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Matrix Representations of the Two-Parameter Deformed Oscillator Algebra

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Abstract. Faithful matrix representations are presented for Polynomially (p, q)-deformed Lie algebra $\mathfrak{l}_{p,q}$: $[K_0, K_+]_{p,q} = rK_+, [K_-, K_0]_{p,q} = rK_-,[K_+, K_-]_{p,q} = F(K_0)$, where F is a real polynomial function, and $p, q, r \in \mathbb{R}^*$, ($\mathbb{R}^*$ is the set of nonzero real numbers). Conditions are set for F and the (p, q)-parameters, where the operators K_+, K_-, and K_0, satisfy certain physical properties.

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Key Words and Phrases: Faithful matrix representation of Lie algebras, deformed Lie algebra, (p, q)-algebraic deformation

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1. Introduction

In many physical contexts, such as quantum optical systems, non-linear quantum Hamiltonians of order higher than quadratic or bilinear forms appear in modelling and investigations of many phenomena. Few examples are: many-body systems and multi-photon processes [1], [2], particles that interpolate between Bosons and Fermions [3], [4], [5], and quantum deformed (q-deformed) oscillators. Such higher orders of non-linear Hamiltonian models are usually associated with two types of non-linear q-deformed algebra, namely, the q-deformed Lie brackets and the polynomially deformed Lie algebra $SU_{pd}(2)$, [6].

Generally speaking, within the quantum mechanical context, generalization of the harmonic oscillator (HO) algebraic commutation relations, called q-deformed HO, is due to basically the mathematical non-linearity/non-ideal nature of the concerned quantum complex system [[7], and refs. therein]. Two particular physical applications of the deformation algebra, namely: (i) The one q-deformation parameter is an indicator of information of

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quantum impurities in many-particle interactions in a condensed matter system [8], (*ii*) The two (p, q)-deformation parameters of the bosonic Fibonacci oscillator algebra [9] mimic the defects and impurities in crystalline lattice structure.

Investigations of appropriate time evolution operators for such nonlinear Hamiltonian systems are generally complicated to handle. The alternative Lie algebraic decomposition approach and its faithful matrix representation (once found!) for the generators of such deformed Lie algebra is relatively less tedious than dealing with the direct method of solving Schrödinger's wave equations, e.g., [10].

On the other hand, the q-deformed Virasoro algebras [11] are ideal physical applications of quantum groups [12], which are non-commutative and co-commutative Hopf algebras. A two-parameter quantum deformation of Lie super algebras in the non-standard simple root system with two odd simple roots is examined in [13].

In the present work, we search for faithful matrix representations of the generators of two-parameter (p, q)-deformed algebraic structure, namely, the q- and the polynomially-deformed Lie algebra, associated with the generalized algebra, $\mathfrak{l}_{p,q}$, mentioned next in Section 2.

2. Preliminaries

In [12], the (p, q)-deformed Lie bracket is given as follows.

Definition 1. *Let p and q be real numbers. If X and Y are square matrices of order n, then the (p, q)-deformed Lie bracket of X and Y, namely $[X, Y]_{p,q}$, is defined as,*

$$[X, Y]_{p,q} = pXY - qYX. \tag{1}$$

For the two particular cases, namely, $p = 1, q = 0$ and $p = 0, q = -1$, we have the usual matrix multiplication of X and Y. So we always consider that here, p and q are supposed to be nonzero real numbers, i.e., $pq \neq 0$. For instance, in [3], the model of Fermion Oscillators has $p = 1$ and $0 \leq q \leq 1$. It should be noted that the case when $p = 1, q = 1$ is the ordinary Lie bracket. Thus, we may write $[X, Y]_{1,1}$ as $[X, Y]$.

Faithful matrix representations of the least degree of the Lie algebra $\mathfrak{l}$ were considered in [10]-[14], [1]-[15] where,

$$\mathfrak{l} : [K_0, K_\pm] = \pm r K_\pm \text{ and } [K_+, K_-] = F(K_0), \tag{2}$$

with F is a real polynomial function and $r \in \mathbb{R}^*$ ($\mathbb{R}^*$, is the set of nonzero real numbers), subject to the physical properties, namely, K_0 is a real diagonal operator and $K_- = K_+^\dagger$, ($\dagger$ is for Hermitian conjugation). The nonzero parameter r is a proportionality factor relating the HO operators $K_\pm$ to the corresponding atomic spin-up and -down operators $S_\pm$ (e.g., [16])

