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A Method Employing Legendre Wavelets and a Finite Iterative Approach for Efficiently Solving Systems of Linear Fredholm Integral Equations

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Abstract. Integral equations are essential in numerous domains of practical mathematics. This article introduces a straightforward, precise, and efficient iterative technique for resolving one-dimensional Fredholm integral equations of the second class. The suggested numerical method relies on Legendre wavelet functions. Utilizing these wavelets, the integral equation system is converted into a duo of interconnected systems of algebraic matrix equations. A finite iterative approach is employed to resolve these systems and ascertain the coefficients that formulate the approximate numerical solutions of the unknown functions. A variety of examples are included to evaluate the proposed numerical approach.

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1. Introduction

Integral equations are essential in the mathematical modeling of numerous applied science and engineering challenges, encompassing areas such as quantum physics, thermal analysis, and signal processing ([1–3]). Due to the frequent inaccessibility of precise solutions for several integral equations, researchers have developed a diverse array of analytical and computational methods to approximate solutions for various types of integral equations ([4–11]). These techniques often include orthogonal polynomials, including Legendre, Bernstein, Jacobi, Chebyshev, and triangular functions, and Laguerre polynomials, which are extensively applied in resolving integrals that involve special functions.

Integral equation systems (IES), especially of the Fredholm variety, typically cannot be resolved in closed form, requiring effective numerical methods. Researchers have suggested a diverse range of such methodologies. Babolian et al. ([12, 13]) presented direct and decomposition methods; Jafarian and colleagues ([14, 15]) utilized neural networks and Bernstein collocation; and Mahmoodi [16] implemented collocation and spectral methods for both Fredholm and Volterra systems. Alipour et al. [17] expanded these concepts to coupled systems, whereas Huang et al. [18] employed Taylor expansion. Maleknejad et al. [19] utilized block pulse functions (BPFs). Ramadan et al. ([20, 21]) proposed extended and generalized finite iterative solution methodologies. Maleknejad et al. [22] achieved novel results utilizing Taylor series for first-kind systems. In [23], Ramadan et al. expanded the triangle function in conjunction with a finite iterative technique. Maleknejad et al. [24] employed second-kind Taylor series systems. H. Almasieh and Roodaki [25] employed

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triangular functions to solve a system of Fredholm integral equations, whereas Golbabai and Keramati [26] proposed an effective computational methodology. Alternative methodologies encompass triangle function techniques and B-spline wavelet frameworks ([27, 28]). Najafi et al. [29] introduced a linear Legendre multi-wavelets method for addressing systems of Fredholm integral equations, while Z. Elahiet et al. [30] applied the Laguerre method for solving linear systems of Fredholm integral equations. Additionally, Elahi et al. ([31, 32]) utilized Laguerre and Bessel polynomials for integro-differential and differential-difference equations. Recently, Ramadan et al. [31] introduced a method utilizing triangular functions to solve systems of linear Fredholm integral equations by an efficient finite iterative algorithm, while Arafa and Ramadan [33] and Legendre wavelet-based methods were presented for addressing coupled Fredholm systems.
The structure of this paper is as follows:Section 1 provides the introduction. In Section 2, we present key definitions and preliminaries, including an overview of Legendre wavelets, their properties, function approximation, and an iterative algorithm for solving coupled system matrix equations. Section 3 outlines our proposed method for addressing linear systems of Fredholm integral equations of the second kind. We provide the convergence analysis and error estimation in Section 4. Section 5 presents three numerical examples that showcase the precision and effectiveness of the proposed method. Finally, Section 5 concludes the paper by summarizing the main findings.

2. Definitions and Preliminaries

2.1. Legendre wavelet and its properties

Considering a single function "mother wavelet" $\psi(t)$, from which wavelets represent a family of functions by dilating and transforming this single function. This family of continuous wavelets [17] has the following form:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \left(\frac{t-b}{a} \right), a, b \in R, a \neq 0 \tag{2.1}$$

The Legendre wavelets on the interval [0,1) defined by

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m+\frac{1}{2}} 2^{\frac{k}{2}} L_m \left(2^k t - \hat{n}\right), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

for which k is positive integer, $n = 1, 2, \ldots, 2^{k-1}$ and $\hat{n} = 2n - 1$, the order of the Legendre Polynomial is denoted by $m = 0, 1, 2, \ldots, M-1$ and the normalized time is denoted by t. The Legendre Polynomials L_m which are obtained in the above definition is proposed as follows:

$$\begin{aligned} &L_0(t) = 1, \\ &L_1(t) = t, \\ &L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t) \quad , \quad m = 1, 2, 3, \ldots \end{aligned} \tag{2.3}$$

which are orthogonal over [-1,1] with weighting function.

2.2. Function Approximation

A function $f(t)$ which is defined on $[0, 1)$ can be extended as Legendre Wavelet infinite series of the following type

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}, \tag{2.4}$$

where $c_{n,m} = \langle f, \psi_{n,m} \rangle$.
After being trimmed, Eq. (2.4) can be rewritten as follows.

$$f(t) \approx \sum_{n=1}^{2^{k-1}} c_{n,m} \sum_{m=1}^{M} \psi_{\mathrm{n},m} = C^T \psi(t), \tag{2.5}$$

where $C = \begin{bmatrix} c_{l,0} & c_{l,1} & \ldots & c_{l,M-1} & \ldots & c_{2^{k-1},0} & c_{2^{k-1},1} & \cdots c_{2^{k-1},M-1} \end{bmatrix}^T$,
and
$\psi(t) = \begin{bmatrix} \psi_{1,0} & \psi_{1,1} & \ldots & \psi_{1,M-1} & \ldots & \psi_{2^{k-1},0} & \psi_{2^{k-1},1} & \ldots & \psi_{2^{k-1},M-1} \end{bmatrix}^T$.

2.3. Iterative algorithm for solving the coupled system matrix equations

Finite iterative methods are a class of numerical algorithms designed to solve matrix equations and coupled matrix equations by reaching the exact solution in a finite number of steps, assuming exact arithmetic. Unlike traditional iterative methods that converge asymptotically, these methods terminate after a predetermined number of iterations, making them highly efficient and predictable in computational cost. For example, see Ramadan et al. [20] Extended and generalized finite iterative solution techniques previously applied to specific Sylvester-type equations. Moreover, Ramdan et al. [21] developed an iterative method tailored for bisymmetric structured solutions, focused on least-norm generalized solutions. Ramadan and Ali [10] used orthogonal triangular functions to convert the linear system of Fredholm integral equations into a coupled matrix equation, then applies a finite iterative algorithm. In addition, Ramadan et al. [23] extended the triangular function coupled with finite iterative method to 2D fuzzy Fredholm integral equations, transforming them into a coupled algebraic system and solving with finite iteration. Recently, Ramadan et al. [33] approximated the solution of system of linear Fredholm integral equations via a set of orthogonal triangular functions, which converts the continuous integral equations into four coupled algebraic matrix equations. These matrix equations are then tackled with an efficient finite-step iterative algorithm. Based on the finite-step iterative algorithms developed in the literature, particularly those introduced by M.A. Ramadan et al. [10,20,2123,33] and other related works, we propose in this subsection a new iterative algorithm for solving a coupled system of linear matrix equations of the forms:

