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Some Fixed Point Results for Monotone Multivalued and Integral Type Contractive Mappings

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Abstract. This study focuses on establishing fixed point results for monotone multivalued mappings within the framework of partially ordered complete G_b-metric spaces. The partial order on the set $(X, \preccurlyeq)$ is defined through a functional pair (κ, Θ). The research further explores conditions under which coupled fixed points exist and are unique, particularly for mappings that meet certain contractive requirements. These investigations are carried out using the notion of integral-type contractions tailored to the structure of partially ordered G_b-metric spaces. In addition to the core results, several corollaries are derived as specific instances. To enhance the reliability and relevance of the findings, the paper includes a number of illustrative examples.

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Key Words and Phrases: Fixed point, G_b-metric space, monotone multivalued functions

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1. Introduction

The concept of a metric space was first introduced by Fréchet [1] in 1906, and later extended by his student Kurepa [2] in 1934 to more abstract spaces where the metric takes values in an ordered vector space. Consider a complete metric space (X, d). A mapping $T : X \to X$ is said to be a contraction if

$$\mathsf{d}(T(u), T(v)) \leq \alpha\, \mathsf{d}(u, v) \quad \text{for all } u, v \in X,$$

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where $\alpha \in (0,1)$. According to the Banach fixed point theorem, such a mapping T posseses a unique fixed point in X. The Banach fixed point principle has undergone significant expansions due to its efficacy in resolving existence and uniqueness problems in integral and differential equations. Researchers have built upon this foundation, introducing novel generalizations. Notably, Edelstein [3] work on subsequences of iterates led to a relaxation of the contraction condition, Subsequently, Boyd and Wong [4] introduced a continuous function $\eth : [0,\infty) \to [0,\infty)$, replacing the linear contraction condition $\mathsf{q}\mathsf{d}(u,v)\ \forall\ \mathsf{q} \in (0,1)$ with $\eth(\mathsf{d}(u,v))$, thereby presenting a more general version of the Banach fixed point theorem. Fixed point theorems in a partially ordered metric space play a vital role in determining the existence and uniqueness of solutions to specific equations. Moreover, multivalued mappings have gained significant attention due to its wide ranging applications in fields such as convex optimization, optimal control theory, and differential inclusions. For more related works, see [5–16]. As a generalization of metric space the concept b-metric space was first introduced by Bakhtin [17]. He also established several fixed point results for mappings satisfying specific contractive conditions within this framework. Later on, Selma Gulyaz Ozyurt [18] defined α-admissible contraction mappings on Branciari b-metric spaces. Conditions for the existence and uniqueness of fixed points for these mappings were discussed, and related theorems were proved. Aydi et al. [19] established a fixed point theorem for set-valued quasi-contraction mappings in b-metric spaces. Further generalizations in such spaces can be found in [20, 21]. In 1976, Caristi [22] formulated a new class of fixed point results based on the concept of weakly inward mappings. A variant of the Banach contraction principle tailored to partially ordered sets was later established by Ran and Reurings [23], which is now widely referred to as the Ran-Reurings fixed point theorem.

However, an unsuccessful attempt to generalize the Banach principle was made by Dhage et al. [24], who gave the concept of D-metric space topology. More precisely, Sedghi et al. [25] proposed a revised framework in 2007, introducing the notion of G-metric spaces as a modification of the original D-metric structure. Since then, numerous fixed point results have been established within this improved framework by various authors [26, 27]. Researchers have investigated coupled fixed point results for mixed monotone mappings in ordered metric spaces [28, 29]. For comprehensive insights into coupled fixed points and n-tupled fixed points theorems, readers can refer to [30]. Recently, Rajagopalan Ramaswamy and Gunaseelan Mani [31] introduced graphical Branciari $\aleph$-metric spaces and proved a fixed point theorem for Ω-Q contractions on complete graphical Branciari $\aleph$-metric spaces. Additionally, fixed point problems have been extensively studied in the setting of partially ordered complete metric spaces. Notably, Al-Jumaili [32] utilized the concept of these spaces to establish coincidence fixed point theorems for functions satisfying certain contractive properties involving monotone increasing η-mappings, thereby advancing the field.

Ghasab et al. [33] used the notion of integral-type contractions to establish coupled fixed point results in ordered G-metric spaces. Majid et al. [34] developed and investigated new fixed point theorems for multivalued functions in partially ordered complete D-metric spaces, where the order is defined by a pair of functions (κ, Θ) while Aghajani et al. [35]

introduced a new type of metric, called the G_b-metric. Ramaswamy et al. [36] introduced a new notion of (β, ϕ)-admissible hybrid contractions in metric spaces and established fixed point results in this setting. Recently, Samuel et al. [37] introduced integral-type contractions on orthogonal S-metric spaces and established common fixed point results. In this article, we aim to develop and explore several novel fixed point results for monotone multivalued mappings within the framework of partially ordered complete G_b-metric spaces. The partial order on the set $(X, \preccurlyeq)$ is defined through a functional pair (κ, Θ). Additionally, we establish existence and uniqueness results for coupled fixed points of mappings that satisfy specific contractive conditions, utilizing the notion of integral type contractions. To support our findings, appropriate examples are provided as practical applications, see related application [38, 39].

2. Preliminaries

The following are some definitions and results which are useful for the proof of our main theorems.

Definition 1. *[35] A function $G_b : X \times X \times X \to [0, \infty)$ is a G_b-metric on X if for all $u, v, w, x \in X$, the following conditions are satisfied:*

(G_{b1}) $G_b(u, v, w) = 0 \Leftrightarrow u = v = w$.

(G_{b2}) $0 < G_b(u, u, v)$ *for all* $u, v \in X$ *with* $u \neq v$.

(G_{b3}) $G_b(u, u, v) \leq G_b(u, v, w)$ *for all* $u, v, w \in X$ *with* $v \neq w$.

(G_{b4}) $G_b(u, v, w)$ *is invariant under permutations of its arguments, i.e.,* $G_b(u, v, w) = G_b(p\{u, v, w\})$ *for any permutation* p.

(G_{b5}) $G_b(u, v, w) \leq l[G_b(u, v, x) + G_b(x, w, w)]$ *for some constant* $l \geq 1$.

The pair (X, G_b) is called a G_b-metric space.

Example 1. *Let $X = [0, \infty)$ and $G_b : X \times X \times X \to [0, \infty)$ be a mapping defined by*

$$G_b(u, v, w) = |u - v|^q + |u - w|^q + |v - w|^q, \text{ where } q \geq 1.$$

We have:

(i) $G_b(u, v, w) \geq 0$.

(ii) $G_b(u, v, w) = 0 \Leftrightarrow |u - v|^q + |u - w|^q + |v - w|^q = 0 \Leftrightarrow |u - v|^q = 0, \ |u - w|^q = 0, \ |v - w|^q = 0 \Leftrightarrow |u - v| = 0$.
It yields that $u = v$.
Also, $|u - w| = 0 \Rightarrow u = w$.
Moreover, $|v - w| = 0 \Rightarrow v = w$. *We have* $u = v = w$.

