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Generalized Inner Structure Spaces: Indefinite Sesquilinear Forms in Fréchet Spaces and Their Applications

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Abstract. We introduce *Generalized Inner Structure Spaces* (GISS), a class of Fréchet spaces equipped with a sesquilinear form $[\cdot,\cdot]$ that generalizes Hilbert spaces by allowing indefinite (possibly sign-changing) forms, unifying Krein spaces, semi-inner product spaces, Gelfand triples, and related structures. We develop a comprehensive theory, establishing a locally convex topology τ_p induced by seminorms $p_t(w) = |[t, w]|$, a Hahn–Banach-type separation theorem, and operator theory for self-adjoint operators with real spectra. A key result is the identification of maximal positive and negative subspaces and the role of neutral vectors in the indefinite geometry. Examples in sequence spaces, Sobolev spaces, Gelfand triples, and quantum state spaces illustrate GISS's versatility. Applications include quantum field theory (e.g., Dirac operator quantization) and hyperbolic PDEs. GISS offers a robust framework for non-Hilbertian analysis in functional analysis and mathematical physics.

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1. Introduction

Inner product structures play a central role in functional analysis and mathematical physics, most notably through Hilbert spaces and their applications in quantum mechanics and partial differential equations [1]. Nevertheless, a wide range of modern problems naturally involve sesquilinear forms that are not positive definite, or topologies that are not induced by a single norm. Typical examples arise in gauge quantum field theory with indefinite metrics [2], in hyperbolic and mixed-type partial differential equations [3], and in distributional frameworks based on Gelfand triples [4].

Several mathematical structures have been developed to address these situations. Krein spaces extend Hilbert spaces by allowing indefinite inner products while retaining a Hilbertian topology and a fixed orthogonal decomposition [5]. Semi-inner product

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spaces relax positivity but may lose non-degeneracy [6]. Gelfand triples and rigged Hilbert spaces provide powerful distributional frameworks, but their underlying sesquilinear forms are tied to duality pairings and specific topological assumptions [7]. Despite their success, these approaches remain specialized and are not designed to form a single unifying framework.

The purpose of this paper is to introduce *Generalized Inner Structure Spaces* (GISS), a class of Fréchet spaces equipped with a continuous, Hermitian sesquilinear form that may be indefinite. The proposed framework retains enough structure to support operator theory, topological decompositions, and separation results, while remaining flexible enough to encompass non-Hilbertian and non-normable settings.

The main contributions of the paper are as follows. We introduce the axiomatic definition of GISS and establish basic continuity properties of the associated sesquilinear form. We construct a locally convex topology induced by the form and show that it is complete. A structural theorem is proved establishing the existence of maximal positive and maximal negative subspaces associated with the sign of $[t, t]$, and clarifying the role of neutral vectors in the indefinite geometry. We further develop an operator theory for self-adjoint operators in this setting and discuss spectral properties under mild assumptions. Finally, several concrete examples and applications are presented, including explicit constructions arising in quantum field theory, partial differential equations, signal processing, and control theory.

2. Preliminaries

This section contains the basic definitions and implications required to study Generalized Inner Structure Spaces (GISS, Sec. 3). We describe Fréchet spaces, locally convex topologies, and sesquilinear forms, defining their essential characteristics in the context of topological (Section 5), operator-theoretic (Section 6), and decomposition (Section 7) analysis. Applications are discussed later in Section 9; the present preliminaries collect the required background. We recall some standard notions from funcational analysis (see [6, 8, 9]

Definition 1. *A topological vector space E over $\mathbb{C}$, if the topology τ is Hausdorff, metrizable, complete, generated by a countable family of seminorms $\{p_n\}_{n\in\mathbb{N}}$, is termed a* Fréchet space.

Definition 2. *In other words, a topology τ on a vector space E is* locally convex *if it is created by a family of seminorms $\{p_\alpha\}_{\alpha\in A}$ in such a way that if $p_\alpha(t) = 0$ for all $\alpha \in A$, then $t = 0$ (separating family).*

Definition 3. *A map $[\cdot, \cdot] : E \times E \to \mathbb{C}$ on a complex vector space E is* sesquilinear *if it is linear in the first argument and conjugate-linear in the second: for all $t, w, z \in E$, $a, b \in \mathbb{C}$,*

$$[at + bw, z] = a[t, z] + b[w, z], \quad [t, aw + bz] = a[t, w] + b[t, z].$$

Let (E, τ) be a Fréchet space. Each Cauchy sequence $\{t_n\}$ in E converges to some $t \in E$ with respect to τ.

Proof. Since (E, τ) is a Fréchet space, it is complete and metrizable by Definition 1. Let $\{t_n\}$ be a Cauchy sequence, that for every seminorm p_n in the countable family generating τ, $p_n(t_n - t_m) \to 0$ as $n, m \to \infty$. By completeness, there exists $t \in E$ such that $p_n(t_n - t) \to 0$ for all n, implying $t_n \to t$ in τ [10, Theorem 1.12].

Proposition 1. *The dual space E^* of a Fréchet space (E, τ), composed of continuous linear functionals $f : E \to \mathbb{C}$, separates points: if $t \neq w$, there is $f \in E^*$ such that $f(t) \neq f(w)$.*

Proof. Because (E, τ) is a Fréchet space, it is locally convex and Hausdorff (Definition 1). The Hahn-Banach theorem for locally convex spaces holds that for $t \neq w$ we would have a continuous linear functional $f \in E^*$ such that $f(t - w) \neq 0$, so $f(t) \neq f(w)$ [10, Theorem 3.5].

Here, let (E, τ) be a Fréchet space, and $[\cdot, \cdot] : E \times E \to \mathbb{C}$ be sesquilinear. If for every $t \in E$, the map $w \mapsto [t, w]$ is continuous in τ, then $[\cdot, \cdot]$ is jointly continuous in the product topology $\tau \times \tau$.

Proof. Fix $t \in E$. In τ, $w \mapsto [t, w]$ has a continuous mapping (by assumption). This implies that $[w, t] = [t, w]$ by conjugate symmetry, and as such, $t \mapsto [t, w]$ is continuous for fixed w. Hence, for a net $(t_\alpha, w_\beta) \to (t, w)$ in $\tau \times \tau$, we require $[t_\alpha, w_\beta] \to [t, w]$. Write

$$[t_\alpha, w_\beta] - [t, w] = [t_\alpha - t, w_\beta] + [t, w_\beta - w].$$

Since $w_\beta \to w$ in τ, $[t, w_\beta - w] \to 0$ by continuity. Note that $\{w_\beta\}$ is convergent and hence bounded in (E, τ). According to the uniform boundedness principle for Fréchet spaces [10, Theorem 2.8], there exists $C > 0$ such that $p_n(w_\beta) \leq C$ for all seminorms p_n. Thus, since $t_\alpha \to t$, $[t_\alpha - t, w_\beta] \to 0$ uniformly in β. Hence $[t_\alpha, w_\beta] \to [t, w]$, providing evidence for joint continuity [10, Theorem 1.35].

3. Generalized Inner Structure Spaces

Here *Generalized Inner Structure Spaces* (GISS) are described, a novel class of topological vector spaces that can generalize Hilbert spaces through the inclusion of indefinite sesquilinear forms. In response to the necessity toward a holistic paradigm for the computation of structures such as Krein spaces, semi-inner product spaces and Gelfand triples, GISS offers such flexibility in setting functional analysis and application to mathematical physics. So, we describe GISS, define their properties to define their fundamental structure, derive the base results regarding functional representations and continuity. We also set the stage for subsequent topological and operator-theoretic analysis.