$$A_1 C_1 + B_1 C_2 = \mathrm{F} \text{ and } A_2 C_1 + B_2 C_2 = \mathrm{G}.$$

Algorithm 2.1
1- Input $A_1, B_1, A_2, B_2, F, \mathrm{G}$
2- Choose arbitrary vectors $C_{1_1} \in C^{n\times 1}$ and $C_{2_1} \in C^{n\times 1}$
3- Set

$$\begin{aligned} R_1 &= \operatorname{diag}\left(F - f\left(C_{1_1}, C_{2_1}\right), \left(\mathrm{G} - g\left(C_{1_1}, C_{2_1}\right)\right), \\ S_1 &= A_1^T \left(\mathrm{F} - f\left(C_{1_1}, C_{2_1}\right)\right) + A_2^T \left(\mathrm{G} - g\left(C_{1_1}, C_{2_1}\right)\right), \\ T_1 &= B_1^T \left(\mathrm{F} - f\left(C_{1_1}, C_{2_1}\right)\right) + B_2^T \left(\mathrm{G} - g\left(C_{1_1}, C_{2_1}\right)\right), \end{aligned}$$

where

$$\begin{aligned} f\left(C_{1_1}, C_{1_1}\right) &= \mathrm{A}_1 C_{1_1} + \mathrm{B}_1 C_{2_1} \\ g\left(C_{1_1}, C_{2_1}\right) &= \mathrm{A}_2 C_{1_1} + \mathrm{B}_2 C_{2_1} \end{aligned}$$

4- If $R_k = 0$ then stop and C_{1_k}, YC_{2_k} is the solution else let $k = k + 1$ go to step 5 .
5- Compute

$$C_{1_{k+1}} = C_{1k} + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k,\ C_{2_{k+1}} = C_{2k} + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k,$$
$$R_{k+1} = \operatorname{diag}\left(\mathrm{F} - f\left(C_{1_{k+1}}, C_{2_{k+1}}\right), \left(\mathrm{G} - g\left(C_{1_{k+1}}, C_{2_{k+1}}\right)\right)\right.$$
$$= R_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} \operatorname{diag}\left(f\left(S_k, T_k\right), g\left(S_k, T_k\right)\right),$$
$$S_{k+1} = A_1^T\left(F - f\left(Y_{1_{k+1}}, Y_{2_{k+1}}\right)\right) + A_2^T\left(\mathrm{G} - g\left(C_{1_{k+1}}, C_{2_{k+1}}\right)\right) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k,$$
$$T_{k+1} = V\left(\ \mathrm{F} - f\left(Y_{1_{k+1}}, Y_{2_{k+1}}\right)\right) + B_2^T\left(\mathrm{G} - g\left(C_{1_{k+1}}, C_{2_{k+1}}\right)\right) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k.$$

3. Legendre wavelet method for linear Fredholm integral equations system of second kind

The proposed approach begins by using the Legendre wavelet method to convert the coupled linear system if Fredholm integral equations into a system of coupled matrix algebraic equations. Next, Algorithm 2.1 is employed to solve the resulting system of Sylvester matrix equations, yielding the solution function for the original problem. First, consider the following equations:

$$\begin{aligned} u(x) &= f(x) + \lambda_1\left(\int_0^1 \left(k_1(x,t)u(t)\right)dt + \int_0^1 \left(k_2(x,t)v(t)\right)\right)dt \\ v(x) &= g(x) + \lambda_2\left(\int_0^1 \left(k_3(x,t)u(t)\right)dt + \int_0^1 \left(k_4(x,t)v(t)\right)\right)dt \end{aligned} \tag{3.1}$$

where $k_1(x,t), k_2(x,t) \in L_2([0,1]\times[0,1])$ and $h(x), g(x) \in L_2([0,1])$
The unknown functions $u(x), v(x)$ can be expanded as

$$\mathrm{u(x)} \approx C_1^T\psi(t), \quad \mathrm{v(x)} \approx \mathrm{C}_2^{\mathrm{T}}\psi(t) \tag{3.2}$$

where C_1 and C_2 are the unknown $2^{k-1}M$ vectors and $\psi(t)$ is given by Eq. (2.5) and (2.6). Likewise, $k_1(x,t), k_2(x,t), k_3(x,t), k_4(x,t), f(x), g(x)$ are also expanded into the LWM as:

$$\begin{aligned} &k_1(\mathrm{x},\mathrm{t}) \approx \psi^{\mathrm{T}}(\mathrm{x})K_1\psi(\mathrm{t}), \quad k_2(\mathrm{x},\mathrm{t}) \approx \psi^{\mathrm{T}}(\mathrm{x})K_2\psi(\mathrm{t}) \\ &k_3(\mathrm{x},\mathrm{t}) \approx \psi^{\mathrm{T}}(\mathrm{x})K_3\psi(\mathrm{t}), \quad k_4(\mathrm{x},\mathrm{t}) \approx \psi^{\mathrm{T}}(\mathrm{x})K_4\psi(\mathrm{t}) \\ &\mathrm{f(x)} \approx F^T\psi(\mathrm{x}), \mathrm{g(x)} \approx G^T\psi(\mathrm{x}) \end{aligned} \tag{3.3}$$

After substituting the approximate equations (3.2) - (3.3) into (3.1) we get,

$$\begin{aligned} \psi^T(x)C_1 &= \psi^T(x)F + \lambda_1\psi^T(x)K_1\int_0^1 \psi(t)\psi^T(t)C_1dt \\ &\quad + \lambda_1\emptyset^T(x)K_2\int_0^1 \psi(t)\psi^T(t)C_2dt \\ \psi^T(x)C_2 &= \psi^T(x)G + \lambda_2\psi^T(x)K_3\int_0^1 \psi(t)\psi^T(t)C_1dt \\ &\quad + \lambda_2\psi^T(x)K_4\int_0^1 \psi(t)\psi^T(t)C_2dt \end{aligned} \tag{3.4}$$

where,

$$\int_0^1 \psi(t)\psi^{\mathrm{T}}(t)dt = \mathrm{I} \tag{3.5}$$

Making use of Eq. (3.5), we get,

$$\begin{aligned} \psi^{\mathrm{T}}(\mathrm{x})C_1 &= F^T\psi(\mathrm{x}) + +\lambda_1\left[\psi^{\mathrm{T}}(\mathrm{x})K_1C_1 + \psi^{\mathrm{T}}(\mathrm{x})K_2C_2\right], \\ \psi^{\mathrm{T}}(\mathrm{x})C_2 &= G^T\psi(\mathrm{x}) + +\lambda_2\left[\psi^{\mathrm{T}}(\mathrm{x})K_3C_1 + \psi^{\mathrm{T}}(\mathrm{x})K_4C_2\right]. \end{aligned} \tag{3.6}$$

Therefore,

$$C_1 \approx F + \lambda_1\left(K_1C_1 + K_2C_2\right) \text{ and } C_2 \approx G + \lambda_2\left(K_3C_1 + K_4C_2\right). \tag{3.7}$$

or,

$$\left(\mathrm{I} - \lambda_1K_1\right)C_1 - \lambda_1K_2C_2 = \mathrm{F} \text{ and } \left(\mathrm{I} - \lambda_2K_3\right)C_2 - \lambda_2K_4C_1 = \mathrm{G}. \tag{3.8}$$

After replacing $\approx$ with $=$, we have a linear system that can be solved with using finite iterative algorithm for the unknown vectors C_1, C_2 then by the use of $u(x) \approx {C_1}^T\psi(x), \mathrm{v(x)} \approx {C_2}^T\psi(\mathrm{x})$ the approximated solution is given.