In the present work, we consider the Hamiltonian H of two coupled harmonic oscillators in an optical cavity, representing 'optical 2-level atom' in the form [16] (we take the reduced Planck constant $\hbar = 1$),

$$H = \omega K_0 + \lambda(t)(K_+ + K_-) = H^\dagger. \tag{3}$$

Note that, the total Hamiltonian operator H in (3) represents the total quantum energy operator of the considered system (coupled quantized optical atoms). For physical requirements, H must be a Hermitian operator, and so, its observed energy eigenvalues must be real, and in turn, its constituent operators K_0, a real diagonal operator, and $K_- = K_+^\dagger$ are two Hermitian conjugate operators.

Here, we examine the faithful matrix representations of the generalized Lie algebra $\mathfrak{l}_{p,q}$ (cf [13]),

$$[K_0, K_+]_{p,q} = rK_+, \tag{4}$$

$$[K_-, K_0]_{p,q} = rK_-, \tag{5}$$

$$[K_+, K_-]_{p,q} = F(K_0), \tag{6}$$

subject to the physical properties mentioned below (3). Note that, for $p = q = 1$, equations (4)-(6) are reduced to the Lie algebra $\mathfrak{l}_{1,1} = \mathfrak{l}$ in (2), [14], [15].

3. Basic Properties of the (p, q)-deformed Lie Bracket

Here, we list some properties of the (p, q)-deformed Lie bracket in the following theorem.

Theorem 1. *Let X, Y and Z be $n \times n$ matrices, and $p, q \in \mathbb{R}^*$, and $\alpha, \beta \in \mathbb{R}$. Then*

1. $tr\left([X, Y]_{p,q}\right) = tr\left([Y, X]_{p,q}\right) = (p - q)\, tr(XY) = (p - q)\, tr(YX).$
2. $[X, Y]_{p,q}^\dagger = \left[Y^\dagger, X^\dagger\right]_{p,q}.$
3. $\left(\left[X, X^\dagger\right]_{p,q}\right)_{ij} = \overline{\left(\left[X, X^\dagger\right]_{p,q}\right)_{ji}}$ *for* $i \neq j$.
4. $[X, Y + Z]_{p,q} = [X, Y]_{p,q} + [X, Z]_{p,q}.$
5. $[X + Y, Z]_{p,q} = [X, Z]_{p,q} + [Y, Z]_{p,q}.$
6. $[\alpha X, \beta Y]_{p,q} = \alpha\beta\, [X, Y]_{p,q}.$
7. $\left[X, [Y, Z]_{p,q}\right]_{p,q} = p^2 XYZ - pqYZX - pqXZY + q^2 ZYX.$
8. $\left[[X, Y]_{p,q}, Z\right]_{p,q} = p^2 XYZ - pqYXZ - pqZXY + q^2 ZYX.$
9. $\left[X, [Y, Z]_{p,q}\right]_{p,q} + \left[Y, [Z, X]_{p,q}\right]_{p,q} + \left[Z, [X, Y]_{p,q}\right]_{p,q}$
   $= (p - q)\left[p(XYZ + YZX + ZXY) - q(XZY + YXZ + ZYX)\right].$

Proof. The proof of parts (1) - (6) is simple and deduced directly from the definition in (1). So, we prove parts (7) - (9).

For the proof of part (7), we have

$\left[X,[Y,Z]_{p,q}\right]_{p,q} = [X,(pYZ-qZY)]_{p,q}$
$= pX\,(pYZ-qZY) - q\,(pYZ-qZY)\,X$
$= p^2XYZ - pqXZY - pqYZX + q^2ZYX.$

Similarly, for the proof of part (8), we have

$\left[[X,Y]_{p,q}\,,Z\right]_{p,q} = [(pXY-qYX)\,,Z]_{p,q}$
$= p\,(pXY-qYX)\,Z - qZ\,(pXY-qYX)$
$= p^2XYZ - pqYXZ - pqZXY + q^2ZYX.$

