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# Fractal Fractional Modelling of Social Media Addiction with Mittag Generalized Decay Stability and Recovery Analysis

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**Abstract.** Social media addiction impacts mental health, social interaction, and productivity levels significantly. The addiction dynamics of this paper are modelled by stratifying the users into general users, mildly addicted, and highly addicted persons in light of recovery-based psychological treatment. This paper uses the fractional mathematical model, the fractal-fractional operator, and the Mittag-Leffler kernel to study addiction paths and recoveries. Reliability of the model presented is established through fixed-point theory and Ulam–Hyers stability. Numerical simulations are conducted to show the influence of fractional-order parameters on addiction and recovery. It gives an idea about how to intervene in an effective manner. Our findings will support the development of targeted rehabilitation programs and digital wellness policies.

**2020 Mathematics Subject Classifications**: 26A33, 34A08, 34D20, 92D25

**Key Words and Phrases**: Fractional-order modelling, fractal calculus, unique solvability, Lipschitz condition, Ulam–Hyers-type stability

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## 1. Introduction

Social media (SM) has had a deep impact on the world, with its effects felt in how people and societies communicate. Services such as Facebook, Instagram, TikTok, and YouTube have redefined communication, entertainment, and exchange of information. SM is utilized for a wide range of applications, including entertainment and business, education, and socialization [1–4]. For learners, SM can be a means of learning and social connection, but excessive use can have negative effects, including compromised academic performance, social exclusion, and mental health [5, 6]. Recent reports have pointed out that during the COVID-19 pandemic, the increase in SM usage in adolescents has been especially concerning, with most youths spending as much as 9 hours per day on these sites [7–9].

Social media addiction (SMA) is presently defined as a serious phenomenon, simulating other types of addiction like alcohol or drug abuse. Such addiction reflects constant use, loss of control, and excess distress experienced when people cannot access their favourite sites. The detrimental effects of SMA are far-reaching and manifest not only in the mental health of individuals but also in interpersonal relationships, academic performance, and general well-being. This has made SMA addressing imperative, particularly in a world where digital platforms govern day-to-day life [10, 11].

Mathematical modelling has become an effective tool for the analysis of intricate behaviours such as addiction. Several researchers have used infectious disease dynamics models to explore different forms of addiction, such as smoking, alcoholism, drug addiction, internet gaming, and social media addiction. Prior research has concentrated on simulating alcoholism [12–16], addiction to online gaming [17, 18], and the effect of social media on academic achievement [19]. In addition, optimal control methods have been utilized to study control and reduction strategies of addiction with minimal implementation costs [20]. Mathematical modelling and optimal control methods have also recently been studied in an effort to better understand and control social media addiction [21].

Fractional Differential Equations (FDE) has gained much attention in disease modeling due to its benefits over standard integer-order models as well as the incorporation of memory effects and inherited properties, enabling more accurate representations of complex temporal and spatial dynamics. The Atangana-Baleanu Caputo (ABC) operator is particularly useful due to its non-singular kernel, which removes the singularities inherent in other fractional operators like Caputo and Riemann-Liouville [22, 23]. This renders the ABC operator more appropriate for the modeling of systems with delayed responses and anomalous diffusion, offering higher flexibility and accuracy. Through the ABC fractional derivative, we can improve our predictions of disease transmission dynamics, enhancing our knowledge and regulation [24, 25].

Current research [26–28] has applied fractional calculus to model social media addiction (SMA), highlighting its ability to capture long-memory effects and complex behaviours. Studies [26] applied Caputo (singular) fractional operators and [27] used Caputo–Fabrizio (non-singular kernels) to validate the role of memory effects in SMA dynamics, while [28] employed fractal–fractional operators to improve model precision through multi-scale behaviour. Unlike previous works, our study introduces a novel SMA model that separates mildly and severely addicted individuals, with an additional compartment for those under psychological treatment. This refinement allows a more realistic analysis of addiction progression and recovery, offering new insights into effective intervention strategies.

Recent studies have highlighted the growing use of fractional calculus in diverse fields. For instance, [29] explored its role in avoiding divergence in Newton-like solvers for nonlinear models, while [30] proposed a modified fractional Newton's solver to improve convergence and stability. The application of fractional operators in epidemiology has also been demonstrated, with [31] analysing bi-modal COVID-19 transmission using Caputo derivatives and [32] investigating HIV/HCV co-infection through a fractional epidemic model. In addition,[33] examined a generalized tumour model with Caputo fractional derivatives, showing the effectiveness of fractional approaches in capturing complex biological dynamics.

In order to solve the intricacies of SMA, mathematical models with fractional order have been established that combine fractal and fractional derivatives. These models assist in simulating the memory-dependent nature of addiction and give insight into the dynamics of user behavior, the effects of intervention, and the efficacy of recovery approaches. The advent of Atangana-Baleanu fractal-fractional derivatives has heightened classical fractional models to improve on the depiction of SMA dynamics, especially in situations where common power-law or exponential decay functions fail.

The current study proposes a fractal-fractional mathematical model for the analysis and regulation of social media addiction, representing an innovative method in this area. In contrast to existing models, the new framework models memory effects, intricate behavioral processes, and nonlinear interactions, aiming at a more realistic and integrated comprehension of the development of addiction.

The subject of this study includes the determination of best treatment approaches that facilitate the recovery of addicted persons, especially students, by integrating psychological interventions into the framework. Moreover, through the application of non-singular and fractal-fractional operators, this study enhances precision in the forecasting of addiction patterns and the effectiveness of interventions. The research outcome is useful in developing evidence-based policies for healthier digital lifestyles and reducing the adverse effects of unhealthy social media usage.

The paper is divided into clear sections: Section 2 brings forth basic definitions, and model formulation is presented in Section 3. Qualitative analysis is discussed in Section 4, while Section 5 goes into studying the existence and uniqueness of the suggested model. Section 6 contains numerical simulations, and Section 7 is reserved for results discussion. Conclusions are then drawn in Section 8.

## 2. Preliminaries

Some of the basic definitions used in fractal-fractional differential equations are provided in Refs [34, 35] and are briefly presented here for completeness.