(iii) Trivial.

(iv) Recall that

$$(a+b)^q \leq 2^{q-1}(a^q+b^q), \quad q \geq 1$$

and

$$2^n(a+b)+c \leq 2^n(a+b+c), \quad n \geq 1.$$

Now,

$$\begin{aligned}
G_b(u,v,w) &= |u-v|^q + |u-w|^q + |v-w|^q \\
&= |u-v|^q + |v-x+x-w|^q + |w-x+x-u|^q \\
&\leq |u-v|^q + 2^{q-1}(|v-x|^q + |x-w|^q) + 2^{q-1}(|w-x|^q + |x-u|^q) \\
&\leq 2^{q-1}(|u-v|^q + |v-x|^q + |x-w|^q + |w-x|^q + |x-u|^q) \\
&= 2^{q-1}(G_b(u,v,x) + G_b(x,w,w)) \\
&= l(G_b(u,v,x) + G_b(x,w,w)).
\end{aligned}$$

Since all conditions are satisfied, therefore (X, G_b) is a G_b-metric space.

Definition 2. *[35] Let (X, G_b) be a G_b-metric space. A sequence (u_s) in X is said to converge to $u \in X$ if and only if $G_b(u_s, u_s, u) = G_b(u, u, u_s) \to 0$ as $s \to \infty$.*

Definition 3. *[35] A sequence $(u_s) \in X$ is called a Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer s_0 such that for all $s, r \geq s_0$, $G_b(u_s, u_s, u_r) < \epsilon$.*

Definition 4. *[35] A G_b-metric space (X, G_b) is a complete G_b-metric if every Cauchy sequence in (X, G_b) converges in (X, G_b).*

Definition 5. *Let $(X, G_b, \preccurlyeq)$ be a G_b-metric space, and $\Theta : X \to [0, \infty)$ be a functional. We define the relation $\preccurlyeq$ as follows:*

$$u \preccurlyeq v \Leftrightarrow \kappa(G_b(u,u,v)) \leq \Theta(u) - \Theta(v) \ \forall \ u, v \in X,$$

where $\kappa : [0, \infty) \to [0, \infty)$ is so that

(i) κ is increasing and continuous.

(ii) $\kappa^{-1}(\{0\}) = \{0\}$.

(iii) $\kappa(l(\mathsf{k}+\mathsf{j})) \leq \kappa(\mathsf{k}) + \kappa(\mathsf{j}) \ \forall \ \mathsf{k}, \mathsf{j} \in [0, \infty)$.

The triplet $(X, G_b, \preccurlyeq)$ with this partial order is called an ordered G_b-metric space induced via (κ, Θ).

Proposition 1. *Suppose that (X, G_b) is a G_b-metric space, then $\preccurlyeq$ is a partial order on X and $(X, \preccurlyeq)$ is a partially ordered set.*

Proof. Let start by showing that the relation $\preccurlyeq$ is reflexive, meaning that every element is $\preccurlyeq$ itself.

Since $\kappa(G_b(u,u,u)) = \Theta(u) - \Theta(u)$ *for all* $u \in X$, *this implies that* $\preccurlyeq$ *is reflexive.*
Next to show that the relation $\preccurlyeq$ *is antisymmetric. If* $u, v \in X$ *with* $u \preccurlyeq v$ *and* $v \preccurlyeq u$, *then*

$$\kappa(G_b(u,u,v)) \le \Theta(u) - \Theta(v)$$

and

$$\kappa(G_b(v,v,u)) \le \Theta(v) - \Theta(u).$$

It implies

$$\kappa(G_b(u,u,v)) + \kappa(G_b(v,v,u)) = 0.$$

Thus,

$$\kappa(G_b(u,u,v)) = \kappa(G_b(v,v,u)) = 0.$$

That is,

$$\kappa(G_b(u,u,v)) = 0,$$

and so $u = v$, *which shows that* $\preccurlyeq$ *is antisymmetric.*
Lastly, we prove that $\preccurlyeq$ *is transitive. If* $u, v, w \in X$ *such that* $u \preccurlyeq v$ *and* $v \preccurlyeq w$, *then*

$$\kappa(G_b(u,u,v)) \le \Theta(u) - \Theta(v). \tag{1}$$

Also,

$$\kappa(G_b(v,v,u)) \le \Theta(v) - \Theta(u). \tag{2}$$

Hence, combining (1) and (2), one writes

$$\kappa(G_b(u,u,v)) + \kappa(G_b(v,v,u)) \le \Theta(u) - \Theta(w).$$

By the definition of G_b*-metric space, one has*

$$\begin{aligned} \kappa(G_b(u,u,w)) &= \kappa[l\,(G_b(u,u,v) + G_b(v,v,w))] \\ &\le \kappa\,(G_b(u,u,v)) + \kappa\,(G_b(v,v,w)) \\ &= \Theta(u) - \Theta(v) + \Theta(v) - \Theta(w) \\ &\le \Theta(u) - \Theta(w). \end{aligned}$$

Thus, $u \preccurlyeq w$.

The triplet $(X, G_b, \preccurlyeq)$ *is called partially ordered* G_b*-metric space induced via* (κ, Θ).

Definition 6. *Let* $(X, G_b, \preccurlyeq)$ *be an ordered* G_b*-metric space induced by* (κ, Θ). *The following defines the ordered intervals in* X*:*

(i) $[u, v] = \{w \in X : u \preccurlyeq w \preccurlyeq v\}$.

(ii) $[u, \infty) = \{w \in X : u \preccurlyeq w\}$.

(iii) $(-\infty, u] = \{w \in X : w \preccurlyeq u\}$.

Definition 7. *[40] Let $u \in X$. u is said to be a fixed point of a multivalued mapping $T : X \to 2^X$ if $u \in T(u)$.*

Definition 8. *[40] Let $T : X \to 2^X$ be a multivalued mapping, then T is termed as upper semi-continuous if whenever $(u_s) \in X$ and $(v_s) \in T(u_s)$ with $u_s \to m \in X$ and $v_s \to e \in X$, then $e \in T(m)$.*

Definition 9. *[41] An element $(\mathsf{a}, \mathsf{b}) \in X \times X$ is said to be a coupled fixed point of a mapping $T : X \times X \to X$ if $T(\mathsf{a}, \mathsf{b}) = \mathsf{a}$ and $T(\mathsf{b}, \mathsf{a}) = \mathsf{b}$.*

Definition 10. *[40] A function $\mathsf{U} : X \to \mathbb{R}$ is called a lower semi-continuous if for any $\{u_n\} \subset X$ and $u \in X$*

$$u_n \to u \Rightarrow \mathsf{U}(u) \leq \lim_{n\to\infty} \inf \mathsf{U}(u_n).$$

Definition 11. *[41] Let $(X, \preccurlyeq)$ be a partial order set, then $T : X \times X \to X$ is said to have mixed monotone property if $T(u, v)$ is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; i.e, for all $u_1, u_2 \in X, u_1 \preccurlyeq u_2 \Rightarrow T(u_1, v) \preccurlyeq T(u_2, v)\ \forall\ v \in X$ and for all $v_1, v_2 \in X, v_1 \preccurlyeq v_2 \Rightarrow T(u, v_1) \succcurlyeq T(u, v_2)\ \forall\ u \in T$.*

Definition 12. *[42] Denote by σ the collection of all functions $\kappa : [0, \infty) \to [0, \infty)$ such that:*

(i) κ is continuous.

