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Exploring Certified Domination Subdivision Numbers in Graph Theory

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Abstract. A *certified dominating set* S is a dominating set of a graph G, if every vertex in S has either zero or at least two neighbours in $V \setminus S$. The minimum cardinality of certified dominating set of G is the *certified domination number* of G denoted by $\gamma_{cer}(G)$. We defined *certified domination subdivision number* $Sd^{+}_{\gamma_{cer}}(G)$ $[Sd^{-}_{\gamma_{cer}}(G)]$ of a graph G to be the minimum number of edges that must be subdivided (where no edge in G can be subdivided more than once) in order to construct a graph with a certified domination number larger [lesser] than the certified domination number of G. In this paper, we determine the values of certified domination subdivision number for certain classes of graphs including circulant graphs [$C_n(1,2)$ and $C_n(1,3)$] and petersen graphs [$P(n,1)$ and $P(n,2)$].

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1. Introduction

Haynes [1] introduced the most fundamental and well studied concepts in graph theory called domination in graphs. A *dominating set* of a graph G is a set $S \subseteq V$ with the property that for each vertex $u \in V \setminus S$ there exists at least a vertex $x \in S$ adjacent to u. The minimum cardinality amongst all dominating sets of G is the *domination number* $\gamma(G)$ and S is called γ-set of G, if S is minimum. Many advanced researches are going

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on in the variety of domination terminology [2],[3]. One latest among these varieties is *certified domination* which was introduced by Magda Dettlaff et al [4]. A *certified dominating set* is defined as $D \subseteq V$ is a dominating set of a graph G and every vertex in D has either zero or at least two neighbours in $V \backslash D$. $\gamma_{cer}(G)$ is the *certified domination number* of G which is defined as the minimum cardinality of certified dominating set of G and D is the γ_{cer}-set of G, if D is minimum. Further results on this parameter seen in [5–9].

An edge $uv \in E(G)$ is subdivided if the edge uv is deleted, but a new vertex called *subdivision vertex* w is added along with two new edges uw and vw. The *domination subdivision number* $Sd(G)$ of a graph G is the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase the domination number. S. Arumugam and J. Paulraj Joseph [10] first defined the domination subdivision number $Sd(G)$ of a graph G and showed that $Sd(T) \leq 3$ for any tree T with at least three vertices. Other results and General bounds of domination subdivision number can be found in [11–16].

Motivated by recent researches focusing on subdivision number, we defined *certified domination subdivision* number of a graph G denoted by $Sd^+_{\gamma_{cer}}(G)$ $[Sd^-_{\gamma_{cer}}(G)]$ to be the minimum number of edges which must be subdivided (where each edge can be subdivided at most once) in order to increase [decrease] the certified domination number of G and also we characterised these parameters for trees in [17]. *Domination* in graphs has applications in a variety of fields. Domination occurs in facility location problems in which the number of facilities (e.g., health centres, police stations) is fixed and an attempt is made to minimize the distance that a person must travel to reach the facility. Certified domination is one such latest parameter, in which a set S is the facility centres and set T is the area of stakeholders. For each area $x \in T$, there must be a facility centre $v \in S$, that can serve for x and whenever such v is serving x, there must also be at least one neighbouring area $y \in T$ that uses the facility centre v. We can determine the minimum number of facility centres either to take care of its area (where it situated) or it takes care of more than one neighbouring areas. Here we introduce subdivision certified domination number, which is to determine how the areas are subdivided according to the convenience of the stakeholders in neighbour areas so as to facilitate them to save time and money without increasing the facility centres [some times the number of facility centres can also be reduced due to the subdivision of areas]. In this paper, we determine the values of certified domination subdivision number for certain classes of graphs including circulant graphs [$C_n(1,2)$ and $C_n(1,3)$] and petersen graphs [$P(n,1)$ and $P(n,2)$].

2. Notation

Let $G = (V, E)$ be a connected, simple graph with order $|V| = n$. We use Harary's [18] for graph theoretic notation. For any vertex $v \in V$, the *open neighbourhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$, the closed neighbourhood of S is $N[S] = N(S) \cup S$ and the *private neighbourhood* $pn(v, S)$ of a vertex $u \in S$ is defined by $pn(v, S) = \{u \in V - S : N(u) \cap S = \{v\}\}$. A *path* is a walk with no repeated vertices. A nontrivial closed path is called a *cycle*. A graph G is *k-partite*, $k \geq 1$ if it is possible to partition $V(G)$ into k subsets, $v_1, v_2 \ldots v_k$ (called *partite set*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. If G is a *1-partite graph* of order n, then $G = K_n$. For $k = 2$, such graphs are called *bipartite graphs*. A *complete bipartite graph* is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has m vertices in one of its bipartition subsets and n vertices in the other is denoted $K_{m,n}$. A *wheel* graph is a graph formed by connecting all vertices of a cycle to a single universal vertex.

3. Main Results

Theorem 1. *[5] If C_n is an n-vertex cycle, $n \geq 3$, then* $\gamma_{cer}(C_n) = \lceil \frac{n}{3} \rceil$

Theorem 2. *For any cycle C_n, $n \geq 4$*

$$Sd^+_{\gamma_{cer}}(C_n) = \begin{cases} 1 & \textit{if } n \equiv 0 \ (mod\ 3) \\ 2 & \textit{if } n \equiv 2 \ (mod\ 3) \\ 3 & \textit{if } n \equiv 1 \ (mod\ 3) \end{cases}$$

Proof. By theorem 1, $\gamma_{cer}(C_n) = \lceil \frac{n}{3} \rceil$, $n \geq 4$. Let D be a γ_{cer}-set of C_n. Consider the following cases.