Definition 4. *A* Generalized Inner Structure Space *(GISS) is a Fréchet space E equipped with a sesquilinear map $[\cdot, \cdot] : E \times E \to \mathbb{C}$ and a Fréchet topology τ such that:*

(G1) Linearity in the first argument: $[at + bw, z] = a[t, z] + b[w, z]$.

(G2) Hermitian symmetry: $[t, w] = \overline{[w, t]}$.

(G3) Non-degeneracy: if $[t, w] = 0$ *for all* $w \in E$*, then* $t = 0$.

(G4) Continuity: for each $t \in E$*, the map* $w \mapsto [t, w]$ *is continuous in* τ.

Definition 5. *The* radical *of* $[\cdot, \cdot]$ *is*

$$\mathrm{Rad}(E) := \{t \in E : \ [t, w] = 0 \text{ for all } w \in E\}.$$

The set of neutral (isotropic) vectors *is*

$$\mathcal{N} := \{t \in E : \ [t, t] = 0\}.$$

Under axiom (G3), $\mathrm{Rad}(E) = \{0\}$. However, $\mathcal{N}$ may be nontrivial when $[\cdot, \cdot]$ is indefinite.

Proposition 2. *Every Hilbert space is a GISS. However, the converse is false: not every GISS is a Hilbert space.*

Let $[t, w] = \langle t, w\rangle$. Then:

- **(G1)** The linearity in the first argument originates from the linearity of the inner product.
- **(G2)** Conjugate symmetry applies since $\langle t, w\rangle = \overline{\langle w, t\rangle}$.
- **(G3)** Non-degeneracy: for $\langle t, w\rangle = 0$ for all w, $t = 0$ by the Riesz representation theorem.
- **(G4)** Continuity: by the Cauchy–Schwarz inequality, $|\langle t, w\rangle| \leq \|t\|\|w\|$ so $w \mapsto \langle t, w\rangle$ is continuous.

Hence, all Hilbert spaces satisfy the GISS axioms.
Let $E = \ell^2(\mathbb{Z})$ and express the sesquilinear form

$$[t, w] := \sum_{n \in \mathbb{Z}} (-1)^n t_n \overline{w_n}.$$

Then:

- $[t, w]$ is a sesquilinear and conjugate symmetric form.
- It is weakly non-degenerate: if $[t, w] = 0$ for all $w \in E$, then $t = 0$.
- For fixed t, the map $w \mapsto [t, w]$ is continuous since it is bounded on ℓ^2.

Accordingly, $(E, [\cdot, \cdot])$ is a GISS. But $[t, t]$ can be negative or zero even when $t \neq 0$, so the form is not positive-definite. Thus, E is not a Hilbert space under this structure.

Proposition 3. *Though every Krein space is a GISS, not every GISS is a Krein space.*

Proof. The sesquilinear form of a Krein space $(E, [\cdot,\cdot])$ satisfies (G1)–(G3) and (G4) [5]. However, a GISS might lack the orthogonal decomposition $E = E_+ \oplus E_-$ that Krein spaces require.

For each $t \in E$, the map $f_t : E \to \mathbb{C}$ defined by $f_t(w) = [t, w]$ is a continuous linear functional on (E, τ). Hence the map $T : E \to E^*$ given by $T(t) = f_t$ is linear and injective.

Proof. Linearity of f_t in w follows from sesquilinearity of $[\cdot,\cdot]$. Continuity of f_t is exactly axiom (G4). To prove injectivity of T, assume $T(t) = 0$. Then $[t, w] = 0$ for all $w \in E$, hence $t = 0$ by (G3).

[Fundamental symmetry representation] Assume that the Fréchet topology τ on E is induced by a Hilbert norm $\|\cdot\|$ with inner product $\langle\cdot,\cdot\rangle$. If $[\cdot,\cdot]$ is continuous with respect to $\|\cdot\|$, i.e. there exists $C > 0$ such that

$$|[t, w]| \le C\|t\|\|w\| \qquad \text{for all } t, w \in E,$$

then there exists a unique bounded operator $J \in \mathcal{B}(E)$ such that

$$[t, w] = \langle Jt, w\rangle \qquad \text{for all } t, w \in E.$$

Moreover, J is self-adjoint with respect to $\langle\cdot,\cdot\rangle$.

Proof. Fix $t \in E$. The map $w \mapsto [t, w]$ is a continuous linear functional on the Hilbert space $(E, \langle\cdot,\cdot\rangle)$ by the assumed bound. By the Riesz representation theorem, there exists a unique element $Jt \in E$ such that

$$[t, w] = \langle Jt, w\rangle \quad \text{for all } w \in E.$$

This defines a linear operator $J : E \to E$. The bound yields

$$\|Jt\| = \sup_{\|w\|=1} |\langle Jt, w\rangle| = \sup_{\|w\|=1} |[t, w]| \le C\|t\|,$$

so J is bounded. Finally, since $[\cdot,\cdot]$ is Hermitian,

$$\langle Jt, w\rangle = [t, w] = \overline{[w, t]} = \overline{\langle Jw, t\rangle} = \langle t, Jw\rangle,$$

hence J is self-adjoint with respect to $\langle\cdot,\cdot\rangle$.

Under the assumptions of Theorem 3, the representing operator J is unique.

Proof. If $\langle J_1 t, w\rangle = \langle J_2 t, w\rangle$ for all $w \in E$, then $\langle (J_1 - J_2)t, w\rangle = 0$ for all w. By non-degeneracy of $\langle\cdot,\cdot\rangle$, $(J_1 - J_2)t = 0$ for all t, hence $J_1 = J_2$.

The sesquilinear form $[\cdot,\cdot] : E \times E \to \mathbb{C}$ in a GISS is jointly continuous in the product topology $\tau \times \tau$.

Proof. Fix $t \in E$. By (G4), the map $w \mapsto [t, w]$ is continuous in τ. Similarly, by (G2), $[w, t] = \overline{[t, w]}$, so $t \mapsto [t, w]$ is continuous for fixed w. To show joint continuity, consider a net $(t_\alpha, w_\beta) \to (t, w)$ in $\tau \times \tau$. We need $[t_\alpha, w_\beta] \to [t, w]$. Write

$$[t_\alpha, w_\beta] - [t, w] = [t_\alpha - t, w_\beta] + [t, w_\beta - w].$$

Since $w_\beta \to w$ in τ, $[t, w_\beta - w] \to 0$ by (G4). For $[t_\alpha - t, w_\beta]$, note that $\{w_\beta\}$ is convergent, hence bounded in the Fréchet space (E, τ). By the uniform boundedness principle [10, Theorem 2.8], there exists a constant C such that $p_z(w_\beta) \leq C$ for all $z \in E$. Since $t_\alpha \to t$, $[t_\alpha - t, w_\beta] \to 0$ uniformly in β. Thus, $[t_\alpha, w_\beta] \to [t, w]$, proving joint continuity.

If τ is normable (e.g., E is a Banach space), the sesquilinear form $[\cdot, \cdot]$ is bounded: there exists $M > 0$ such that $|[t, w]| \leq M\|t\|_\tau \|w\|_\tau$ for all $t, w \in E$.

Proof. By Theorem 3, $[\cdot, \cdot]$ is jointly continuous in $\tau \times \tau$. If τ is normable, let $\|\cdot\|_\tau$ be the norm. By the continuity of $[\cdot, \cdot] : (E \times E, \|\cdot\|_\tau \times \|\cdot\|_\tau) \to \mathbb{C}$, there exists $M > 0$ such that $|[t, w]| \leq M\|t\|_\tau \|w\|_\tau$ for all $t, w \in E$ [10, Theorem 1.35].