**Definition 1.** *Let $h \in H^1(a,b)$, $b > a$, and $\eta \in (0,1)$. The Atangana–Baleanu fractional derivative in Caputo sense is defined as*

$$^{ABC}\mathcal{D}^{\eta}_{0+}h(x) = \frac{\mathcal{N}(\eta)}{1-\eta}\int_a^x h'(\tau)E_\eta\left(-\frac{\eta}{1-\eta}(x-\tau)^\eta\right)d\tau,$$

*where $\mathcal{N}(\eta)$ is a normalization function satisfying $\mathcal{N}(0) = \mathcal{N}(1) = 1$, and $E_\eta(\cdot)$ is the Mittag–Leffler function.*

**Definition 2.** *The fractal-fractional derivative with Mittag–Leffler kernel (FFM) of order $\eta \in (0,1)$ and fractal dimension $\vartheta > 0$ is given by*

$$^{FFM}\mathcal{D}^{\eta,\vartheta}_{0+}U(t) = \frac{\mathcal{N}(\eta)}{1-\eta}\frac{d}{dt^\vartheta}\int_0^t E_\eta\left(-\frac{\eta}{1-\eta}(t-\psi)^\eta\right)U(\psi)d\psi \tag{1}$$

*with corresponding integral operator*

$$^{FFM}\mathcal{I}^{\eta,\vartheta}_{0+}U(t) = \frac{\vartheta(1-\eta)t^{\vartheta-1}U(t)}{\mathcal{N}(\eta)} + \frac{\eta\vartheta}{\mathcal{N}(\eta)\Gamma(\eta)}\int_0^t \psi^{\vartheta-1}(t-\psi)^{\eta-1}U(\psi)d\psi \tag{2}$$

*where $\mathcal{N}(\eta) = 1-\eta+\frac{\eta}{\Gamma(\eta)}$.*

## 3. Materials and Methods

In this study, the mathematical model of social media addiction is formulated using a fractal–fractional derivative with the generalized Mittag-Leffler kernel. The Mittag-Leffler kernel offers significant advantages compared to the classical power-law and exponential kernels: the power-law kernel captures long-term memory with slow decay but may exaggerate past influences, while the exponential kernel accounts only for short-term memory with rapid decay. In contrast, the Mittag-Leffler kernel provides a more balanced description by incorporating both short- and long-range memory effects, which makes it particularly suitable for modelling social media addiction where user behaviour is influenced by both recent interactions and long-term usage patterns. Furthermore, the inclusion of the fractal dimension ($\vartheta$) enables the model to account for complex irregularities in user behavior, while the fractional-order derivative ($0 < \eta < 1$) captures the hereditary properties of addiction dynamics with greater accuracy than classical integer-order models.

The proposed framework divides the population into seven subpopulations: susceptible individuals using social media in a normal or typical manner ($\mathbb{S}$), exposed individuals ($\mathbb{E}$) who spend unusually long or extended periods of time on social media compared to typical usage and are therefore at higher risk of transitioning into addiction, mildly addicted individuals ($\mathbb{A}_1$), severely addicted individuals ($\mathbb{A}_2$), individuals undergoing psychological treatment ($\mathbb{P}$), recovered individuals ($\mathbb{R}$), and non-users ($\mathbb{Q}$). The recruitment of new susceptible individuals is represented by $\Lambda$, while natural death occurs at rate $\mu$. The transmission of social media addiction is governed by the parameter $\alpha$, which represents the rate at which susceptible individuals become addicted due to exposure and peer influence, along with the contact rate $\delta$ between susceptible and addicted individuals. Susceptible individuals may develop mild or severe addiction at rates $\lambda_1$ and $\lambda_2$, respectively, while susceptible and recovered individuals re-enter the non-user category at rates $\tau_1$ and $\tau_2$. Mildly and severely addicted individuals may recover naturally, without clinical intervention, at rates $\xi_1$ and $\xi_2$, reflecting self-discipline, family encouragement, and awareness. In addition, mildly and severely addicted individuals may undergo psychological or behavioural treatment at rates $\phi_1$ and $\phi_2$, respectively, and subsequently recover at rate $\xi_3$. Recovered individuals may relapse and return to the susceptible class at rate $\gamma$. Finally, mildly addicted individuals may progress to severe addiction at rate $\varepsilon$. The parameter transmission dynamics of the proposed model are explained in Fig. 1.

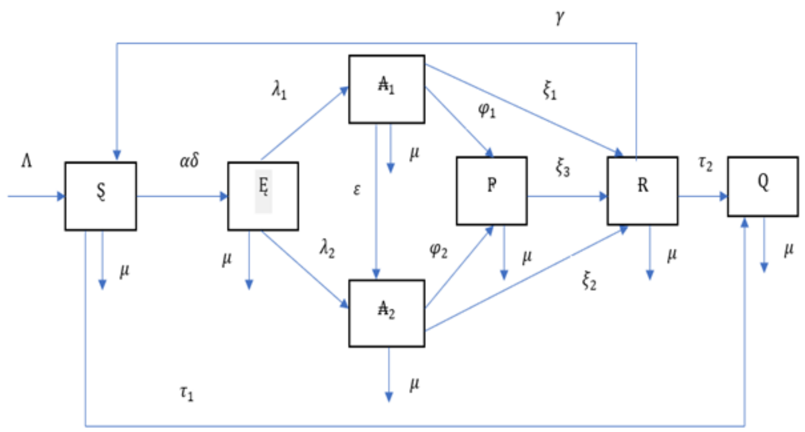


Figure 1: Flowchart of the proposed fractional-order social media addiction model illustrating the transitions among susceptible, mildly addicted, severely addicted, under psychological treatment, and recovered compartments.

In this study, the parameters are considered as constants, representing average transition and treatment rates over the observation period. In extended formulations, these parameters could alternatively be expressed as functions of time (to capture variations due to seasonal trends or intervention campaigns) or as functions of the system state (to account for behavioral saturation, peer influence, or treatment capacity limitations). For analytical tractability, the present model employs constant values.