Case (i) $n \equiv 0 \pmod 3$
In this case, each vertex in D dominates exactly 3 vertices, including itself. Subdividing an edge in C_n results C_{n+1}. Since $\gamma_{cer}(C_n) = \lceil \frac{n}{3} \rceil$, for $n \equiv 0 \pmod 3$. So we need to add one more vertex in D to dominate C_{n+1}. Hence $\gamma_{cer}\ (C_{n+1}) > \gamma_{cer}\ (C_n)$. Therefore $Sd^+_{\gamma_{cer}}(C_n) = 1$.

Case (ii) $n \equiv 2 \pmod 3$
In this case, each vertex in D dominates exactly 3 vertices including itself except one which dominates 2 vertices including itself. Subdividing an edge in C_n results C_{n+1}, where $n + 1 \equiv 0 \pmod 3$. Now by case (i) we notice that $\gamma_{cer}(C_{n+1}) = \gamma_{cer}(C_n)$. This implies that $Sd^+_{\gamma_{cer}}(G) > 1$. By case (i) we need to subdivide one more edge in C_{n+1} results C_{n+2}. Hence $\gamma_{cer}(C_{n+2}) > \gamma_{cer}(C_{n+1})$. Therefore $Sd^+_{\gamma_{cer}}(C_n) = 2$.

Case (iii) $n \equiv 1 \pmod 3$
In this case subdividing an edge in C_n results $n \equiv 2 \pmod 3$. By case (ii) we clearly see that $Sd^+_{\gamma_{cer}}(G) = 3$. Hence the proof.

Theorem 3. *[4] Let G be a connected graph of order at least three vertices . Then $\gamma(G) = \gamma_{cer}(G))$ if and only if G has a γ-set D such that every vertex in D has at least two neighbours in $V_G - D$.*

Theorem 4. *[5] If $K_{m,n}$ is a complete bipartite graph with $1 \leq m \leq n$, then*

$$\gamma_{cer}(K_{m,n}) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n > 1 \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 5. *For complete bipartite graph $G = K_{m,n}$,*

$$Sd^{+}_{\gamma_{cer}}(G) = \begin{cases} 3 & \text{if } m = 2 \text{ and } n \geq 2 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $G = K_{m,n}$, Let V_1, V_2 be the vertex partition of G . Let D be a γ_{cer}-set of G. By theorem 3, for all vertex $v \in D$, $|N(v)| \geq 2$, we clearly see that $\gamma_{cer}(G) = 2$. Consider the following cases.

Case (i) $m = 2$
In this case, we have the following subcases.

Subcase (i) $n = 2$
Here $K_{2,2} = C_4$, by theorem 2 we have $Sd^{+}_{\gamma_{cer}}(C_4) = 3$. Hence $Sd^{+}_{\gamma_{cer}}(K_{2,2}) = 3$.

Subcase (ii) $n > 2$
Let $v_1, v_2 \in V_1 \cap D$ and $u_1, u_2, \ldots u_n \in V_2$ and let G' be a graph derived from G through subdividing an edge in G say $e = v_1u_i$ for some i by a subdivision vertex x_1. All vertices in V_2 are dominated by v_2 and x_1 is dominated by v_1. We clearly see that D is a γ_{cer}-set of G. Hence, $\gamma_{cer}(G) = \gamma_{cer}(G')$, this implies $Sd^{+}_{\gamma_{cer}}(G) > 1$.

Let G'' be the graph obtained from G', by subdividing an edge in G' say $e = v_2u_i$ by a subdivision vertex x_2 results $D_1 = \{D - \{v_2\} \cup \{x_2\}\}$ is a γ_{cer}-set of G''. Hence, $Sd^{+}_{\gamma_{cer}}(G) > 2$.

Let G''' be the graph obtained from G'' by subdividing an edge say $e = v_2u_2$ in G'' by a subdivision vertex x_3 results $D_2 = D_1 \cup \{v_2\}$ is a γ_{cer}-set of G'''. Here $|D_2| > |D_1|$, that is $\gamma_{cer}(G) < \gamma_{cer}(G''')$. Hence $Sd^{+}_{\gamma_{cer}}(G) = 3$.

Case (ii) $m = 1$ and $n > 1$
Let $v_1 \in V_1 \cap D$ and $u_1, u_2, u_3, \ldots u_n \in V_2$. We know that $\gamma_{cer}(G) = 1$. Subdividing the edge $e = v_1u_i$, for some $1 < i < n$ by subdivision vertex x results $D_1 = D - \{u_i\}$. Hence $Sd^{+}_{\gamma_{cer}}(G) = 1$.

Case (iii) $m > 2$ and $n > 2$
Let $v_1, v_2, v_3, \ldots v_m \in V_1$ and $u_1, u_2, u_3, \ldots u_n \in V_2$ and by theorem [4], $\gamma_{cer}(G) = 2$. Let $v_i, u_j \in D$ for some $1 < i < m$ and $1 < j < n$. Let G' be a graph obtained by subdividing an edge e in G say $e = v_1u_1$ by a subdivision vertex x_1, results a new configuration of

γ_{cer}-set say $D_1 = \{v_1, u_1\}$ which implies $|D| = |D_1|$. Therefore $Sd^+_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G' by subdividing the edge $e = u_1v_2$ by a subdivision vertices x_2. Here x_1 and x_2 are dominated by u_1 and $v_2 \notin N[v]$ for all $v \in D$. Now $D_2 = D_1 \cup \{v_2\}$. Hence $|D_2| > |D_1|$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$.

We observe that, for all graphs G which is isomorphic to complete graphs, wheel graphs and grid graphs ($P_2 \times P_n$, $n \geq 2$), $Sd^+_{\gamma_{cer}}(G) = 1$.