4. Comparison with Existing Frameworks

Generalized Inner Structure Spaces (GISS) developed in Section 3 can also be used to provide a unified structure for topological vector spaces, which can be sesquilinear, freeing them from the strict axioms of positive-definiteness and orthogonal decomposition in traditional structures. We compare the GISS against a number of important systems—Hilbert spaces, Krein spaces, semi-inner product spaces, Gelfand triples, Banach spaces with indefinite metrics and rigged Hilbert spaces—to stress the generality of GISS whilst correcting some of the problems with extant state-of-the-art. They are described, a comparison table drawn, and mathematical results are presented formalizing the relationships. It emphasizes GISS's applicability for cases such as quantum field theory and partial differential equations (Section 9).

- *Hilbert Spaces*: A Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a complete inner product space with a positive-definite inner product: $\langle t, t \rangle \geq 0$ and $\langle t, t \rangle = 0$ implies $t = 0$ [10]. The induced norm topology is Fréchet, and the inner product is jointly continuous. GISS takes this one step further and enables indefinite shapes of the sesquilinear form, such as $[t, t]$, to be negative, and Fréchet topologies to not be induced by a norm. Such versatility is especially important for applications where positive-definiteness breaks down, for example, indefinite metric space in QM [2]. But Hilbert spaces are well supported by a broad spectral theory and GISS might be a missing link without such a theoretical framework (Section 6).

- *Krein Spaces*: A Krein space $(E, [\cdot, \cdot])$ is a vector space with an indefinite sesquilinear form and an orthogonal decomposition $E = E_+ \oplus E_-$, where $[t, t] > 0$ on E_+ and $[t, t] < 0$ on E_- [8]. The topology is typically Hilbertian, which allows for a consistent topology. As we discussed in Section 3, GISS does not impose this decomposition (Theorem 7) which allows for non-Hilbertian Fréchet topologies. This also allows GISS to be more general, but possibly less organized for spectral analysis [5].

- *Semi-Inner Product Spaces*: They contain a positive but potentially degenerate semi-inner product, satisfying $[t, t] \geq 0$ with $[t, t] = 0$ for $t \neq 0$ [6]. It is the normability of the topology, but not a complete topology. GISS demands non-degeneracy (G3) and indefinite forms, making it suitable for those situations where semi-inner product space constraints are strict, such as quantum field theory (Example 8).

- *Gelfand Triples*: A Gelfand triple consists of a Hilbert space H with a dense subspace $\Phi \subset H \subset \Phi'$, where Φ' is the dual space, and a sesquilinear form that is defined as the duality pairing $\langle \cdot, \cdot \rangle_{\Phi' \times \Phi}$ [4]. Φ is likely to have a stronger topological order than H. GISS applies Gelfand triples by treating duality pairing as a sesquilinear (see e.g. 2 in Section 8) and then generalises the topology to all Fréchet spaces.

- *Banach Spaces with Indefinite Metrics*: Banach spaces containing an indefinite sesquilinear form is also often used in operator theory [9]. They need a normable topology unlike GISS, which has a more general scope than the Fréchet topology of GISS. But their norm structure allows bounded operator analysis and therefore GISS can be used with non-normable distributions.

- *Rigged Hilbert Spaces*: They generalize Gelfand triples to include Φ as part of the lattice which has a nuclear or Schwartz topology that is used in quantum mechanics for distributions [7]. GISS allows for flexible topologies as well as nondegenerate sesquilinear forms of Hilbert space for more complex arrangements, but they do not necessarily inherit any of their nuclear properties unless permitted.

Summary of the key features of these frameworks compared to GISS are provided in table 1.

Table 1: Comparison of frameworks.

Framework	**Sesquilinear Form**	**Topology**	**Decomposition**	**Non-degeneracy**
Hilbert Space	Positive-definite	Normed (Fréchet)	Not required	Yes
Krein Space	Indefinite	Hilbertian	$E = E_+ \oplus E_-$	Yes
Semi-inner product	Positive, possibly degenerate	Normable	Not required	No
Gelfand triple	Duality pairing	Fréchet (on Φ)	Not required	Yes
Banach (indefinite)	Indefinite	Normed	Not required	Varies
Rigged Hilbert space	Duality pairing	Nuclear/Schwartz	Not required	Yes
GISS	Indefinite	Fréchet	Optional (Thm. 7)	Yes

GISS overcomes the shortcomings of current structures, including the need for positive-definiteness in Hilbert, orthogonal decompositions in Krein, or normable topologies in Banach. Since it is flexible it is well tailored to applications where data are indefinite, e.g. quantum field theory [2] or hyperbolic PDEs [3].

4.1. Mathematical Relationships

With its generalism, GISS provides a structured and dynamic dataset of the number of quantum states. To formalize the relationship between GISS and other structures, results link their sesquilinear forms and topologies. Let $(E, [\cdot, \cdot], \tau)$ be a GISS. If τ is normable and $[t, t] \geq 0$ with $[t, t] = 0$ implying $t = 0$, then $(E, [\cdot, \cdot])$ embeds isometrically into a Hilbert space.

Proof. Assume that τ is induced by a norm $\|\cdot\|_\tau$, and that $[t,t] \geq 0$ with $[t,t] = 0$ and therefore $t = 0$. We define a norm on X as $\|t\| = \sqrt{[t,t]}$. It is well-defined from the positive-definiteness. To illustrate equivalence with $\|\cdot\|_\tau$, see Corollary 3 (Section 3) where $M > 0$ means that $|[t,w]| \leq M\|t\|_\tau\|w\|_\tau$. Therefore, $[t,t] = \|t\|^2$, and therefore $\|t\|^2 \leq M\|t\|_\tau^2$ and $\|t\| \leq \sqrt{M}\|t\|_\tau$. On the other hand, τ and $[\cdot,\cdot]$ can be normable and continuous (Theorem 3), implying $c > 0$ and thus $\|t\|_\tau \leq c\|t\|$, thus the norms are the same. Finishing E under $\|\cdot\|$ gives us a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with $\langle t,w\rangle = [t,w]$, and E isometrically embeds into H [10, Theorem 1.38].

Assume that an explicit topological splitting

$$E = E_+ \oplus E_-$$

is available (as in Example 8), where E_+ and E_- are closed subspaces, $[t,t] > 0$ for $t \in E_+ \setminus \{0\}$, and $[t,t] < 0$ for $t \in E_- \setminus \{0\}$. Assume moreover that the restrictions $\tau|_{E_+}$ and $\tau|_{E_-}$ are normable and that $[\cdot,\cdot]$ is continuous with respect to these norms. Then $(E, [\cdot,\cdot])$ admits a continuous injective embedding into a Krein space.

Proof. Let $\|\cdot\|_+$ be a norm inducing $\tau|_{E_+}$ and $\|\cdot\|_-$ a norm inducing $\tau|_{E_-}$. Define on E_+ the inner product

$$\langle x_+, y_+\rangle_+ := [x_+, y_+], \qquad x_+, y_+ \in E_+,$$

which is positive definite by assumption. Similarly, define on E_- the inner product

$$\langle x_-, y_-\rangle_- := -[x_-, y_-], \qquad x_-, y_- \in E_-,$$

which is positive definite because $[x_-, x_-] < 0$ for $x_- \neq 0$.

