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Hyers-Ulam Stability of Generalized Quartic Mapping in Non-Archimedean (n, β)-Normed Spaces

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Abstract. In this article, we introduce a novel structure termed as the non-Archimedean (n, β)-normed space, formulated over a non-Archimedean field. This generalization extends the concept of classical normed spaces by integrating a parameterized framework involving n-tuples and an exponent β. We delve into the fundamental characteristics of these spaces, demonstrating how they connect to standard non-Archimedean n-normed and n-quasi-normed structures. Moreover, we provide examples that support the theory and help show some fixed point results, making these spaces easier to use in real problems.

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1. Introduction

Due to its extensive applicability in many other domains and its deep consequences in mathematical analysis, the study of functional equations and their stability qualities has attracted a lot of attention. Ulam [1] raised a basic query about the stability of group homomorphisms in 1940, which is when this topic first emerged. The theory of functional

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equation stability was developed as a result of this investigation. Hyers [2] in 1941 provided a crucial answer by establishing the first stability result for linear functional equations, which is now known as the Hyers-Ulam stability. Later, Rassias developed Hyers-Ulam-Rassias stability theory by adding a perturbation term that was dependent on the norm, which was a substantial generalization.

Notable among the many functional equation classes are quartic functional equations, which generalize the behavior of fourth-degree polynomials. Approximation theory, information theory, theoretical physics, and differential equations all naturally produce these types of equations. The quartic functional equation

$$\tau(\mathsf{t}_1 + 2\mathsf{t}_2) + \tau(\mathsf{t}_1 - 2\mathsf{t}_2) = 4[\tau(\mathsf{t}_1 + \mathsf{t}_2) + \tau(\mathsf{t}_1 - \mathsf{t}_2)] + 6\tau(\mathsf{t}_1) - 24\tau(\mathsf{t}_2)$$

serves as a classical example whose general solution often involves quartic mappings such as $\tau(\mathsf{t}) = a\mathsf{t}^4$. A lot of research has been done on how stable these kinds of equations are in both standard and extended normed settings [3–9].

A notable development in this field pertains to the application of non-Archimedean normed spaces. These spaces satisfy the strong triangle inequality:

$$\|\mathsf{t}_1 + \mathsf{t}_2\| \leq \max\{\|\mathsf{t}_1\|, \|\mathsf{t}_2\|\},$$

which endows them with distinct topological and algebraic properties (see, [10, 11]). Such spaces frequently occur in p-adic analysis, number theory, and information theory. Building on this framework, the notion of (n, β)-normed spaces was introduced to capture a broader class of normed structures (Ref. [12–14]).These spaces provide a more comprehensive analytical investigation by generalizing n-normed spaces ($\beta = 1$) and β-normed spaces ($n = 1$) (Ref. [15–17]).

Recent research has expanded classical stability results to non-Archimedean (n, β)-normed spaces, analyzing quadratic, cubic, and quartic equations through novel fixed point methods and contractive conditions. These advancements offer enhanced understanding of the structure of functional equations in the context of perturbations within ultrametric environments ([18–27]).

This paper aims to investigate the Hyers-Ulam stability of a generalized quartic functional equation within non-Archimedean (n, β)-normed spaces. By employing direct analytical methods alongside fixed point techniques, we derive new stability results and outline conditions for the existence and uniqueness of quartic solutions. The findings present a notable extension of current research and pave the way for future investigations in abstract analysis and p-adic functional theory.

The purpose of this study is to explore the Hyers-Ulam stability of the generalized quartic functional equation

$$\begin{aligned}\phi\left(\sum_{i=1}^{r} \mathsf{t}_i\right) &= \sum_{1\leq i<j<k<l\leq r} \phi(\mathsf{t}_i + \mathsf{t}_j + \mathsf{t}_k + \mathsf{t}_l) + (-r+4) \sum_{1\leq i<j<k\leq r} \phi(\mathsf{t}_i + \mathsf{t}_j + \mathsf{t}_k) \\ &\quad + \left(\frac{r^2-7r+12}{2}\right) \sum_{1=i;i\neq j}^{r} \phi(\mathsf{t}_i + \mathsf{t}_j) - \sum_{i=1}^{r} \phi(2\mathsf{t}_i)\end{aligned}$$

$$+\left(\frac{-r^3+9r^2-26r+120}{6}\right)\sum_{i=1}^{r}\left(\frac{\phi(\mathsf{t}_i)+\phi(-\mathsf{t}_i)}{2}\right), \tag{1}$$

where $r \geq 4$, in non-Archimedean (n, β)-normed spaces.

2. Preliminaries

The following are some ideas and findings that will be utilized in the upcoming sections. Here, we denote $\mathbb{N}$ as the set of non-negative integers, with $n, t, p, i \in \mathbb{N}$, and fix $2 \leq n$.

