing on $(0,1)$. As a consequence, the function $\mathcal{G}$ achieves its maximum value at $\epsilon_2 = 1$, and thus we have

$$\max \mathcal{G}(\epsilon_2) = \mathcal{G}(1) = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4.$$

Case (ii): When $\Lambda_3 + \Lambda_4 < 0$. Given that $\Lambda_2 + 2(\Lambda_3 + \Lambda_4) \geq 0$ for $\epsilon_2 \in (0,1)$ and any fixed $c \in [0,2)$, it follows from the inequality

$$\Lambda_2 + 2(\Lambda_3 + \Lambda_4) < 2(\Lambda_3 + \Lambda_4)\epsilon_2 + \Lambda_2 < \Lambda_2$$

that $\mathcal{G}'(\epsilon_2) > 0$. Therefore, the function $\mathcal{G}(\epsilon_2)$ is increasing, and it attains its maximum at $\epsilon_2 = 1$. Additionally, when $c = 2$, the expression for $\mathcal{F}(\epsilon_1, \epsilon_2)$ simplifies to

$$\mathcal{F}(\epsilon_1, \epsilon_2)\Big|_{c=2} = \left| \frac{\vartheta_q^4\left(6p^3(1+6q)+6p^2(5+18q)\mu+3p\mu(-1+(11+36q)\mu)+\mu\left(-2+\mu(-3+(11+36q)\mu)\right)\right)}{6(p+\mu)^4(p+p^2+p^3+\mu)} \right|. \tag{51}$$

Taking into account the analysis in both subcases (i) and (ii), we deduce that for $\epsilon_2 \in [0,1)$ and any $c \in [0,2]$, the maximum of $\mathcal{G}(\epsilon_2)$ is again given by

$$\max \mathcal{G}(\epsilon_2) = \mathcal{G}(1) = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4.$$

Now, considering the boundary condition $\epsilon_1 = 1$ and $0 \le \epsilon_2 \le 1$ (similarly, $\epsilon_2 = 1$ and $0 \le \epsilon_1 \le 1$), the function takes the form

$$\mathcal{F}(1,\epsilon_2) = \mathcal{H}(\epsilon_2) = (\Lambda_3 + \Lambda_4)\epsilon_2^2 + (\Lambda_2 + 2\Lambda_4)\epsilon_2 + \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4.$$

Following the same logic applied earlier for the cases involving $\Lambda_3 + \Lambda_4$, we find that $\mathcal{H}(\epsilon_2)$ reaches its maximum at $\epsilon_2 = 1$, leading to

$$\max \mathcal{H}(\epsilon_2) = \mathcal{H}(1) = \Lambda_1 + 2\Lambda_2 + 2\Lambda_3 + 4\Lambda_4.$$

Since it holds that $\mathcal{G}(1) = \mathcal{H}(1)$ for all $c \in [0,2]$, the maximum value of the function $\mathcal{F}(\epsilon_1,\epsilon_2)$ over the boundary of the square $\mathbf{\Lambda}$ is attained at the point $(1,1)$. Consequently, the maximum of $\mathcal{F}$ within the closed square $\mathbf{\Lambda}$ is achieved precisely at this corner. Moreover, we define the function $\mathcal{Y} : [0,2] \to \mathbb{R}$ by

$$\mathcal{Y}(c) = \max \mathcal{F}(\epsilon_1,\epsilon_2) = \mathcal{F}(1,1) = \Lambda_1 + 2\Lambda_2 + 2\Lambda_3 + 4\Lambda_4, \tag{52}$$

which expresses the maximum value of $\mathcal{F}$ in terms of c. By substituting the explicit expressions of $\Lambda_1, \Lambda_2, \Lambda_3$, and Λ_4 into the function $\mathcal{Y}$ as defined in equation (52), we arrive at a more detailed representation of $\mathcal{Y}(c)$ in terms of the parameters involved:

$$\mathcal{Y}(c) = \frac{\vartheta_q^2}{(p+p^2+\mu)^2} + \frac{\mathcal{X}_1 c^4 + 24\mathcal{X}_2 c^2}{96(p+\mu)^4(p+p^2+\mu)^2(p+p^2+p^3+\mu)},$$