The coupled linear system (3.8) can be further written in the form

$$A_1C_1 + B_1C_2 = \mathrm{F} \text{ and } A_2C_1 + B_2C_2 = \mathrm{G},$$

where $A_1 = I - \lambda_1K_1, B_1 = -\lambda_1K_2, A_2 = I - \lambda_2K_3, B_2 = -\lambda_2K_4$.

4. Convergence Analysis and Error Estimation

This section provides a brief study of the convergence behavior and error bounds of the proposed numerical method, ensuring its accuracy and reliability.

4.1. Convergence analysis

In this section, we will discuss the convergence analysis for our proposed numerical approach.

Theorem 3.1 Using our proposed numerical method, Legendre wavelets coupled with a finite iterative method, the solution of (3.1) converges to $\breve{\boldsymbol{u}}(t) := (\breve{u}(\mathrm{t}), \breve{v}(\mathrm{t}))$ defined in (3.2).

Proof. Let $\{\psi_{n,m}(t)\}_{n,m}$ be the Legendre wavelets forming an orthonoral basis of $L^2(R)$ and let $L^2(R)$ be a Hilbert space. Since $\breve{\boldsymbol{u}}(t)$ is a vector-valued function with two components in $L^2(R)$, we expand it component -wise

$$\breve{u}(t) \approx \sum_{n=1}^{2^{K-1}}\sum_{m=0}^{M-1} {c^{(1)}}_{n,m}\psi_{n,m}(t), \breve{v}(t) \approx \sum_{n=1}^{2^{K-1}}\sum_{m=0}^{M-1} {c^{(2)}}_{n,m}\psi_{n,m}(t),$$

where

$${c^{(1)}}_{n,m} =< \breve{u}(t), \psi_{n,m}(t) >, \quad {c^{(2)}}_{n,m} =< \breve{v}(\mathrm{t}), \psi_{n,m}(t) > .$$

Then, the full approximation of the vector function $\breve{\boldsymbol{u}}(t)$ is

$$\widetilde{\boldsymbol{u}}_{\boldsymbol{N}}(t) \approx \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} c_{n,m}\boldsymbol{\psi}_{n,m}(t), \text{ where } c_{n,m} := \begin{pmatrix} c^{(1)}{}_{n,m} \\ c^{(2)}{}_{n,m} \end{pmatrix},$$

and

$$\boldsymbol{\psi}_{n,m}(t) := \left(\begin{array}{c} \psi_{n,m}^{(1)} \\ \psi_{n,m}^{(2)} \end{array} \right).$$

Relabeling (n, m) into a single index j, we write:

$$\widetilde{\boldsymbol{u}}_N(t) = \sum_{j=1}^{N} \zeta_j \psi\left(t_j\right), \text{ where } \zeta_j = \begin{pmatrix} \zeta^{(1)}{}_j \\ \zeta^{(2)}{}_j \end{pmatrix},$$

here we denote

$$\zeta^{(1)}{}_j :=< \breve{u}(t), \psi_j(t) > \quad \text{and} \quad \zeta^{(2)}{}_j :=< \breve{v}(\mathrm{t}), \psi_j(t) > .$$

To prove convergence, consider the partial sums:

$$S_N = \sum_{j=1}^{N} \zeta_j \psi_j(\mathrm{t}).$$

Let $N > M$. Then,

$$\|S_N - S_M\|^2 = \left\| \sum_{j=M+1}^{N} \zeta_j \psi_j(\mathrm{t}) \right\|^2 = \sum_{j=1}^{n} \|\zeta_j\|^2,$$

due to the orthonomality of $\psi_j(\mathrm{t})$.
Hence,

$$\|S_N - S_M\|^2 = \| \sum_{j=M+1}^{N} \left(\left|\zeta_j^{(1)}\right|^2 + \left|\zeta_j^{(2)}\right|^2 \right),$$

which converges as $M.\ N \to \infty$ by Bessel's inequality. Therefore, $\{S_n\}$ is a cauchy sequence in $L^2(R) \times L^2(R)$, and thus converges to some limit $'s(t)'$.
To show that $s(t) = \breve{u}(t)$, consider

$$\begin{aligned} &< s - \breve{u}, \psi_j > = \lim_{N\to\infty} < S_N - \breve{u}, \psi_j >= \lim_{N\to\infty} (< S_N - \psi_j > - < \breve{u}, \psi_j >) \\ &= \zeta_j - \zeta_j = 0 \end{aligned}$$

Thus, the difference $s(t) - \breve{u}(t)$ is orthogonal to all basis functions $\psi_j(\mathrm{t})$, and therefore:

$$s(t) = \breve{\boldsymbol{u}}(t).$$

Hence, the approximation series $\sum_{j=1}^{\infty} \zeta_j \psi_j(\mathrm{t})$ converges to $\breve{\boldsymbol{u}}(t)$.
Consequently, we have $\breve{\boldsymbol{u}}(t) = s$ and $\sum_{j=1}^{n} \zeta_j \psi\left(t_j\right)$ converges to $\breve{\boldsymbol{u}}(t)$ in $L^2(R) \times L^2(R)$, completing the proof.

4.2. Error Estimation

Suppose that $\breve{\boldsymbol{u}}(t) := (\breve{u}(\mathrm{t}), \breve{v}(\mathrm{t}))$ is the approximate solution and $\boldsymbol{u}(t) := (u(t), \mathrm{v(t)})$ is the exact solution, then the error function $E_n(t)$ is given by the following relation:

$$E_n(t) = \boldsymbol{u}(t) - \breve{\boldsymbol{u}}(t),$$

hence

$$\breve{\boldsymbol{u}}(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) + H_n(t) = C^T\psi(t) + H_n(t),$$

where, $H_n(t)$ is the perturbation term.

$$H_n(t) = \breve{\boldsymbol{u}}(t) - C^T\psi(t) \tag{3.11}$$

so, its easily to see that

$$E_n(t) + C^T\psi(t) = -H_n(t),$$

Hence the stability of our proposed method is established through this convergence theorem and error estimation.

5. Numerical examples

In this section, some numerical examples are provided to illustrate the efficiency and accuracy of our method, which integrates Legendre wavelets with a proposed finite iterative algorithm. These examples are drawn from recent existing literature, allowing us to compare the numerical results of our approach with both exact solutions and those reported in previous studies. All computations were performed using a program developed in MATLAB R2015a.