(ii) κ is non-decreasing with $\kappa(\tau) = 0 \Leftrightarrow \tau = 0$.

Definition 13. *[42] Denote by $\aleph$ the collection of all functions $\Theta : [0, \infty) \to [0, \infty)$ such that:*

(i) Θ is lower semi-continuous.

(ii) $\Theta(\tau) > 0$ for all $\tau > 0$ and $\Theta(0) = 0$.

3. Multivalued Functions and G_b-Metric Spaces

This section introduces and explores new fixed point theorems for monotone multivalued functions, with a specific focus on their applications within partially ordered complete G_b-metric spaces.

Theorem 1. *Let $(X, G_b, \preccurlyeq)$ be a partially ordered complete G_b-metric space generated by (κ, Θ), where $\Theta : X \to [0, \infty)$ is a mapping which is bounded below. Let $T : X \to 2^X$ be a multivalued mapping and $\mathcal{M} = \{u \in X : T(u) \cap [u, \infty) \neq \emptyset\}$. Assume that:*

(i) T is upper semi-continuous.

(ii) If $u \in \mathcal{M}$, then $v \in \mathcal{M}$ for all $v \in T(u) \cap [u, \infty)$.

(iii) $T(m) \cap [m, \infty) \neq \emptyset$ *for some* $m \in X$.

Then there is a sequence (u_s) *such that* $u_{s-1} \preccurlyeq u_s \in T(u_{s-1})$ *for all* $s \in \mathbb{N}$, *and* T *has a fixed point* u_0 *such that* $u_s \to u_0$. *In addition, if* Θ *is lower semi-continuous, then* $u_s \preccurlyeq u_0$ *for all* s.

Proof. By using (iii), there is $m \in X$ that belongs to $\mathcal{M}$. Then choose $n \in T(m) \cap [m, \infty)$, and we have $m \preccurlyeq n$. By condition (ii), $n \in \mathcal{M}$. Choose $\tau \in T(n) \cap [n, \infty)$ such that $n \preccurlyeq \tau$. By repeating the process, we get a sequence $(u_s) \in X$ such that $u_{s-1} \preccurlyeq u_s \in T(u_{s-1})\ \forall\ s \in \mathbb{N}$.
Since $(X, G_b, \preccurlyeq)$ is a partially ordered G_b-metric space induced via (κ, Θ)

$$\kappa(G_b(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s). \tag{3}$$

The mapping κ is non-negative, so for all $s \in \mathbb{N}$,

$$\Theta(u_{s-1}) - \Theta(u_s) \geq 0.$$

That is, for all $s \in \mathbb{N}$,

$$\Theta(u_{s-1}) \geq \Theta(u_s).$$

Since Θ is bounded below, the sequence $\Theta(u_s)$ is both decreasing and bounded below. Therefore, by the completeness property of $\mathbb{R}$, $\lim_{s\to\infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}$. Thus, by equation (3),

$$\lim_{s,r\to\infty} \kappa(G_b(u_s, u_s, u_r)) \leq \lim_{s\to\infty} \Theta(u_s) - \lim_{r\to\infty} \Theta(u_r).$$

Therefore,

$$\lim_{s,r\to\infty} \kappa(G_b(u_s, u_s, u_r)) = 0.$$

By exploiting the continuity of κ and the fact that $\kappa^{-1}(\{0\}) = \{0\}$, it follows that

$$\lim_{s,r\to\infty} G_b(u_s, u_s, u_r) = 0.$$

Therefore, (u_s) is a Cauchy sequence in X. Since X is a complete G_b-metric space, there is $\exists\, u_0 \in X$ such that (u_s) is G_b-convergent to u_0. Since $u_{s-1} \in X, u_s \in T(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of T, we have $u_0 \in T(u_0)$.
Now, assuming Θ is lower semi-continuous, then for each $s \in \mathbb{N}$,

$$\begin{aligned}
\kappa(G_b(u_s, u_s, u_0)) &= \lim_{r\to\infty} \kappa(G_b(u_s, u_s, u_r)) \\
&\leq \lim_{r\to\infty} \inf\{\Theta(u_s) - \Theta(u_r)\} \\
&= \Theta(u_s) - \lim_{r\to\infty} \inf \Theta(u_r) \\
&\leq \Theta(u_s) - \Theta(u_0).
\end{aligned}$$

Thus, $u_s \leq u_0$ for all $s \in \mathbb{N}$.

Corollary 1. *Suppose that* $(X, G_b, \preccurlyeq)$ *is a partially ordered complete* G_b*-induced via* (κ, Θ)*, where* $\Theta : X \to [0, \infty)$ *is a bounded below mapping, and let* $T : X \to 2^X$ *be a multivalued mapping be so that:*

(i) T *is upper semi-continuous.*

(ii) T *satisfies the condition of monotonic sequence: for all* $u, v \in X$ *and* $u \preccurlyeq v$ *and every* $\alpha \preccurlyeq T(u)$*, there exists* $\beta \preccurlyeq T(v)$ *such that* $\alpha \preccurlyeq \beta$*.*

(iii) There is $\exists\ m \in X$ *such that* $T(m) \cap [0, \infty) \neq \emptyset$*.*

Then there exists a sequence $(u_s) \in X$ *with* $u_{s-1} \preccurlyeq u_s \in T(u_{s-1})$ *for all* $s \in \mathbb{N}$*, and* T *has a fixed point* u_0 *such that* $u_s \to u_0$*. Furthermore, if* Θ *is lower semi-continuous, then* $u_s \preccurlyeq u_0$ *for all* s*.*

Proof. By property (ii), $m \in \mathcal{M}$. Now, consider $v \in T(m) \cap [0, \infty)$, then by the condition of T, there exists $w \in T(v)$ such that $v \preccurlyeq w$. Equivalently, $w \in T(v) \cap [0, \infty) \neq \emptyset$. This implies that $v \in \mathcal{M}$ and then by Theorem 1, the proof is completed.