4. Circulant Graphs

The *circulant graph* $C_n(S_c)$ is the graph with the vertex set $V(C_n(S_c)) = \{v_i : 0 \leq i \leq n-1\}$ and the edge set $E(C_n(S_c)) = \{v_iv_j : 0 \leq i, j \leq n-1, (i-j)(\text{mod } n) \in S_c\}$. Here $S_c \subseteq \{1, 2, 3, \ldots, \lceil \frac{n}{2} \rceil\}$ where subscripts are taken modulo n.

In this section we find the value of $Sd^+_{\gamma_{cer}}(G)$ for the circulant graphs $C_n(1,2)$ and $C_n(1,3)$

Theorem 6. *[19] For any integer $n \geq 5$, $\gamma(C_n(1,2)) = \lceil \frac{n}{5} \rceil$*

Theorem 7. *For any circulant graph $G \cong C_n(1,2)$, $n \geq 6$,*

$$Sd^+_{\gamma_{cer}}(G) = \begin{cases} 1 & \text{if } n \equiv 0, 4 \ (mod\ 5) \\ 2 & \text{if } n \equiv 2, 3 \ (mod\ 5) \\ 3 & \text{if } n \equiv 1 \ (mod\ 5) \end{cases}$$

Proof. Let $G \cong C_n(1,2)$. Let $V(G) = \{v_1, v_2, v_3, \ldots v_n\}$ be the vertex set of G and $D = \{v_{5k-4} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$ is a γ_{cer}-set of G. For every vertex $v \in D$, $N(v) \geq 2$. By theorem 6 and theorem 3 , we clearly see that $\gamma(G) = \gamma_{cer}(G) = \lceil \frac{n}{5} \rceil$.

Case (i) $n \equiv 0, 4 \pmod 5$

If $n \equiv 0 \pmod 5$ then $|pn(v, D)| = 4$ for each vertex $v \in D$. If $n \equiv 4 \pmod 5$ then $|pn(v, D)| = 4$ for all vertex $v \in D$ except v_1 and v_{n-3} for which $|pn(v_1, D)| = 3$ and $|pn(v_{n-3}, D)| = 3$. Here, $v_{n-1} \in pn(v_1, D) \cap pn(v_{n-3}, D)$.

For $n \equiv 0, 4 \pmod 5$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ (say) by a subdivision vertex x. Here, we clearly see that $v_n \notin N_{G'}(D)$. Hence $D' = D \cup \{v_n\}$ is a γ_{cer}-set of G'. Therefore, $Sd^+_{\gamma_{cer}}(G) = 1$.

Case (ii) $n \equiv 2, 3 \pmod 5$

If $n \equiv 2 \pmod 5$ then $|pn(v, D)| = 4$ for all vertex $v \in D$ except v_1 and v_{n-1} for which $|pn(v_1, D)| = 2$ and $|pn(v_{n-1}, D)| = 2$. Here v_n is a non-private neighbour of the vertices v_1 and v_{n-1}. If $n \equiv 3 \pmod 5$ then $|pn(v, D)| = 4$ for all vertex $v \in D$ except v_1 and v_{n-2} for which $|pn(v_1, D)| = 2$ and $|pn(v_{n-2}, D)| = 2$. Here, $v_n \notin pn(v_1, D)$ *and* $v_{n-1} \notin pn(v_{n-2}, D)$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ [or v_1v_{n-1}] by a subdivision vertex x_1 (or y_1) . Here, D is the γ_{cer}-set of G'. Therefore, $Sd^+_{\gamma_{cer}}(G) > 1$. Now G'' be a graph obtained from G' by subdividing an edge $e = v_nv_{n-1}$

by a subdivision vertex x_2. For $n \equiv 2 \pmod 5$, we see that $v_n \notin N_{G''}(D)$. Hence $D' = D \cup \{v_n\}$ is a γ_{cer}-set of G''. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$. Now for $n \equiv 3 \pmod 5$ $x_2 \notin N_{G''}(D)$. Hence $D' = D \cup \{x_2\}$ is a γ_{cer}-set of G''. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$, refer Figure 1 .

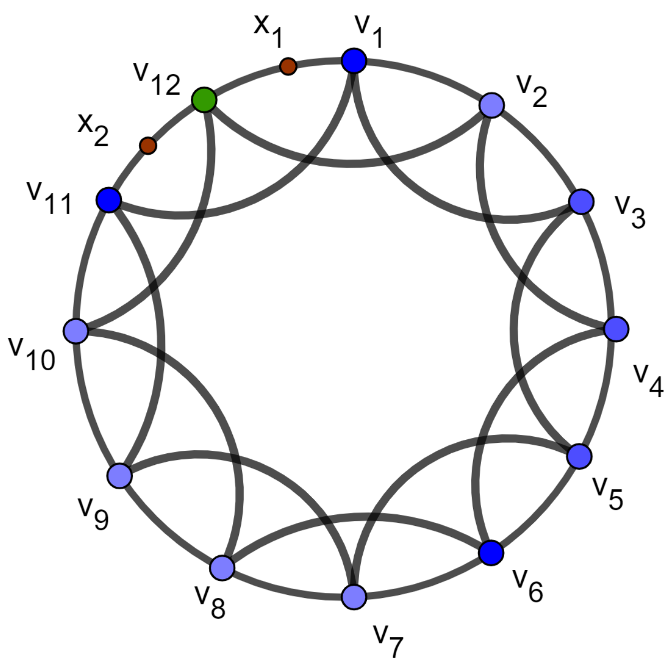

Figure 1: **A graph illustrating case (ii) of theorem 7,** $Sd^+_{\gamma_{cer}}(G) = 2$

Case (iii) $n \equiv 1 \pmod 5$
If $n \equiv 1 \pmod 5$, then $|pn(v, D)| = 4$ for all vertex $v \in D$ except v_n and v_1 for which $|pn(v_1, D)| = 2$ and $|pn(v_n, D)| = 1$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ [or v_1v_{n-1}] by a subdivision vertex y. Here y is dominated by v_1 and v_{n-1} is dominated by v_n. Hence, D is the γ_{cer}-set of G'. Therefore, $Sd^+_{\gamma_{cer}}(G) > 1$. Now let G'' be a graph obtained from G by subdividing the 2 edges e_1 and e_2 by subdivision vertices x_1 and x_2 respectively. Now we have the following subcases.