Definition 1. *[28] Let E be a linear space with dim $E \geq n$, and let $0 < \beta \leq 1$. A mapping $\|\cdot, \cdots, \cdot\|_\beta : E^n \to \mathbb{R}$ is called an (n, β)-norm on E if it satisfies the following conditions for every $\nu_1, \cdots, \nu_n, u, \mathsf{t} \in E$ and every $\lambda \in \mathbb{R}$:*

(i) $\|\nu_1, \cdots, \nu_n\|_\beta = 0$ *if and only if* $\nu_1, \cdots, \nu_n$ *are linearly dependent;*

(ii) $\|\nu_1, \cdots, \nu_n\|_\beta$ *is invariant under any permutations of its arguments;*

(iii) $\|\lambda\nu_1, \cdots, \nu_n\|_\beta = |\lambda|^\beta \ \|\nu_1, \cdots, \nu_n\|_\beta$;

(iv) $\|\nu_1, \cdots, \nu_{n-1}, u + \mathsf{t}\|_\beta \leq \|\nu_1, \cdots, \nu_{n-1}, u\|_\beta + \|\nu_1, \cdots, \nu_{n-1}, \mathsf{t}\|_\beta$.

The pair $(E, \|\cdot, \cdots, \cdot\|_\beta)$ is then called a linear (n, β)-normed spaces or simply (n, β)-normed space.

The linear (n, β)-normed space is defined as an integration of a linear n-normed space, applicable when $\beta = 1$, and a β-normed space, relevant when $n = 1$.

Definition 2. *[28] Let E be a real vector space over a scalar field K equipped with a non-Archimedean non-trivial valuation $|\cdot|$, and assume that dim $E \geq n$, where $n \in \mathbb{Z}^+$. Let $0 < \beta \leq 1$ be a fixed constant. A function $\|\cdot, \cdots, \cdot\|_\beta : E^n \to \mathbb{R}$ is called an (n, β)-norm on E if it satisfies the following conditions for all $\lambda \in K$ and all $\nu_0, \nu_1, \cdots, \nu_n \in E$.*

(a) $\|\nu_1, \cdots, \nu_n\|_\beta = 0$ *if and only if* $\nu_1, \cdots, \nu_n$ *are linearly dependent;*

(b) $\|\nu_1, \cdots, \nu_n\|_\beta$ *is invariant under permutations of its arguments;*

(c) $\|\lambda\nu_1, \cdots, \nu_n\|_\beta = |\lambda|^\beta \ \|\nu_1, \cdots, \nu_n\|_\beta$;

(d) $\|\nu_0 + \nu_1, \cdots, \nu_n\|_\beta \leq \max\{\|\nu_0, \nu_2, \cdots, \nu_n\|_\beta, \|\nu_1, \nu_2 \cdots, \nu_n\|_\beta\}$.

If these conditions are satisfies, then the pair $(E, \|\cdot, \cdots, \cdot\|_\beta)$ is called a non-Archimedean (n, β)-normed space.

Example 1. *[25] Let p denote a prime integer. If $x = \frac{a}{b}p^r$ is a nonzero rational number, where a and b are coprime to the prime number p, then the p-adic absolute value is defined as $\|x\|_p := p^r$, and $r \in Z$. On $\mathbb{Q}$, the norm $\|\cdot\|_p$ is classified as a non-Archimedean norm. The field $\mathbb{Q}_p$ represents the completion of the rational numbers $\mathbb{Q}$ under the p-adic norm $\|\cdot\|_p$. The p-adic number field is also referred to as such.*

If $p > 3$, then $\|2^n\| = 1$ for all integer n.

Remark 1. *[28] A non-Archimedean (n, β) containing a sequence $\{\mathsf{t}_m\}$ if and only if the negative absolute value of t_{m+1} converges to zero, then normed space E is a Cauchy sequence.*

Lemma 1. *[28] Consider $\{\mathsf{t}_p\}$ is a convergent sequence in a linear (n, β)-normed space E,*

$$\lim_{p\to\infty} \|\mathsf{t}_p, \kappa_1, \kappa_2, \cdots, \kappa_{n-1}\|_\beta = \left\| \lim_{p\to\infty} \mathsf{t}_p, \kappa_1, \kappa_2, \cdots, \kappa_{n-1} \right\|_\beta$$

for all $\kappa_1, \kappa_2, \cdots, \kappa_{n-1} \in E$.

Lemma 2. *[28] Let $(E, \|\cdot, \cdots, \cdot\|_\beta)$ be a linear (n, β)-normed space, $0 < \beta \leq 1$ and $n \geq 2$. If $\mathsf{t}_1 \in E$ and $\|\mathsf{t}_1, \kappa_1, \cdots, \kappa_{n-1}\|_\beta = 0$ for all $\kappa_1, \cdots, \kappa_{n-1} \in E$, then $\mathsf{t}_1 = 0$.*

Theorem 1. *[29] If a mapping $\phi : E \to F$ satisfies the functional equation (1) for all $\mathsf{t}_1, \mathsf{t}_2, \cdots, \mathsf{t}_r \in E$, then the function $\phi : E \to F$ is quartic.*

2.1. Structural Examples and Fundamental Properties of Non-Archimedean (n, β)-Normed Spaces

In this subsection, we provide illustrative examples to demonstrate the structure and behavior of non-Archimedean (n, β)-normed spaces. These examples underline how such spaces extend the classical notions of normed vector spaces under ultrametric constraints.