Based on the given parameters, the mathematical model is derived that predicts the dynamics between social media addiction and recovery with the inclusion of psychological interventions aimed at understanding their effects on controlling addiction. These model equations are given below:

$$\begin{aligned}
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{S}(t) &= \Lambda - \alpha\delta\mathbb{S} - (\tau_1+\mu)\mathbb{S} + \gamma\mathbb{R},\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{E}(t) &= \alpha\delta\mathbb{S} - (\lambda_1+\lambda_2+\mu)\mathbb{E},\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{A}_1(t) &= \lambda_1\mathbb{E} - (\varepsilon+\phi_1+\xi_1+\mu)\mathbb{A}_1,\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{A}_2(t) &= \lambda_2\mathbb{E} + \varepsilon\mathbb{A}_1 - (\phi_2+\xi_2+\mu)\mathbb{A}_2,\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{P}(t) &= \phi_1\mathbb{A}_1 + \phi_2\mathbb{A}_2 - (\xi_3+\mu)\mathbb{P},\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{R}(t) &= \xi_1\mathbb{A}_1 + \xi_2\mathbb{A}_2 + \xi_3\mathbb{P} - (\gamma+\tau_2+\mu)\mathbb{R},\\
{}^{FFM}\mathcal{D}_{0+}^{\eta,\vartheta}\mathbb{Q}(t) &= \tau_1\mathbb{S} + \tau_2\mathbb{R} - \mu\mathbb{Q}.
\end{aligned} \tag{3}$$

These are the equations for the dynamic development of variables

$$(\mathbb{S}(t),\ \mathbb{E}(t),\ \mathbb{A}_1(t),\ \mathbb{A}_2(t),\ \mathbb{P}(t),\ \mathbb{R}(t),\ \mathbb{Q}(t)) \geq 0$$

over time in the context of the suggested model. The adopted framework employs Caputo-type concepts, where the fractal–fractional derivative with a generalized Mittag-Leffler kernel (${}_0^{FFM}D_t^{\eta,\vartheta}$) is used, combining both the fractional order ($\eta$) and fractal dimension ($\vartheta$) to enhance the realism and descriptive power of the model.

## 4. Qualitative Analysis

**Theorem 1.** The proposed fractional-order system (3) admits a unique, positive, and bounded solution in the region $\mathbb{R}_+^7$.

**Proof.** To establish existence, uniqueness, and positivity of the solutions of system (3) over the interval $(0,\infty)$, it is sufficient to show that the non-negative region $\mathbb{R}_+^7$ is invariant under the system dynamics. In other words, if the initial conditions are non-negative, then the trajectories of the system remain in $\mathbb{R}_+^7$ for all $t>0$. Furthermore, by analysing the governing equations, it can be shown that all state variables are bounded above by biologically/socially feasible limits, ensuring that the solutions are not only positive but also uniformly bounded.

$$\begin{aligned}
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{S}(t)\big|_{\mathbb{S}=0} &= \Lambda + \gamma\mathbb{R} \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{E}(t)\big|_{\mathbb{E}=0} &= \alpha\delta\mathbb{S} \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{A}_1(t)\big|_{\mathbb{A}_1=0} &= \lambda_1\mathbb{E} \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{A}_2(t)\big|_{\mathbb{A}_2=0} &= \lambda_2\mathbb{E} + \varepsilon\mathbb{A}_1 \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{P}(t)\big|_{\mathbb{P}=0} &= \phi_1\mathbb{A}_1 + \phi_2\mathbb{A}_2 \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{R}(t)\big|_{\mathbb{R}=0} &= \xi_1\mathbb{A}_1 + \xi_2\mathbb{A}_2 + \xi_3\mathbb{P} \geq 0,\\
{}_0^{FFM}D_t^{\eta,\vartheta}\mathbb{Q}(t)\big|_{\mathbb{Q}=0} &= \tau_1\mathbb{S} + \tau_2\mathbb{R} \geq 0.
\end{aligned} \tag{4}$$

The solution remains confined within the hyperplane if the initial values are $(\mathbb{S}_p(0), T_p(0), T_p(0), K_p(0), B_m(0), \mathbb{S}_v(0), T_v(0)) \in \mathbb{R}^7_+$, as specified in Equation (4). In addition, the vector field always points to the region $\mathbb{R}^7_+$ inside every hyperplane enclosing the non-negative space. Simply, the domain $\mathbb{R}^7_+$ as a whole function to be positively invariant set.

## 5. Existence and Uniqueness of Suggested Model

The system in equations (3) can be reformulated using the Fractal–Fractional Operator in the Atangana–Baleanu Caputo sense. By applying fixed-point theorems, we establish the existence and uniqueness of at least one solution. The model is expressed as

$$
\begin{aligned}
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{S}(t) &= \vartheta t^{\vartheta-1} g_1(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{E}(t) &= \vartheta t^{\vartheta-1} g_2(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{A}_1(t) &= \vartheta t^{\vartheta-1} g_3(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{A}_2(t) &= \vartheta t^{\vartheta-1} g_4(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{P}(t) &= \vartheta t^{\vartheta-1} g_5(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{R}(t) &= \vartheta t^{\vartheta-1} g_6(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}),\\
{}_0^{ABC}D_t^{\eta,\vartheta}\mathbb{Q}(t) &= \vartheta t^{\vartheta-1} g_7(t, \mathbb{S}, \mathbb{E}, \mathbb{A}_1, \mathbb{A}_2, \mathbb{P}, \mathbb{R}, \mathbb{Q}).
\end{aligned}
$$

The system can be further written as

$$
{}_0^{ABC}D_t^{\eta,\vartheta}\theta(t) = \vartheta t^{\vartheta-1}\Omega(t, \theta(t)), \quad \theta(0) = \theta_0.
$$

By applying the fractional integral operator, the system can be equivalently represented in its integral form

$$
\theta(t) = \theta(0) + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t, \theta(t)) \quad + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}\,\Omega(t, \theta(t))\,dm. \tag{5}
$$

Where

$$
\begin{aligned}
\theta(t) &= (\mathbb{S}(t), \mathbb{E}(t), \mathbb{A}_1(t), \mathbb{A}_2(t), \mathbb{P}(t), \mathbb{R}(t), \mathbb{Q}(t)),\\
\theta(0) &= (\mathbb{S}(0), \mathbb{E}(0), \mathbb{A}_1(0), \mathbb{A}_2(0), \mathbb{P}(0), \mathbb{R}(0), \mathbb{Q}(0)).
\end{aligned}
$$

We let the Banach Space $I = C \times C \times C \times C \times C \times C \times C$, consist of component functions in $C[0, K]$ equipped with the norm

$$
\|\theta\| = \max_{t\in[0,K]} |\mathbb{S}(t), \mathbb{E}(t), \mathbb{A}_1(t), \mathbb{A}_2(t), \mathbb{P}(t), \mathbb{R}(t), \mathbb{Q}(t)|
$$

to facilitate the analysis of the existence theory. An operator $\psi : I \to I$ is defined as follows:

$$
\psi(\theta)(t) = \theta(0) + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t, \theta(t)) + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}\Omega(t, \theta(t))\,dm.
$$