Example 1
Consider the system of two linear Fredholm integral equations ([12],[29],[33],[34]):

$$\begin{aligned} u_1(t) &= \frac{t}{18} - \frac{17}{36} + \int_{x=0}^{1} \frac{x+t}{3}\left(u_1(x) + u_2(x)\right)dx, \\ u_2(t) &= t^2 - \frac{19}{12}t + 1 + \int_{t=0}^{1} xt\left(u_1(x) + u_2(x)\right)dx, \end{aligned} \tag{4.1}$$

with exact solution $(u_1(t), u_2(t)) = \big(t+1, t^2+1\big)$.
This example has been addressed by several researchers. Initially, Babolian et al. [12] solved it using the Adomian decomposition method. Subsequently, Ramadan et al. [33] tackled it using the triangular basis functions method with $m = 32$ triangular basis functions. Najafi et al. [29] also analyzed the problem using the linear Legendre multi-wavelets method. More recently, Arafa and Ramadan [34] investigated it using Bernoulli wavelets method with parameters $M = 3, k = 2$.

Applying our proposed method (LWM), the unknown functions $u_1(t), u_2(t)$ can be expanded as

$$u_1(t) \approx C_1^T\Psi(t), \quad u_2(t) \approx C_2^T\Psi(t) \tag{4.2}$$

where C_1, C_2 are the unknown $2^{k-1}M$, $M = 3$, $k = 2$ vectors with
$C_1 = \begin{bmatrix} c_{1,0} & c_{1,1} & c_{1,2} & c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}^T$, $C_2 = \begin{bmatrix} c'_{1,0} & c'_{1,1} & c'_{1,2} & c'_{2,0} & c'_{2,1}c'_{2,2} \end{bmatrix}^T$ and $\Psi(t)$ is Legendre wavelets for $\mathrm{M} = 3, \mathrm{k} = 2$:

$$\Psi(t)=\begin{cases}\left.\begin{cases}\Psi_{1,0}=\sqrt{2}\\ \Psi_{1,1}=\sqrt{6}(4t-1)\\ \Psi_{1,2}=\sqrt{10}\left(\frac{3}{2}(4t-1)^2-\frac{1}{2}\right)\end{cases}\right\}, 0\le t<\frac{1}{2},\\ \left.\begin{cases}\Psi_{2,0}=\sqrt{2}\\ \Psi_{2,1}=\sqrt{6}(4t-3)\\ \Psi_{2,2}=\sqrt{10}\left(\frac{3}{2}(4t-3)^2-\frac{1}{2}\right)\end{cases}\right\}, \frac{1}{2}\le t<1.\end{cases}$$

Likewise,

$$k_1(x,t)=\frac{x+t}{3},\ k_2(x,t)=\frac{x+t}{3},\ k_3(x,t)=xt, k_4(x,t)=xt$$
$$f(t)=\frac{t}{18}-\frac{17}{36},\ g(t)=t^2-\frac{19}{12}t+1,$$

are also expanded into the LWM as:

$$\begin{aligned}&k_1(x,t)\approx\Psi^T(x)K_1\Psi(t),\quad k_2(x,t)\approx\Psi^T(x)K_2\Psi(t),\\&k_3(x,t)\approx\Psi^T(x)K_3\Psi(t),\quad k_4(x,t)\approx\Psi^T(x)K_4\Psi(t),\\&f(t)\approx F^T\Psi(t),\quad g(t)\approx G^T\Psi(t).\end{aligned}\tag{4.3}$$

After substituting (4.2), (4.3) into (4.1) we get,

$$\begin{aligned}\Psi^T(t)C_1&=F^T\Psi(t)+\left[\Psi^T(t)K_1C_1+\Psi^T(t)K_2C_2\right],\\ \Psi^T(t)C_2&=G^T\Psi(t)+\left[\Psi^T(t)K_3C_1+\Psi^T(t)K_4C_2\right],\end{aligned}\tag{4.4}$$

which can be written in the coupled system of matrix equations,

$$(I-K_1)\,C_1-K_2C_2=F\ \text{ and }\ (I-K_3)\,C_2-K_4C_1=G.\tag{4.5}$$

We can write (4.5) further in the form:

$$A_1C_1+B_1C_2=F\ \text{ and }\ A_2C_1+B_2C_2=G,\tag{4.6}$$

where,

$$A_1 = I - K_1 = \begin{pmatrix} 0.91667 & -0.024056 & 0 & -0.16667 & -0.024056 & 0 \\ -0.024056 & 1 & 0 & -0.024056 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.16667 & -0.024056 & 0 & 0.75 & -0.024056 & 0 \\ -0.024056 & 0 & 0 & -0.024056 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = -K_2 = \begin{pmatrix} -0.083333 & -0.024056 & 0 & -0.16667 & -0.024056 & 0 \\ -0.024056 & 0 & 0 & -0.024056 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.16667 & -0.024056 & 0 & -0.25 & -0.024056 & 0 \\ -0.024056 & 0 & 0 & -0.024056 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = -K_4 = \begin{pmatrix} -0.03125 & -0.018042 & 0 & -0.09375 & -0.018042 & 0 \\ -0.018042 & -0.010417 & 0 & -0.054127 & -0.010417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.09375 & -0.054127 & 0 & -0.28125 & -0.054127 & 0 \\ -0.018042 & -0.010417 & 0 & -0.054127 & -0.010417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = I - K_3 = \begin{pmatrix} 0.96875 & -0.018042 & 0 & -0.09375 & -0.018042 & 0 \\ -0.018042 & 0.98958 & 0 & -0.054127 & -0.010417 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.09375 & -0.054127 & 0 & 0.71875 & -0.054127 & 0 \\ -0.018042 & -0.010417 & 0 & -0.054127 & 0.98958 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F = \begin{bmatrix} 0.34373 & 0.0056701 & 0 & 0.36337 & 0.0056701 & 0 \end{bmatrix}^T,$$

$$G = \begin{bmatrix} -0.48614 & -0.11057 & 0.013176 & 0.2799 & -0.0085052 & 0.013176 \end{bmatrix}^T.$$

Using our proposed finite iterative algorithm 2.1 to solve the coupled matrix system given in equation (4.6), we obtain the coefficient vectors

$C_1 = \begin{bmatrix} 0.8839, & 0.1021, & 0, & 1.237, & 0.1021, & 0\end{bmatrix}^T$

and

$C_2 = [0.766032, 0.051031, 0.0131762, 1.11959, 0.153093, 0.0131762]^T$.

Substituting these coefficient vectors into a coupled system of equations (4.6) provides the approximate solutions at the corresponding time points, as shown in Table 1.