By the assumed continuity of $[\cdot,\cdot]$ on $E_\pm$ with respect to $\|\cdot\|_\pm$, both sesquilinear forms $\langle\cdot,\cdot\rangle_+$ and $\langle\cdot,\cdot\rangle_-$ are continuous on the normed spaces $(E_+, \|\cdot\|_+)$ and $(E_-, \|\cdot\|_-)$. Hence they define pre-Hilbert structures on E_+ and E_-. Let H_+ and H_- denote the Hilbert space completions of $(E_+, \langle\cdot,\cdot\rangle_+)$ and $(E_-, \langle\cdot,\cdot\rangle_-)$, respectively.

Set $K := H_+ \oplus H_-$ and equip K with the indefinite inner product

$$[(u_+, u_-), (v_+, v_-)]_K := \langle u_+, v_+\rangle_+ - \langle u_-, v_-\rangle_-.$$

Then $(K, [\cdot,\cdot]_K)$ is a Krein space (see [8, Section 2.4]).

Define the linear map $i : E \to K$ by $i(x_+ + x_-) = (x_+, x_-)$, where $x_\pm \in E_\pm$ are the components of x in the fixed splitting. Since the splitting is topological, i is continuous and injective. Moreover, for $x = x_+ + x_-$ and $y = y_+ + y_-$,

$$[i(x), i(y)]_K = \langle x_+, y_+\rangle_+ - \langle x_-, y_-\rangle_- = [x_+, y_+] + [x_-, y_-] = [x, y],$$

where the last equality uses the assumed orthogonality of the splitting in the model (as in the explicit examples). Thus i preserves the indefinite form, hence $(E, [\cdot,\cdot])$ embeds continuously and injectively into the Krein space $(K, [\cdot,\cdot]_K)$.

Summary of inclusions and trade-offs. Every Hilbert space is a GISS with $[\cdot,\cdot] = \langle\cdot,\cdot\rangle$, and every Krein space is a GISS once its indefinite form is viewed on the underlying locally convex structure. The advantage of GISS is that it allows indefinite forms together with general Fréchet topologies (including non-normable ones), which is essential in distribution-type settings. The cost of this generality is that classical Hilbert-space tools (e.g. full spectral decompositions) do not automatically extend without additional assumptions (compactness, fundamental symmetry, or a Krein-type structure).

5. Topology and Continuity in GISS

Below you will look at the topological properties of Generalized Inner Structure Spaces (GISS) defined in Section 3, paying attention to the locally convex topology τ_p generated by the sesquilinear form $[\cdot,\cdot]$. We show that the τ_p is separating and metrizable, investigate its relationship to the original Fréchet topology τ, and provide evidence of completeness, function continuity, and subspace structures. These results lay substantial groundwork for operator theory (Section 6) and decomposition theorems (Section 7), which has some applications to quantum field theory and to partial differential equations (Section 9).

Assume $(E,[\cdot,\cdot],\tau)$ is a GISS. The family $\{p_t(w) = |[t,w]| : t \in E\}$ is separating and induces a locally convex topology τ_p on E coarser than τ.

Proof. For every $t \in E$, $p_t(w) = |[t,w]|$ is a seminorm since $[t,\cdot]$ is linear and continuous by (G4). If $p_t(w) = 0$ for every t, then $[t,w] = 0$ for all t, so $w = 0$ by (G3). Thus, $\{p_t\}$ is separating. Because E is Fréchet, τ_p is locally convex and metrizable [10, Theorem 1.24].

Let $(E,[\cdot,\cdot],\tau)$ be a GISS and let τ_p be the locally convex topology generated by $p_t(w) = |[t,w]|$. Assume in addition that the identity map

$$\mathrm{id} : (E,\tau_p) \longrightarrow (E,\tau)$$

is continuous (equivalently, $\tau \subseteq \tau_p$). Then (E,τ_p) is complete. In particular, if $\tau = \tau_p$ (e.g. under the boundedness/normability hypothesis of Proposition 4), then (E,τ_p) is a Fréchet space.

Proof. Let (w_n) be a Cauchy sequence in (E,τ_p). By continuity of $\mathrm{id} : (E,\tau_p) \to (E,\tau)$, the sequence (w_n) is Cauchy in (E,τ). Since (E,τ) is Fréchet, it is complete, hence there exists $w \in E$ such that $w_n \to w$ in τ.

Fix $t \in E$. By axiom (G4), the map $w \mapsto [t,w]$ is continuous in τ, so

$$p_t(w_n - w) = |[t, w_n - w]| \longrightarrow 0.$$

Since the seminorms p_t generate τ_p, this implies $w_n \to w$ in τ_p. Therefore (E,τ_p) is complete.

Finally, if $\tau = \tau_p$ and τ is Fréchet, then τ_p is also Fréchet.

Proposition 4. *If the sesquilinear form $[\cdot,\cdot]$ on a GISS $(E,[\cdot,\cdot],\tau)$ is bounded with respect to a norm inducing τ, then $\tau_p = \tau$.*

Proof. Suppose τ is induced by a norm $\|\cdot\|_\tau$, and there exists $M > 0$ such that $|[t,w]| \leq M\|t\|_\tau\|w\|_\tau$ for all $t, w \in E$ (e.g., as in Corollary 3 where τ is normable). By Lemma 5, τ_p is generated by $\{p_t(w) = |[t,w]| : t \in E\}$. For each $t, w \in E$, $p_t(w) = |[t,w]| \leq M\|t\|_\tau\|w\|_\tau$, so the seminorm p_t is continuous in τ. Hence, $\tau_p \subseteq \tau$. On the other hand, since τ_p is coarser than τ (Lemma 5), and $[\cdot,\cdot]$ is continuous in $\tau \times \tau$ (Theorem 3), the identity map $id : (E, \tau_p) \to (E, \tau)$ is continuous if τ_p contains enough seminorms to generate τ. Since $\{p_t\}$ is separating and τ is normable, the boundedness of $[\cdot,\cdot]$ guarantees $\tau_p = \tau$ [10, Theorem 1.35].

[Hahn-Banach-Type Separation] Let $(E, [\cdot,\cdot], \tau_p)$ be a GISS and let $C \subset E$ be convex and closed in τ_p. If $t \notin C$, there exists a continuous linear functional $f : E \to \mathbb{C}$ such that

$$\mathrm{Re}(f(t)) < \inf_{w \in C} \mathrm{Re}(f(w)).$$

Proof. Since (E, τ_p) is locally convex (Lemma 5) and C is convex and τ_p-closed, the Hahn–Banach separation theorem for locally convex spaces applies. Hence there exists a nonzero continuous linear functional $f \in (E, \tau_p)^*$ such that

$$\mathrm{Re}\, f(t) < \inf_{w \in C} \mathrm{Re}\, f(w).$$

No representation of f in the form $f(\cdot) = [z, \cdot]$ is required (and is not available in general).

Let (E, τ_p) be a locally convex space. The weak topology $\sigma(E, (E, \tau_p)^*)$ on E is coarser than τ_p. In particular, every functional $f \in (E, \tau_p)^*$ is continuous for the weak topology $\sigma(E, (E, \tau_p)^*)$.

Proof. By definition, the weak topology $\sigma(E, (E, \tau_p)^*)$ is the coarsest topology on E for which all functionals in $(E, \tau_p)^*$ are continuous. Since all those functionals are already continuous in τ_p, it follows immediately that $\sigma(E, (E, \tau_p)^*) \subseteq \tau_p$ (i.e. the weak topology is coarser than τ_p).

The dual space E^* of a GISS $(E, [\cdot,\cdot], \tau_p)$ is dense in the weak-* topology induced by E.

Proof. By Theorem 5, every continuous linear functional on (E, τ_p) is weak-* continuous. Since E is a Fréchet space (Lemma 5), E^* is non-empty and separates points (by non-degeneracy, G3). The weak-* topology on E^* is Hausdorff, and the set of continuous functionals is dense in E^* under the weak-* topology [10, Theorem 3.12]. Thus, E^* is dense in itself.

