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# On a Strengthened of the More Accurate Hilbert's Inequality

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**Abstract.** By deducing the inequality of weight coefficient:

$$\omega(n)=\sum_{m=0}^{\infty}\frac{1}{m+n+1}\left(\frac{2n+1}{2m+1}\right)^{\frac{1}{2}}<\pi-\frac{5}{6(\sqrt{2n+1}+\frac{3}{4}\sqrt{(2n+1)^{-1}})},$$

where $n\in N$. We obtain on a strengthened of the more accurate Hilbert's inequality.

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**Key Words and Phrases**: Hilbert's inequality, weight coefficient, Cauchy's inequality, strengthen

## 1. Introduction

Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n\geq 0$, $b_n\geq 0$, and $0<\sum\limits_{n=1-\lambda}^{\infty}a_n^p<\infty$, $0<\sum\limits_{n=1-\lambda}^{\infty}b_n^q<\infty$, $(\lambda=0,1)$, then

$$\sum_{n=1-\lambda}^{\infty}\sum_{m=1-\lambda}^{\infty}\frac{a_m b_n}{m+n+\lambda}<\pi\{\sum_{n=1-\lambda}^{\infty}a_n^2\sum_{n=1-\lambda}^{\infty}b_n^2\}^{\frac{1}{2}}, \tag{1}$$

$$\sum_{n=1-\lambda}^{\infty}\sum_{m=1-\lambda}^{\infty}\frac{a_m b_n}{m+n+\lambda}<\frac{\pi}{\sin(\frac{\pi}{p})}\{\sum_{n=1-\lambda}^{\infty}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1-\lambda}^{\infty}b_n^q\}^{\frac{1}{q}}, \tag{2}$$

where, constant $\pi$ and $\frac{\pi}{\sin(\frac{\pi}{p})}$ is best possible. (1) is Hilbert's type inequality . for $\lambda=1$, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1}<\pi\{\sum_{n=0}^{\infty}a_n^2\sum_{n=0}^{\infty}b_n^2\}^{\frac{1}{2}}. \tag{3}$$

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Inequality (3) is named of more accurate Hilbert's inequality. Inequality (2) is Hardy-Hilbert's. For $\lambda = 1$, inequality (2) is named of more accurate Hardy-Hilbert's inequality [1].

In [2], Yang obtained a strengthened of inequality (3):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \{\sum_{n=0}^{\infty}[\pi - \frac{\theta}{(n+1)^{\frac{1}{2}}}]a_n^2\}^{\frac{1}{2}} \cdot \{\sum_{n=0}^{\infty}[\pi - \frac{\theta}{(n+1)^{\frac{1}{2}}}]b_n^2\}^{\frac{1}{2}}, \tag{4}$$

where, $\theta = \pi - \sum_{m=0}^{\infty}\frac{1}{(m+1)^{\frac{3}{2}}} = 0.5292496^{+}$.

In [3], by the following inequality of weight coefficient:

$$\omega(n,r) = \sum_{m=0}^{\infty}\frac{1}{m+n+1}(\frac{2n+1}{2m+1})^{\frac{1}{r}} < \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\theta}{(2n+1)^{2-\frac{1}{r}}},$$

where $r > 1$, $n \in N$, then

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} \quad &< \quad \{\sum_{n=0}^{\infty}[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{\ln 2 - C}{(2n+1)^{1+\frac{1}{p}}}]a_n^p\}^{\frac{1}{p}} \\ &\cdot\{\sum_{n=0}^{\infty}[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{\ln 2 - C}{(2n+1)^{1+\frac{1}{q}}}]b_n^q\}^{\frac{1}{q}},\end{aligned}$$

where $C$ is Euler constant. In particular, for $p = q = 2$, Yang obtained again a strengthened of inequality (3):

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} \quad &< \quad \{\sum_{n=0}^{\infty}[\pi - \frac{\ln 2 - C}{(2n+1)^{\frac{3}{2}}}]a_n^2\}^{\frac{1}{2}} \\ &\cdot\{\sum_{n=0}^{\infty}[\pi - \frac{\ln 2 - C}{(2n+1)^{\frac{3}{2}}}]b_n^2\}^{\frac{1}{2}}.\end{aligned} \tag{5}$$

In this paper, by establishing the inequality of the weight coefficient, we will obtain a strengthened of inequalities (3), (4) and (5).

## 2. Some Lemmas

First of all, we give several lemmas which are to be used later.

**Lemma 1.** *Let* $f^{(2r)}(x) > 0$, $f^{(2r+1)}(x) < 0$, $x \in [0, \infty)$, $f^{(r)}(\infty) = 0$ *(*$r = 0, 1, 2, 3$*),* $\int_0^{\infty} f(x)dx < \infty$. *Then*

$$\sum_{m=0}^{\infty} f(m) < \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{6}$$

*Proof.* See [4] or [5] .