Example 2. *[28, 30, 31] Let $X = \mathbb{K}^n$ be the n-dimensional vector space over a non-Archimedean field $\mathbb{K}$. Define the mapping $\|\cdot, \ldots, \cdot\|_\beta : X^n \to \mathbb{R}_+$ as*

$$\|\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n\|_\beta := |\det(\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n)|^\beta,$$

where $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ are vectors in X and the determinant is computed by treating them as rows of an $n \times n$ matrix. It can be verified that this function satisfies all the conditions of a non-Archimedean (n, β)-norm.

Example 3. *[28, 30, 31] Consider the vector space $X = c_0(\mathbb{K})$, the space of sequences converging to zero over a non-Archimedean field $\mathbb{K}$. Define the (n, β)-norm by*

$$\|\mathsf{x}_1, \ldots, \mathsf{x}_n\|_\beta := \sup_{\mathsf{m}\in\mathbb{N}} \left| \det\left(\mathsf{x}_1^{(\mathsf{m})}, \mathsf{x}_2^{(\mathsf{m})}, \ldots, \mathsf{x}_n^{(\mathsf{m})} \right) \right|^\beta,$$

where $\mathsf{x}_i^{(m)}$ denotes the m-th component of the sequence x_i. This function defines a valid (n, β)-norm due to the ultrametric inequality and properties of determinants over $\mathbb{K}$.

We now list some fundamental properties that hold in any non-Archimedean (n, β)-normed space $(X, \|\cdot, \ldots, \cdot\|_\beta)$.

Proposition 1. *[28] Let $\mathsf{x}_1, \ldots, \mathsf{x}_n, \mathsf{y} \in X$. Suppose that X is non-Archimedean (n, β)-normed space. Then:*

(i) If x_1 *is linearly dependent on* $\{\mathsf{x}_2, \ldots, \mathsf{x}_n\}$*, then* $\|\mathsf{x}_1, \ldots, \mathsf{x}_n\|_\beta = 0$.

(ii) If all vectors $\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n$ *are linearly independent, then* $\|\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n\|_\beta > 0$.

(iii) The (n, β)*-norm is symmetric in all arguments.*

(iv) For any scalar $\zeta \in \mathbb{K}$*,*

$$\|\zeta\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n\|_\beta = |\zeta|^\beta \cdot \|\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n\|_\beta.$$

(v) The strong triangle inequality holds:

$$\|\mathsf{x}_1 + \mathsf{y},\ \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n\|_\beta \leq \max\left\{\|\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n\|_\beta, \|\mathsf{y}, \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n\|_\beta\right\}.$$

3. Stability of the Generalized Quartic Functional Equation

Consider E as a vector space and $(F, \|\cdot, \ldots, \cdot\|_\beta)$ as an element of it. Rest assured that the space (n, β) is non-Archimedean, with $n \geq 2$ and $0 < \beta,\ \beta_1 \leq 1$.

We consider the generalized quartic functional equation (1) defined via the following difference operator:

$$\begin{aligned}
\Delta\phi(\mathsf{t}_1, \ldots, \mathsf{t}_r) \quad = \quad & -\phi\left(\sum_{i=1}^{r} \mathsf{t}_i\right) + \sum_{1 \leq i < j < k < l \leq r} \phi(\mathsf{t}_i + \mathsf{t}_j + \mathsf{t}_k + \mathsf{t}_l) \\
& + (-r + 4) \sum_{1 \leq i < j < k \leq r} \phi(\mathsf{t}_i + \mathsf{t}_j + \mathsf{t}_k) \\
& + \left(\frac{r^2 - 7r + 12}{2}\right) \sum_{i=1,\, i \neq j}^{r} \phi(\mathsf{t}_i + \mathsf{t}_j) - \sum_{i=1}^{r} \phi(2\mathsf{t}_i) \\
& + \left(\frac{-r^3 + 9r^2 - 26r + 120}{6}\right) \sum_{i=1}^{r} \left(\frac{\phi(\mathsf{t}_i) + \phi(-\mathsf{t}_i)}{2}\right)
\end{aligned}$$

for any $\mathsf{t}_1, \ldots, \mathsf{t}_r \in E$, with $r \geq 4$.

Theorem 2. *Let* $\mu \in [0, \infty)$ *and* $s \in (0, \infty)$ *with* $s\beta_1 > \beta$*, and let* $\varpi : F^{n-1} \to [0, \infty)$ *be a control function. Assume that* $\phi : E \to F$ *is a function such that*

$$\|\Delta\phi\,(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_r)\,, \nu_1, \ldots, \nu_{n-1}\|_\beta \leq \mu \sum_{j=1}^{r} \|\mathsf{t}_j\|_{\beta_1}^{s}\, \varpi(\nu_1, \ldots, \nu_{n-1}) \tag{2}$$

for all $\mathsf{t}_1, \ldots, \mathsf{t}_r \in E$ *and* $\nu_1, \ldots, \nu_{n-1} \in F$*. Then there exists a unique quartic mapping* $Q_4 : E \to F$ *satisfying*