Subcase (a) $e_1 = v_1v_n$ and $e_2 = v_1v_2$ [adjacent edges].
In this subcase, $x_1, x_2 \in N(v_1)$ and $v_2 \in N(v_n)$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$.

Subcase (b) $e_1 = v_1v_n$ and $e_2 = v_6v_7$ [non adjacent edges in the outer cycle]
Here $x_1 \in N(v_1)$ and $x_2 \in N(v_6)$ and to dominate v_7, $D_1 = \{D - \{(v_{5k+6}) : 1 \leq k \leq \frac{n-6}{5}\}\} \cup \{v_{5k+4} : 1 \leq k \leq \frac{n-6}{5}\}$ is the new configuration of the certified dominating set D

and $|D| = |D_1|$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$.

Subcase (c) $e_1 = v_1v_n$ and $e_2 = v_1v_{n-1}$ [adjacent edges with one edge in the outer cycle]
In this subcase, $x_1, x_2 \in N(v_1)$ and $v_{n-1} \in N(v_n)$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$.

Subcase (d) $e_1 = v_1v_n$ and $e_2 = v_6v_8$ [non adjacent edges with one edge in the outer cycle]
In this subcase, $x_1 \in N(v_1)$ and $x_2 \in N(v_6)$ as in subcase (b), D_1 is the certified dominating set of G'' and $v_8 \in N(v_9)$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$.

Subcase (e) $e_1 = v_1v_{n-1}$ and $e_2 = v_1v_3$ [adjacent edges, which are not in the outer cycle]
Here, $x_1, x_2 \in N(v_1)$. In order to dominate v_3, $D_1 = \{v_{5k} : 1 \leq k \leq \frac{n-1}{5}\}$ is the new configuration of the certified dominating set D, and $|D| = |D_1|$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$.

Subcase (f) $e_1 = v_1v_3$ and $e_2 = v_4v_6$ [Non adjacent edges, which are not in the outer cycle]
In this subcase, $x_1 \in N(v_1)$ and $x_2 \in N(v_4)$. Hence $\gamma_{cer}(G) = \gamma_{cer}(G'')$. From all the above subcases we see that $Sd^+_{\gamma_{cer}}(G) > 2$. Let G''' be the graph obtained from G by subdividing 3 edges $e_1 = v_1v_n$, $e_2 = v_1v_2$ and $e_3 = v_2v_3$ by the subdivision vertices x_1, x_2 and x_3 respectively. Here $x_3 \notin N(v)$ for every vertex $v \in D$. To dominate x_3, set $D_1 = \{v_{5k-2} : 1 \leq k \leq \frac{n-1}{5}\} \cup \{v_1\}$ which is the new configuration of the certified dominating set D. But $\{v_n, v_2\} \notin N_{G'''}(D_1)$. So D_1 is not a certified dominating set of G'''. Hence $D_2 = D_1 \cup \{v_n\}$ is the certified dominating set of G'''. Therefore, $Sd^+_{\gamma_{cer}}(G) = 3$.

Theorem 8. *[19] For any integer $n \geq 6$,* $\gamma(C_n(1,3)) = \begin{cases} \lceil \frac{n}{5} \rceil, & n \notin 4 \ (mod\ 5) \\ \lceil \frac{n}{5} \rceil + 1, & n \equiv 4 \ (mod\ 5) \end{cases}$

Theorem 9. *For any Circulant graph $G \cong C_n(1,3)$, $n \geq 6$,*

$$Sd^+_{\gamma_{cer}}(G) = \begin{cases} 1 & \textit{if } n \equiv 0, 3 (mod\ 5) \\ 2 & \textit{otherwise} \end{cases}$$

Proof. Let $G \cong C_n(1,3)$. Let $V(G) = \{v_1, v_2, v_3, \ldots v_n\}$ be the vertex set of G and $D = \{v_{5k-4} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$ be a γ_{cer}-set of G. For every vertex $v \in D$, $N(v) \geq 2$. By theorem 3 and theorem 8, we clearly see that $\gamma(G) = \gamma_{cer}(G) = \lceil \frac{n}{5} \rceil$.

Case (i) $n \equiv 0$ (mod 5)
Here $|pn(v, D)| = 4$ for each vertex $v \in D$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_2$ (say) by a subdivision vertex x, Here x is dominated by v_1, and $v_2 \notin N(v)$ for all vertex $v \in D$. Hence $D_1 = D \cup \{v_2\}$ is the γ_{cer}-set of G'. Therefore, $Sd^+_{\gamma_{cer}}(G) = 1$, refer Figure 2.