Let $\Omega(t,\theta(t))$ be a nonlinear function with the Lipschitz condition and growth restrictions. Then, for any $\theta \in I$, there are constants $C_\phi > 0$ and $G_\phi$ such that

$$|\Omega(t,\theta(t))| \leq C_\phi|\theta(t)| + G_\phi. \tag{6}$$

In addition, for any $\theta, \bar{\theta} \in I$, there is a constant $H_\phi > 0$ such that

$$|\Omega(t,\theta(t)) - \Omega(t,\bar{\theta}(t))| \leq H_\phi|\theta(t) - \bar{\theta}(t)|. \tag{7}$$

Through this formulation, we analyze the uniqueness and stability properties of the proposed fractal–fractional scheme under the Atangana–Baleanu fractional operator framework.

**Theorem 2.** Assume that conditions (4) are fulfilled. A continuous function $\Omega : [0,K] \times I \to L$ is identified, ensuring that the proposed model possesses a unique solution.

**Proof.** Firstly, we want to show the full continuity of the operator $\psi$. With the constancy of $\Omega$, the continuity of $\psi$ is also guaranteed.

Let $\Phi = \{\theta \in I : \|\theta\| \leq L, L > 0\}$. Whenever $\theta \in I$, we obtain

$$\begin{aligned}
|\psi(\theta)| = \max_{t\in[0,K]} \bigg| & \theta(0) + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t,\theta(t)) \\
& + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}\Omega(m,\theta(m))\,dm \bigg| \\
& \leq \theta(0) + \frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)}(C_\phi\|\theta\| + G_\phi) \\
& \quad + \max_{t\in[0,K]} \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}\Omega(m,\theta(m))\,dm \bigg| \\
& \leq \theta(0) + \frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)}(C_\phi\|\theta\| + G_\phi) \\
& \quad + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}(C_\phi\|\theta\| + G_\phi)K^{\eta+\vartheta-1}\Phi(\eta,\vartheta) \leq \mathscr{L}.
\end{aligned} \tag{8}$$

Accordingly, whenever $\Phi(\eta,\vartheta)$ represents a function, the operator $\psi$ is restricted by homogeneity. As $\psi$ is equi-continuous, Suppose that $t_1, t_2 \leq K$. Then consider

$$\begin{aligned}
& |\psi(\theta)(t_2) - \psi(\theta)(t_1)| \\
& = \bigg| \frac{\vartheta t_2^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t_2,\theta(t_2)) + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^{t_2} m^{\vartheta-1}(t_2-m)^{\eta-1}\Omega(m,\theta(m))\,dm \\
& \quad - \frac{\vartheta t_1^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t_1,\theta(t_1)) - \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^{t_1} m^{\vartheta-1}(t_1-m)^{\eta-1}\Omega(m,\theta(m))\,dm \bigg|.
\end{aligned}$$

$$\leq \bigg| \frac{\vartheta t_2^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t_2,\theta(t_2)) + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^{t_2} m^{\vartheta-1}(t_2-m)^{\eta-1}\Omega(m,\theta(m))\,dm$$

$$- \frac{\vartheta t_1^{\vartheta-1}(1-\eta)}{ABC(\eta)}\Omega(t_1,\theta(t_1)) - \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^{t_1} m^{\vartheta-1}(t_1-m)^{\eta-1}\Omega(m,\theta(m))\,dm\Bigg|.$$

when $t_1, t_2$ then $|\psi(\theta)(t_2) - \psi(\theta)(t_1)| \to 0$. As a result, we have $\|\psi(\theta)(t_2) - \psi(\theta)(t_1)\| \to 0$ when $t_1 \to t_2$. Since $\psi$ is also continuous, the theory of Arzela-Ascoli is also completely continuous. Hence, Schauder's fixed point theorem shows that the suggested model has at least one solution.

**Theorem 3.** Assume that condition (8) holds with $\varpi < 1$, where

$$\varpi = \left(\frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)} + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}K^{\eta+\vartheta-1}\Phi(\eta,\vartheta)\right)H_\phi$$

Then, model (7) gives a definite answer.

**Proof.** Let $\theta, \bar{\theta} \in I$, we have

$$\begin{aligned}
|\psi(\theta) - \psi(\bar{\theta})| = \max_{t\in[0,K]} \Bigg| &\frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}[\Omega(t,\theta(t)) - \Omega(t,\bar{\theta}(t))] \\
&+ \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}[\Omega(m,\theta(m)) - \Omega(m,\bar{\theta}(m))]\,dm\Bigg| \\
&\leq \left[\frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)} + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)\rceil}K^{\eta+\vartheta-1}\Phi(\eta,\vartheta)\right]\|\theta - \bar{\theta}\| \\
&\leq \varpi\|\theta - \bar{\theta}\|.
\end{aligned}$$

As a result, $\psi$ is a contraction mapping, and the proposed model possesses a solution by virtue of the Banach contraction principle.

### 5.1. Ulam-Hyers Stability

**Definition:** The considered model is Ulam-Hyers stable if for every small perturbation $Q(t)$ with $|Q(t)| \leq \varepsilon$ for $\varepsilon > 0$, there is an exact unique solution $P(t)$ in such a way that the perturbed solution $\theta(t)$ fulfills the inequality

$$|\theta(t) - P(t)| \leq \psi_{\eta,\vartheta}\varepsilon, \quad \text{for every } t \in [0,K]$$

where $\psi_{\eta,\vartheta}$ is a positive constant depending on the system's parameters.