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\|_\beta \leq \mu|2^{-4\beta}|\, \|\mathsf{t}\|_{\beta_1}^{s}\, \varpi(\nu_1, \ldots, \nu_{n-1}) \tag{3}$$

for all $\mathsf{t} \in E$ *and all* $\nu_1, \ldots, \nu_{n-1} \in F$*.*

Proof. Replacing $(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_r)$ by $(\mathsf{t}, 0, \ldots, 0)$ in (2), we obtain

$$\left\|\phi(2\mathsf{t}) - 2^4\phi(\mathsf{t}),\ \nu_1, \ldots, \nu_{n-1}\right\|_\beta \le \mu \,\|\mathsf{t}\|_{\beta_1}^s \,\varpi(\nu_1, \ldots, \nu_{n-1}). \tag{4}$$

Dividing both sides of (4) by $|2^{4\beta}|$ gives

$$\left\|\frac{\phi(2\mathsf{t})}{2^4} - \phi(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta \le |2^{-4\beta}|\mu \,\|\mathsf{t}\|_{\beta_1}^s \,\varpi(\nu_1, \ldots, \nu_{n-1}). \tag{5}$$

Replacing t by $2^p\mathsf{t}$ in (5) yields

$$\begin{aligned}\left\|\frac{\phi(2^{p+1}\mathsf{t})}{2^{4(p+1)}} - \frac{\phi(2^p\mathsf{t})}{2^{4p}}, \nu_1, \ldots, \nu_{n-1}\right\|_\beta &\le |2^{-4(p+1)\beta}|\mu\|2^p\mathsf{t}\|_{\beta_1}^s \varpi(\nu_1, \ldots, \nu_{n-1}) \\ &\le |2^{-4\beta}| \left|2^{s\beta_1 - 4\beta}\right|^p \mu\|\mathsf{t}\|_{\beta_1}^s \varpi(\nu_1, \ldots, \nu_{n-1}). \end{aligned} \tag{6}$$

Since $s\beta_1 > \beta$ and $|2| \neq 1$, the R.H.S. of (6) tends to zero as $p \to \infty$. Hence, the sequence $\left\{\frac{\phi(2^p\mathsf{t})}{2^{4p}}\right\}$ is Cauchy in F, which is complete. Therefore, we define

$$Q_4(\mathsf{t}) := \lim_{p\to\infty} \frac{\phi(2^p\mathsf{t})}{2^{4p}} \quad \text{for all } \mathsf{t} \in E. \tag{7}$$

To prove that Q_4 is quartic, apply (2) and Lemma 1:

$$\begin{aligned}\|\Delta Q_4(\mathsf{t}_1, \ldots, \mathsf{t}_r), \nu_1, \ldots, \nu_{n-1}\|_\beta &= \lim_{p\to\infty} \left|2^{-4p\beta}\right| \|\Delta\phi(2^p\mathsf{t}_1, \ldots, 2^p\mathsf{t}_r), \nu_1, \ldots, \nu_{n-1}\|_\beta \\ &\le \lim_{p\to\infty} \mu \left|2^{s\beta_1 - 4\beta}\right|^p \sum_{j=1}^r \|\mathsf{t}_j\|_{\beta_1}^s \varpi(\nu_1, \ldots, \nu_{n-1}) = 0.\end{aligned}$$

Thus, by Lemma 2, Q_4 satisfies $\Delta Q_4 = 0$, and so Q_4 is quartic.

To estimate the difference between ϕ and Q_4, we observe from (5) and similar recursive steps that

$$\left\|\phi(\mathsf{t}) - \frac{\phi(2^p\mathsf{t})}{2^{4p}}, \nu_1, \ldots, \nu_{n-1}\right\|_\beta \le |2^{-4\beta}|\mu\|\mathsf{t}\|_{\beta_1}^s \varpi(\nu_1, \ldots, \nu_{n-1}). \tag{8}$$

Letting $p \to \infty$ in (8) and applying the definition of Q_4, we obtain the inequality (3).

Finally, we show uniqueness. Consider another quartic mapping Q_4' satisfying (3). Then

$$\begin{aligned}\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta &= \left|2^{-4p\beta}\right| \left\|Q_4(2^p\mathsf{t}) - Q_4'(2^p\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta \\ &\le \left|2^{-4p\beta}\right| \cdot \max\Big\{\|Q_4(2^p\mathsf{t}) - \phi(2^p\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\|_\beta, \\ &\qquad \left\|\phi(2^p\mathsf{t}) - Q_4'(2^p\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta\Big\} \\ &\le \mu \left|2^{-4\beta}\right| \left|2^{s\beta_1 - 4\beta}\right|^p \|\mathsf{t}\|_{\beta_1}^s \varpi(\nu_1, \ldots, \nu_{n-1}).\end{aligned}$$

Taking the limit as $p \to \infty$, we conclude $Q_4(\mathsf{t}) = Q_4'(\mathsf{t})$ for all $\mathsf{t} \in E$. Hence, Q_4 is the only one quartic function satisfying (3).