Case (ii) $n \equiv 3$ (mod 5)

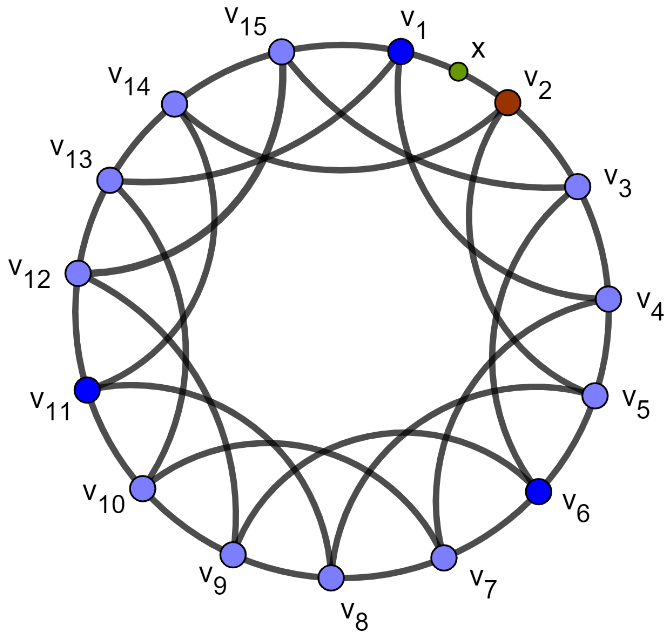

Figure 2: **A graph illustrating case (i) of theorem 9,** $Sd^{+}_{\gamma_{cer}}(G) = 1$

Here $|pn(v, D)| = 4$ for all vertex $v \in D$ except the vertices v_1 and v_{n-2} for which $|pn(v_1, D)| = 3$ and $|pn(v_{n-2}, D)| =3$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_2$ (say) by a subdivision vertex x, here $x \in N(v_1)$ in G' and $v_2 \notin N(v)$, for all $v \in D$. Hence $D_1 = D \cup \{v_2\}$ is the γ_{cer}-set of G'. Therefore, $Sd^{+}_{\gamma_{cer}}(G) = 1$.

Case (iii) $n \equiv 1 \pmod 5$
Here $|pn(v, D)| = 4$ for all vertex $v \in D$ except the vertices v_1 and v_n for which $|pn(v_1, D)| = 2$ and $|pn(v_n, D)| = 2$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ (or $e = v_1v_{n-2}$) by a subdivision vertex x. Here, $x \in N(v_1)$, $v_{n-2} \in N[v_{n-5}]$. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore $Sd^{+}_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G' by subdividing an edge $e = v_1v_2$ (say) by a subdivision vertex y and $y \in N(v_1)$. Now $v_2 \notin N_{G''}(D)$. Here $D_1 = D \cup \{v_2\}$ is a γ_{cer} set of G''. Hence, $\gamma_{cer}(G'') > \gamma_{cer}(G')$. Therefore $Sd^{+}_{\gamma_{cer}}(G) = 2$.

Case (iv) $n \equiv 2 \pmod 5$
Here $|pn(v, D)| = 4$ for all vertex $v \in D$ except the vertices v_1 and v_{n-1} for which $|pn(v_1, D)| = 1$ and $|pn(v_{n-1}, D)| = 1$ and v_n is a non-private neighbour. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ [or $e = v_1v_{n-2}$] by a subdivision vertex x. Here $x \in N(v_1)$,(or $x \in N(v_{n-1})$). Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing the edges $e_1 = v_1v_{n-2}$ and $e_2 = v_1v_4$ by a subdivision vertices x and y respectively, and $x, y \in N(v_1)$. Now

$D_1 = D \cup \{v_4\}$ is a γ_{cer} set of G''. Hence $\gamma_{cer}(G'') > \gamma_{cer}(G')$. That is $|D_1| > |D|$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$.

Case (v) $n \equiv 4 \pmod 5$

In this case $D \cup \{v_{n-1}\}$ is the γ_{cer}-set of G . Here $|pn(v, D)| = 4$ for all vertex $v \in D$ except the vertices v_1, v_{n-1} and v_{n-3} for which $|pn(v_1, D)| = 2$, $|pn(v_{n-1}, D)| = 0$ and $|pn(v_{n-3}, D)| = 2$. Here $v_n, v_{n-2} \notin pn(v, D)$ for all $v \in D$. Let G' be a graph obtained from G by subdividing an edge $e = v_1v_n$ (or $e = v_1v_{n-2}$) by a subdivision vertex x. Here $x \in N(v_1)$, $v_n \in N(v_{n-1})$, $[v_{n-2} \in N(v_{n-3})]$. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore $Sd^+_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing the edges $e_1 = v_1v_n$ and $e_2 = v_5v_6$ by a subdivision vertices x and y respectively, $x \in N(v_1)$, and $v_4 \notin N(v)$ for all $v \in D$. Hence we have $D_1 = D \cup \{v_4\}$ is a γ_{cer} set of G''. So $\gamma_{cer}(G'') > \gamma_{cer}(G')$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$.

5. Generalized Petersen Graphs

The *generalized Petersen graph* $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i, u_i : 0 \le i \le n-1\}$ and $E(P(n, k)) = \{v_i, v_{i+1,}, v_i, u_i, u_iu_{i+k} : 0 \le i \le n-1\}$ subscipts taken modulo n. The edges u_iv_i for $0 \le i \le n-1$ are called the spokes.