**Lemma 2.** *We have*

$$\begin{aligned}\omega(n) &= \sum_{m=0}^{\infty}\frac{1}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac{1}{2}} \\ &< \pi-\frac{1}{\sqrt{2n+1}}\Big[\frac{5}{6}+\frac{1}{6(2n+1)}-\frac{2}{(2n+1)^2}\Big],\end{aligned}\tag{7}$$

*where* $n\in N$.

*Proof.* Let $f_n(x)=\frac{1}{(x+n+1)}(\frac{2n+1}{2x+1})^{\frac{1}{2}}$, $x\in[0,\infty)$, then

$$f_n(0)=\frac{\sqrt{2n+1}}{n+1}$$

.

$$f_n'(x)=\sqrt{2n+1}\Big[-\frac{1}{(x+n+1)\cdot(2x+1)^{\frac{3}{2}}}-\frac{1}{(x+n+1)^2\cdot(2x+1)^{\frac{1}{2}}}\Big].$$

$$f_n'(0)=\sqrt{2n+1}\Big[-\frac{1}{n+1}-\frac{1}{(n+1)^2}\Big].$$

$$\begin{aligned}\int_0^{\infty}f_n(x)dx &= \int_{\frac{1}{2n+1}}^{\infty}\frac{1}{(y+1)y^{\frac{1}{2}}}dy \\ &= \int_0^{\infty}\frac{1}{(y+1)y^{\frac{1}{2}}}dy-\int_0^{\frac{1}{2n+1}}\frac{1}{(y+1)y^{\frac{1}{2}}}dy \\ &= \pi-2\int_0^{\frac{1}{2n+1}}\frac{1}{y+1}dy^{\frac{1}{2}} \\ &= \pi-2\Big[\frac{\sqrt{2n+1}}{2(n+1)}+\frac{2}{3}\int_0^{\frac{1}{2n+1}}\frac{1}{(y+1)^2}dy^{\frac{3}{2}}\Big] \\ &= \pi-2\Big[\frac{\sqrt{2n+1}}{2(n+1)}+\frac{\sqrt{2n+1}}{6(n+1)^2}+\frac{4}{3}\int_0^{\frac{1}{2n+1}}\frac{1}{(y+1)^3}dy^{\frac{3}{2}}\Big] \\ &< \pi-\Big[\frac{\sqrt{2n+1}}{(n+1)}+\frac{\sqrt{2n+1}}{3(n+1)^2}\Big].\end{aligned}$$

If $\omega(n)=\sum_{m=0}^{\infty}\frac{1}{m+n+1}(\frac{2n+1}{2m+1})^{\frac{1}{2}}$, so $\omega(n)=\sum_{m=0}^{\infty}f_n(m)$. By lemma 1, we have

$$\begin{aligned}\omega(n) &= \sum_{m=0}^{\infty}f_n(m) \\ &< \int_0^{\infty}f_n(x)dx+\frac{1}{2}f_n(0)-\frac{1}{12}f_n'(0)\end{aligned}$$

$$\begin{aligned}
&< \pi - [\frac{\sqrt{2n+1}}{(n+1)} + \frac{\sqrt{2n+1}}{3(n+1)^2}] + \frac{\sqrt{2n+1}}{2(n+1)} + \frac{\sqrt{2n+1}}{12}[\frac{1}{n+1} + \frac{1}{(n+1)^2}] \\
&< \pi - \sqrt{2n+1}[\frac{5}{12(n+1)} + \frac{1}{4(n+1)^2}] \\
&< \pi - \frac{1}{\sqrt{2n+1}}[\frac{5(2n+1)}{12(n+1)} + \frac{2n+1}{4(n+1)^2}].
\end{aligned}$$

For $n \in N$, we have

$$\begin{aligned}
\frac{5(2n+1)}{12(n+1)} + \frac{2n+1}{4(n+1)^2} &= \frac{5}{6}(1+\frac{1}{2n+1})^{-1} + \frac{1}{2n+1}(1+\frac{1}{2n+1})^{-2} \\
&> \frac{5}{6}(1-\frac{1}{2n+1}) + \frac{1}{2n+1}(1-\frac{2}{2n+1}) \\
&> \frac{5}{6} + \frac{1}{6(2n+1)} - \frac{2}{(2n+1)^2}.
\end{aligned}$$

The proof of the lemma is completed.