According to the above definition of Ulam–Hyers stability, we now use the concept to investigate the behaviour of the provided fractional-order system:

$${}_0^{FFM}D_t^{\eta,\vartheta}\theta(t) = \Omega(t,\theta(t)) + Q(t), \quad \theta(0) = \theta_0.$$

**Lemma 1.** *Let $P(t)$ be the exact solution of the unperturbed system, and let $\theta(t)$ be the solution of the perturbed system. If $|Q(t)| \leq \varepsilon$, then*

$$|\theta(t) - P(t)| \leq \alpha_{\eta,\vartheta}\varepsilon + \varpi|\theta(t) - P(t)|.$$

*Proof.* This is a very basic demonstration. It can be illustrated by taking system (7) into consideration:

$$ {}_0^{FFM}D_t^{\eta,\vartheta}(\theta(t) - P(t)) = \Omega(t, \theta(t)) + Q(t) - P(t). $$

Using Lipschitz condition for $\Omega$:

$$ |{}_0^{FFM}D_t^{\eta,\vartheta}(\theta(t) - P(t))| \leq \mathscr{L}|\theta(t) - P(t)| + |Q(t)|, $$

where $\mathscr{L}$ is Lipschitz condition. The fractional derivative expansion gives

$$ |\theta(t)-P(t)| = \left|\theta(t)-\left(P(0)+\frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\Omega(t,P(t))+\frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)}\int_0^t m^{\vartheta-1}(t-m)^{\eta-1}\,\Omega(m,P(m))\,dm\right)\right| $$

$$ \leq \frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)} + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)}K^{\eta+\vartheta-1}\Phi(\eta,\vartheta) + \varpi|\theta(t) - P(t)|. $$

Thus, the condition holds:

$$ |\theta(t) - P(t)| \leq \alpha_{\eta,\vartheta}\varepsilon + \varpi|\theta(t) - P(t)|, $$

where

$$ \alpha_{\eta,\vartheta}\varepsilon = \frac{\vartheta K^{\vartheta-1}(1-\eta)}{ABC(\eta)} + \frac{\eta\vartheta}{\lceil(\eta)ABC(\eta)}K^{\eta+\vartheta-1}\Phi(\eta,\vartheta). $$

Therefore, the proof is verified.

**Lemma 2.** *If $\varpi < 1$, the suggested model's Ulam-Hyers reliable result interacts with Lemma 1. The Ulam-Hyers stability of the fractional-order system is assured if $\varpi < 1$. Under this condition, the perturbed solution $\Upsilon(t)$ satisfies:*

$$ |\theta(t) - P(t)| \leq \frac{\alpha_{\eta,\vartheta}}{1-\varpi}. $$

*Proof.* From Lemma 1,

$$ |\theta(t) - P(t)| \leq \alpha_{\eta,\vartheta}\varepsilon + \varpi|\theta(t) - P(t)|. $$

A different form to write the above relation is given as

$$ |\theta(t) - P(t)| \leq \psi_{\eta,\vartheta}\varepsilon, $$

where

$$ \psi_{\eta,\vartheta} = \frac{\alpha_{\eta,\vartheta}}{1-\varpi}. $$

With the above lemmas, it is established that the system is Ulam-Hyers stable. This result ensures that any small perturbation $Q(t)$ leads to bounded deviations in the solution, thereby preserving the stability of the system.

## 6. Fractal-based numerical framework – fractional representation

Now, since the numerical scheme can be obtained in $t = t_{d+1}$, we can obtain the approximate solution in system (3) using the Caputo sense Atangana–Baleanu fractal–fractional operators [36, 37].

$$
\begin{aligned}
\mathbb{S}^{d+1} &= \mathbb{S}_p^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_1(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_1(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds \\
\mathbb{E}^{d+1} &= I_p^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_2(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_2(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds \\
\mathbb{A}_1^{d+1} &= T_p^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_3(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_3(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds, \\
\mathbb{A}_2^{d+1} &= R_p^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_4(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_4(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds \\
\mathbb{P}^{d+1} &= B_m^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_5(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_5(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds, \\
\mathbb{R}^{(d+1)} &= \mathbb{S}_v^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_6(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_6(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds, \\
\mathbb{Q}^{d+1} &= I_v^0 + \frac{\vartheta t^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_7\big(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d\big) \\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)\,ABC(\eta)}\int_0^t s^{\vartheta-1}(s-t)^{\eta-1}\mathbb{K}_7\big(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q}\big)\,ds.
\end{aligned} \tag{9}
$$

Applying the approximation of the integrals from the right-hand side of Equation (9), we have

$$
\begin{aligned}
\mathbb{S}^{d+1} &= \mathbb{S}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_1(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_1(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{E}^{d+1} &= \mathbb{E}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_2(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_2(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{A}_1^{d+1} &= \mathbb{A}_1^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_3(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_3(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{A}_2^{d+1} &= \mathbb{A}_2^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_4(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_4(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{P}^{d+1} &= \mathbb{P}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_5(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_5(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{R}^{d} &= \mathbb{R}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_6(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_6(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds,\\
\mathbb{Q}^{d} &= \mathbb{Q}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_7(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\
&\quad + \frac{\eta\vartheta}{\Gamma(\eta)ABC(\eta)}\sum_{c=0}^{d}\int_{t_c}^{t_{c+1}} s^{\vartheta-1}(t_{d+1}-s)^{\eta-1}\mathbb{K}_7(t,\mathbb{S},\mathbb{E},\mathbb{A}_1,\mathbb{A}_2,\mathbb{P},\mathbb{R},\mathbb{Q})\,ds.
\end{aligned}
\tag{10}
$$

Afterward, we have

$$\begin{aligned}
Q_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_1(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_1(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
U_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_2(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_2(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
V_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_3(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_3(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
W_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_4(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_4(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
X_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_5(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_5(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
Y_\xi(s) &= \frac{s-t_{\xi-1}}{t_\xi - t_{\xi-1}} t_\xi^{\vartheta-1}\mathbb{K}_6(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{s-t_\xi}{t_\xi - t_{\xi-1}} t_{\xi-1}^{\vartheta-1}\mathbb{K}_6(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}), \\
Z_\xi(s) &= \frac{(s-t_{\xi-1})}{(t_\xi - t_{\xi-1})} t_\xi^{\vartheta-1}\mathbb{K}_7(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad - \frac{(s-t_\xi)}{(t_\xi - t_{\xi-1})} t_{\xi-1}^{\vartheta-1}\mathbb{K}_7(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}).
\end{aligned} \tag{11}$$

By solving the Lagrangian polynomial in segments in Equation (11), we have the following outcomes, which are the fundation for the further numerical analysis.