Assume that $\tau = \tau_p$ and that E admits a decomposition $E = E_+ \oplus E_-$ as in the explicit examples of Section 8. Then E_+ is dense in (E, τ_p).

Proof. Since $\tau = \tau_p$, it suffices to prove that E_+ is dense in (E, τ). Assume, for contradiction, that E_+ is not dense in (E, τ). Then its closure $\overline{E_+}^{\tau}$ is a proper closed subspace of E. By the Hahn–Banach separation theorem (equivalently, by the standard duality for locally convex spaces), there exists a nonzero continuous linear functional $f \in E^*$ such that

$$f(w) = 0 \qquad \text{for all } w \in \overline{E_+}^{\tau},$$

and hence in particular $f|_{E_+} = 0$.

In the explicit examples of Section 8, the topology τ is Hilbertizable (indeed induced by a Hilbert norm), so by the Riesz representation theorem there exists $u \in E$, $u \neq 0$, such that

$$f(w) = \langle w, u \rangle \qquad \text{for all } w \in E,$$

where $\langle \cdot, \cdot \rangle$ is the Hilbert inner product generating τ. The condition $f|_{E_+} = 0$ implies $\langle w, u \rangle = 0$ for all $w \in E_+$, i.e.

$$u \in E_+^{\perp}.$$

Moreover, in these explicit decomposable cases the splitting $E = E_+ \oplus E_-$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$, hence $E_+^{\perp} = E_-$. Therefore $u \in E_-$. Now take $w = u \in E_-$. Since $E_- = E_- \cap E_+^{\perp}$, we have $\langle u, u \rangle = 0$, hence $u = 0$, contradicting $u \neq 0$.

This contradiction shows that $\overline{E_+}^{\tau} = E$, i.e. E_+ is dense in (E, τ). Because $\tau = \tau_p$, it follows that E_+ is dense in (E, τ_p) as well.

6. Operators on GISS

This section describes the operator theory for Generalized Inner Structure Spaces (GISS) which we defined in Section 3 with a focus on properties of linear operators in the indefinite sesquilinear form $[\cdot, \cdot]$ and the induced topology τ_p (Section 5). We first consider self-adjoint operators, their spectra, adjoint properties, and subspace invariance (in addition to the topological framework presented earlier). These results are important and critical to applications in quantum mechanics and partial differential equations (Section 9), especially where metrics become indefinite. The section also focuses on spectral decomposition issues and extends the analysis to compact operators and invariant subspaces.

Definition 6. *An operator $T : E \to E$ on a GISS $(E, [\cdot, \cdot])$ is* self-adjoint *if $[Tt, w] = [t, Tw]$ for all $t, w \in E$.*

Let $(E, [\cdot, \cdot], \tau_p)$ be a GISS. If $T : E \to E$ is a self-adjoint operator, then T is bounded in τ_p if and only if it is bounded in τ.

Proof. According to Lemma 5 (Section 5), τ_p is smaller than τ, thus the identity map $id : (E, \tau) \to (E, \tau_p)$ is continuous. If T is bounded in τ then for some constant C, $\|Tt\|_\tau \leq C\|t\|_\tau$. As $\tau_p \subseteq \tau$, T is bounded with τ_p as the boundary. Contrarily, if T is bounded in τ_p, that is, when $C' > 0$ it is $p_z(Tt) = |[z, Tt]| \leq C' p_z(t) = C'|[z, t]|$ for all $z, t \in X$. According to self-adjointness $[z, Tt] = [Tz, t]$. Via Theorem 3 (Section 3), there is a constant M such that $[Tz, t] \leq M\|Tz\|_\tau \|t\|_\tau$. If $p_z(Tt) \leq C' p_z(t)$, and $\{p_z\}$ gives τ_p, the uniform boundedness principle [10, Theorem 2.8] implies that T is bounded in τ since (E, τ) is Fréchet.

Proposition 5. *For each bounded linear operator $T : (E, \tau_p) \to (E, \tau_p)$ on a GISS, a unique adjoint operator $T^* : E \to E$ exists where $[Tt, w] = [t, T^* w]$ for all $t, w \in E$, and T^* is bounded in τ_p.*

Proof. For fixed $t \in E$, consider the functional $f_t(w) = [Tt, w]$. In the application of Theorem 3 (Section 3), $[\cdot, \cdot]$ is continuous in $\tau_p \times \tau_p$, and since T is bounded in τ_p, $f_t(w)$ is continuous in τ_p. According to Lemma 3 (Section 3) there exists a unique $z \in E$ such that $f_t(w) = [z, w]$. We define $T^*t = z$, so $[Tt, w] = [t, T^*w]$. The linearity of T^* is derived from the linearity of $[Tt, \cdot]$. Therefore, to show boundedness, we have $p_s(T^*t) = |[s, T^*t]| = |[Ts, t]| \leq Cp_s(t)$ for some C, as T is bounded. For this reason, T^* is bounded in τ_p. Uniqueness derives from non-degeneracy (G3): if $[T_1^*t, w] = [T_2^*t, w]$ then $[(T_1^* - T_2^*)t, w] = 0$, and subsequently $T_1^* = T_2^*$ [11, Theorem 5.1].

Suppose $T : E \to E$ is a bounded linear operator on a GISS $(E, [\cdot, \cdot], \tau_p)$. If T is self-adjoint with $\sigma(T) \subset \mathbb{R}$, then the resolvent set $\rho(T) = \mathbb{C} \setminus \sigma(T)$ is open in the weak topology arising from $\{w \mapsto [t, w] : t \in E\}$.

Proof. The resolvent $R_\lambda = (T - \lambda I)^{-1}$ exists and is bounded on (E, τ_p) along the operator norm for $\lambda \notin \sigma(T)$. The weak topology is produced by functionals $w \mapsto [t, w]$. Since R_λ is continuous in τ_p, the map $\lambda \mapsto [t, R_\lambda w]$ is continuous for the fixed t, w. This implies that, in the weak topology, $\rho(T)$ is open [11, Theorem 5.10].

The spectrum $\sigma(T)$ of a self-adjoint operator in a GISS is real. Decomposition of spectral information requires some form of extra structure like Krein or similar decomposition.

Let $T : (E, \tau_p) \to (E, \tau_p)$ be a bounded self-adjoint operator on a GISS. The spectral radius $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ satisfies $r(T) \leq \sup\{|[Tt, t]| : p_t(t) \leq 1\}$.

Proof. As T is self-adjoint, $\sigma(T) \subset \mathbb{R}$ (Corollary 6). For $\lambda \in \sigma(T)$, there is a sequence $\{t_n\}$ with $p_{t_n}(t_n) = |[t_n, t_n]| \leq 1$ such that $(T - \lambda I)t_n \to 0$ in τ_p. By Theorem 3, $[Tt_n, t_n] - \lambda[t_n, t_n] = [(T - \lambda I)t_n, t_n] \to 0$. Therefore, $|[Tt_n, t_n] - \lambda| \leq Cp_{t_n}((T - \lambda I)t_n) \to 0$, so $|[Tt_n, t_n]| \to |\lambda|$. Because $p_{t_n}(t_n) \leq 1$, $|\lambda| \leq \sup\{|[Tt, t]| : p_t(t) \leq 1\}$. Thus, $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \sup\{|[Tt, t]| : p_t(t) \leq 1\}$ [11, Theorem 5.13].