(iii) Now, for part (9), we have

$\left[X,[Y,Z]_{p,q}\right]_{p,q} + \left[Y,[Z,X]_{p,q}\right]_{p,q} + \left[Z,[X,Y]_{p,q}\right]_{p,q}$
$= [X,(pYZ-qZY)]_{p,q} + [Y,(pZX-qXZ)]_{p,q} + [Z,(pXY-qYX)]_{p,q}$
$= \{pX\,(pYZ-qZY) - q\,(pYZ-qZY)\,X\} + \{pY\,(pZX-qXZ) - q\,(pZX-qXZ)\,Y\} + \{pZ\,(pXY-qYX) - q\,(pXY-qYX)\,Z\}$
$= p^2\,(XYZ+YZX+ZXY) - pq\,(XZY+YZX+YXZ+ZXY+ZYX+XYZ) + q^2\,(ZYX+XZY+YXZ)$
$= p^2\,(XYZ+YZX+ZXY) - pq\,(XYZ+YZX+ZXY+XZY+YXZ+ZYX) + q^2\,(XZY+YXZ+ZYX)$
$= p^2\,(XYZ+YZX+ZXY) - pq\,\{(XYZ+YZX+ZXY)+(XZY+YXZ+ZYX)\} + q^2\,(XZY+YXZ+ZYX)$
$= p^2\,(XYZ+YZX+ZXY) - pq\,(XYZ+YZX+ZXY) - pq\,(XZY+YXZ+ZYX) + q^2\,(XZY+YXZ+ZYX)$
$= \left(p^2-pq\right)(XYZ+YZX+ZXY) - \left(pq-q^2\right)(XZY+YXZ+ZYX)$
$= p\,(p-q)\,(XYZ+YZX+ZXY) - q\,(p-q)\,(XZY+YXZ+ZYX)$
$= (p-q)\,[p\,(XYZ+YZX+ZXY) - q\,(XZY+YXZ+ZYX)]\,.$

Part (9) of Theorem 1 is the Jacobi identity, where its right-hand side should be zero for $p=q=1$.

Now, using part (2) of Theorem 1 and the fact that $K_+^\dagger = K_-$, we get the next theorem.

Theorem 2. *The defining relations of* $\mathfrak{l}_{p,q}$ *can be either:*

$$[K_0,K_+]_{p,q} = rK_+,\ \text{and}\ [K_+,K_-]_{p,q} = F\,(K_0) \tag{7}$$

or in its Hermitian form,

$$[K_-,K_0]_{p,q} = rK_-,\ \text{and}\ [K_+,K_-]_{p,q} = F\,(K_0)\,.$$

Proof. Consider the Hermitian conjugate of (4). From part (2) of Theorem 1 and the facts that $K_+^\dagger = K_-$ and $K_0^\dagger = K_0$, we have $\left([K_0,K_+]_{p,q}\right)^\dagger = \left[K_+^\dagger,K_0^\dagger\right]_{p,q} = \left[K_-^\dagger,K_0\right]_{p,q} = (rK_+)^\dagger = rK_-$. So, we have the second defining relation of $\mathfrak{l}_{p,q}$, (5). Alternatively, (4) is actually the Hermitian conjugate of (5).

4. Faithful representations of $\mathfrak{l}_{p,q}$

Some basic and necessary definitions are presented at the beginning of this section.

Definition 2. *Let K be a field and $M_n(K)$ be the set of all $n \times n$ matrices of entries from K. A matrix representation of the degree n of the (p, q)-deformed Lie algebra, $\mathfrak{l}_{p,q}$, is a mapping $\rho : \mathfrak{l}_{p,q} \to M_n(K)$ satisfying the following properties, for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathfrak{l}_{p,q}$ and all α and β in K:*

(i) $\rho(\mathbf{u}+\mathbf{v}) = \rho(\mathbf{u}) + \rho(\mathbf{v})$,

(ii) $\rho(\mathbf{uv}) = \rho(\mathbf{u})\rho(\mathbf{v})$,

(iii) $\rho(\alpha\mathbf{u}) = \alpha\rho(\mathbf{v})$, *and*

(iv) $\rho\left([\mathbf{u}, \mathbf{v}]_{p,q}\right) = p\rho(\mathbf{uv}) - q\rho(\mathbf{vu})$.

The matrix $\rho(\mathbf{u})$ is called the representation matrix of $\mathbf{u}$ in $\mathfrak{l}_{p,q}$. If ρ is a one-to-one mapping, then the representation is said to be faithful.