Table 1: The numerical results for Example 1 with proposed Legendre wavelets for $M = 3$, $k = 2$

t	Exact solution $(u_1(t), u_2(t))$		Presented Method $(u_1(t), u_2(t))$		Absolute Error	
0	1.0	1.0	0.9999999999999529	1.000000000000972	4.7073e − 14	9.7167e − 13
0.1	1.1	1.01	1.099999999999799	1.01000000000094	2.0117e − 13	9.3969e − 13
0.2	1.2	1.04	1.199999999999645	1.040000000000885	3.5505e − 13	8.8507e − 13
0.3	1.3	1.09	1.2999999999999491	1.090000000000808	5.0915e − 13	8.0802e − 13
0.4	1.4	1.16	1.399999999999337	1.160000000000709	6.6303e − 13	7.0877e − 13
0.5	1.5	1.25	1.499999999998751	1.249999999997582	1.2488e − 12	2.4176e − 12
0.6	1.6	1.36	1.599999999998597	1.359999999997396	1.4029e − 12	2.6037e − 12
0.7	1.7	1.49	1.699999999998443	1.489999999997188	1.5568e − 12	2.812e − 12
0.8	1.8	1.64	1.799999999998289	1.639999999996957	1.7109e − 12	3.0429e − 12
0.9	1.9	1.81	1.899999999998135	1.809999999996703	1.8647e − 12	3.2967e − 12

Table 2: Comparison between absolute errors of Example 4.1 for our presented method with $M = 3, k = 2$, the T.F. method with $m = 32$ [33] and Adomian decomposition method proposed in [12].

t	TF method [33]	Absolute Error	Method in [12]	Absolute Error	Presented Method	Absolute Error
Results of $u_1(t)$						
0	1.000077	7.74578E − 05	0.988498	1.15020E − 02	0.99999999999995	4.7073e − 14
0.1	1.100097	9.70027E − 05	1.086632	1.33680 E -02	1.0999999999998	2.0117e − 13
0.2	1.200117	1.16548E − 04	1.184766	1.52340E − 02	1.1999999999996	3.5505e − 13
0.3	1.300136	1.36093E − 04	1.282899	1.71010 E -02	1.2999999999995	5.0915e − 13
0.4	1.400156	1.55637E − 04	1.381033	1.89670 E -02	1.3999999999993	6.6303e − 13
0.5	1.500175	1.75182E − 04	1.479167	2.08330 E -02	1.4999999999988	1.2488e − 12
0.6	1.600195	1.94727E − 04	1.577301	2.26990 E -02	1.5999999999986	1.4029e − 12
0.7	1.700214	2.14272E − 04	1.675435	2.45650 E -02	1.6999999999984	1.5568e − 12
0.8	1.800234	2.33817E − 04	1.773569	2.64310 E -02	1.7999999999983	1.7109e − 12
0.9	1.900253	2.53362 E -04	1.9871702	8.71702 E -02	1.8999999999981	1.8647e − 12
Results of $u_2(t)$						
t	TF method [33]	Absolute Error	Method in [12]	Absolute Error	Presented Method	Absolute Error
0	1.070363	7.03625E − 02	1.000000	0.00000E + 00	71.000000000001	9.7167e − 13
0.1	1.067708	5.77080E − 02	1.006549	3.45100E − 03	1.0100000000009	9.3969e − 13
0.2	1.086082	4.60824E − 02	1.033099	6.90100E − 03	1.0400000000009	8.8507e − 13
0.3	1.125486	3.54856E − 02	1.079648	1.03520E − 02	1.0900000000008	8.0802e − 13
0.4	1.185918	2.59177E − 02	1.146198	1.38020E − 02	1.1600000000007	7.0877e − 13
0.5	1.267379	1.73787E − 02	1.232747	1.72530E − 02	1.2499999999976	2.4176e − 12
0.6	1.369869	9.86860E − 03	1.339296	2.07040E − 02	1.35999999999974	2.6037e − 12
0.7	1.493387	3.38735E − 03	1.465846	2.41540E − 02	1.4899999999972	2.812e − 12
0.8	1.637935	2.06503E − 03	1.612695	2.73050E − 02	1.639999999997	3.0429e − 12
0.9	1.803511	6.48854E − 03	1.778945	3.10550E − 02	1.8099999999967	3.2967e − 12

As shown in Tables 2 and 3, our proposed method yields more accurate results compared to those reported in references [12], [33], and [29]. Furthermore, Table 4 indicates that the accuracy of our method is nearly equivalent to that of the approach presented in reference [34]. In addition to its superior or comparable accuracy, our method is also highly efficient, as it eliminates the need for the computationally intensive matrix inversion when determining the coefficient vectors C_1, C_2.

Example 2

Consider the system of two linear Fredholm integral equations ([12],[25,[33]).

$$\begin{aligned} u_1(t) &= t - \tfrac{5}{18} + \tfrac{1}{3} \int_{x=0}^{1} (u_1(x) + u_2(x))\, dx, \\ u_2(t) &= t^2 - \tfrac{5}{12} + \tfrac{1}{2} \int_{x=0}^{1} (u_1(x) + u_2(x))\, dx, \end{aligned} \tag{4.7}$$

Table 3: Comparison between absolute errors for results of Example 1 for our presented method with $M = 3, k = 2$ and presented method [29].

t	Method in [29] Absolute Error	Presented Method Absolute error	Method in [29] Absolute error	Presented Method Absolute error
Results of $u_1(t)$			Results of $u_2(t)$	
0	0	4.7073e − 14	8.437524563517452E − 01	9.7167e − 13
0.1	0	2.0117e − 13	4.583333669999856E − 03	9.3969e − 13
0.2	0	3.5505e − 13	4.166662600000315E − 04	8.8507e − 13
0.3	0	5.0915e − 13	4.166661900000257E − 04	8.0802e − 13
0.4	0	6.6303e − 13	4.583333879999874E − 03	7.0877e − 13
0.5	0	1.2488e − 12	1.041666597500002E − 02	2.4176e − 12
0.6	0	1.4029e − 12	4.583334199999900E − 03	2.6037e − 12
0.7	0	1.5568e − 12	4.166656999999852E − 04	2.812e − 12
0.8	0	1.7109e − 12	4.166658000002155E − 04	3.0429e − 12
0.9	0	1.8647e − 12	4.583334299999686E − 03	3.2967e − 12

Table 4: Comparison between absolute errors for results of Example 1 for our presented method with $M = 3, k = 2$ and presented method in [34].

t	Method in[34] Absolute error	Presented Method Absolute error	Method in[34] Absolute error	Presented Method Absolute error
	Results of $u_1(t)$		Results of $u_2(t)$	
0	4.7073e − 14	4.7073e − 14	9.7167e − 13	9.7167e − 13
0.1	2.0117e − 13	2.0117e − 13	9.3969e − 13	9.3969e − 13
0.2	3.5505e − 13	3.5505e − 13	8.8507e − 13	8.8507e − 13
0.3	5.0915e − 13	5.0915e − 13	8.0802e − 13	8.0802e − 13
0.4	6.6303e − 13	6.6303e − 13	7.0877e − 13	7.0877e − 13
0.5	1.2488e − 12	1.2488e − 12	2.4176e − 12	2.4176e − 12
0.6	1.4029e − 12	1.4029e − 12	2.6037e − 12	2.6037e − 12
0.7	1.5568e − 12	1.5568e − 12	2.812e − 12	2.812e − 12
0.8	1.7109e − 12	1.7109e − 12	3.0429e − 12	3.0429e − 12
0.9	1.8647e − 12	1.8647e − 12	3.2967e − 12	3.2967e − 12

with exact solution $(u_1(t), u_2(t)) = \left(t, t^2\right)$.