Theorem 3. *Let* $\mu \in [0,\infty)$ *and* $s \in (0,\infty)$ *with* $s\beta_1 < \beta$. *Let* $\varpi : F^{n-1} \to [0,\infty)$ *be a control function. Assume that the mapping* $\phi : E \to F$ *satisfies*

$$\|\Delta\phi(\mathsf{t}_1,\ldots,\mathsf{t}_r),\nu_1,\ldots,\nu_{n-1}\|_\beta \le \mu\sum_{j=1}^{r}\|\mathsf{t}_j\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}) \tag{9}$$

for all $\mathsf{t}_1,\ldots,\mathsf{t}_r \in E$ *and all* $\nu_1,\ldots,\nu_{n-1} \in F$. *Then there exists a unique quartic function* $Q_4 : E \to F$ *satisfying*

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t}),\nu_1,\ldots,\nu_{n-1}\|_\beta \le \mu|2^{-s\beta_1}|\,\|\mathsf{t}\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}) \tag{10}$$

for all $\mathsf{t} \in E$ *and all* $\nu_1,\ldots,\nu_{n-1} \in F$.

Proof. Replacing $(\mathsf{t}_1,\ldots,\mathsf{t}_r)$ by $(\mathsf{t},0,\ldots,0)$ in (9), we have

$$\left\|\phi(2\mathsf{t}) - 2^4\phi(\mathsf{t}),\nu_1,\ldots,\nu_{n-1}\right\|_\beta \le \mu\,\|\mathsf{t}\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}). \tag{11}$$

Replacing t by $\mathsf{t}/2$ in (11), we obtain

$$\left\|\phi(\mathsf{t}) - 2^4\phi\left(\frac{\mathsf{t}}{2}\right),\nu_1,\ldots,\nu_{n-1}\right\|_\beta \le \mu|2^{-s\beta_1}|\,\|\mathsf{t}\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}). \tag{12}$$

Switching t by $\mathsf{t}/2^p$ in (12), we have

$$\left\|2^{4p}\phi\left(\frac{\mathsf{t}}{2^p}\right) - 2^{4(p+1)}\phi\left(\frac{\mathsf{t}}{2^{p+1}}\right),\nu_1,\ldots,\nu_{n-1}\right\|_\beta \le \mu|2^{-s\beta_1}|\left|2^{4\beta-s\beta_1}\right|^p\|\mathsf{t}\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}). \tag{13}$$

Since $s\beta_1 < \beta$ and $|2| \neq 1$, the R.H.S. of (13) tends to zero as $p \to \infty$. Thus, the sequence $\{2^{4p}\phi(\mathsf{t}/2^p)\}$ is Cauchy in F, which is complete. Hence, define

$$Q_4(\mathsf{t}) := \lim_{p\to\infty} 2^{4p}\phi\left(\frac{\mathsf{t}}{2^p}\right) \tag{14}$$

for all $\mathsf{t} \in E$.

We now show that Q_4 is quartic. From (9) and Lemma 1, we get

$$\begin{aligned}\|\Delta Q_4(\mathsf{t}_1,\ldots,\mathsf{t}_r),\nu_1,\ldots,\nu_{n-1}\|_\beta &= \lim_{p\to\infty}\left|2^{4p\beta}\right|\left\|\Delta\phi\left(\frac{\mathsf{t}_1}{2^p},\ldots,\frac{\mathsf{t}_r}{2^p}\right),\nu_1,\ldots,\nu_{n-1}\right\|_\beta\\ &\le \lim_{p\to\infty}\mu\left|2^{4\beta-s\beta_1}\right|^p\sum_{j=1}^{r}\|\mathsf{t}_j\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}) = 0.\end{aligned}$$

Hence, by Lemma 2, the function Q_4 is quartic.

To prove inequality (10), note from (12) and similar reasoning (induction or recursion) that

$$\left\|\phi(\mathsf{t}) - 2^{4p}\phi\left(\frac{\mathsf{t}}{2^p}\right),\nu_1,\ldots,\nu_{n-1}\right\|_\beta \le \mu|2^{-s\beta_1}|\,\|\mathsf{t}\|_{\beta_1}^{s}\,\varpi(\nu_1,\ldots,\nu_{n-1}). \tag{15}$$

Taking the limit as $p \to \infty$ in (15) and using the definition of Q_4 in (14), we obtain (10).

To prove uniqueness, assume that another quartic function $Q_4' : E \to F$ satisfying (10). Then

$$\begin{aligned}\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \dots, \nu_{n-1}\right\|_\beta &= \left|2^{4p\beta}\right| \left\|Q_4\left(\frac{\mathsf{t}}{2^p}\right) - Q_4'\left(\frac{\mathsf{t}}{2^p}\right), \nu_1, \dots, \nu_{n-1}\right\|_\beta \\ &\leq \left|2^{4p\beta}\right| \max\left\{ \left\|Q_4\left(\frac{\mathsf{t}}{2^p}\right) - \phi\left(\frac{\mathsf{t}}{2^p}\right)\right\|_\beta, \right. \\ &\qquad \left. \left\|\phi\left(\frac{\mathsf{t}}{2^p}\right) - Q_4'\left(\frac{\mathsf{t}}{2^p}\right)\right\|_\beta \right\} \\ &\leq \mu|2^{-s\beta_1}| \left|2^{4\beta - s\beta_1}\right|^p \|\mathsf{t}\|_{\beta_1}^s \varpi(\nu_1, \dots, \nu_{n-1}).\end{aligned}$$

As $p \to \infty$, the R.H.S. tends to zero. Thus,

$$\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \dots, \nu_{n-1}\right\|_\beta = 0,$$

which implies, by Lemma 2, that $Q_4 = Q_4'$. Hence, Q_4 is unique.