$$\begin{aligned}\mathbb{S}^{d+1} &= \mathbb{S}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_1(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\ &\quad+\frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)}\sum_{\xi=0}^{d}\Big[t_\xi^{\vartheta-1}\mathbb{K}_1(t_\xi,\mathbb{S}^\xi,\mathbb{E}^\xi,\mathbb{A}_1^\xi,\mathbb{A}_2^\xi,\mathbb{P}^\xi,\mathbb{R}^\xi,\mathbb{Q}^\xi)\\ &\quad\times\big((d+1-\xi)^\eta(d-\xi+2+\eta)-(d-\xi)(2+2\eta+d-\xi)\big)\\ &\quad- t_{\xi-1}^{\vartheta-1}\mathbb{K}_1(t_{\xi-1},\mathbb{S}^{\xi-1},\mathbb{E}^{\xi-1},\mathbb{A}_1^{\xi-1},\mathbb{A}_2^{\xi-1},\mathbb{P}^{\xi-1},\mathbb{R}^{\xi-1},\mathbb{Q}^{\xi-1})\\ &\quad\times\big((d+1-\xi)^{\eta+1}-(d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].\end{aligned}$$

$$\begin{aligned}\mathbb{E}^{d+1} &= \mathbb{E}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_2(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\ &\quad+\frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)}\sum_{\xi=0}^{d}\Big[t_\xi^{\vartheta-1}\mathbb{K}_2(t_\xi,\mathbb{S}^\xi,\mathbb{E}^\xi,\mathbb{A}_1^\xi,\mathbb{A}_2^\xi,\mathbb{P}^\xi,\mathbb{R}^\xi,\mathbb{Q}^\xi)\\ &\quad\times\big((d+1-\xi)^\eta(d-\xi+2+\eta)-(d-\xi)(2+2\eta+d-\xi)\big)\\ &\quad- t_{\xi-1}^{\vartheta-1}\mathbb{K}_2(t_{\xi-1},\mathbb{S}^{\xi-1},\mathbb{E}^{\xi-1},\mathbb{A}_1^{\xi-1},\mathbb{A}_2^{\xi-1},\mathbb{P}^{\xi-1},\mathbb{R}^{\xi-1},\mathbb{Q}^{\xi-1})\\ &\quad\times\big((d+1-\xi)^{\eta+1}-(d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].\end{aligned}$$

$$\begin{aligned}\mathbb{A}_1^{d+1} &= \mathbb{A}_1^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_3(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\ &\quad+\frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)}\sum_{\xi=0}^{d}\Big[t_\xi^{\vartheta-1}\mathbb{K}_3(t_\xi,\mathbb{S}^\xi,\mathbb{E}^\xi,\mathbb{A}_1^\xi,\mathbb{A}_2^\xi,\mathbb{P}^\xi,\mathbb{R}^\xi,\mathbb{Q}^\xi)\\ &\quad\times\big((d+1-\xi)^\eta(d-\xi+2+\eta)-(d-\xi)(2+2\eta+d-\xi)\big)\\ &\quad- t_{\xi-1}^{\vartheta-1}\mathbb{K}_3(t_{\xi-1},\mathbb{S}^{\xi-1},\mathbb{E}^{\xi-1},\mathbb{A}_1^{\xi-1},\mathbb{A}_2^{\xi-1},\mathbb{P}^{\xi-1},\mathbb{R}^{\xi-1},\mathbb{Q}^{\xi-1})\\ &\quad\times\big((d+1-\xi)^{\eta+1}-(d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].\end{aligned}$$

$$\begin{aligned}\mathbb{A}_2^{d+1} &= \mathbb{A}_2^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_4(t_d,\mathbb{S}^d,\mathbb{E}^d,\mathbb{A}_1^d,\mathbb{A}_2^d,\mathbb{P}^d,\mathbb{R}^d,\mathbb{Q}^d)\\ &\quad+\frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)}\sum_{\xi=0}^{d}\Big[t_\xi^{\vartheta-1}\mathbb{K}_4(t_\xi,\mathbb{S}^\xi,\mathbb{E}^\xi,\mathbb{A}_1^\xi,\mathbb{A}_2^\xi,\mathbb{P}^\xi,\mathbb{R}^\xi,\mathbb{Q}^\xi)\\ &\quad\times\big((d+1-\xi)^\eta(d-\xi+2+\eta)-(d-\xi)(2+2\eta+d-\xi)\big)\\ &\quad- t_{\xi-1}^{\vartheta-1}\mathbb{K}_4(t_{\xi-1},\mathbb{S}^{\xi-1},\mathbb{E}^{\xi-1},\mathbb{A}_1^{\xi-1},\mathbb{A}_2^{\xi-1},\mathbb{P}^{\xi-1},\mathbb{R}^{\xi-1},\mathbb{Q}^{\xi-1})\\ &\quad\times\big((d+1-\xi)^{\eta+1}-(d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}^{d+1} &= \mathbb{P}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_5(t_d, \mathbb{S}^d, \mathbb{E}^d, \mathbb{A}_1^d, \mathbb{A}_2^d, \mathbb{P}^d, \mathbb{R}^d, \mathbb{Q}^d) \\
&\quad + \frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)} \sum_{\xi=0}^{d} \Big[ t_\xi^{\vartheta-1}\mathbb{K}_5(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad \times \big((d+1-\xi)^\eta(d-\xi+2+\eta) - (d-\xi)(2+2\eta+d-\xi)\big) \\
&\quad - t_{\xi-1}^{\vartheta-1}\mathbb{K}_5(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}) \\
&\quad \times \big((d+1-\xi)^{\eta+1} - (d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{R}^{d+1} &= \mathbb{R}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_6(t_d, \mathbb{S}^d, \mathbb{E}^d, \mathbb{A}_1^d, \mathbb{A}_2^d, \mathbb{P}^d, \mathbb{R}^d, \mathbb{Q}^d) \\
&\quad + \frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)} \sum_{\xi=0}^{d} \Big[ t_\xi^{\vartheta-1}\mathbb{K}_6(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad \times \big((d+1-\xi)^\eta(d-\xi+2+\eta) - (d-\xi)(2+2\eta+d-\xi)\big) \\
&\quad - t_{\xi-1}^{\vartheta-1}\mathbb{K}_2(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}) \\
&\quad \times \big((d+1-\xi)^{\eta+1} - (d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Q}^{d+1} &= \mathbb{Q}^0 + \frac{\vartheta t_d^{\vartheta-1}(1-\eta)}{ABC(\eta)}\,\mathbb{K}_7(t_d, \mathbb{S}^d, \mathbb{E}^d, \mathbb{A}_1^d, \mathbb{A}_2^d, \mathbb{P}^d, \mathbb{R}^d, \mathbb{Q}^d) \\
&\quad + \frac{\vartheta(\delta t)^\eta}{(\eta+2)ABC(\eta)} \sum_{\xi=0}^{d} \Big[ t_\xi^{\vartheta-1}\mathbb{K}_7(t_\xi, \mathbb{S}^\xi, \mathbb{E}^\xi, \mathbb{A}_1^\xi, \mathbb{A}_2^\xi, \mathbb{P}^\xi, \mathbb{R}^\xi, \mathbb{Q}^\xi) \\
&\quad \times \big((d+1-\xi)^\eta(d-\xi+2+\eta) - (d-\xi)(2+2\eta+d-\xi)\big) \\
&\quad - t_{\xi-1}^{\vartheta-1}\mathbb{K}_7(t_{\xi-1}, \mathbb{S}^{\xi-1}, \mathbb{E}^{\xi-1}, \mathbb{A}_1^{\xi-1}, \mathbb{A}_2^{\xi-1}, \mathbb{P}^{\xi-1}, \mathbb{R}^{\xi-1}, \mathbb{Q}^{\xi-1}) \\
&\quad \times \big((d+1-\xi)^{\eta+1} - (d-\xi)^\eta(1+\eta+d-\xi)\big)\Big].
\end{aligned} \tag{12}
$$