It can be shown that the representation matrices of a linearly independent set of elements in $\mathfrak{l}_{p,q}$ are linearly independent. Here, we assume that A, B, and C are representation matrices for the generators of $\mathfrak{l}_{p,q}$, namely, K_+, K_-, and K_0, respectively. All representations under consideration are supposed to satisfy the physical properties: $B = A^\dagger$, and C is a real diagonal matrix. Also, $p, q, r \in \mathbb{R}^*$ and $F(x)$ is a polynomial function in $\mathbb{R}[x]$. Faithful matrix representations of $\mathfrak{l}_{p,q}$ of the least degree are the main purpose of this work. Mind that the representation matrices A, B, and C are supposed to be linearly independent in the case of faithful representations. Also, we use $\mathbf{0}$ for the zero matrix of appropriate size, and $\mathbf{o}$ for the zero element of $\mathfrak{l}_{p,q}$.

For a representation of degree n, the following equations (8)-(12) are necessary relations for A, B, and C, which are obtained from (4) and (6), respectively. For $i, j = 1, 2, ..., n$, we have

$$[r - (pc_{ii} - qc_{jj})]\, a_{ij} = 0 \text{ for } i \neq j, \tag{8}$$

$$a_{ii}\,[r - (p - q)\, c_{ii}] = 0, \tag{9}$$

and

$$\sum\nolimits_{t=1}^{n} (pa_{it}\bar{a}_{jt} - qa_{tj}\bar{a}_{ti}) = 0 \text{ for } i \neq j, \tag{10}$$

$$\sum\nolimits_{t=1}^{n} \left(p\,|a_{it}|^2 - q\,|a_{ti}|^2\right) = F(c_{ii}), \tag{11}$$

$$(p - q)\sum\nolimits_{i=1}^{n}\sum\nolimits_{t=1}^{n} |a_{it}|^2 = \sum\nolimits_{i=1}^{n} F(c_{ii}). \tag{12}$$

5. Faithful matrix representations of $\mathfrak{l}_{p,q}$

Since $\mathfrak{l}_{p,q}$ is generated by 3 generators, namely, $K_{\pm}$ and K_0, then the least possible degree of a faithful matrix representation is 2. So, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = A^{\dagger} \text{ and } C = diag\,(c_1, c_2)\,, \text{ where } c_1, c_2 \in \mathbb{R}, \text{ while } a, b, c, d \in \mathbb{C} \quad (13)$$

be linearly independent 2×2 matrices.

Thus, from (13) in (6), we have,

$[A, B]_{p,q} = pAA^{\dagger} - qA^{\dagger}A$

$= p \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} - q \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= p \begin{bmatrix} |a|^2 + |b|^2 & a\bar{c} + b\bar{d} \\ \bar{a}c + \bar{b}d & |c|^2 + |d|^2 \end{bmatrix} - q \begin{bmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ a\bar{b} + c\bar{d} & |b|^2 + |d|^2 \end{bmatrix}$

$= \begin{bmatrix} (p - q)\,|a|^2 + p\,|b|^2 - q\,|c|^2 & p\left(a\bar{c} + b\bar{d}\right) - q\,(\bar{a}b + \bar{c}d) \\ \overline{p\left(a\bar{c} + b\bar{d}\right) - q\,(\bar{a}b + \bar{c}d)} & p\,|c|^2 - q\,|b|^2 + (p - q)\,|d|^2 \end{bmatrix}$

$= F\,(C)\,.$

Thus, we have

$$\begin{aligned} [A, B]_{p,q} &= \begin{bmatrix} (p - q)\,|a|^2 + p\,|b|^2 - q\,|c|^2 & p\left(a\bar{c} + b\bar{d}\right) - q\,(\bar{a}b + \bar{c}d) \\ \overline{p\left(a\bar{c} + b\bar{d}\right) - q\,(\bar{a}b + \bar{c}d)} & p\,|c|^2 - q\,|b|^2 + (p - q)\,|d|^2 \end{bmatrix} \\ &= diag\,(F\,(c_1)\,, F\,(c_2))\,. \end{aligned} \quad (14)$$

Similarly, using (13) in (4), we have,

$[C, A]_{p,q} = p \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - q \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = rA.$

So, we have,

$$[C, A]_{p,q} = \begin{bmatrix} (p - q)\,c_1 a & (pc_1 - qc_2)\,b \\ (pc_2 - qc_1)\,c & (p - q)\,c_2 d \end{bmatrix} = rA. \quad (15)$$

From (14), as $tr\left([A, B]_{p,q}\right) = F\,(c_1) + F\,(c_2)\,.$

Thus, we have,

$$(p - q)\left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right) = F\,(c_1) + F\,(c_2)\,. \quad (16)$$

Similarly, from (15), we have, $tr\left([C, A]_{p,q}\right) = ra + rd.$

So,

$$(p - q)\,(ac_1 + dc_2) = r\,(a + d)\,. \quad (17)$$

Lemma 1. *If $C = kI_2$, a scalar matrix and $p \neq q$, then $k = \frac{r}{p-q}$.*

Proof. Since $[C, A]_{p,q} = (p-q)\, kA = rA$. Since $A \neq \mathbf{0}$, then $k = \frac{r}{p-q}$.