In this section we find the value of $Sd^+_{\gamma_{cer}}(G)$ for the generalised Petersen graphs $P(n, 1)$ and $P(n, 2)$

Theorem 10. *[20] For* $n \ge 3$*,*

$$\gamma(P(n,1)) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 0,1,3 \ (mod\ 4) \\ \lceil \frac{n}{2} \rceil + 1, & n \equiv 2 \ (mod\ 4) \end{cases}$$

Theorem 11. *For any Petersen graph* $G \cong P(n, 1)$*,* $n \ge 4$*,*

$$Sd^+_{\gamma_{cer}}(G) = \begin{cases} 1 & if\ n \equiv 0,1 \ (mod\ 4) \\ 2 & if\ n \equiv 3 \ (mod\ 4) \\ 3 & if\ n \equiv 2 \ (mod\ 4) \end{cases}$$

Proof. Let $G \cong P(n, 1)$. Let C' and C'' be the inner and outer cycles of G respectively. Let $V(C') = \{v_1, v_2, v_3, \ldots v_n\}$ and $V(C'') = \{u_1, u_2, u_3, \ldots u_n\}$, by theorem 3 and theorem 10, hence, $\gamma(P(n, 1)) = \gamma_{cer}(P(n, 1))$. Let D be a γ_{cer}-set of G and $D = D' \cup D''$ where $D' = D \cap V(C')$ and $D'' = D \cap V(C'')$. Let $D' = \{v_{4k+1} : 0 \le k \le \lfloor \frac{n}{4} \rfloor\}$ for all n and $D'' = \begin{cases} u_{4k+3}, 0 \le k < \lceil \frac{n}{4} \rceil \cup \{u_n\} & \text{for } n \equiv 2 \pmod 4 \\ u_{4k+3}, 0 \le k < \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$

Now consider the following cases

Case (i) $n \equiv 0 \pmod 4$
In this case each vertex in D dominates exactly 4 vertices including itself. Let G' be a graph obtained from G by subdividing an edge $e = u_1u_n$ by a subdivision vertex x, we clearly see that $x \notin N(v)$ for all $v \in D$. Hence $D_1 = D \cup \{x\}$ is a γ_{cer}- set of G'. This implies that $\gamma_{cer}(G') > \gamma_{cer}(G)$. Hence $Sd^+_{\gamma_{cer}}(G) = 1$.

Case (ii) $n \equiv 1 \pmod 4$
Let G' be a graph obtained from G by subdividing an edge $e = u_1v_1$ by a subdivision vertex x, we clearly see that $x \in N(v_1)$. In order to dominate u_1, the position of the D will be changed to D' where the dissimilar sets are $D' = \{x\} \cup \{v_{4k-2}, 1 \leq k < \frac{n+2}{4}\} \cup \{u_{4k}, 1 \leq k \leq \frac{n-1}{4}\}\}$ and $D'' = \{\{u_{3k-2}, 1 \leq k \leq \frac{n+2}{4}\} \cup \{u_{4k-2}, 1 \leq k \leq \frac{n+1}{4}\}\}$. Hence $\gamma_{cer}(G') > \gamma_{cer}(G)$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 1$.

Case (iii) $n \equiv 3 \pmod 4$

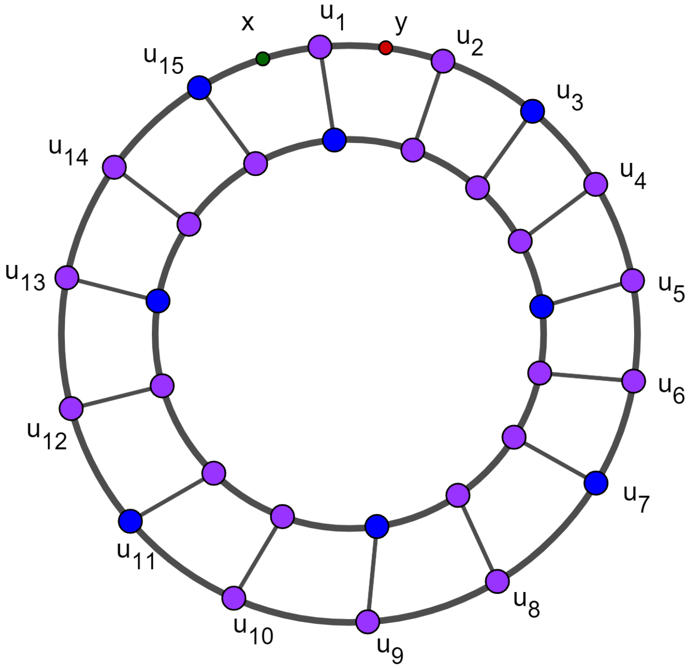

Figure 3: **A graph illustrating case (iii) of theorem 11,** $Sd^+_{\gamma_{cer}}(G) = 2$

Let G' be a graph obtained from G by subdividing an edge $e = u_1u_n$ [or $e = u_nv_n$, or $e = v_1v_n$] by a subdivision vertex x, here $x \in N(u_n)$ [or $N(u_n)$ or $N(v_1)$]. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore, $Sd^+_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing two edges (say) $e_1 = u_1u_n$ and $e_2 = u_1u_2$ by a subdivision vertices x and y respectively, $x \in N(u_n)$ here $y \notin N[D]$. Now $D' = D \cup \{y\}$ is a γ_{cer}-set of G''. Hence, $Sd^+_{\gamma_{cer}}(G) = 2$, refer Figure 3.

Case (iv) $n \equiv 2 \pmod 4$
Let G' be a graph obtained from G by subdividing an edge $e = u_1u_n$ [or $e = u_nv_n$ or $e = v_1v_n$] by a subdivision vertex x, here $x \in N(u_n)$ and $u_1 \in N(v_1)$ [or $N(u_n)$ or $N(v_1)$] respectively. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore $Sd^+_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing two edges (say) e_1 and e_2 by a subdivision vertices x and y respectively. We have the following subcases.

Subcase (a) $e_1 = u_1u_n$ and $e_2 = u_nu_{n-1}$ (adjacent edges in the outer cycle)
In this subcase, $x, y \in N(u_n)$ and $u_{n-1} \in N(v_{n-1})$ and $u_1 \in N(v_1)$, hence $\gamma_{cer}(G'') = \gamma_{cer}(G)$.