## 7. Results and Discussion

This section presents MATLAB-based simulations for system (12) using the fractal–fractional derivative approach and compares its efficiency with existing fractional-order operators. The data used, taken from [21], reflect the social media addiction and recovery dynamics described in Table 1. The results highlight key trends and parameter variations, offering insights into system sensitivity and transmission dynamics, consistent with earlier studies [26, 27].

The results of the simulation show the social media addiction dynamics under various fractional orders, which provide information about the intervention strategies' effectiveness. The trends seen throughout the figures highlight the need for early intervention,

Table 1: Model parameters for social media addiction dynamics

| Parameter | Description | Value |
|---|---|---|
| $\Lambda$ | Recruitment rate into susceptible class | 0.975 |
| $\delta$ | Addiction transmission rate | 0.1–0.8 |
| $\theta$ | Contact rate (susceptible–addicted) | 0.2 |
| $\lambda_1$ | Transition to mild addiction | 0.5 |
| $\lambda_2$ | Transition to severe addiction | 0.7 |
| $\mu$ | Natural exit (death) rate | 0.05 |
| $\tau_1$ | Non-users staying non-users | 0.4 |
| $\tau_2$ | Recovered becoming non-users | 0.35 |
| $\xi_1$ | Recovery from mild addiction | 0.02 |
| $\xi_2$ | Recovery from severe addiction | 0.03 |
| $\xi_3$ | Recovery under treatment | 0.25 |
| $\phi_1$ | Treatment rate (mild) | 0.01 |
| $\phi_2$ | Treatment rate (severe) | 0.05 |
| $\gamma$ | Relapse rate (recovered $\rightarrow$ susceptible) | 0.01 |
| $\varepsilon$ | Progression (mild $\rightarrow$ severe addiction) | 0.7 |

awareness campaigns, and mental support in decreasing the negative effects of addiction.

Figure 2 illustrates a consistent decrease in the susceptible population for various fractional orders. This behaviour indicates that as social media influence grows, more people move from the susceptible class into the exposed or addicted classes. The different rates of decrease for various fractional orders indicate that addiction propagates with different intensities based on susceptibility and external factors like peer pressure, online activity, and social trends.

Figure 3 demonstrates a downward trend in the exposed population, showing that people either advance to the stage of addiction or get cured with intervention. The rate of this change depends on psychological aspects, efficacy of intervention, and external motivational forces like social awareness programs. The reduction in the exposed group suggests that in the absence of early intervention, people tend to develop addiction to social media, emphasizing the necessity for proactive psychological and behavioural interventions.

Figures 4 and 5 demonstrate the decline in the mildly and severely addicted groups over a period. The pattern indicates the successful outcome of psychological therapies, behavioural counselling, and awareness campaigns in managing social media addiction. The decline in the groups indicates that targeted interventions are successful in reducing addiction levels step by step, avoiding long-term dependency, and promoting recovery. The fractional-order dynamics also suggest memory effects are important in addiction behaviour since those with a history of extensive social media usage take longer to recover.

Figures 6 to 8 show a significant increase in the population being treated for psy-

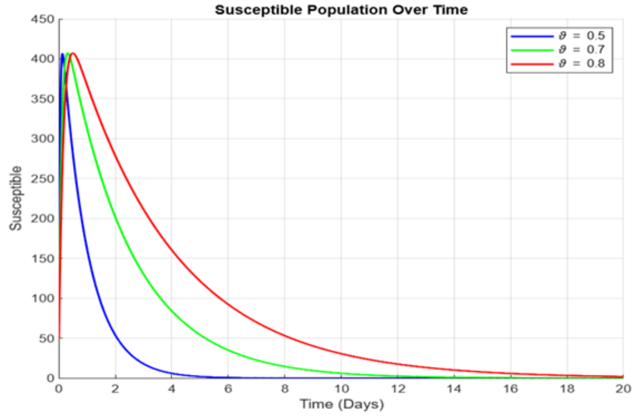


Figure 2: Simulation of population under Different Fractional Orders for Susceptible.

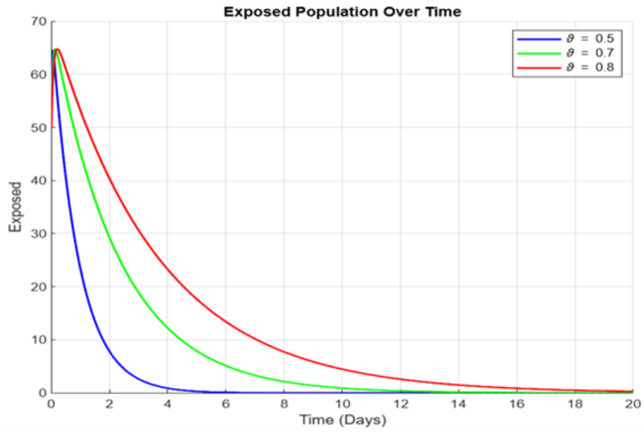


Figure 3: Simulation of population under Different Fractional Orders for Exposed

chological issues, recovered users, and non-users. This increasing trend highlights the success of intervention efforts at enabling recovery and individuals to have control over their social media behaviour. The rise in the population of non-users indicates that if the right treatment protocols are followed, not only can people recover, but they can also give up on excessive social media usage entirely. This trend points towards the importance of awareness campaigns, self-regulation methods, and support mechanisms in minimizing addiction relapse threats. In addition, initiating treatment early results strongly in enhanced recovery rates, further justifying the importance of early intervention programs.