Theorem 4. *Let a function $\psi : E^r \to [0, \infty)$ such that*

$$\lim_{p\to\infty} \left|\frac{1}{2^{4p\beta}}\right| \psi\left(2^p \mathsf{t}_1, \dots, 2^p \mathsf{t}_r\right) = 0 \tag{16}$$

for all $\mathsf{t}_1, \dots, \mathsf{t}_r \in E$, and let $\varpi : F^{n-1} \to [0, \infty)$ be a control function. Assume that the mapping $\phi : E \to F$ satisfies

$$\|\Delta\phi(\mathsf{t}_1, \dots, \mathsf{t}_r), \nu_1, \dots, \nu_{n-1}\|_\beta \leq \psi(\mathsf{t}_1, \dots, \mathsf{t}_r)\, \varpi(\nu_1, \dots, \nu_{n-1}) \tag{17}$$

for all $\mathsf{t}_1, \dots, \mathsf{t}_r \in E$ and $\nu_1, \dots, \nu_{n-1} \in F$. Then there exists a unique quartic mapping $Q_4 : E \to F$ satisfying

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t}), \nu_1, \dots, \nu_{n-1}\|_\beta \leq \tilde{\psi}(\mathsf{t})\, \varpi(\nu_1, \dots, \nu_{n-1}) \tag{18}$$

where

$$\tilde{\psi}(\mathsf{t}) := \lim_{p\to\infty} \max\left\{\left|2^{-4i\beta}\right| \psi\left(2^{i-1}\mathsf{t}, 0, 0, \dots, 0\right) : 1 \leq i \leq p\right\} \tag{19}$$

for all $\mathsf{t} \in E$. Moreover, if

$$\lim_{t\to\infty} \lim_{p\to\infty} \max\left\{\left|2^{-4i\beta}\right| \psi\left(2^{i-1}\mathsf{t}, 0, 0, \dots, 0\right) :\ 1 + t \leq i \leq p + t\right\} = 0 \tag{20}$$

for every $\mathsf{t} \in E$, then the function Q_4 is unique.

Proof. Replacing $(\mathsf{t}_1, \dots, \mathsf{t}_r)$ by $(\mathsf{t}, 0, \dots, 0)$ in (17) and dividing both sides by $|2^{4\beta}|$ gives

$$\left\|\frac{\phi(2\mathsf{t})}{2^4} - \phi(\mathsf{t}), \nu_1, \dots, \nu_{n-1}\right\|_\beta \leq |2^{-4\beta}|\, \psi(\mathsf{t}, 0, \dots, 0)\, \varpi(\nu_1, \dots, \nu_{n-1}). \tag{21}$$

Replacing t with $2^i\mathsf{t}$ and dividing by $|2^{4i\beta}|$, we get

$$\left\| \frac{\phi(2^{i+1}\mathsf{t})}{2^{4(i+1)}} - \frac{\phi(2^i\mathsf{t})}{2^{4i}}, \nu_1, \ldots, \nu_{n-1} \right\|_\beta \leq |2^{-4(i+1)\beta}|\, \psi\left(2^i\mathsf{t}, 0, \ldots, 0\right)\, \varpi(\nu_1, \ldots, \nu_{n-1}). \quad (22)$$

By (16), the right-hand side tends to zero as $i \to \infty$, so the sequence $\left\{ \frac{\phi(2^m\mathsf{t})}{2^{4m}} \right\}$ is Cauchy. As F is complete, define

$$Q_4(\mathsf{t}) := \lim_{m\to\infty} \frac{\phi(2^m\mathsf{t})}{2^{4m}}.$$

We next show Q_4 is quartic. From (17), Lemma 1, and the definition of Q_4, we have:

$$\begin{aligned}\|\Delta Q_4(\mathsf{t}_1, \ldots, \mathsf{t}_r), \nu_1, \ldots, \nu_{n-1}\|_\beta &= \lim_{p\to\infty} \left\| \frac{1}{2^{4p}} \Delta\phi\left(2^p\mathsf{t}_1, \ldots, 2^p\mathsf{t}_r\right), \nu_1, \ldots, \nu_{n-1} \right\|_\beta \\ &\leq \lim_{p\to\infty} |2^{-4p\beta}|\, \psi\left(2^p\mathsf{t}_1, \ldots, 2^p\mathsf{t}_r\right)\, \varpi(\nu_1, \ldots, \nu_{n-1}) = 0.\end{aligned}$$

Hence, by Lemma 2, Q_4 is quartic.