Subcase (b) $e_1 = u_nu_{n-1}$ and $e_2 = v_nv_{n-1}$ (an edge in the outer cycle and an edge in the inner cycle)
In this subcase, $x \in N(u_n)$ and $y \in N(v_{n-1})$ and $v_{n-1} \in N(u_{n-1})$, hence $\gamma_{cer}(G'') = \gamma_{cer}(G)$.

Subcase (c) $e_1 = u_1v_1$ and $e_2 = u_nv_n$ (edges in the spokes)
In this subcase, $x \in N(v_1)$ and $y \in N(u_n)$, $u_1 \in N(u_n)$ and $v_n \in N(v_1)$. Hence $\gamma_{cer}(G'') = \gamma_{cer}(G)$.

Subcase (d) $e_1 = v_1v_n$ and $e_2 = v_nv_{n-1}$ (adjacent edges in the inner cycles)
In this subcase, $x \in N(v_1)$ and $y \in N(v_{n-1})$, $v_n \in N(u_n)$. Hence $\gamma_{cer}(G'') = \gamma_{cer}(G)$.

Subcase (e) $e_1 = u_1u_n$ and $e_2 = u_{n-1}v_{n-1}$ (non adjacent edges with one edge in the outer cycle and another edge in the spoke)
In this subcase, $x \in N(u_n)$ and $y \in N(v_{n-1})$, $u_{n-1} \in N(u_n)$, $u_1 \in N(v_1)$. Hence $\gamma_{cer}(G'') = \gamma_{cer}(G)$.

Subcase (f) $e_1 = u_nv_n$ and $e_2 = v_{n-1}v_n$ (an edge in spoke and an edge in inner cycle)
In this subcase $x \in N(u_n)$ and $y \in N(v_{n-1})$, $u_{n-1} \in N(u_n)$, $v_n \in N(v_1)$. Hence, $\gamma_{cer}(G'') = \gamma_{cer}(G)$.
From all the cases mentioned above clearly, we see that $Sd^+_{\gamma_{cer}}(G) > 2$. Let G''' be a graph obtained from G by subdividing the edges $e_1 = u_nu_1$, $e_2 = u_nu_{n-1}$ and $e_3 = u_1u_2$ by the subdivision vertices x, y and z respectively, where $x, y \in N(u_n)$ and $z \notin N(v)$. Hence, $D' = D \cup \{z\}$ is a γ_{cer}-set of G'''. Therefore, $Sd^+_{\gamma_{cer}}(G) = 3$.

Theorem 12. *[20] For $n \geq 5$, we have $\gamma(P(n, 2)) = \lceil \frac{3n}{5} \rceil$*

Theorem 13. *For any Petersen graph $G \cong P(n,2)$, $n \geq 5$,*

$$Sd^{+}_{\gamma_{cer}}(G) = \begin{cases} 1 & \text{if } n \equiv 0,3 \ (mod\ 5) \\ 2 & \text{otherwise} \end{cases}$$

Proof. Let $G \cong P(n,2)$, C' and C'' be the inner and outer cycles of G. Let $V(C') = \{v_1, v_2, v_3, \ldots v_n\}$ and $V(C'') = \{u_1, u_2, u_3, \ldots u_n\}$ by theorem 3 and theorem 12, $\gamma(P(n,2)) = \gamma_{cer}(P(n,2))$. Let D be a γ_{cer}-set of G and $D = D' \cup D''$, where $D' = D \cap V(C')$ and $D'' = D \cap V(C'')$. Now consider the following cases

Case (i) $n \equiv 0 \pmod 5$
Let $D' = \{v_{5k-4} : 1 \leq k \leq \frac{n}{5}\} \cup \{v_{5k-3} : 1 \leq k \leq \frac{n}{5}\}$ and $D'' = \{u_{5k-1} : 1 \leq k \leq \frac{n}{5}\}$ and G' be a graph obtained from G by subdividing an edge $e = u_1u_2$ by a subdivision vertex x, where $x \notin N(v)$ for all $v \in D$. Now $D_1 = D \cup \{x\}$ is the γ_{cer}-set of G'. We clearly see that $|D_1| > |D|$, Hence $\gamma_{cer}(G') > \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) = 1$.

Case (ii) $n \equiv 1 \pmod 5$
Let $D' = \{v_{5k-4} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\} \cup \{v_{5k-3} : 1 \leq k \leq \lfloor \frac{n}{5} \rfloor\}$ and $D'' = \{u_{5k-1} : 1 \leq k \leq \lfloor \frac{n}{5} \rfloor\}$ and G' be a graph obtained from G by subdividing an edge $e = u_3u_4$ (or $e = v_1v_3$), (or $e = u_4v_4$) by a subdivision vertex x, here $x \in N(u_4)$ or $N(v_1)$ or $N(v_6)$ respectively. For the first two category in order to dominate u_3 (or v_3) the configuration of D has to be changed to $D_1 = D - \{v_2\} \cup \{u_3\}$ again $|D_1| = |D|$. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing the two edges (say) $e_1 = u_3u_4$ and $e_2 = u_4u_5$ by a subdivision vertices x and y respectively. Here, $x \in N(u_3)$, $y \in N(u_4)$. $u_5 \notin N(v)$. Now $D' = D \cup \{u_5\}$ is the γ_{cer}-set of G''. Hence $\gamma_{cer}(G'') > \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) = 2$.