5.1. Representation matrices of degree 2 of $\mathfrak{l}_{q,q}$

In this subsection, we consider the special case when $p = q$.

Theorem 3. *$\mathfrak{l}_{p,q}$ is a Lie algebra iff $p = q$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathfrak{l}_{q,q}$, then from definition (1), we have $[\mathbf{x}, \mathbf{y}]_{q,q} = -[\mathbf{y}, \mathbf{x}]_{q,q}$ and $[\mathbf{x}, \mathbf{x}]_{q,q} = \mathbf{o}$. From parts (4) and (5), the bilinearity is satisfied, while the Jacobi identity is satisfied from part (9) of Theorem 1, respectively.

Conversely, from definition (1), $\forall \mathbf{x} \in \mathfrak{l}_{p,q}$, we have $[\mathbf{x}, \mathbf{x}]_{p,q} = (p-q)\, \mathbf{x}^2 = \mathbf{o}$, only if $p = q$ or $\mathbf{x}^2 = \mathbf{o}$, $\forall \mathbf{x} \in l_{p,q}$. Since C is a diagonal matrix, then $C^2 = \mathbf{0}$, if and only if, $C = \mathbf{0}$, which is impossible since C is a representation matrix of a basis element of $\mathfrak{l}_{p,q}$.

As $\mathfrak{l}_{q.q}$ is a Lie algebra, we get the following Corollary.

Corollary 1. *In $\mathfrak{l}_{q,q}$, $tr\,(A) = tr\,(F\,(C)) = 0$.*

Proof. From part (1) of Theorem 1, as $p = q$, we have $tr\,([X, Y]) = 0$ for every $X, Y \in \mathfrak{l}_{q,q}$, then the corollary results from (14) and (15).

Lemma 2. *If C is a scalar matrix, then the matrix representation of $\mathfrak{l}_{q,q}$ is not faithful.*

Proof. Let C be a scalar matrix, then from (15), as $p = q$ and since $r \neq 0$, then $A = \mathbf{0}$.

Lemma 3. *For faithful representation of $\mathfrak{l}_{q,q}$, the representation matrix $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where b is a nonzero complex number.*

Proof. From (9), if $i = 1$, we have $a\,[r - (q-q)\, c_1] = 0$, then $a = 0$ and similarly, if $i = 2$, we have $d = 0$. From (8), if $i = 1$ and $j = 2$, we have $b\,[r - q\,(c_1 - c_2)] = 0$, and similarly, for $i = 2$ and $j = 1$, we have $c\,[r - q\,(c_2 - c_1)] = 0$. Thus, suppose $bc \neq 0$, then we have $q\,(c_1 - c_2) = r = q\,(c_2 - c_1)$. As $q \neq 0$, one gets that $c_1 = c_2$, i.e., C is a scalar matrix. From Lemma 2, the representation is not faithful. Therefore, $bc = 0$. The case where $b = c = 0$, implies that $A = \mathbf{0}$ which is rejected since A is a basis element. Hence, the lemma.

Theorem 4. *The Lie algebra $\mathfrak{l}_{q,q}$, where $q \in \mathbb{R}^*$, has a faithful representation of degree 2 as the least degree, if and only if, there exists $t \in \mathbb{R}$, such that $F\,(t) = -F\left(t - \frac{r}{q}\right)$ with $\frac{F(t)}{q} > 0$. Moreover, the representation matrices of K_+, K_-, and K_0 are $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$, and $C = diag\left(t, t - \frac{r}{q}\right)$, respectively, such that $|b|^2 = \frac{F(t)}{q}$.*

Proof. From Lemma 3, $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, with $b \neq 0$, that is $|b|^2 > 0$. From (15), we have $bq\,(c_1 - c_2) = rb$. Thus, $c_2 = c_1 - \frac{r}{q}$, since $b \neq 0$.