From (21), we obtain:

$$\left\| \phi(\mathsf{t}) - \frac{\phi(2^4\mathsf{t})}{2^8}, \nu_1, \ldots, \nu_{n-1} \right\|_\beta \leq \max\left\{ |2^{-4\beta}|\, \psi(\mathsf{t}, 0, \ldots, 0),\ |2^{-8\beta}|\, \psi(2\mathsf{t}, 0, \ldots, 0) \right\} \varpi(\nu_1, \ldots, \nu_{n-1}).$$

Inductively, for all $p \in \mathbb{N}$, we get:

$$\left\| \phi(\mathsf{t}) - \frac{\phi(2^p\mathsf{t})}{2^{4p}}, \nu_1, \ldots, \nu_{n-1} \right\|_\beta \leq \max\left\{ \left|2^{-4t\beta}\right|\, \psi\left(2^{t-1}\mathsf{t}, 0, \ldots, 0\right) : 1 \leq t \leq p \right\} \varpi(\nu_1, \ldots, \nu_{n-1}). \quad (23)$$

Letting $p \to \infty$ in (23), we obtain (18) by the definition of $\tilde{\psi}(\mathsf{t})$ in (19).

To prove uniqueness, suppose another quartic mapping Q_4' also satisfies (18). Then

$$\begin{aligned}&\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta \\ &= \left|2^{-4t\beta}\right| \left\|Q_4(2^t\mathsf{t}) - Q_4'(2^t\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta \\ &\leq \left|2^{-4t\beta}\right| \max\left\{ \left\|Q_4(2^t\mathsf{t}) - \phi(2^t\mathsf{t})\right\|_\beta,\ \left\|\phi(2^t\mathsf{t}) - Q_4'(2^t\mathsf{t})\right\|_\beta \right\} \\ &\leq \left|2^{-4t\beta}\right| \tilde{\psi}(2^t\mathsf{t})\, \varpi(\nu_1, \ldots, \nu_{n-1}).\end{aligned}$$

By assumption (20), the last term tends to zero as $t \to \infty$. Therefore,

$$\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta = 0,$$

and by Lemma 2, we conclude $Q_4 = Q_4'$. Thus, Q_4 is unique.

Theorem 5. *Let $\psi : E^r \to [0,\infty)$ be a control function satisfying*

$$\lim_{p\to\infty} \left|2^{4p\beta}\right| \psi\left(\frac{\mathsf{t}_1}{2^p}, \frac{\mathsf{t}_2}{2^p}, \dots, \frac{\mathsf{t}_r}{2^p}\right) = 0 \tag{24}$$

for all $\mathsf{t}_1, \dots, \mathsf{t}_r \in E$, and let $\varpi : F^{n-1} \to [0,\infty)$ be a control mapping. Assume that a function $\phi : E \to F$ satisfies

$$\|\Delta\phi(\mathsf{t}_1, \dots, \mathsf{t}_r), \nu_1, \dots, \nu_{n-1}\|_\beta \le \psi(\mathsf{t}_1, \dots, \mathsf{t}_r)\varpi(\nu_1, \dots, \nu_{n-1}) \tag{25}$$

for all $\mathsf{t}_1, \dots, \mathsf{t}_r \in E$ and $\nu_1, \dots, \nu_{n-1} \in F$. Then there exists a unique quartic mapping $Q_4 : E \to F$ satisfying

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t}), \nu_1, \dots, \nu_{n-1}\|_\beta \le \tilde{\psi}(\mathsf{t})\varpi(\nu_1, \dots, \nu_{n-1}), \tag{26}$$

where

$$\tilde{\psi}(\mathsf{t}) := \lim_{p\to\infty} \max\left\{\left|2^{4(i-1)\beta}\right| \psi\left(2^{-i}\mathsf{t}, 0, 0, \dots, 0\right) : 1 \le i \le p\right\}. \tag{27}$$

Moreover, if

$$\lim_{t\to\infty}\lim_{p\to\infty} \max\left\{\left|2^{4(i-1)\beta}\right| \psi\left(2^{-i}\mathsf{t}, 0, 0, \dots, 0\right) : 1 + t \le i \le p + t\right\} = 0 \tag{28}$$

for all $\mathsf{t} \in E$, then the quartic mapping Q_4 is unique.

Proof. Setting $(\mathsf{t}_1, \mathsf{t}_2, \dots, \mathsf{t}_r)$ by $(\mathsf{t}, 0, \dots, 0)$ in (25), we obtain

$$\left\|\phi(2\mathsf{t}) - 2^4\phi(\mathsf{t}), \nu_1, \nu_2, \dots, \nu_{n-1}\right\|_\beta \le \psi(\mathsf{t}, 0, \dots, 0)\varpi(\nu_1, \nu_2, \dots, \nu_{n-1}). \tag{29}$$

Replacing t by $\frac{\mathsf{t}}{2}$ and multiplying by $|2^{4\beta}|$ repeatedly, we define a sequence:

$$\left\{2^{4p}\phi\left(\frac{\mathsf{t}}{2^p}\right)\right\}. \tag{30}$$

Using (24) and similar arguments as in Theorem 4, we conclude this sequence is Cauchy in F and hence convergent, due to completeness. Define

$$Q_4(\mathsf{t}) := \lim_{p\to\infty} 2^{4p}\phi\left(\frac{\mathsf{t}}{2^p}\right), \quad \text{for all } \mathsf{t} \in E.$$

Following the structure of (30), we can show

$$\|Q_4(\mathsf{t}) - \phi(\mathsf{t}), \nu_1, \nu_2, \dots, \nu_{n-1}\|_\beta \le \tilde{\psi}(\mathsf{t})\varpi(\nu_1, \nu_2, \dots, \nu_{n-1}),$$

establishing (26).