where

$$\begin{aligned}
\mathcal{X}_1 :=\ & \big|\vartheta_q^4\big(6q^3(1+6q) + 6q^2(5+18q)\mu + 3q\mu(-1+(11+36q)\mu) \\
&+\mu\big(-2+\mu(-3+(11+36q)\mu)\big)\big)\big|\,(q+q^2+\mu)^2 \\
&- 6(q+\mu)^2\Big[q^3(1+q(3+q))\vartheta_q^2 - q(1+q)\big(1+2q(1+q(1+q)(3+4q))\big)\vartheta_q^3 \\
&- \Big(-3q^2(1+2q)\vartheta_q^2 + \vartheta_q^3 + q\big(3+q(15+18q+4q^2+8(1+q)(3+q)q)\big)\vartheta_q^3\Big)\mu \\
&- \Big(q(-3+(-3+q)q)\vartheta_q^2 + \vartheta_q^3 + 4q(3+2q)(1+2q)\vartheta_q^3\Big)\mu^2 \\
&+ \Big(\vartheta_q^2 - 4(\vartheta_q^3 + 2q\vartheta_q^3)\Big)\mu^3\Big],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{X}_2 :=\ & -(q+\mu)^2\Big[2q^5(3+4q)\vartheta_q^3 + q\vartheta_q^3\big(1+3\mu+12(1+2q)\mu^2\big) \\
&+ \vartheta_q^3\mu\big(1+\mu+(4+8q)\mu^2\big) + 2q^3\big(\vartheta_q^2(-2+\mu)\mu + \vartheta_q^3(4+4q+9\mu+16q\mu)\big)
\end{aligned}$$

$$+ q^2\big(-2\vartheta_q^2\mu^2 + \vartheta_q^3(3 + 3(5 + 8q)\mu + 8(1 + 2q)\mu^2)\big)$$
$$+ 2q^4\big(\vartheta_q^2(-1 + \mu) + 2\vartheta_q^3(3 + \mu + 2q(2 + \mu))\big)\Big],$$

Let us suppose that the function $\mathcal{Y}(c)$ attains its maximum at an interior point in the interval $c \in [0, 2]$. Through straightforward computations, we obtain the following expression:

$$\mathcal{Y}'(c) = \frac{(\mathcal{X}_1 c^2 + 12q)c}{24(q + \mu)^4(q + q^2 + \mu)^2(q + q^2 + q^3 + \mu)}.$$

In the subsequent analysis, we investigate the sign of $\mathcal{Y}'(c)$ by considering various combinations of the signs of $\mathcal{X}_1$ and q, as detailed below:

(i) Let $\mathcal{X}_1 \geq 0$ and $\mathcal{X}_2 \geq 0$, then $\mathcal{Y}'(c) \geq 0$, so $\mathcal{Y}(c)$ is an increasing function. Therefore,

$$\begin{aligned} \max\big\{\mathcal{Y}(c) : c \in (0, 2)\big\} &= \mathcal{Y}(2^-) \\ &= \frac{\vartheta_q^2}{(q + q^2 + \mu)^2} + \frac{\mathcal{X}_1 + 6\mathcal{X}_2 c^2}{6(q + \mu)^4(q + q^2 + \mu)^2(q + q^2 + q^3 + \mu)}, \end{aligned} \tag{53}$$

that is,

$$\max\Big\{\max\big\{F(\epsilon_1, \epsilon_2) : \epsilon_1, \epsilon_2 \in [0, 1]\big\} : c \in (0, 2)\Big\} = \mathcal{Y}(2^-).$$

(ii) Let $\mathcal{X}_1 > 0$ and $\mathcal{X}_2 < 0$, then $c_0 = \sqrt{\frac{-12\mathcal{X}_2}{\mathcal{X}_1}}$ is a critical point of the function $\mathcal{Y}(c)$. We assume that $c_0 \in (0, 2)$. Since $\mathcal{Y}''(c) > 0$, c_0 is a local minimum point of the function $\mathcal{Y}(c)$. That is, the function $\mathcal{Y}(c)$ cannot have a local maximum.