From (14), $F\,(c_1) = q\,|b|^2$, and $F\,(c_2) = -q\,|b|^2 = -F\,(c_1)$. Thus, for a faithful representation of $\mathfrak{l}_{q,q}$, the polynomial function F should satisfy that $F\,(t) = -F\left(t - \frac{r}{q}\right)$ for some real number t. Actually, $c_1 = t$. Thus $C = diag\left(t, t - \frac{r}{q}\right)$. If there is no such t, then the representation matrix C cannot be found and hence, $\mathfrak{l}_{q,q}$ has no matrix representation. Also, from (14), we have $|b|^2 = \frac{F(t)}{q}$ must be positive, because if it is negative, A does not exist, and if it is 0, then $A = \mathbf{0}$, and the representation is not faithful. Hence the theorem.

The following examples demonstrate the method for calculating the representation matrices.

Example 1. *Given that $p = q = 2, r = 4$ and $F\,(x) = x^3 + x$. First we consider the equation $F\left(t - \frac{r}{q}\right) = -F\,(t)$. Thus, $(t-2)^3 + (t-2) = -\left(t^3 + t\right)$, which is $2t^3 - 5t^2 + 13t - 10 = 0$. Choose the real solution $t = 1$, as it satisfies that $\frac{F(t)}{q} > 0$, since $\frac{F(t)}{q} = \frac{1^3+1}{2} = 1 > 0$. So, $|b|^2 = \frac{F(t)}{q} = 1$. Take $b = 1$. Therefore, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$ and $C = diag\,(1, -1)$ are representation matrices of K_+, K_- and K_0, respectively. Since*

$$[A, B]_{2,2} = 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = diag\,(F\,(1)\,, F\,(-1))\,.$$

And

$$[C, A]_{2,2} = 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} = 4A.$$

Clearly, the representation is faithful, because the matrices $A, B,$ and C are linearly independent.

Another representation matrices can be considered as $|b|^2 = 1$, let $A = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$, $B = A^\dagger$ and $C = diag\,(1, -1)$. Such a representation does not satisfy the condition $K_+ + K_-$ is real. Actually, this condition is only satisfied when A is a real matrix.

Example 2. *Let $F\,(x) = x^2 + x + 1$. Then $\mathfrak{l}_{q,q}$ has no faithful representation since $F\,(t) > 0$ for any $t \in \mathbb{R}$.*

Example 3. *The model of light amplifier, namely, $[K_+, K_-] = -2K_0$, $[K_0, K_\pm] = \pm K_\pm$ [17], where $p = q = 1, r = 1$ and $F\,(x) = -2x$. Solving $F\,(t) = -F\left(t - \frac{r}{q}\right)$ for t, we get $t = \frac{1}{2}$ is the only real solution, but $\frac{F(t)}{q} = \frac{-2\left(\frac{1}{2}\right)}{1} = -1$ not positive, i.e., $|b|^2 < 0$, which is impossible. So, A does not exist. So, the light amplifier model has no matrix representation satisfying the physical conditions, as previously proved in [17].*

5.2. Representation matrices of degree 2 of $\mathfrak{l}_{-q,q}$

In this subsection, we consider the special case when $p = -q$.

Lemma 4. *In $\mathfrak{l}_{-q,q}$, if C is a scalar matrix, then A is of zero diagonal elements.*

Proof. Let A as in (13), and C is a scalar matrix. So, let $C = tI_2$. For the linear independence of the generators, and since t is a scalar matrix, we assume that $d = 0$. Suppose $a \neq 0$. We have from (14), $F(t) = -q\left(2|a|^2 + |b|^2 + |c|^2\right)$ and $F(t) = -q\left(|b|^2 + |c|^2\right)$. Then $a = 0$ as $q \neq 0$.

Theorem 5. *The $\mathfrak{l}_{-q,q}$ has faithful matrix representation with C a scalar matrix, if $F(x)$ satisfies, that $-\frac{1}{q}F\left(\frac{r}{-2q}\right) > 0$, and the representation matrices of K_+, K_-, and K_0 are, $A, A^\dagger$, and C, respectively, where*

(i) $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, A^\dagger$, *and* $C = \frac{r}{-2q}I_2$, *respectively, where* $|b|^2 + |c|^2 = -\frac{1}{q}F\left(\frac{r}{-2q}\right) > 0$ *such that* $|b|^2 \neq |c|^2$, *with b and c are of equal imaginary parts to satisfy that $K_+ + K_-$ is real.*

(ii) $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, A^\dagger$, *and* $C = \frac{r}{-2q}I_2$, *respectively, where* $|b|^2 = -\frac{1}{q}F\left(\frac{r}{-2q}\right) > 0$. *In this representation, the condition $K_+ + K_-$ is real, and will only be satisfied for a real matrix A.*

Proof. Let $C = tI_2$ scalar matrix and from Lemma 4, $a = d = 0$. Thus $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. From (14), $-q\left(|b|^2 + |c|^2\right) = F(t)$.