If $T : (E, \tau_p) \to (E, \tau_p)$ is a compact self-adjoint operator on a GISS, then $\sigma(T) \setminus \{0\}$ has at most countably many isolated eigenvalues with finite multiplicity.

Proof. Compact and self-adjoint T, thus $\sigma(T) \subset \mathbb{R}$ (Corollary 6). For a Fréchet space (E, τ_p) (Lemma 5), compact operators have a spectrum consisting of zero and at most countably many isolated eigenvalues with finite multiplicity [10, Theorem 4.25]. For every non-zero eigenvalue λ, an eigenspace $\{t : Tt = \lambda t\}$ is finite-dimensional due to compactness, and self-adjointness provides coherence to the geometry $[Tt, w] = [t, Tw]$. Thus, $\sigma(T) \setminus \{0\}$ can be claimed as stated.

6.1. Spectral Theory Challenges

Unlike Hilbert spaces, GISS may not provide full spectral decomposition for self-adjoint operators due to the indefinite nature of $[\cdot, \cdot]$. For a bounded self-adjoint operator T, the spectrum is real (Theorem 6), but without positive-definiteness, a direct analogue to the Hilbert space spectral theorem is not possible. We surmise that such a Krein-like decomposition $E = E_+ \oplus E_-$ (Section 7), with compactness assumed on T, could result in a partial spectral decomposition in which eigenvectors corresponding to positive and negative eigenvalues are orthogonal in E_+ and E_-, respectively [9]. Conditions for such decompositions will require further research.

7. Decomposition Theorem

This section defines the decomposition theory for Generalized Inner Structure Spaces (GISS) based on Section 3, including the existence of maximal positive and maximal negative subspaces associated with the sign of $[t, t]$ using positive, negative, and isotropic subspaces as parameters to the sesquilinear form $[\cdot, \cdot]$ and to the induced topology τ_p (Section 5). The decomposition theorem forms the basis for GISS structure (using operators/operator theory) as well as application in quantum field theory/partial differential equations (Section 9). We validate decomposition properties such as the closedness of subspaces and continuity of projections using state of the art procedures.

Definition 7. *Two vectors* $t, w \in E$ *are* GISS-orthogonal *if* $[t, w] = 0$. *The* radical *is* $\mathrm{Rad}(E) = \{t \in E : [t, w] = 0 \ \forall w \in E\}$ *and the set of neutral vectors is* $\mathcal{N} = \{t \in E : [t, t] = 0\}$.

[Maximal positive and negative subspaces] Let $(E, [\cdot, \cdot], \tau_p)$ be a GISS. Then there exist subspaces $E_+, E_- \subseteq E$ such that:

(i) $[t, t] > 0$ for all $t \in E_+ \setminus \{0\}$ and E_+ is maximal with this property.

(ii) $[t, t] < 0$ for all $t \in E_- \setminus \{0\}$ and E_- is maximal with this property.

Moreover, by (G3) the radical $\mathrm{Rad}(E) = \{t : [t, w] = 0 \ \forall w \in E\}$ is $\{0\}$.

Proof. Let $\mathcal{P}$ be the family of subspaces $M \subseteq E$ such that $[t, t] > 0$ for all $t \in M \setminus \{0\}$, ordered by inclusion. The union of any chain in $\mathcal{P}$ is again a subspace in $\mathcal{P}$, hence by Zorn's lemma $\mathcal{P}$ has a maximal element E_+. The construction of E_- is analogous. Finally $\mathrm{Rad}(E) = \{0\}$ is exactly axiom (G3).

at (0,0) E; [-¿] (0.5,0) – (2,1) node[right] E_+; [-¿] (0.5,0) – (2,-1) node[right] E_-; [-¿] (0.5,0) – (2,0) node[right] E_0; at (0.5,-1.5) E_+, $E_- \subseteq E$ maximal sign subspaces;

Figure 1: Decomposition of a GISS into positive (E_+), negative (E_-), and isotropic (E_0) spaces, similar to Theorem 7.

In the explicit decomposable examples of Section 8 (e.g. Example 8), where E admits a topological direct sum decomposition $E = E_+ \oplus E_-$ and the projections $\pi_\pm : E \to E_\pm$ are continuous, the subspaces E_+ and E_- are closed in (E, τ_p).

Proof. Assume $E = E_+ \oplus E_-$ is a topological direct sum and $\pi_\pm$ are continuous in τ_p. Then $E_+ = \pi_+(E)$ and $E_- = \pi_-(E)$. Since π_+ is continuous and $E_+ = \ker(\pi_-)$, we have that E_+ is the kernel of a continuous linear map, hence E_+ is closed in (E, τ_p). Similarly, $E_- = \ker(\pi_+)$ is closed in (E, τ_p).

Proposition 6. *In the explicit decomposable examples of Section 8 (e.g. Example 8), the splitting $E = E_+ \oplus E_-$ is $[\cdot,\cdot]$-orthogonal, i.e.*

$$[t_+, t_-] = 0 \qquad \textit{for all } t_+ \in E_+,\ t_- \in E_-.$$

Proof. In Example 8, E_+ consists of sequences supported on even indices and E_- consists of sequences supported on odd indices. Hence, for $t_+ \in E_+$ and $t_- \in E_-$,

$$[t_+, t_-] = \sum_{n\in\mathbb{Z}} (-1)^n (t_+)_n \, \overline{(t_-)_n} = 0,$$

because for each n at least one factor $(t_+)_n$ or $(t_-)_n$ is zero. The same verification applies to the other explicit examples where the decomposition is defined by disjoint supports (or by the given fundamental symmetry).

(Example-level uniqueness.) In the explicit decomposable examples of Section 8 (e.g. Example 8), where E_+ and E_- are defined by an explicit rule (such as even/odd support), the resulting decomposition $E = E_+ \oplus E_-$ is uniquely determined by that rule.

Proof. In Example 8, E_+ is defined as the set of sequences supported on even indices and E_- as those supported on odd indices. These definitions fix E_+ and E_- uniquely. Every $t \in E$ decomposes uniquely as $t = \pi_+ t + \pi_- t$ with $\pi_\pm$ defined componentwise, so the decomposition is unique for this construction.

In the explicit decomposable examples of Section 8, if $\tau = \tau_p$ and $E = E_+ \oplus E_-$, then E_+ is dense in (E, τ_p).

Proof. This is exactly Theorem 5.

In the explicit decomposable examples of Section 8 (e.g. Example 8), the projections $\pi_+ : E \to E_+$ and $\pi_- : E \to E_-$ associated with the decomposition $E = E_+ \oplus E_-$ are continuous in τ_p.

Proof. In Example 8, the projections are given explicitly by

$$(\pi_+ t)_n = \begin{cases} t_n, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad (\pi_- t)_n = \begin{cases} 0, & n \text{ even}, \\ t_n, & n \text{ odd}. \end{cases}$$

Let $t^{(k)} \to t$ in τ_p. For any fixed $s \in E$,

$$p_s(\pi_+ t^{(k)} - \pi_+ t) = |[s, \pi_+(t^{(k)} - t)]| \le |[s, t^{(k)} - t]| = p_s(t^{(k)} - t) \to 0,$$

because π_+ only removes coordinates and does not create new ones in this example, and $[\cdot,\cdot]$ is computed componentwise. Hence $\pi_+ t^{(k)} \to \pi_+ t$ in τ_p, so π_+ is continuous. The same argument applies to π_-.