This example has been addressed by several researchers. Initially, Ramadan et al. [33] tackled it using the Triangular Basis Functions Method with m = 32 and m = 10 using triangular basis functions, also authors in [25] analyzed this problem using Triangular functions method, besides , E. Babolian et al. [12] investigated the same problem using Adomian decomposition method.

By taking $M = 3, k = 2$, applying our proposed method (LWM), the unknown functions $u_1(t), u_2(t)$ can be expanded as

$$u_1(t) \approx C_1^T \Psi(t), \quad u_2(t) \approx C_2^T \Psi(t), \tag{4.8}$$

where C_1, C_2 are the unknown $2^{k-1}M, M = 3, k = 2$ vectors with $C_1 = \begin{bmatrix} c_{1,0} & c_{1,1} & c_{1,2} & c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}^T$, $C_2 = \begin{bmatrix} c'_{1,0} & c'_{1,1} & c'_{1,2} & c'_{2,0} & c'_{2,1} & c'_{2,2} \end{bmatrix}^T$ and $\Psi(t)$ is Legendre wavelets for M = 3, k = 2 :

$$\Psi(t) = \begin{cases} \left\{ \begin{array}{c} \Psi_{1,0} = \sqrt{2} \\ \Psi_{1,1} = \sqrt{6}(4t-1) \\ \Psi_{1,2} = \sqrt{10}\left(\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right) \end{array} \right\}, 0 \le t < \frac{1}{2}, \\ \left\{ \begin{array}{c} \Psi_{2,0} = \sqrt{2} \\ \Psi_{2,1} = \sqrt{6}(4t-3) \\ \Psi_{2,2} = \sqrt{10}\left(\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right) \end{array} \right\}, \frac{1}{2} \le t < 1. \end{cases}$$

Likewise,

$$k_1(x,t) = \frac{1}{3},\ k_2(x,t) = \frac{1}{3},\ k_3(x,t) = \frac{1}{2},\ k_4(x,t) = \frac{1}{2}, f(t) = t - \frac{5}{18}$$

$g(t) = t^2 - \frac{5}{12}$, are also expanded into the LWM as:

$$\begin{aligned} &k_1(x,t) \approx \Psi^T(x) K_1 \Psi(t), \quad k_2(x,t) \approx \Psi^T(x) K_2 \Psi(t), \\ &k_3(x,t) \approx \Psi^T(x) K_3 \Psi(t), \quad k_4(x,t) \approx \Psi^T(x) K_4 \Psi(t), \\ &f(t) \approx F^T \Psi(t), g(t) \approx G^T \Psi(t). \end{aligned} \tag{4.9}$$

After substituting (4.8), (4.9) into (4.7) we get,

$$\begin{aligned} \Psi^T(t) C_1 &= F^T \Psi(t) + \left[\Psi^T(t) K_1 C_1 + \Psi^T(t) K_2 C_2\right], \\ \Psi^T(t) C_2 &= G^T \Psi(t) + \left[\Psi^T(t) K_3 C_1 + \Psi^T(t) K_4 C_2\right], \end{aligned} \tag{4.10}$$

which can be written in the coupled system of matrix equations,

$$(I - K_1) C_1 - K_2 C_2 = F \ \text{ and } \ (I - K_3) C_2 - K_4 C_1 = G. \tag{4.11}$$

We can write (4.11) further in the form:

$$A_1C_1 + B_1C_2 = F \ \text{ and } \ A_2C_1 + B_2C_2 = G, \tag{4.12}$$

where,

$$A_1 = I - K_1 = \begin{pmatrix} 0.83333 & 0 & 0 & -0.16667 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.16667 & 0 & 0 & 0.83333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = -K_2 = \begin{pmatrix} -0.16667 & 0 & 0 & -0.16667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.16667 & 0 & 0 & -0.16667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = -K_4 = \begin{pmatrix} -0.25 & 0 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 0 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = I - K_3 = \begin{pmatrix} 0.75 & 0 & 0 & -0.25 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.25 & 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$F = \left[\frac{-\sqrt{2}}{72}, \frac{\sqrt{6}}{24}, 0, \frac{17\sqrt{2}}{72}, \frac{\sqrt{6}}{24}, 0\right]^T$,
$G = \left[\frac{-\sqrt{2}}{6}, \frac{\sqrt{6}}{48}, \frac{\sqrt{10}}{240}, \frac{\sqrt{2}}{12}, \frac{\sqrt{6}}{16}, \frac{\sqrt{10}}{240}\right]^T$.

Using our proposed finite iterative algorithm 2.1 to solve the coupled matrix system given in equation (4.12), we obtain the coefficient vectors

$$C_1 = [0.17679, 0.10206, 0, 0.53034, 0.10206, 0]^T,$$

and

$$C_2 = [0.058937, 0.051031, 0.013176, 0.41249, 0.15309, 0.013176]^T.$$

Substituting these coefficient vectors into coupled system of equations (4.12) provides the approximate solutions at the corresponding time points, as shown in Table 5 below.

As shown in Tables 6 and 7, our proposed method yields more accurate results compared to those reported in references [33], [12], and [25]. In addition to its superior or comparable accuracy, our method is also highly efficient, as it eliminates the need for the computationally intensive matrix inversion when determining the coefficient vectors C_1, C_2

Example 3
Consider the system of two coupled linear Fredholm integral equations [30]

Table 5: The numerical results for Example 2 with proposed Legendre wavelets for $M = 3, k = 2$

t	Exact solution		Presented Method $(u_1(t), u_2(t))$		Absolute Error	
0	0	0	0.00002389254361656518	0.00001576411199376885	2.3893e − 05	1.5764e − 05
0.1	0.1	0.01	0.1000218618029359	0.01001620490521751	2.1862e − 05	1.6205e − 05
0.2	0.2	0.04	0.2000198310622553	0.04001640751462307	1.9831e − 05	1.6408e − 05
0.3	0.3	0.09	0.3000178003215747	0.09001637194021046	1.78e − 05	1.6372e − 05
0.4	0.4	0.16	0.400015769580894	0.1600160981819797	1.577e − 05	1.6098e − 05
0.5	0.5	0.25	0.5000190975197177	0.2500227380709804	1.9098e − 05	2.2738e − 05
0.6	0.6	0.36	0.6000170667790372	0.3600201683276863	1.7067e − 05	2.0168e − 05
0.7	0.7	0.49	0.7000150360383566	0.490017360400574	1.5036e − 05	1.736e − 05
0.8	0.8	0.64	0.8000130052976759	0.6400143142896436	1.3005e − 05	1.4314e − 05
0.9	0.9	0.81	0.9000109745569953	0.8100110299948949	1.0975e − 05	1.103e − 05