Corollary 2. *Let* $(X, G_b, \preccurlyeq)$ *be a partially ordered complete* G_b*-metric space induced by* (κ, Θ) *such that* $\Theta : X \to [0, \infty)$ *is a bounded below mapping, and let* $S : X \to X$ *satisfy the following:*

(i) S *is a continuous function.*

(ii) For any $\alpha \in S(u)$*, there is* $\beta \in S(v)$ *such that* $\alpha \preccurlyeq \beta$*.*

(iii) There is $m \in X$ *such that* $m \preccurlyeq S(m)$*.*

Then there is a sequence $(u_s) \in X$ *with* $u_{s-1} \preccurlyeq u_s \in S(u_{s-1})$ *for all* $s \in \mathbb{N}$*, and* S *has a fixed point* u_0 *such that* $u_s \to u_0$*. Also, if* Θ *is lower semi-continuous, then* $u_s \preccurlyeq u_0$ *for all* s*.*

Proof. Define the multivalued mapping $T : X \to 2^X$ via $T(u) = \{S(u)\}$, then T and X satisfy all the conditions of Theorem 1. Therefore, the proof follows from Theorem 1.

By replacing the conditions bounded below with the conditions of bounded above, we obtain the following results.

Theorem 2. *Let* $(X, G_b, \preccurlyeq)$ *be a partially ordered complete* G_b*-metric space induced via* (κ, Θ)*, where* $\Theta : X \to (-\infty, 0]$ *is a bounded above mapping. Presume that* $T : X \to 2^X$ *is a multivalued mapping and* $\mathcal{M} = \{u \in X : T(u) \cap (-\infty, u] \neq \emptyset\}$*. Assume that*

(i) T *is upper semi-continuous.*

(ii) For all $u \in \mathcal{M}$, $T(u) \cap \mathcal{M} \cap (-\infty, u] \neq \emptyset$*.*

Then there is a sequence (u_s) such that $u_{s-1} \succcurlyeq u_s \in T(u_{s-1})$ for all $s \in \mathbb{N}$, and T has a fixed point u_0 such that $u_s \to u_0$. Also, if Θ is lower semi-continuous, then $u_s \succcurlyeq u_0$ for all s.

Proof. By using condition (ii), there exists $m \in X$ such that $m \in \mathcal{M}$. By choosing $n \in T(m) \cap (-\infty, m]$, and we get $m \succcurlyeq n$. By condition (ii), $n \in \mathcal{M}$. Choose $\tau \in T(n) \cap (-\infty, n]$,
$\Rightarrow n \succcurlyeq \tau$.
By proceeding in this way, there is a sequence $(u_s) \in X$ s.t. $u_{s-1} \succcurlyeq u_s \in T(u_{s-1})$ for all $s \in \mathbb{N}$.
Since $(X, G_b, \preccurlyeq)$ is a partially ordered G_b metric space induced via (κ, Θ),

$$\Rightarrow \kappa(G_b(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s).$$

Given that κ is non-negative mapping, $\Theta(u_{s-1}) - \Theta(u_s) \geq 0 \; \forall \; s \in \mathbb{N}$.
$\Rightarrow \Theta(u_{s-1}) \geq \Theta(u_s) \; \forall \; s \in \mathbb{N}$.
As Θ is bounded above, we get $\Theta(u_s)$ is an increasing sequence which is bounded above. By the completeness of $\mathbb{R}$, $\lim_{s \to -\infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}$, thus

$$\lim_{s,r \to -\infty} \kappa(G_b(u_s, u_s, u_r)) \leq \lim_{s \to -\infty} \Theta(u_s) - \lim_{r \to -\infty} \Theta(u_r).$$

Therefore, $\lim_{s,r \to -\infty} \kappa(G_b(u_s, u_s, u_r)) = 0$.
Now, since κ is continuous $\kappa^{-1}(\{0\}) = \{0\}$, we get $\lim_{s,r \to -\infty} G_b(u_s, u_s, u_r) = 0$. Therefore, (u_s) is a Cauchy sequence in X. Since X is complete, there exists $u_0 \in X$ such that (u_s) is G_b convergent. to u_0. Since $u_{s-1} \in X, u_s \in T(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of T, we have $u_0 \in T(u_0)$.
Now, if Θ is lower semi-continuous, then for all $s \in \mathbb{N}$,

$$\begin{aligned}
\kappa(G_b(u_s, u_s, u_0)) &= \lim_{r \to \infty} \kappa(G_b(u_s, u_s, u_r)) \\
&\leq \lim_{r \to \infty} \{\inf \Theta(u_s) - \Theta(u_r)\} \\
&= \Theta(u_s) - \lim_{r \to \infty} \Theta(u_r) \\
&\leq \Theta(u_s) - \Theta(u_0).
\end{aligned}$$

Thus, $u_s \succcurlyeq u_0$ for all $s \in \mathbb{N}$.

Corollary 3. *Suppose that $(X, G_b, \preccurlyeq)$ is a partially ordered complete G_b-metric space induced via (κ, Θ), where $\Theta : X \to (-\infty, 0]$ is bounded above, and let $T : X \to 2^X$ be a multivalued mapping so that:*

(i) T is upper semi-continuous.

(ii) For all $u, v \in X$ and $u \succcurlyeq v$ and every $\alpha \in T(u)$, there exists $\beta \in T(v)$ such that $\alpha \succcurlyeq \beta$.

(iii) There is $m \in X$ *such that* $T(m) \cap [0, \infty) \neq \emptyset$.

Then there exists a sequence $(u_s) \in X$ *with* $u_{s-1} \succcurlyeq u_s \in T(u_{s-1})$ $\forall$ $s \in \mathbb{N}$, *and* T *has a fixed point* u_0 *such that* $u_s \to u_0$. *Furthermore, if* Θ *is lower semi-continuous, then* $u_s \succcurlyeq u_0$ *for all* s.

Corollary 4. *Assume that* $(X, G_b, \preccurlyeq)$ *is a partially ordered complete* G_b*-metric space induced via* (κ, Θ) *such that* $\Theta : X \to (-\infty, 0]$ *is bounded above, and let* $S : X \to X$ *satisfy the following:*

(i) S *is continuous.*

(ii) For any $\alpha \in S(u)$, *there exists* $\beta \in S(v)$ *such that* $\alpha \succcurlyeq \beta$.

(iii) There is $m \in X$ *such that* $m \succcurlyeq S(m)$.

Then there is a sequence $(u_s) \in X$ *with* $u_{s-1} \succcurlyeq u_s \in S(u_{s-1})$ *for all* $s \in \mathbb{N}$, *and* T *has a fixed point* u_0 *such that* $u_s \to u_0$. *Also, if* Θ *is lower semi-continuous, then* $u_s \succcurlyeq u_0$ *for all* s.

4. Coupled Fixed Point Theorems in G_b-Metric Spaces

Theorem 3. *Assume that* $(X, G_b, \preccurlyeq)$ *is a partially ordered complete* G_b*-metric space, and let*
$T : X \times X \to X$ *be a continuous mapping with the mixed monotone property on* X *such that*

$$\int_0^{G_b(T(u,v),T(m,n),T(f,w))} g(t)dt \leq \sigma \left(\int_0^{G_b(u,m,f)+G_b(v,n,w)} g(t)dt \right), \tag{4}$$

where $u, v, w, m, n, f \in X$ *and* $g : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping with* $f \preccurlyeq m \preccurlyeq u$ *and* $v \preccurlyeq n \preccurlyeq w$, *where either* $m \neq f$ *or* $n \neq w$. *If there exist* $u_0, v_0 \in X$ *such that* $u_0 \preccurlyeq T(u_0, v_0)$ *and* $T(v_0, u_0) \preccurlyeq v_0$, *then* T *has a coupled fixed point in* X.