Consider the two cases. Case 1: If $bc = 0$. Let $b \neq 0, c = 0$. Then from (8), $(r + 2qt) = 0$. Hence, $t = \frac{r}{-2q}$.

Then $F\left(\frac{r}{-2q}\right) = -q|b|^2$. Therefore, $|b|^2 = -\frac{1}{q}F\left(\frac{r}{-2q}\right) > 0$.

Case 2: If $bc \neq 0$, then from (8), we have $b(r + 2qt) = 0$ and $c(r + 2qt) = 0$. Then $t = \frac{r}{-2q}$. Then $|b|^2 + |c|^2 = -\frac{1}{q}F\left(\frac{r}{-2q}\right) > 0$.

To satisfy that $K_+ + K_-$ is a real operator, one must have, $b + \bar{c} \in \mathbb{R}$.

If $|b|^2 = |c|^2$, then A and B are linearly dependent, and hence the representation is not faithful.

Example 4. *For $\mathfrak{l}_{-2,2}$, if $r = 8$ and $F(x) = x - 18$, then $C = -2I_2$. To get a representation of type in 1, we can take, for instance, $b = 3$ and $c = \pm 1$, since $|b|^2 + |c|^2 = 10$.*

Also, we can take $b = \sqrt{2} + 2i$ and $c = 2i$.

Example 5. *For $\mathfrak{l}_{-2,2}$, if $r = 8$ and $F(x) = x - 18$, then $C = -2I_2$ and take $b = \pm\sqrt{10}$, for a representation of type in 2.*

Now consider the case where $C = diag\,(c_1, c_2)$, which is not a scalar matrix. From (14) and (15), we have the following theorem as $p = -q$.

Theorem 6. *Let $p = -q$. Then the $\mathfrak{l}_{-q,q}$ has faithful matrix representations, with C not a scalar matrix, namely,*

(i) if there exists $t \in \mathbb{R}$, such that $F\,(t) = -q\left(|b|^2 + |c|^2\right)$ and $F\left(\frac{-qt-r}{q}\right) = F\,(t)$ where $C = diag\left(t, -\frac{qt+r}{q}\right)$ and $A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, where $|c|^2 = -\frac{q|b|^2+F(t)}{q} > 0$,

and for the special case, when $c = 0$,

(ii) if there exists $t \in \mathbb{R}$, such that $|b|^2 = -\frac{F(t)}{q} > 0$ and $F\left(-\frac{qt+r}{q}\right) = F\,(t)$ where $C = diag\left(t, -\frac{qt+r}{q}\right)$ and $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$.

Proof. Let $p = -q$, then we have from (14), with $a = d = 0$, we get $F\,(c_1) = F\,(c_2) = -q\left(|b|^2 + |c|^2\right)$. So, Let $c_1 = t$, then $|c|^2 = -\frac{q|b|^2+F(t)}{q} > 0$, if $bc \neq 0$. From (15), we have, $(-qt - qc_2)\,b = rb$, thus, if $bc \neq 0$, then $c_2 = -\frac{qt+r}{q}$.

5.3. Representation matrices of degree 2 of $\mathfrak{l}_{p,q}$ where $p^2 \neq q^2$

In this subsection, we consider the case where $p^2 \neq q^2$, and we seek a faithful representation for $\mathfrak{l}_{p,q}$ of degree 2 as the least degree.

Lemma 5. *If $bc \neq 0$, and $p + q \neq 0$, then C is a scalar matrix.*

Proof. Comparing the elements on both sides of equation (15), we get:
if $b \neq 0$, then, $r = pc_1 - qc_2$ and similarly, when $c \neq 0$, then $r = pc_2 - qc_1$. Thus, $(p + q)\,(c_1 - c_2) = 0$. Therefore, C is a scalar matrix.