Table 6: Comparison between absolute errors of Example 4.2 for our presented method with $M = 3, k = 2$, the T.F. method with $m = 32$ [33] and when $m = 10$ [33]

t	TF method [33] $(m = 10)$	Absolute Error	TF method [33] ($m = 32$)	Absolute Error	Presented Method	Absolute Error
Results of $u_1(t)$						
0	0.00333333	2.49088e − 003	0.00032552	3.25520839e − 04	0.00002389254361656518	2.3893e − 05
0.1	0.10333333	2.49083e − 003	0.10032552	3.25520839e − 04	0.1000218618029359	2.1862e − 05
0.2	0.20333333	2.49078e − 003	0.20032552	3.25520839e − 04	0.2000198310622553	1.9831e − 05
0.3	0.30333333	2.49073e − 003	0.30032552	3.25520839e − 04	0.3000178003215747	1.78e − 05
0.4	0.40333333	2.49068e − 003	0.40032552	3.25520839e − 04	0.400015769580894	1.577e − 05
0.5	0.50333333	2.49063e − 003	0.50032552	3.25520839e − 04	0.5000190975197177	1.9098e − 05
0.6	0.60333333	2.49058e − 003	0.60032552	3.25520839e − 04	0.6000170667790372	1.7067e − 05
0.7	0.70333333	2.49053e − 003	0.70032552	3.25520839e − 04	0.7000150360383566	1.5036e − 05
0.8	0.80333333	2.49047e − 003	0.80032552	3.25520839e − 04	0.8000130052976759	1.3005e − 05
0.9	0.90333333	2.49042e − 003	0.90032552	3.25520839e − 04	0.9000109745569953	1.0975e − 05
Results of $u_2(t)$						
t	TF method [33] For m=10	Absolute Error	TF method [33]	Absolute Error	Presented Method	Absolute
0	0.00500000	4.15743e − 003	-0.00146484	1.46484374e − 03	0.00001576411199376885	1.5764e − 05
0.1	0.01500000	4.15743e − 003	0.00908203	9.17968744e − 04	0.01001620490521751	1.6205e − 05
0.2	0.04500000	4.15741e − 003	0.03955078	4.49218744e − 04	0.04001640751462307	1.6408e − 05
0.3	0.09500000	4.15739e − 003	0.08994141	5.85937439e − 05	0.09001637194021046	1.6372e − 05
0.4	0.16500000	4.15735e − 003	0.16025391	2.53906256e − 04	0.1600160981819797	1.6098e − 05
0.5	0.25500000	4.15731e − 003	0.25048828	4.88281256e − 04	0.2500227380709804	2.2738e − 05
0.6	0.36500000	4.15725e − 003	0.36064453	6.44531256e − 04	0.3600201683276863	2.0168e − 05
0.7	0.49500000	4.15718e − 003	0.49072266	7.22656256e − 04	0.490017360400574	1.736e − 05
0.8	0.64500000	4.15711e − 003	0.64072266	7.22656256e − 04	0.6400143142896436	1.4314e − 05
0.9	0.81500000	4.15702e − 003	0.81064453	6.44531256e − 04	0.8100110299948949	1.103e − 05

$$
\begin{aligned}
u_1(t) &= \sin(t) - \cos(1) + \sin(1) - t\sin(1) + \int_{x=0}^{1} [(t-s)u_1(s) + tsu_2(s)]\, ds, \\
u_2(t) &= \cos(t) - (1-\cos(1))t^2 + \cos(1) - 3\sin(1) - t\sin(1) + 1 + \\
&\int_{x=0}^{1} \left[\left(t^2 + 2s\right) u_1(s) + (s+t)u_2(s) \right] ds,
\end{aligned} \tag{4.13}
$$

with exact solution $(u_1(t), u_2(t)) = (\sin(t), \cos(t))$.
Noting that, Zaffer Elahi et al. [30] tackled this system using Laguerre method.

By taking $M = 3, k = 2$, applying our proposed method (LWM), the unknown functions $u_1(t), u_2(t)$ can be expanded as

$$
u_1(t) \approx C_1^T \Psi(t), \quad u_2(t) \approx C_2^T \Psi(t) \tag{4.14}
$$

where C_1, C_2 are the unknown $2^{k-1}M, M = 3, k = 2$ vectors with
$C_1 = \begin{bmatrix} c_{1,0} & c_{1,1} & c_{1,2} & c_{2,0} & c_{2,1} & c_{2,2} \end{bmatrix}^T$ and $C_2 = \begin{bmatrix} c'_{1,0} & c'_{1,1} & c'_{1,2} & c'_{2,0} & c'_{2,1} & c'_{2,2} \end{bmatrix}^T$ and $\Psi(t)$ is Legendre wavelets for $M = 3, k = 2$:

Table 7: Comparison between absolute errors for results of Example 4.2 for our presented method with $M = 3, k = 2$ and Absolute Error for TF method [25] and Absolute Error for Adomian method in [12]

t	AE for TF method [25]	AE in [12]	AE for our presented	AE in [12]	AE for our presented
	Results of $u_1(t)$			Results of $u_2(t)$	
0	2.170e − 04	2.30e − 02	2.3893e − 05	4.43e − 02	1.5764e − 05
0.1	2.170e − 04	2.30e − 02	2.1862e − 05	4.43e − 02	1.6205e − 05
0.2	2.170e − 04	2.30e − 02	1.9831e − 05	4.43e − 02	1.6408e − 05
0.3	2.170e − 04	2.30e − 02	1.78e − 05	4.43e − 02	1.6372e − 05
0.4	2.170e − 04	2.30e − 02	1.577e − 05	4.43e − 02	1.6098e − 05
0.5	2.170e − 04	2.30e − 02	1.9098e − 05	4.43e − 02	2.2738e − 05
0.6	2.170e − 04	2.30e − 02	1.7067e − 05	4.43e − 02	2.0168e − 05
0.7	2.170e − 04	2.30e − 02	1.5036e − 05	4.43e − 02	1.736e − 05
0.8	2.170e − 04	2.30e − 02	1.3005e − 05	4.43e − 02	1.4314e − 05
0.9	2.170e − 04	2.30e − 02	1.0975e − 05	4.43e − 02	1.103e − 05

$$\Psi(t) = \begin{cases} \left.\begin{cases} \Psi_{1,0} = \sqrt{2} \\ \Psi_{1,1} = \sqrt{6}(4t-1) \\ \Psi_{1,2} = \sqrt{10}\left(\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right) \end{cases}\right\}, 0 \le t < \frac{1}{2}, \\ \left.\begin{cases} \Psi_{2,0} = \sqrt{2} \\ \Psi_{2,1} = \sqrt{6}(4t-3) \\ \Psi_{2,2} = \sqrt{10}\left(\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right) \end{cases}\right\}, \frac{1}{2} \le t < 1. \end{cases}$$