Proof. By hypothesis, there are $u_0, v_0 \in X$ such that $u_0 \preccurlyeq T(u_0, v_0)$ and $T(v_0, u_0) \preccurlyeq v_0$.
Define $u_1, v_1 \in X$ as

$$u_0 \preccurlyeq T(u_0, v_0) = u_1 \text{ and } v_1 = T(v_0, u_0) \preccurlyeq v_0.$$

Suppose that $u_2 = T(u_1, v_1)$ and $v_2 = T(v_1, u_1)$, therefore

$$u_2 = T(u_1, v_1) = T(T(u_0, v_0), T(v_0, u_0)) = T^2(u_0, v_0).$$

$$v_2 = T(v_1, u_1) = T(T(v_0, u_0), T(u_0, v_0)) = T^2(v_0, u_0).$$

Utilizing the mixed monotonicity for the mapping T, one writes

$$u_2 = T^2(u_0, v_0) = T(u_1, v_1) \succcurlyeq T(u_0, v_0) = u_1 \succcurlyeq u_0,$$
$$v_2 = T^2(v_0, u_0) = T(v_1, u_1) \preccurlyeq T(v_0, u_0) = v_1 \preccurlyeq v_0.$$

Repeatedly applying the above process for all $s \geq 0$ leads to

$$u_0 \leq u_1 \preccurlyeq u_2 \preccurlyeq \dots \preccurlyeq u_{s+1} \preccurlyeq \dots,$$
$$v_0 \succcurlyeq v_1 \succcurlyeq v_2 \succcurlyeq \dots \succcurlyeq v_{s+1} \succcurlyeq \dots$$

such that

$$u_{s+1} = T^{s+1}(u_0, v_0) = T(T^s(u_0, v_0), T^s(v_0, u_0)),$$
$$v_{s+1} = T^{s+1}(v_0, u_0) = T(T^s(v_0, u_0), T^s(u_0, v_0)).$$

If $(u_{s+1}, v_{s+1}) = (u_0, v_0)$, then a coupled fixed point exists for the mapping T.
Now, we assume that $(u_{s+1}, v_{s+1}) \neq (u_s, v_s)$ for all $s \geq 0$, that is, let either $u_{s+1} = T(u_s, v_s) \neq u_s$ or $v_{s+1} = T(v_0, u_0) \neq v_s$. By equation (4), it follows that

$$\begin{aligned} \int_0^{G_b(u_s,u_s,u_{s+1})} g(t)dt &= \int_0^{G_b(T(u_{s-1},v_{s-1}),T(u_{s-1},v_{s-1}),T(u_s,v_s))} g(t)dt \\ &\leq \sigma\left(\int_0^{G_b(u_{s-1},u_{s-1},u_s),G_b(v_{s-1},v_{s-1},v_s)} g(t)dt\right). \end{aligned} \tag{5}$$

In the same way, it can be proved that

$$\begin{aligned} \int_0^{G_b(v_s,v_s,v_{s+1})} g(t)dt &= \int_0^{G_b(T(v_{s-1},u_{s-1}),T(v_{s-1},u_{s-1}),T(v_s,u_s))} g(t)dt \\ &\leq \sigma\left(\int_0^{G_b(u_{s-1},u_{s-1},u_s),G_b(v_{s-1},v_{s-1},v_s)} g(t)dt\right). \end{aligned} \tag{6}$$

Since g is non-increasing mapping, then for each $\mathsf{k}, \mathsf{j} \geq 0$,

$$\int_0^{\mathsf{k}+\mathsf{j}} g(t)dt \leq \int_0^{\mathsf{k}} g(t)dt + \int_0^{\mathsf{j}} g(t)dt. \tag{7}$$

Additionally, since σ is a linear and monotonically increasing mapping, it follows from (4), (5) and (7) that for all $s \geq 0$

$$\begin{aligned} \int_0^{G_b(u_s,u_s,u_{s+1})} g(t)dt &= \int_0^{G_b(T(u_{s-1},v_{s-1}),T(u_{s-1},v_{s-1}),T(u_s,v_s))} g(t)dt \\ &\leq \sigma\left(\int_0^{G_b(u_{s-1},u_{s-1},u_s)+G_b(v_{s-1},v_{s-1},v_s)} g(t)dt\right) \\ &\leq \sigma\left(\int_0^{G_b(u_{s-1},u_{s-1},u_s)} g(t)dt\right) + \sigma\left(\int_0^{G_b(v_{s-1},v_{s-1},v_s)} g(t)dt\right) \\ &= \sigma\left(\int_0^{G_b(T(u_{s-2},v_{s-2}),T(u_{s-2},v_{s-2}),T(u_{s-1},v_{s-1}))} g(t)dt\right) \\ &+ \sigma\left(\int_0^{G_b(T(v_{s-2},u_{s-2}),T(v_{s-2},u_{s-2}),T(v_{s-1},u_{s-1}))} g(t)dt\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \sigma\left(\sigma\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})+G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\right)\\
&+\sigma\left(\sigma\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})+G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\right)\\
&\leq \sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&+2\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt+\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&\leq 2\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})+G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&\vdots\\
&\leq s\sigma^s\left(\int_0^{G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)} g(t)dt\right).
\end{aligned}
$$

Following the same steps, it can be proved that

$$
\begin{aligned}
\int_0^{G_b(v_s,v_s,v_{s+1})} g(t)dt &= \int_0^{G_b(T(v_{s-1},u_{s-1}),T(v_{s-1},u_{s-1}),T(v_s,u_s))} g(t)dt\\
&\leq \sigma\left(\int_0^{G_b(v_{s-1},v_{s-1},v_s)+G_b(u_{s-1},u_{s-1},u_s)} g(t)dt\right)\\
&\leq \sigma\left(\int_0^{G_b(v_{s-1},v_{s-1},v_s)} g(t)dt\right)+\sigma\left(\int_0^{G_b(u_{s-1},u_{s-1},u_s)} g(t)dt\right)\\
&= \sigma\left(\int_0^{G_b(T(v_{s-2},u_{s-2}),T(v_{s-2},u_{s-2}),T(v_{s-1},u_{s-1}))} g(t)dt\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sigma\left(\int_0^{G_b(T(u_{s-2},v_{s-2}),T(u_{s-2},v_{s-2}),T(u_{s-1},v_{s-1}))} g(t)dt\right)\\
&\leq \sigma\left(\sigma\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})+G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\right)\\
&+\sigma\left(\sigma\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})+G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\right)\\
&\leq \sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&+\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&=2\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt\right)\\
&+2\sigma^2\left(\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&=2\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})} g(t)dt+\int_0^{G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&\leq 2\sigma^2\left(\int_0^{G_b(v_{s-2},v_{s-2},v_{s-1})+G_b(u_{s-2},u_{s-2},u_{s-1})} g(t)dt\right)\\
&\vdots\\
&\leq s\sigma^s\left(\int_0^{G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)} g(t)dt\right).
\end{aligned}
$$