8. Examples

A variety of different examples describe the applicability of Generalized Inner Structure Spaces (GISS), as presented in Section 3, in both functional analysis as well as mathematical physics. These examples illustrate how GISS is possible for indefinite and non-degenerate sesquilinear forms through many topological options, and can be applied to sequence spaces, Sobolev spaces, Gelfand triples, quantum mechanical state spaces and distribution spaces. We validate the GISS axioms in each example, discuss the decomposition in Theorem 7, and draw related comparisons to applications in quantum field theory and partial differential equations (Section 9). Further, we present the math used to formalize the properties observed in such examples for a solid grounding. Let $E = \ell^2(\mathbb{Z})$ have the sesquilinear form $[t, w] = \sum_{n\in\mathbb{Z}}(-1)^n t_n\overline{w_n}$. Verify GISS axiomatic (G1). Linearity maintains as the sum for t_n is linear and the sum is conjugate-linear for w_n. (G4) Continuity: The τ topology is in standard ℓ^2 norm topology and $p_t(w) = |[t, w]| \leq \|t\|_2\|w\|_2$ according to Cauchy-Schwarz, thus $\tau_p \subseteq \tau$. For $t = \delta_{n,1}$, $[t, t] = (-1)^1 = -1$. E is NOT a Hilbert space. Define $E_+ = \{t : t_n = 0 \text{ for } n \text{ odd}\}$, $E_- = \{t : t_n = 0 \text{ for } n \text{ even}\}$ and $E_0 = \{0\}$ (non-degeneracy) by Theorem 7. Then $E = E_+ \oplus E_-$, with $[t, t] = \sum_{n \text{ even}} t_n\overline{t_n} > 0$ on E_+ and $[t, t] = -\sum_{n \text{ odd}} t_n\overline{t_n} < 0$ on E_-. However, this structure can be used in sequence space analysis [1].

In Example 8, the projections are explicit:

$$(\pi_+t)_n = \begin{cases} t_n, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad (\pi_-t)_n = \begin{cases} 0, & n \text{ even}, \\ t_n, & n \text{ odd}. \end{cases}$$

Then $t = \pi_+t + \pi_-t$ and $[t, t] = \|\pi_+t\|_2^2 - \|\pi_-t\|_2^2$.

Consider a Gelfand triple $\Phi \subset H \subset \Phi'$, where H is Hilbert space with the inner product $\langle\cdot,\cdot\rangle_H$ and Φ a dense Fréchet subspace with τ. Let $[t, w] = \langle t, w\rangle_{\Phi'\times\Phi}$ through duality pairing. Axiom (G1) is dependent on linearity of the pairing. (G2) Symmetry in conjugates follows from properties of the pairing. (G3) Non-degeneracy: if $[t, w] = 0$ for all $w \in \Phi$, then $t = 0 \in \Phi'$, in similar fashion for the second argument. (G4) Continuity: the map $w \mapsto \langle t, w\rangle_{\Phi'\times\Phi}$ is continuous in τ because of the dual. Therefore, $(\Phi, [\cdot,\cdot], \tau)$ is a GISS. One may identify maximal positive/negative subspaces depending on the pairing; an explicit global orthogonal decomposition is not assumed in general.

Take $E = H^1(\mathbb{R})$, the Sobolev space of square-integrable functions with square-integrable derivatives, whose topology τ proceeds from the norm $\|f\|_{H^1} = \sqrt{\int |f|^2 + \int |f'|^2}$. Define $[f, g] = \int_{\mathbb{R}} f(t)\overline{g(t)}dt - \int_{\mathbb{R}} f'(t)\overline{g'(t)}dt$. (G1) From the integrals, linearity and (G2) conjugate symmetry results. (G3) Non-degeneracy: $[f, g] = 0$ for all $g \in H^1$, therefore $\int f\overline{g} = \int f'\overline{g'}$, which means $f = 0$ for $g \in C_c^\infty(\mathbb{R})$. (G4) Continuity: $[f, g] \leq \|f\|_2\|g\|_2 + \|f'\|_2\|g'\|_2 \leq \sqrt{2}\|f\|_{H^1}\|g\|_{H^1}$, hence the form is continuous in τ. Therefore, $H^1(\mathbb{R})$ is a GISS. Decomposition (Theorem 7) gives E_+ where one is forced to consider first integral dominance, which is used in PDE analysis [3]. In quantum mechanics, let E behave as space of states in a Gupta-Bleuler quantization with $[t, w]$ denoted with the Lorentz metric. Taking $E = L^2(\mathbb{R}^3, \mathbb{C}^4)$ as a Dirac operator:

where $[t, w] = \int t^\dagger \eta w \, dt$, where $\eta = \text{diag}(1, 1, -1, -1)$ is Minkowski metric. (G1) and (G2) are determined by linearity and Hermitian properties of η. (G3) Non-degeneracy: for all w, if $[t, w] = 0$ then $t = 0$. (G4) Continuity: with an L^2 topology τ, continuity is guaranteed.

In this Hilbertizable example, an explicit decomposition $E = E_+ \oplus E_-$ is induced by η (positive/negative spectral subspaces).

Let $E = \ell^2(\mathbb{N})$ and $[t, w] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} t_n \overline{w_n}$. (G1) Linearity and (G2) conjugate symmetry are maintained. (G3) Non-degeneracy: if $[t, w] = 0$ for all w, then $\sum \frac{(-1)^n}{n} t_n \overline{w_n} = 0$, so $t_n = 0$. (G4) Continuity: the weights $\frac{(-1)^n}{n}$ satisfy $|\frac{(-1)^n}{n}| \leq \frac{1}{n}$, so $[t, w] \leq \sum \frac{1}{n} |t_n||w_n| \leq \left(\sum \frac{1}{n^2}\right)^{1/2} \|t\|_2 \|w\|_2$, maintaining a continuous structure across ℓ^2 topology τ. Decomposition $E = E_+ \oplus E_-$ is for $E_+ = \{t : t_n = 0 \text{ for } n \text{ even}\}$, $E_- = \{e : e_n = 0 \text{ for } n \text{ odd}\}$, and $E_0 = \{0\}$ in signal processing [12].

Proposition 7. *In Examples 8 and 8, the multiplication operator $Tt_n = nt_n$ is self-adjoint with respect to $[\cdot, \cdot]$.*

Proof. In Example 8, for $t, w \in \ell^2(\mathbb{Z})$, $[Tt, w] = \sum (-1)^n (nt_n) \overline{w_n} = \sum (-1)^n n t_n \overline{w_n}$. Similarly, $[t, Tw] = \sum (-1)^n t_n \overline{n w_n} = \sum (-1)^n n t_n \overline{w_n}$. So, $[Tt, w] = [t, Tw]$, so T is self-adjoint. In Example 8, $[Tt, w] = \sum \frac{(-1)^n}{n} (nt_n) \overline{w_n} = \sum (-1)^n t_n \overline{w_n}$ and $[t, Tw] = \sum \frac{(-1)^n}{n} t_n \overline{n w_n} = \sum (-1)^n t_n \overline{w_n}$, so T is self-adjoint. Boundedness holds in the normable case (Example 8) although it necessitates domain restrictions in Example 8 [11].

In Example 8, the self-adjoint operator $Tt_n = nt_n$ on $\ell^2(\mathbb{Z})$ has spectrum $\sigma(T) = \mathbb{Z}$, which is real, and is in accordance with Theorem 6 (Section 6).

Proof. Assume $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $Tt_n = nt_n$. From Proposition 7, T is self-adjoint. For $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ has $(T - \lambda I)t_n = (n - \lambda)t_n$. If $\lambda = k \in \mathbb{Z}$, then for $t = e_k$, $(T - kI)e_k = (k - k)e_k = 0$, that is, $\lambda = k$ is an eigenvalue, and $\sigma(T) \supseteq \mathbb{Z}$. If $\lambda \notin \mathbb{Z}$, then $(T - \lambda I)^{-1} t_n = \frac{t_n}{n - \lambda}$, which is bounded since $|n - \lambda| \geq \text{dist}(\lambda, \mathbb{Z})$. So $\sigma(T) = \mathbb{Z} \subset \mathbb{R}$, consistent with Theorem 6 [11].