Likewise,

$$\begin{aligned} k_1(s,t) = (t-s), \quad k_2(s,t) = ts, \ k_3(s,t) = \left(t^2 + 2s\right) \\ k_4(s,t) = (s+t), \ f(t) = \sin(t) - \cos(1) + \sin(1) - t\sin(1) \\ g(t) = \cos(t) - (1-\cos(1))t^2 + \cos(1) - 3\sin(1) - t\sin(1) + 1, \end{aligned}$$

are also expanded into the LWM as:

$$\begin{aligned} &k_1(s,t) \approx \Psi^T(s)K_1\Psi(t), \quad k_2(s,t) \approx \Psi^T(s)K_2\Psi(t) \\ &k_3(s,t) \approx \Psi^T(s)K_3\Psi(t), \quad k_4(s,t) \approx \Psi^T(s)K_4\Psi(t) \\ &f(t) \approx F^T\Psi(t), \ g(t) \approx G^T\Psi(t) \end{aligned} \tag{4.15}$$

After substituting (4.14), (4.15) into (4.13) we get,

$$\begin{aligned} \Psi^T(t)C_1 = F^T\Psi(t) + \left[\Psi^T(t)K_1C_1 + \Psi^T(t)K_2C_2\right] \\ \Psi^T(t)C_2 = G^T\Psi(t) + \left[\Psi^T(t)K_3C_1 + \Psi^T(t)K_4C_2\right] \end{aligned} \tag{4.16}$$

which can be written in the coupled system of matrix equations,

$$(I - K_1)\,C_1 - K_2C_2 = F \ \text{ and } \ (I - K_3)\,C_2 - K_4C_1 = G. \tag{4.17}$$

We can write (4.3.5) further in the form,

$$A_1C_1 + B_1C_2 = F \ \text{ and } \ A_2C_1 + B_2C_2 = G, \tag{4.18}$$

where,

$$A_1 = I - K_1 = \begin{pmatrix} 1 & 0.072169 & 0 & 0.25 & 0.0721690 & 0 \\ -0.072169 & 1 & 0 & -0.072169 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.25 & 0.072169 & 0 & 1 & 0.072169 & 0 \\ -0.072169 & 0 & 0 & -0.072169 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = -K_2 = \begin{pmatrix} -0.03125 & -0.018042 & 0 & -0.09375 & -0.018042 & 0 \\ -0.018042 & -0.010417 & 0 & -0.054127 & -0.010417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.09375 & -0.054127 & 0 & -0.28125 & -0.054127 & 0 \\ -0.018042 & -0.010417 & 0 & -0.054127 & -0.010417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = -K_4 = \begin{pmatrix} -0.29167 & -0.14434 & 0 & -0.79167 & -0.14434 & 0 \\ -0.036084 & 0 & 0 & -0.036084 & 0 & 0 \\ -0.0093169 & 0 & 0 & -0.0093169 & 0 & 0 \\ -0.54167 & -0.14434 & 0 & -1.0417 & -0.14434 & 0 \\ -0.10825 & 0 & 0 & -0.10825 & 0 & 0 \\ -0.0093169 & 0 & 0 & -0.0093169 & 0 & 0 \end{pmatrix},$$

$$B_2 = I - K_3 = \begin{pmatrix} 0.75 & -0.072169 & 0 & -0.5 & -0.072169 & 0 \\ -0.072169 & 1 & 0 & -0.072169 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.5 & -0.072169 & 0 & 0.25 & -0.072169 & 0 \\ -0.072169 & 0 & 0 & -0.072169 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F = \begin{bmatrix} 0.23733 & 0.01239 & -0.0016227 & 0.24369 & -0.01167 & -0.0044707 \end{bmatrix}^T,$$

$$G = \begin{bmatrix} -0.1937 & -0.13443 & -0.012412 & -0.81973 & -0.22539 & -0.010856 \end{bmatrix}^T.$$

Using our proposed finite iterative algorithm 2.1 to solve the coupled matrix system given in equation (4.18), we obtain the coefficient vectors

$$C_1 = [0.1731, 0.0983, -0.0016, 0.477, 0.0742, -0.0045]^T,$$

and

$$C_2 = [0.678, -0.0251, -0.0064, 0.512, -0.0691, -0.0048]^T.$$

Substituting these coefficient vectors into coupled system of equations (4.18) provides the approximate solutions at the corresponding time points, as shown in Table 8

Table 8: The numerical results for Example 3 with proposed Legendre wavelets for $M = 3, k = 2$

T	Exact solution $(u_1(t), u_2(t))$		Presented Method $(u_1(t), u_2(t))$		Absolute Error $(u_1(t), u_2(t))$	
0	0	1.0	-0.001044118325073061	1.000080410807739	0.00104	8.04e-5
0.2	0.19866933	0.98006658	0.198869642776424	0.9800382076887642	$2.0e-4$	$2.84e-5$
0.4	0.38941834	0.92106099	0.3890688869058838	0.9211379366816407	$3.49e-4$	$7.69e-5$
0.6	0.56464247	0.82533561	0.5649593759244289	0.8250260313600629	$3.17e-4$	$3.1e-4$
0.8	0.71735609	0.69670671	0.7171916068020021	0.6969041261080368	$1.64e-4$	$1.97e-4$
1	0.84147098	0.54030231	0.8421017586957205	0.5396386699398988	$6.31e-4$	$6.64e-4$

Table 9: Comparison between absolute errors of Example 3 for our presented method with $M = 3, k = 2$, and Laguerre method in [30] for $N = 2$

t	AE in [30] taking N = 2	AE for Presented Method	AE in [30]	AE for Presented Method
Results of $u_1(t)$			Results of $u_2(t)$	
0	6.12984 e -4	0.00104	1.71016 e -3	$8.04e-5$
0.2	7.9168 e -3	$2.0e-4$	5.33819 e -3	$2.84e-5$
0.4	4.36011 e -3	$3.49e-4$	4.18645 e -3	$7.69e-5$
0.6	2.45251 e -3	$3.17e-4$	6.07311 e -4	$3.1e-4$
0.8	5.53543 e -3	$1.64e-4$	1.58297 e -3	$1.97e-4$
1	1.19956 e -3	$6.31e-4$	2.74364 e -3	$6.64e-4$

As shown in Table 9, our proposed method yields more accurate results compared to obtained results in [30]. In addition to its superior or comparable accuracy, our method is also highly efficient, as it eliminates the need for the computationally intensive matrix inversion when determining the coefficient vectors C_1, C_2.

6. Conclusion

This study presents an efficient and direct iterative method for addressing one-dimensional Fredholm integral equations of the second order, utilizing Legendre wavelet functions and a finite iterative framework. The method transforms integral equations into systems of algebraic matrix equations, providing a practical and efficient approach for calculating approximation solutions. The method circumvents the necessity of inverting block matrices, thereby improving its precision and computational efficiency. Numerous numerical cases exemplify the method's performance, showcasing its potential for diverse applications in scientific and technical domains.

Dedication

The first author, Mohamed A. Ramadan, dedicates this work to Professor Mohamed Asaad, Professor Emeritus at Cairo University, on his 80th birthday. His pioneering research in abstract algebra and finite groups is a true inspiration. Though my research lies outside his field, his generous support and encouragement have been invaluable.

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