Let $r, s \in \mathbb{N}$ such that $r > s$, then from the definition of G_b-metric space,

$$
\begin{aligned}
\int_0^{G_b(u_s,u_s,u_r)} g(t)dt &\leq \int_0^{l[G_b(u_s,u_s,u_{s+1})+G_b(u_{s+1},u_{s+1},u_r)]} g(t)dt\\
&=\int_0^{lG_b(u_s,u_s,u_{s+1})+lG_b(u_{s+1},u_{s+1},u_r)} g(\mathtt{t})dt\\
&\leq \int_0^{lG_b(u_s,u_s,u_{s+1})} g(t)dt+\int_0^{lG_b(u_{s+1},u_{s+1},u_r)} g(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^{l G_b(u_s,u_s,u_{s+1})} g(t)dt \\
&+ \int_0^{l[l(G_b(u_{s+1},u_{s+1},u_{s+2})+G_b(u_{s+2},u_{s+2},u_r))]} g(t)dt \\
&\leq \int_0^{l G_b(u_s,u_s,u_{s+1})} g(t)dt + \int_0^{l^2 G_b(u_{s+1},u_{s+1},u_{s+2})} g(t)dt \\
&+ \int_0^{l^2 G_b(u_{s+2},u_{s+2},u_r)} g(t)dt \\
&\leq \int_0^{l G_b(u_s,u_s,u_{s+1})} g(t)dt + \int_0^{l^2 G_b(u_{s+1},u_{s+1},u_{s+2})} g(t)dt \\
&+ \int_0^{l^3 G_b(u_{s+2},u_{s+2},u_{s+3})} g(t)dt + \ldots + \int_0^{l^{r-s} G_b(u_{r-2},u_{r-2},u_{r-1})} g(t)dt \\
&+ \int_0^{l^{r-s} G_b(u_{r-1},u_{r-1},u_r)} g(t)dt \\
&\leq s\sigma^s \left(\int_0^{l[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&+ (s+1)\sigma^{s+1} \left(\int_0^{l^2[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&+ (s+2)\sigma^{s+2} \left(\int_o^{l^3[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&+ \ldots + (r-2)\sigma^{r-2} \left(\int_0^{l^{r-s}[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&+ (r-1)\sigma^{r-1} \left(\int_0^{l^{r-s}[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&= \sum_{i=s}^{i=r-1} i\sigma^i \left(\int_0^{l^{i-s+1}[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right) \\
&\leq \sum_{i=s}^{\infty} i\sigma^i \left(\int_0^{l^{i-s+1}[G_b(u_0,u_0,u_1)+G_b(v_0,v_0,v_1)]} g(t)dt \right).
\end{aligned}
$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty$ for all $t > 0$, this implies that $\lim_{s,r\to\infty} G_b(u_s,u_s,u_r) = 0$ and (u_s) is a Cauchy sequence in X. In a similar manner, the following result can be obtained

$$
\int_0^{G_b(v_s,v_s,v_r)} g(t)dt \leq \int_0^{l[G_b(v_s,v_s,v_{s+1})+G_b(v_{s+1},v_{s+1},v_r)]} g(t)dt
$$

$$
\begin{aligned}
&= \int_0^{lG_b(v_s,v_s,v_{s+1})+lG_b(v_{s+1},v_{s+1},v_r)} g(t)dt \\
&\leq \int_0^{lG_b(v_s,v_s,v_{s+1})} g(t)dt + \int_0^{lG_b(v_{s+1},v_{s+1},v_r)} g(t)dt \\
&\leq \int_0^{lG_b(v_s,v_s,v_{s+1})} g(t)dt \\
&+ \int_0^{l[l(G_b(v_{s+1},v_{s+1},v_{s+2})+G_b(v_{s+2},v_{s+2},v_r))]} g(t)dt \\
&\leq \int_0^{lG_b(v_s,v_s,v_{s+1})} g(t)dt + \int_0^{l^2G_b(v_{s+1},v_{s+1},v_{s+2})} g(t)dt \\
&+ \int_0^{l^2G_b(v_{s+2},v_{s+2},v_r)} g(t)dt \\
&\leq \int_0^{lG_b(v_s,v_s,v_{s+1})} g(t)dt + \int_0^{l^2G_b(v_{s+1},v_{s+1},v_{s+2})} g(t)dt \\
&+ \int_0^{l^3G_b(v_{s+2},v_{s+2},v_{s+3})} g(t)dt + \ldots + \int_0^{l^{r-s}G_b(v_{r-2},v_{r-2},v_{r-1})} g(t)dt \\
&+ \int_0^{l^{r-s}G_b(v_{r-1},v_{r-1},v_r)} g(t)dt \\
&\leq s\sigma^s \left(\int_0^{l[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&+ (s+1)\sigma^{s+1} \left(\int_0^{l^2[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&+ (s+2)\sigma^{s+2} \left(\int_o^{l^3[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&+ \ldots + (r-2)\sigma^{r-2} \left(\int_0^{l^{r-s}[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&+ (r-1)\sigma^{r-1} \left(\int_0^{l^{r-s}[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&= \sum_{i=s}^{i=r-1} i\sigma^i \left(\int_0^{l^{i-s+1}[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right) \\
&\leq \sum_{i=s}^{\infty} i\sigma^i \left(\int_0^{l^{i-s+1}[G_b(v_0,v_0,v_1)+G_b(u_0,u_0,u_1)]} g(t)dt \right).
\end{aligned}
$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty$ for all $t \in [0,+\infty)$, then $\lim_{s,r\to\infty} G_b(v_s,v_s,v_r) = 0$ and (v_s) is a

Cauchy sequence in X, which is a complete G_b-metric space, there exist $u, v \in X$ such that $\lim_{s\to\infty} u_s = u$ and $\lim_{s\to\infty} v_s = v$. Since T is continuous, it follows that $T(u, v) = u$ and $T(v, u) = v$, that is, (u, v) is a coupled fixed point of T.

Theorem 4. *Let $(X, G_b, \preccurlyeq)$ be a partially ordered complete G_b-metric space satisfying the following conditions:*

(i) If (u_s) is non-decreasing sequence which converges to $u \in X$, then $u_s \preccurlyeq u\ \forall\ s$.

(ii) If (v_s) is non-increasing sequence which converges to $v \in X$, then $v_s \succcurlyeq v\ \forall\ s$.

Also suppose that $T : X \times X \to X$ is a continuous function having the mixed monotone property on X such that

$$\begin{aligned}&\int_0^{G_b(T(u,v),T(m,n),T(f,w))} g(t)dt\\&\leq \sigma\left(\int_0^{G_b(u,m,f)+G_b(v,n,w)} g(t)dt\right),\end{aligned} \tag{8}$$

where $u, v, w, m, n, f \in X$ and $g : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in X$ such that $u_0 \preccurlyeq H(u_0, v_0)$ and $(v_0, u_0) \preccurlyeq v_0$, then T has a coupled fixed point in X.