(iii) Let $\mathcal{X}_1 \leq 0$ and $\mathcal{X}_2 \leq 0$, then $\mathcal{Y}'(c) \leq 0$, so $\mathcal{Y}(c)$ is a decreasing function on the interval $(0, 2)$. Therefore,

$$\max\{\mathcal{Y}(c) : c \in (0, 2)\} = \mathcal{Y}(0^+) = 4\Lambda_4 = \frac{\vartheta_q^2}{(q + q^2 + \mu)^2}. \tag{54}$$

(iv) Let $\mathcal{X}_1 < 0$ and $\mathcal{X}_2 > 0$, then c_0 is a critical point of the function $\mathcal{Y}(c)$. We assume that $c_0 \in (0, 2)$. Since $\mathcal{Y}''(c) < 0$, c_0 is a local maximum point of the function $\mathcal{Y}(c)$, and the maximum value occurs at $c = c_0$. Therefore,

$$\max\{\mathcal{Y}(c) : c \in (0, 2)\} = \mathcal{Y}(c_0), \tag{55}$$

where

$$\mathcal{Y}(c_0) = \frac{\vartheta_q^2}{(q + q^2 + \mu)^2} - \frac{3\mathcal{X}_2^2}{2\mathcal{X}_1(q + \mu)^4(q + q^2 + \mu)^2(q + q^2 + q^3 + \mu)}$$

Thus, from equations (51) to (55), the proof is complete.

If $\mu = 0$, we obtain the following results for the class $\mathsf{SL}_\Sigma(\Upsilon(z;q))$ defined in Example (2)

Corollary 1. *Let f given by* (1) *be in the class* $\mathsf{SL}_\Sigma(\Upsilon(z);q)$. *Then*

$$\left|a_2\right| \leq \frac{\left|\vartheta_q\right|}{q\sqrt{1-2q\vartheta_q}}. \tag{56}$$

$$\left|a_3\right| \leq \frac{\left|\vartheta_q\right|\left(q-(1+q+2q^2)\vartheta_q\right)}{q^2(1+q)\left(1-2q\vartheta_q\right)}. \tag{57}$$

$$\left|a_3 - \eta\, a_2^2\right| \leq \begin{cases} \frac{|\vartheta_q|}{q(1+q)}, & \left|1-\alpha\right| \leq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \\ \frac{|1-\alpha|\vartheta_q^2}{q^2\left(1-2q\vartheta_q\right)}, & \left|1-\alpha\right| \geq \frac{q\left(1-2q\vartheta_q\right)}{(1+q)|\vartheta_q|} \end{cases} \tag{58}$$

If $q \mapsto 1^-$ and $\mu = 0$, we obtain the following results for the class $\mathsf{SL}_\Sigma(\Upsilon(z))$ defined in Example (3)

Corollary 2. *[35] Let f given by* (1) *be in the class* $\mathsf{SL}_\Sigma(\Upsilon(z))$. *Then*

$$\left|a_2\right| \leq \frac{|\vartheta|}{\sqrt{1-2\vartheta}}, \qquad \left|a_3\right| \leq \frac{|\vartheta|\left(1-4\vartheta\right)}{2\left(1-2\vartheta\right)}.$$

and

$$\left|a_3 - \eta\, a_2^2\right| \leq \begin{cases} \frac{|\vartheta|}{2}, & \left|1-\alpha\right| \leq \frac{1-2\vartheta}{2|\vartheta|} \\ \frac{(1-\alpha)\vartheta^2}{1-2\vartheta}, & \left|1-\alpha\right| \geq \frac{1-2\vartheta}{2|\vartheta|} \end{cases}$$

4. Conclusion

In this study, we introduced and analyzed a subclass of bi-univalent functions associated with shell-like curves, formulated via the q-analogue of Fibonacci numbers, within the framework of the Bazilevič-type class. By employing the subordination principle, we obtained sharp bounds for the initial coefficients of functions in this class. Additionally, we derived Fekete–Szegö-type inequalities and estimates for the second Hankel determinant, thereby contributing to the deeper understanding of bi-univalent function theory and its interplay with special function spaces.

These findings not only enrich the structural theory of bi-univalent functions but also highlight the influence of the underlying q-calculus in shaping their analytic and geometric behavior.

Future investigations may focus on extending the current results to higher-order coefficients, refining extremal characterizations of these subclasses, and studying their geometric features in more depth. Further directions include exploring sharp bounds associated with the Zalcman conjecture and analyzing the third-order Hankel determinant within the same analytic framework.

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