To prove that Q_4 is quartic, observe that:

$$\|\Delta Q_4(\mathsf{t}_1, \dots, \mathsf{t}_r), \nu_1, \dots, \nu_{n-1}\|_\beta = \lim_{p\to\infty} \left|2^{4p\beta}\right| \left\|\Delta\phi\left(\frac{\mathsf{t}_1}{2^p}, \dots, \frac{\mathsf{t}_r}{2^p}\right), \nu_1, \dots, \nu_{n-1}\right\|_\beta,$$

which tends to zero by (24), hence $\Delta Q_4 = 0$ and Q_4 is quartic.

For uniqueness, suppose another quartic mapping Q_4' satisfies (26). Then

$$\left\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\right\|_\beta = \left|2^{4t\beta}\right| \left\|Q_4\left(\frac{\mathsf{t}}{2^t}\right) - Q_4'\left(\frac{\mathsf{t}}{2^t}\right), \nu_1, \ldots, \nu_{n-1}\right\|_\beta$$
$$\leq \left|2^{4t\beta}\right| \tilde{\psi}\left(\frac{\mathsf{t}}{2^t}\right) \varpi(\nu_1, \ldots, \nu_{n-1}) \to 0$$

as $t \to \infty$ by assumption (28). Hence, $\|Q_4(\mathsf{t}) - Q_4'(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\|_\beta = 0$ for all $\mathsf{t} \in E$, implying $Q_4 = Q_4'$ by Lemma 2.

4. Consequences and Illustrative Example in $\mathbb{Q}_p$ Corollaries and Examples

Corollary 4.1 (Classical Non-Archimedean Case)

Let E and F be non-Archimedean normed spaces, i.e., (n, β)-normed spaces with $n = 1$, $\beta = 1$. Assume that a mapping $\phi : E \to F$ fulfills

$$\|\Delta\phi(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_r)\| \leq \mu \sum_{j=1}^{r} \|\mathsf{t}_j\|^s$$

for some constants $\mu \geq 0$, $s > 1$, and all $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_r \in E$. Then there is a unique quartic mapping $Q_4 : E \to F$ fulfilling

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t})\| \leq \mu|2^{-4}|\|\mathsf{t}\|^s, \quad \text{for every } \mathsf{t} \in E.$$

Corollary 4.2 (Stability in β-Normed Ultrametric Spaces)

Let E be a β-normed space and F a complete non-Archimedean (n, β)-normed space with $0 < \beta < 1$. If a function $\phi : E \to F$ satisfies

$$\|\Delta\phi(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_r), \nu_1, \ldots, \nu_{n-1}\|_\beta \leq \mu \sum_{j=1}^{r} \|\mathsf{t}_j\|^{s\beta}$$

for all $\mathsf{t}_j \in E$, $\nu_k \in F$, then there is a unique quartic function $Q_4 : E \to F$ fulfilling

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t}), \nu_1, \ldots, \nu_{n-1}\|_\beta \leq \mu|2^{-4\beta}|\|\mathsf{t}\|^{s\beta}.$$

Example 4.3 (Mapping on a $\mathbb{Q}_p$ Space)

Let $p > 3$ be a prime number and $E = \mathbb{Q}_p$, the field of p-adic numbers. Define the function $\phi : \mathbb{Q}_p \to \mathbb{Q}_p$ by

$$\phi(\mathsf{t}) = \mathsf{t}^4 + \epsilon(\mathsf{t}),$$

where $|\epsilon(\mathsf{t})|_p \leq \delta|\mathsf{t}|_p^s$ for some $\delta > 0$ and $s > 4$. Then ϕ satisfies the condition of Theorem 2 for suitable μ, and there is only one quartic function $Q_4(\mathsf{t}) = \mathsf{t}^4$ fulfilling

$$\|\phi(\mathsf{t}) - Q_4(\mathsf{t})\|_\beta \leq \mu\|t\|^s.$$

5. Conclusion

This work examines the Hyers-Ulam stability of a generalized quartic functional equation within non-Archimedean (n, β)-normed spaces. These spaces, which generalize traditional normed and ultrametric structures, offer a comprehensive framework for examining the behaviour of functional equations under perturbations.

Theorem 2 and Theorem 3 examined stability in the context of a non-Archimedean β_1-normed space as the domain and a full non-Archimedean (n, β)-normed space as the codomain. Theorem 4 and Theorem 5 broadened these findings to encompass more complex control functions, hence permitting enhanced flexibility in the assumptions regarding perturbations.

Our results establish the existence and uniqueness of quartic mappings that resemble the original functional equation, therefore validating its Ulam-type stability in this extended non-Archimedean context.

6. Conflict of interest

The authors declare that they have no competing interests.

Availability of data and materials

Not applicable.

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Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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