Theorem 7. *If C is a scalar matrix and $p + q \neq 0$, then the representation of $\mathfrak{l}_{p,q}$ is not faithful.*

Proof. For the linear independence of the generators, and as C is a scalar matrix, we assume that $d = 0$. So, from (14), we have $p\,(a\bar{c}) - q\,(\bar{a}b) = 0$. Suppose $a \neq 0$, then $b = \dfrac{p\bar{c}a}{q\bar{a}}$ and hence $|b|^2 = \frac{p^2|c|^2}{q^2}$. Since C is scalar, then $F\,(c_1) = F\,(c_2)$, and hence from (14), $(p - q)\,|a|^2 + p\,|b|^2 - q\,|c|^2 = p\,|c|^2 - q\,|b|^2$. If $p + q \neq 0$, we have, $|c|^2 = -\frac{q^2|a|^2}{(p+q)^2}$ which is impossible unless $c = 0$, and in such a case $a = 0$, contradicting with $a \neq 0$. So, $a = 0$. Then from (14), we get $F\,(c_1) = p\,|b|^2 - q\,|c|^2$ and similarly, $F\,(c_2) = p\,|c|^2 - q\,|b|^2$. Therefore, as C is scalar, we get $(p + q)\left(|b|^2 - |c|^2\right) = 0$. Then either $A = \mathbf{0}$ or $|b|^2 = |c|^2$. In both cases, the representation is not faithful, since the generators $A, B = A^\dagger$, and C are linearly independent, and $A + B$ is a real matrix.

Theorem 8. *The Lie algebra $\mathfrak{l}_{p,q}$, where $p^2 \neq q^2$ with $pq \neq 0$, has a faithful representation of degree 2 as the least degree, iff there exists $t \in \mathbb{R}$, such that $\frac{F(t)}{p} > 0$ and $F\left(\frac{pt-r}{q}\right) = -\frac{q}{p}F(t)$.*

Moreover, the representation matrices of the generators K_+, K_-, and K_0 are $A, A^\dagger$, and $C = diag\left(t, \frac{pt-r}{q}\right)$, respectively, where $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, such that $|b|^2 = \frac{F(t)}{p} > 0$.

Proof. From Lemma 5 and Theorem 7, we must have that $bc = 0$. So, we can choose $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, with $b \neq 0$. Thus, from (14), we have $F(c_1) = p|b|^2$ and $F(c_2) = -q|b|^2$. Thus, $|b|^2 = \frac{F(c_1)}{p} = -\frac{F(c_2)}{q}$. From (15), we have $(pc_1 - qc_2) = r$. So, $pc_1 - r = qc_2$. Thus, $c_2 = \frac{pc_1 - r}{q}$. Let $c_1 = t$ a real number, then $c_2 = \frac{pt-r}{q}$.

Thus, $|b|^2 = \frac{F(t)}{p} = -\frac{F\left(\frac{pt-r}{q}\right)}{q} > 0$. Therefore, the function F should satisfy that $F\left(\frac{pt-r}{q}\right) = -\frac{q}{p}F(t)$.

Example 6. *For $\mathfrak{l}_{2,3}$, if $r = -1$ and $F(x) = 3x^2 - x$, then the solutions of the equation $F\left(\frac{pt-r}{q}\right) = -\frac{q}{p}F(t)$, are $t = 0$ or $t = \frac{1}{7}$.*

For $t = 0$, we have $|b|^2 = 0$, rejected.

For $t = \frac{1}{7}$, we have $|b|^2 = -\frac{2}{49} < 0$, rejected.

Example 7. *For $\mathfrak{l}_{2,3}$, if $r = -1$ and $F(x) = 2x^2 - x$, then the solutions of the equation $F\left(\frac{pt-r}{q}\right) = -\frac{q}{p}F(t)$, are $t = \frac{2}{5}$ or $t = -\frac{1}{14}$.*

For $t = -\frac{1}{14}$, $|b|^2 = \frac{2}{49} > 0$ and $C = diag\left(-\frac{1}{14}, \frac{2}{7}\right)$.

For $t = \frac{2}{5}$, $|b|^2 = -\frac{1}{25} < 0$, rejected.

6. Conclusion

Our main purpose in this work was to find faithful matrix representations of the two-parameter (p,q)-deformed Lie algebra $\mathfrak{l}_{p,q}$ defined in (4)-(6) with p and q being nonzero real numbers. The derived representations associated with the operator generators $K_{\pm,0}$ in the Hamiltonian model (3) can be utilized to investigate the quantum state evolution, similar to our earlier work in the case of ordinary Lie bracket, where $p = q = 1$ [10].

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