Proof. Using the similar approach to that in the proof of Theorem 3, gives two Cauchy sequences (u_s) and $(v_s) \in X$.
Conditions (i) and (ii) implies that there exist $u, v \in X$ such that $u_s \preccurlyeq u$ and $v_s \succcurlyeq v$ for all $s \geq 0$.
If $u_s = u$ and $v_s = v$ for some s, then $u_{s+1} = u$ and $v_{s+1} = v$; that is, (u, v) is a coupled fixed point . Now, without loss of generality, let either $u_s \neq u$ or $v_s \neq v$. By using (8),

$$\begin{aligned}&\int_0^{G_b(T(u,v),T(u,v),u)} g(t)dt\\&\leq \int_0^{G_b(T(u,v),T(u,v),T(u_s,v_s))+G_b(T(u_s,v_s),T(u_s,v_s),u)} g(t)dt\\&\leq \int_0^{G_b(T(u,v),T(u,v),T(u_s,v_s))} g(t)dt + \int_0^{G_b(T(u_s,v_s),T(u_s,v_s),u)} g(t)dt\\&\leq \sigma\left(\int_0^{G_b(u,u,u_s)+G_b(v,v,v_s)} g(t)dt\right) + \int_0^{G_b(u_{s+1},u_{s+1},u)} g(t)dt.\end{aligned}$$

Hence, from above with $s \to \infty$, we get $G(T(u, v), T(u, v), u) = 0$, which gives that $T(u, v) = u$. Likewise, the same approach can be applied to write

$$\begin{aligned}
\int_0^{G_b(T(v,u),T(v,u),v)} g(t)dt &\leq \int_0^{G_b(T(v,u),T(v,u),T(v_s,u_s))+G_b(T(v_s,u_s),T(v_s,u_s),v)} g(t)dt \\
&\leq \int_0^{G_b(\mathsf{H}(v,u),\mathsf{H}(v,u),\mathsf{H}(v_s,u_s))} g(t)dt \\
&+ \int_0^{G_b(T(v_s,u_s),T(v_s,u_s),v)} g(t)dt \\
&\leq \sigma\left(\int_0^{G_b(v,v,v_s)+G_b(u,u,u_s)} g(t)dt\right) \\
&+ \int_0^{G_b(v_{s+1},v_{s+1},v)} g(t)dt.
\end{aligned} \tag{9}$$

Hence, via (9) with $s \to \infty$, we get $G_b(T(v,u),H(v,u),v) = 0$ and thus $H(v,u) = v$. Therefore, (u,v) is a coupled fixed point of the mapping T.

The following theorem shows that the coupled fixed point of T can be unique.

Theorem 5. *Suppose that $(X, G_b, \preccurlyeq)$ is a partially ordered complete G_b-metric space satisfying the following:*

(i) If (u_s) is a non-decreasing sequence that converges to some point $u \in X$, then

$$u_s \preccurlyeq u \quad \textit{for all } s \in \mathbb{N}.$$

(ii) If (v_s) is a non-increasing sequence that converges to a point $v \in X$, then

$$v_s \succcurlyeq v \quad \textit{for all } s \in \mathbb{N}.$$

(iii) For any two pairs $(u,v),(u_1,v_1) \in X \times X$, there exists a pair $(w_1,w_2) \in X \times X$ that is comparable with both (u,v) and (u_1,v_1).

Assume that $T : X \times X \to X$ is a continuous mapping which satisfies the mixed monotone property on X such that

$$\int_0^{G_b(T(u,v),T(m,n),T(f,w))} g(t)dt \leq \sigma\left(\int_0^{G_b(u,m,f)+G_b(v,n,w)} g(t)dt\right), \tag{10}$$

where $u,v,w,m,n,f \in X$ and $g : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If $\exists\ u_0, v_0 \in X$ such that $u_0 \preccurlyeq T(u_0,v_0)$ and $T(v_0,u_0) \preccurlyeq v_0$, then T has a unique coupled fixed point in (X,G).

Proof. Suppose that (u_1, v_1) is another fixed point of T. The following cases are now considered.

Case 1: Let (u, v) and (u_1, v_1) be elements in $X \times X$ that are comparable that $(u, v) \preccurlyeq (u_1, v_1)$ i.e. $u \preccurlyeq u_1$ and $v \preccurlyeq v_1$. Now, by using (10),

$$\int_0^{G_b(T^s(u,v),T^s(u_1,v_1),T^s(u_1,v_1))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(u,u_1,u_1)+G_b(v,v_1,v_1)} g(t)dt \right). \quad (11)$$

Taking $\lim_{s\to\infty}$ and by the use of (11), $u = u_1$. Following a similar pattern, it is clear that

$$\int_0^{G_b(T^s(v,u),T^s(v_1,u_1),T^s(v_1,u_1))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(v,v_1,v_1)+G_b(u,u_1,u_1)} g(t)dt \right). \quad (12)$$

Taking $\lim_{s\to\infty}$ and by the use of (12), we obtain $u = u_1$.

Case 2: Let (u, v) be not comparable with (u_1, v_1). So by condition (iii) there exist $(w_1, w_2) \in X \times X$, which is comparable to (u, v) and (u_1, v_1). We can assume that $w_1 \preccurlyeq u, w_2 \preccurlyeq v, w_1 \preccurlyeq u_1$ and $w_2 \preccurlyeq v_1$. Again, by using (10),

$$\int_0^{G_b(T^s(u,v),T^s(w_1,w_2),T^s(w_1,w_2))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(u,w_1,w_1)+G_b(v,w_2,w_2)} g(t)dt \right). \quad (13)$$

Taking $s \to \infty$ and by (13),

$$G_b(T^s(u, v), T^s(w_1, w_2), T^s(w_1, w_2)) = 0.$$

That is,

$$\lim_{s\to\infty} T^s(u, v) = \lim_{s\to\infty} T^s(w_1, w_2) = u.$$

$$\int_0^{G_b(T^s(u_1,v_1),T^s(w_1,w_2),T^s(w_1,w_2))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(u_1,w_1,w_1)+G_b(v_1,w_2,w_2)} g(t)dt \right). \quad (14)$$

From (14) $\lim_{s\to\infty} T^s(u_1, v_1) = \lim_{s\to\infty} T^s(w_1, w_2) = u_1$, and so $u = u_1$. Preceding in the same way, one has

$$\int_0^{G_b(T^s(v,u),T^s(w_2,w_1),T^s(w_2,w_1))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(v,w_2,w_2)+G_b(u,w_1,w_1)} g(t)dt \right). \quad (15)$$

Assume that $s \to \infty$, then (15) gives $G_b(T^s(v, u), T^s(w_2, w_1), T^s(w_2, w_1)) = 0$. We have

$$\lim_{s\to\infty} T^s(v, u) = \lim_{s\to\infty} T^s(w_2, w_1) = v.$$

Similarly, it can be proved that

$$\int_0^{G_b(T^s(v_1,u_1),T^s(w_2,w_1),T^s(w_2,w_1))} g(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{G_b(v_1,w_2,w_2)+G_b(u_1,w_1,w_1)} g(t)dt \right). \quad (16)$$

Also $\lim_{s\to\infty} T^s(v_1, u_1) = \lim_{s\to\infty} T^s(w_2, w_1) = v_1$ by using (16). That is, $v = v_1$.
Hence, in all cases, $(u, v) = (u_1, v_1)$, which means that the coupled fixed point of the mapping T is unique.