9. Applications

Use of Generalized Inner Structure Spaces (GISS) is discussed in Section 3 and applications range across quantum fields, partial differential equations, signal processing, and control theory. Building on the naturality of GISS's indefinite sesquilinear formats and Fréchet topology (Section 5), these applications utilize the sign structure of $[\cdot, \cdot]$ together with maximal positive/negative subspaces and, in several concrete models, explicit decompositions such as $E = E_+ \oplus E_-$. Theorem 7 and operator theory (Section 6) to solve problems where traditional Hilbert space architectures are inadequate. Under each section, we outline the GISS properties to enable analysis and then we present mathematical results to formalize the relationships between them. Furthermore, we cover limitations to point out existing open issues, so that functional and mathematical space applications have a complete frame of reference.

9.1. Quantum Field Theory

Indefinite inner products arise naturally in quantum field theory, particularly in covariant quantization schemes such as the Gupta–Bleuler formalism [2]. A concrete realization of a GISS can be constructed using a fundamental symmetry on a Hilbert space.

Let $E = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with the standard Hilbert inner product

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}^3} \psi(x)^\dagger \phi(x)\, dx.$$

Define the Hermitian matrix $\eta = \mathrm{diag}(1, 1, -1, -1)$ and the sesquilinear form

$$[\psi, \phi] = \langle \eta\psi, \phi \rangle.$$

This form is continuous, Hermitian, and non-degenerate, hence $(E, [\cdot, \cdot])$ is a GISS.

Defining $E_+ = \{\psi : \eta\psi = \psi\}$ and $E_- = \{\psi : \eta\psi = -\psi\}$ yields the explicit decomposition

$$E = E_+ \oplus E_-, \qquad [\psi, \psi] = \|\psi_+\|^2 - \|\psi_-\|^2.$$

Physical state spaces are obtained by imposing constraints and quotienting out isotropic vectors, producing a positive-definite Hilbert space of observables.

9.2. Partial Differential Equations

GISS provides a natural framework for hyperbolic equations, where conserved quantities often have mixed sign. Consider the wave equation $\partial_t^2 u - \Delta u = 0$ and define the first-order system $U = (u, \partial_t u)$.

Let

$$E = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$$

and define

$$[(u, v), (\varphi, \psi)] = \int_{\mathbb{R}^d} \nabla u \cdot \overline{\nabla\varphi}\, dx - \int_{\mathbb{R}^d} v\overline{\psi}\, dx.$$

Then

$$[(u, v), (u, v)] = \|\nabla u\|_2^2 - \|v\|_2^2,$$

which is indefinite but conserved along solutions of the wave equation. This explicit identity illustrates how GISS captures natural energy structures beyond Hilbert-space settings.

9.3. Signal Processing

GISS can be applied in signal processing, notably in the sequence space $\ell^2(\mathbb{N})$ equipped with weighted indefinite forms, as in Example 8. Signals with alternating-frequency structure can be modeled by

$$[t, w] = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, t_n \overline{w_n}.$$

In this example, an explicit splitting $E = E_+ \oplus E_-$ given by even/odd indices separates positive and negative components.

Moreover, the multiplication operator $(Tt)_n := n\, t_n$ is self-adjoint with respect to $[\cdot, \cdot]$ on its natural domain

$$\mathcal{D}(T) := \left\{ t \in \ell^2(\mathbb{N}) : (nt_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \right\},$$

as shown in Proposition 7. Such operators are standard in frequency-domain analysis and support filtering interpretations in discrete-time systems [12].

9.4. Control Theory

Indefinite quadratic functionals arise in control and optimization, for instance in settings with sign-indefinite storage functions or cost functionals [13]. Let $(E, [\cdot, \cdot], \tau_p)$ be a GISS and consider the autonomous system

$$\dot{t} = At,$$

where $A : E \to E$ is linear and self-adjoint with respect to $[\cdot, \cdot]$, i.e. $[At, w] = [t, Aw]$ for all $t, w \in E$.

Assume that an explicit positive/negative splitting $E = E_+ \oplus E_-$ is available in the model under consideration (as in Example 8) and that $A : E \to E$ is self-adjoint with respect to $[\cdot, \cdot]$. If

$$[At, t] \le 0 \qquad \text{for all } t \in E_+,$$

then the functional $V(t) := [t, t]$ is non-increasing along trajectories $t(\tau)$ that remain in E_+, i.e.

$$V(t(\tau)) \le V(t(0)) \quad \text{for all } \tau \ge 0.$$

In particular, the dynamics is Lyapunov stable on E_+ with respect to the indefinite energy V.

Proof. Let $t(\tau)$ be a (classical) solution with $t(0) \in E_+$ and assume $t(\tau) \in E_+$ for all $\tau \ge 0$. Define $V(t) = [t, t]$ on E_+. Using sesquilinearity and self-adjointness,

$$\frac{d}{d\tau} V(t(\tau)) = [\dot{t}(\tau), t(\tau)] + [t(\tau), \dot{t}(\tau)] = [At(\tau), t(\tau)] + [t(\tau), At(\tau)] = 2\,\mathrm{Re}\,[At(\tau), t(\tau)].$$

Since $[At, t] \le 0$ for all $t \in E_+$, we obtain $\frac{d}{d\tau} V(t(\tau)) \le 0$ and hence $V(t(\tau)) \le V(t(0))$ for all $\tau \ge 0$ [13, Section 3.4].

9.5. Limitations

Although GISS is a flexible framework, its Fréchet space structure does not suit small or non-locally-convex spaces, thus restricting its applicability to practical applications. Relaxing non-degeneracy (G3) to accept degenerate forms may broaden GISS, but it creates issues regarding uniqueness of decomposition (Theorem 7). Moreover, decomposition of spectral components in GISS (Section 6.1) requires more structure, like compactness (Corollary 6).

10. Conclusion

This paper introduced the concept of *Generalized Inner Structure Spaces* (GISS), providing a flexible framework for Fréchet spaces equipped with continuous, Hermitian sesquilinear forms that may be indefinite. The theory unifies and extends several classical constructions, including Hilbert spaces, Krein spaces, semi-inner product spaces, and Gelfand triples.

We established fundamental structural properties of GISS, including the existence of maximal positive and maximal negative subspaces and the role of neutral vectors in the associated indefinite geometry. We also investigated the locally convex topology τ_p generated by the seminorms $p_t(w) = |[t, w]|$, and clarified completeness and continuity properties under natural additional hypotheses (in particular, in the important case $\tau = \tau_p$). In Hilbertizable settings, we showed that a bounded sesquilinear form admits a representation via a unique bounded self-adjoint operator through a background Hilbert inner product.

A range of examples demonstrated that GISS naturally accommodates both normable and non-normable contexts, and supports explicit positive/negative splittings in concrete models such as sequence spaces. Applications were presented in quantum field theory and partial differential equations, where indefinite forms arise intrinsically and yield conserved or sign-indefinite energy identities, as well as in signal processing and control-theoretic models where an explicit positive/negative splitting is available.

Future work will focus on identifying additional assumptions (e.g., fundamental symmetries or Krein-type structures) under which stronger global decompositions and spectral results can be obtained, and on extensions to settings that allow degeneracy or broader classes of locally convex spaces.

Declarations

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Competing Interests
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