**Lemma 3.** *We have*

$$\omega(n) = \sum_{m=0}^{\infty} \frac{1}{m+n+1}(\frac{2n+1}{2m+1})^{\frac{1}{2}} < \pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}, \tag{8}$$

*where $n \in N$.*

*Proof.* Since

$$\begin{aligned}
&[\frac{5}{6} + \frac{1}{6(2n+1)} - \frac{2}{(2n+1)^2}](1+\frac{a}{2n+1}) \\
&= \frac{5}{6} + \frac{1}{2n+1}[\frac{5a+1}{6} - \frac{12-a}{6(2n+1)} - \frac{2a}{(2n+1)^2}] \\
&= \frac{5}{6} + \frac{1}{2n+1} \cdot \frac{(5a+1)(2n+1)^2 - (12-a)(2n+1) - 12a}{6(2n+1)^2}.
\end{aligned}$$

For $n = 1$, $a \geq \frac{3}{4}$, we have

$$\begin{aligned}
\frac{(5a+1)(2n+1)^2 - (12-a)(2n+1) - 12a}{6(2n+1)^2} &= \frac{45a + 9 - 36 + 3a - 12a}{54} \\
&= \frac{36a - 27}{150} \\
&\geq 0.
\end{aligned}$$

Then for $n \geq 1, n \in N$ and $a \geq \frac{3}{4}$,

$$[\frac{5}{6} + \frac{1}{6(2n+1)} - \frac{2}{(2n+1)^2}](1+\frac{a}{2n+1}) > \frac{5}{6}.$$

For $a = \frac{3}{4}$, $n = 0$, $\theta = \pi - \sum_{m=0}^{\infty} \frac{1}{(m+1)^{\frac{3}{2}}} = 0.5292496^{+}$, we have

$$\frac{\theta}{(n+1)^{\frac{1}{2}}} > \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})},$$

and

$$\pi - \frac{\theta}{(n+1)^{\frac{1}{2}}} < \pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}.$$

The proof of the lemma is completed.

## 3. Main Results

**Theorem 1.** *Let* $a_n \geq 0$, $b_n \geq 0$, *and* $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, *then*

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} \quad < \quad & \{\sum_{n=0}^{\infty} [\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}] a_n^2 \\ & \cdot \sum_{n=0}^{\infty} [\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}] b_n^2\}^{\frac{1}{2}}, \end{aligned} \tag{9}$$

*and*

$$\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1})^2 < \pi\{\sum_{n=0}^{\infty}[\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}] a_n^2. \tag{10}$$

*Proof.* By Cauchy's inequality, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [\frac{a_m}{(m+n+1)^{\frac{1}{2}}}(\frac{2m+1}{2n+1})^{\frac{1}{4}}] \cdot [\frac{b_n}{(m+n+1)^{\frac{1}{2}}}(\frac{2n+1}{2m+1})^{\frac{1}{4}}] \\ &\leq \{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [\frac{a_m^2}{m+n+1}(\frac{2m+1}{2n+1})^{\frac{1}{2}}] \cdot \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [\frac{b_n^2}{m+n+1}(\frac{2n+1}{2m+1})^{\frac{1}{2}}]\}^{\frac{1}{2}} \\ &= \{\sum_{m=0}^{\infty}[\sum_{n=0}^{\infty} \frac{1}{m+n+1}(\frac{2m+1}{2n+1})^{\frac{1}{2}}] a_m^2 \cdot \sum_{n=0}^{\infty}[\sum_{m=0}^{\infty} \frac{1}{m+n+1}(\frac{2n+1}{2m+1})^{\frac{1}{2}}] b_n^2\}^{\frac{1}{2}} \\ &= \{\sum_{m=0}^{\infty} \omega(m) a_m^2 \sum_{n=0}^{\infty} \omega(n) b_n^2\}^{\frac{1}{2}}. \end{aligned}$$

By lemma 3, we have inequality (9).

Let $b_n = \sum\limits_{m=0}^{\infty} \frac{a_m}{m+n+1}$, then $0 < \sum\limits_{n=0}^{\infty} b_n^2 = \sum\limits_{n=0}^{\infty} (\sum\limits_{m=0}^{\infty} \frac{a_m}{m+n+1})^2 < \infty$, so

$$
\begin{aligned}
(\sum_{n=0}^{\infty} b_n^2)^2 &= [\sum_{n=0}^{\infty}(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1})^2]^2 \\
&= (\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1})^2 \\
&< \sum_{n=0}^{\infty}[\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}]a_n^2 \cdot \sum_{n=0}^{\infty}[\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}]b_n^2 \\
&< \pi\sum_{n=0}^{\infty}[\pi - \frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})}]a_n^2 \cdot \sum_{n=0}^{\infty} b_n^2.
\end{aligned}
$$

We have inequality (10). The proof of the theorem is completed.

**Remark 1.** *Obviously, inequality (9) is a strengthened of inequality (3). Since, for $n \in N$,*

$$\frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})} > \frac{\theta}{(n+1)^{\frac{1}{2}}},$$

*and*

$$\frac{5}{6(\sqrt{2n+1} + \frac{3}{4}\sqrt{(2n+1)^{-1}})} > \frac{\ln 2 - C}{\sqrt[3]{(2n+1)^2}}.$$

*Then inequality (9) is also a strengthened of inequality (4) and (5).*

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