Theorem 6. *Let $(X, G_b, \preccurlyeq)$ be a partially ordered complete G_b-metric space satisfying the following conditions:*

(i) If (u_s) is a non-decreasing sequence which converges to $u \in X$, then $u_s \preccurlyeq u \;\forall\; s$.

(ii) If (v_s) is a non-increasing sequence which converges to $v \in X$, then $v_s \succcurlyeq v \;\forall\; s$.

(iii) Every pair of the element X has an upper bound and a lower bound in X.

Also, let $T : X \times X \to X$ be a continuous having the mixed monotone property on X such that

$$\int_0^{G_b(T(u,v),T(m,n),T(f,w))} g(t)dt \le \sigma\left(\int_0^{G_b(u,m,f)+G_b(v,n,w)} g(t)dt\right), \tag{17}$$

where $u, v, w, m, n, f \in X$ and $g : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in X$ such that $u_0 \preccurlyeq T(u_0, v_0)$ and $T(v_0, u_0) \preccurlyeq v_0$, then $u = v$.

Proof. Assume that u and v are comparable under the partial ordering $\preccurlyeq$ in X, allowing us to assume that $u \preccurlyeq v$ and $v \preccurlyeq v$. Using the same argument as in Theorem 3, we arrive at $u = v$. Next assume that u and v are incomparable. Then there is a common upper bound $w \in X$ that is comparable with both u and v. So suppose that $u \preccurlyeq w$ and $v \preccurlyeq w$. By applying Theorem 3, $(u, v) = (w, w)$. Thus, $w = v$.

The following examples validate our result.

Example 2. *Consider the set $X = [0, 1]$ and define a mapping $G_b : X \times X \times X \to \mathbb{R}^+$ by $G_b(u, v, w) = |u - v|^2 + |u - w|^2 + |v - w|^2 \;\forall\; u, v, w \in X$. Therefore (X, G_b) is a complete G_b-metric space. Now, assume that $\sigma(t) = \frac{t}{2}$ for all $t \in [0, \infty)$, and let $T : X \times X \to X$ be a mapping defined by $T(\mathsf{g}, \mathsf{h}) = \frac{3(\mathsf{g}+\mathsf{h})}{16}$. Thus the conditions of Theorem 3 are satisfied. That is,*

$$\begin{aligned}
\int_0^{G_b(T(\mathsf{g},\mathsf{h}),T(m,n),T(c,k))} g(t)dt &= \int_0^{|T(\mathsf{g},\mathsf{h})-T(m,n)|^2+|T(\mathsf{g},\mathsf{h})-T(c,k)|^2+|T(m,n)-T(c,k)|^2} g(t)dt \\
&= \int_0^{|\frac{3(\mathsf{g}+\mathsf{h})}{16}-\frac{3(m+n)}{16}|^2+|\frac{3(\mathsf{g}+\mathsf{h})}{16}-\frac{3(c+l)}{16}|^2+|\frac{3(m+n)}{16}-\frac{3(c+k)}{16}|^2} g(t)dt \\
&\le \int_0^{\frac{9(2)}{256}(|\mathsf{g}-m|^2+|\mathsf{h}-n|^2+|\mathsf{g}-c|^2+|\mathsf{h}-k|^2+|m-c|^2+|n-k|^2)} g(t)dt \\
&\le \frac{1}{256}\int_0^{l[|\mathsf{g}-m|^2+|\mathsf{h}-n|^2+|\mathsf{g}-c|^2+|\mathsf{h}-k|^2+|m-c|^2+|n-k|^2]} g(t)dt \\
&\le \sigma\left(\int_0^{l[G_b(\mathsf{g},m,c)+G_b(\mathsf{h},n,k)]} g(t)dt\right)
\end{aligned}$$

where $\mathsf{g}, \mathsf{h}, c, m, n, k \in X$. *Thus* T *has a coupled fixed point.*

Example 3. *Let* $X = [0, \infty)$ *and* $G_b : X \times X \times X \to \mathbb{R}^+$ *be a mapping defined by:*

$$G_b(u, v, w) = |u - v|^q + |v - w|^q + |w - u|^q.$$

Then G_b *is* G_b*-metric space (by Example 1). Now, suppose that* $\sigma(t) = \frac{1}{2}t$ *for all* $t \in [0, \infty]$, *and let* $T : X \times X \to X$ *be a mapping defined by* $T(\mathsf{g}, \mathsf{h}) = \frac{\mathsf{g}+\mathsf{h}}{16}$. *We have*

$$\begin{aligned}
\int_0^{G_b(T(\mathsf{g},\mathsf{h}),T(m,n),T(c,k))} g(t)dt &= \int_0^{|T(\mathsf{g},\mathsf{h})-T(m,n)|^q+|T(\mathsf{g},\mathsf{h})-T(c,k)|^q+|T(m,n)-T(c,k)|^q} g(t)dt \\
&= \int_0^{|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{m+n}{16}|^q+|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{c+k}{16}|^q+|\frac{m+n}{16}-\frac{c+k}{16}|^q} g(t)dt \\
&\leq \int_0^{(\frac{1}{16})^q(2)^{q-1}(|\mathsf{g}-m|^q+|\mathsf{h}-n|^q+|\mathsf{g}-c|^q+|\mathsf{h}-k|^q+|m-c|^q+|n-k|^q)} g(t)dt \\
&\leq \left(\frac{1}{16}\right)^q \int_0^{l(|\mathsf{g}-m|^q+|\mathsf{h}-n|^q+|\mathsf{g}-c|^q+|\mathsf{h}-k|^q+|m-c|^q+|n-k|^q)} g(t)dt \\
&\leq \sigma\left(\int_0^{l[G_b(\mathsf{g},m,c)+G_b(\mathsf{h},n,k)]} g(t)dt\right)
\end{aligned}$$

where $\mathsf{g}, \mathsf{h}, c, m, n, k \in X$. *Clearly,* T *fulfills all the axioms of Theorem 3, so* T *possesses a coupled fixed point.*

5. Conclusion

The concepts of partial order and contractive conditions are redefined within the framework of G_b-metric spaces and it is observed that the contractive condition of Majid et al. [43] is a special case of our results. In order to show the applicability of our results, non-trivial examples are provided. In future the generalization of this work can be done by using doubled controlled metric spaces or triple controlled metric spaces. One can also modify the contractive condition to obtain more general results.

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Authors' Contributions

All authors contribute equally in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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