Case (iii) $n \equiv 2 \pmod 5$
Let $D' = \{v_{5k-4} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\} \cup \{v_{5k-3} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$ and $D'' = \{u_{5k-1} : 1 \leq k \leq \lfloor \frac{n}{5} \rfloor\}$ and G' be a graph obtained from G by subdividing an edge $e = u_3u_4$ (or $e = u_4v_4$, or $e = v_2v_4$) by a subdivision vertex x. Here $x \in N(u_4)$ and In order to dominate u_3 the position of the D is changed to D_1, where $D_1 = (D - \{v_2\}) \cup \{u_2\}$, (or $x \in N(u_4)$ or $x \in N(v_2)$). Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) > 1$. Let G'' be a graph obtained from G by subdividing two edges $e_1 = u_1v_1$ and $e_2 = u_nv_n$ by a subdivision vertices x and y respectively. Here $x \in N(v_1)$ and $y \in N(v_n)$. Now $u_1, u_n \notin N[D_1]$. Now $D_2 = D_1 \cup \{u_n\}$. Hence $\gamma_{cer}(G'') > \gamma_{cer}(G)$. Therefore, $Sd^{+}_{\gamma_{cer}}(G) = 2$.

Case (iv) $n \equiv 4 \pmod 5$
Let $D' = \{v_{5k-4} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\} \cup \{v_{5k-3} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$ and $D'' = \{u_{5k-1} : 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$ and G' be a graph obtained from G by subdividing an edge $e = u_3u_4$ [or $e = u_4v_4$, or $e = v_2v_4$] by a subdivision vertex x. Here, $x \in N(u_4)$ and in order to dominate u_3 the position of the D is changed to D_1, where $D_1 = (D - \{v_2\}) \cup \{u_2\}$,[or $x \in N(u_4)$ or $x \in N(v_2)$]. Hence $\gamma_{cer}(G') = \gamma_{cer}(G)$. Therefore $Sd^{+}_{\gamma_{cer}}(G) > 1$. Let G'' be a graph

obtained from G by subdividing two edges $e_1 = u_2u_3$, $e_2 = u_3u_4$ by a subdivision vertices x and y respectively. Here, $x \in N(u_2)$ and $y \in N(u_4)$. Now $u_3 \notin N[D_1]$. Now $D_2 = D_1 \cup \{u_3\}$ is a γ_{cer}-set of G'. Hence $\gamma_{cer}(G'') > \gamma_{cer}(G)$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 2$.

Case (v) $n \equiv 3 \pmod 5$
Let D be a γ_{cer}-set of G. $D = D' \cup D''$ and $|D| = \lceil \frac{3n}{5} \rceil$, where $D' = \{v_{5k-4}, 1 \leq k \leq \lceil \frac{n}{5} \rceil\}$

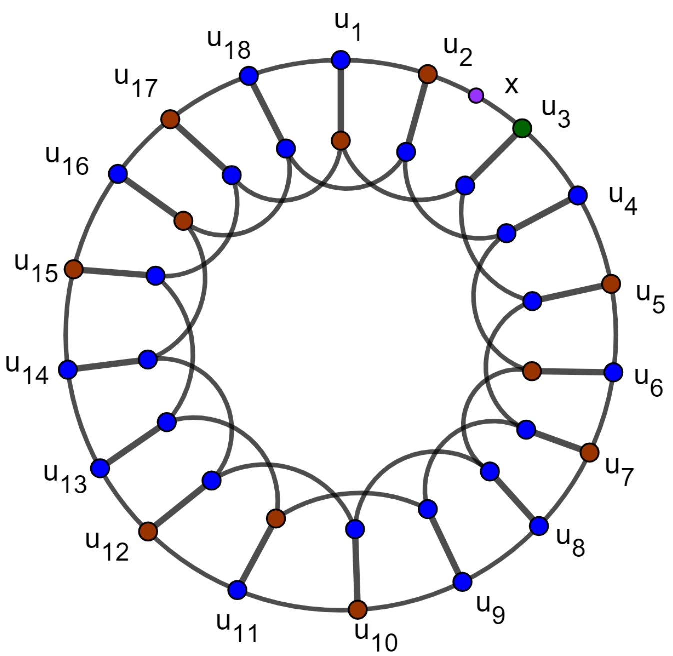

Figure 4: **A graph illustrating case (v) of theorem 13,** $Sd^+_{\gamma_{cer}}(G) = 1$

and $D'' = \{u_{5i-3}, 1 \leq i \leq \lceil \frac{n}{5} \rceil\} \cup \{u_{5j}, 1 \leq j \leq \lfloor \frac{n}{5} \rfloor\}$. Let G' be a graph obtained from G by subdividing an edge $e = u_2u_3$ by a subdivision vertex x . Here, $x \in N(u_2)$, $u_3 \notin N[D]$. Hence $D_1 = D \cup \{u_3\}$ is the γ_{cer}-set of G'. Hence $\gamma_{cer}(G') > \gamma_{cer}(G)$. Therefore, $Sd^+_{\gamma_{cer}}(G) = 1$.

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6. Conclusion

In conclusion, this study establishes the certified domination subdivision number as a measure of how the certified domination number of a graph responds to edge subdivisions, revealing distinct behaviors across different graph classes. For circulant graphs $C_n(1,2)$ and $C_n(1,3)$, as well as Petersen graphs $P(n,1)$ and $P(n,2)$, we identify the precise number of subdivisions required to alter the parameter, thereby offering new insights into structural properties of these graphs with potential applications in network design and optimization. The findings further suggest that strategic edge subdivisions can reduce the number of facility centers without compromising service quality, highlighting their relevance to urban planning and resource management. Beyond these results, the concept of certified domination subdivision numbers opens promising avenues for future research, including extensions to bipartite graphs, hypercubes, grid graphs, chordal graphs, and random graphs, as well as the development of efficient algorithms for computing $Sd^{+}_{\gamma_{cer}}(G)$ and $Sd^{-}_{\gamma_{cer}}(G)$ in large-scale networks. Comparative analyses with related parameters such as bondage, reinforcement, and classical subdivision numbers, along with empirical validation on synthetic and real-world networks, may yield deeper theoretical insights and strengthen the practical significance of these concepts in dynamic and weighted network models.

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