Figure 9 illustrates the comparison between integer-order ($\eta = 1$) and fractional-order ($0 < \eta < 1$) formulations of the social media addiction model. The integer-order dynamics show sharper transitions and faster stabilization across compartments, reflecting only the present state. In contrast, the fractional-order case exhibits smoother and slower changes, highlighting the role of memory in addiction and recovery processes. This demonstrates that fractional-order modelling provides a more realistic description of social media addic-

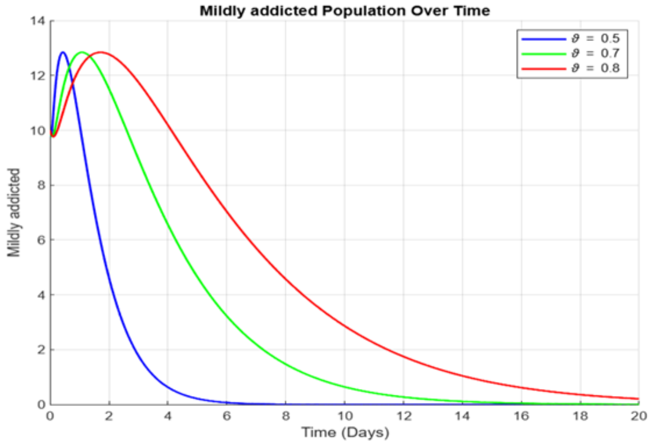


Figure 4: Simulation of population under Different Fractional Orders for Mildly addicted

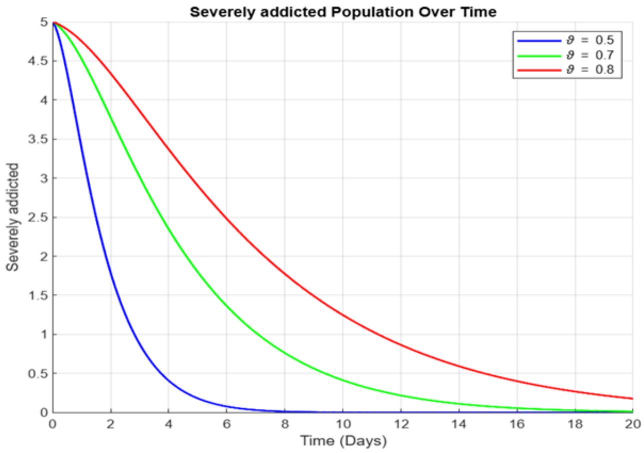


Figure 5: Simulation of population under Different Fractional Orders for Mildly addicted

tion dynamics than the classical integer-order model.

Generally, these results affirm that psychological interventions combined with fractional-order modelling offer a robust model for grasping and counteracting social media addiction. The findings highlight that early intervention, organized treatment programs, and social campaigns are the most important elements in combating and managing addiction dynamics.

## 8. Conclusions

Social media addiction has a strong impact on mental well-being, interpersonal relationships, and productivity and as such, it is crucial to model its dynamics for efficient intervention. This research utilizes a fractional mathematical model based on the fractal-

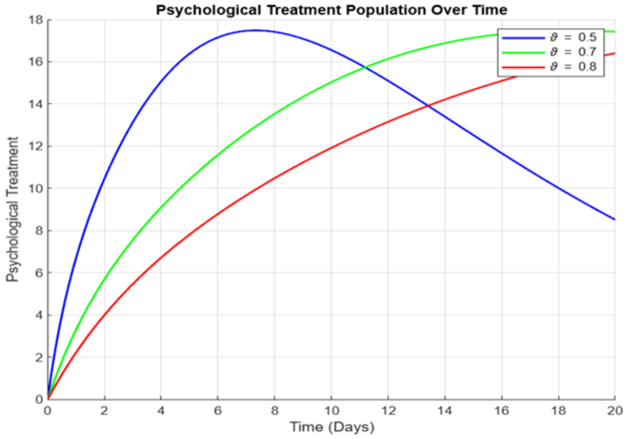


Figure 6: Simulation of population under Different Fractional Orders for Psychological treatment

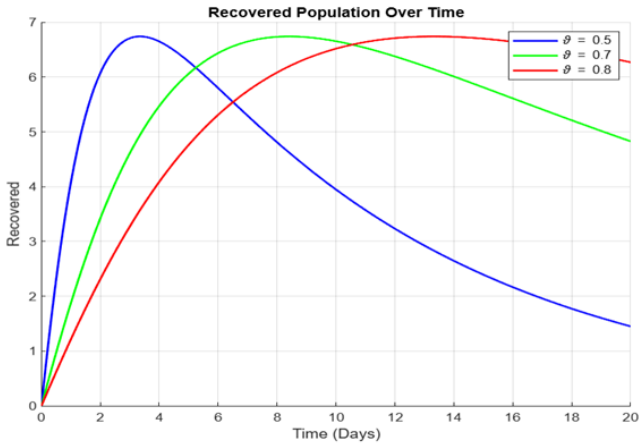


Figure 7: Simulation of population under Different Fractional Orders for Mildly Recovered

fractional operator and Mittag-Leffler kernel to study the evolution of addiction and recovery. By dividing users into general, mildly addicted, and heavily addicted classes, the model successfully exhibits how psychological treatment decreases the severity of addiction. The model's reliability is guaranteed by fixed-point theory and Ulam–Hyers stability, ensuring its validity for practical applications.

Numerical analysis brings out the memory-dependent character of addiction and offers a versatile, accurate framework for explaining long-term patterns of behaviour. This method increases predictive power, enabling improved strategies for intervention. The implications conclude that early intervention and structured rehabilitation programs are significantly important in minimizing addiction relapse and enhancing recovery levels.Future research will investigate the effect of social habits, digital detox methods, and AI-driven predictions to develop more effective intervention strategies. The social implication of this study is in informing digital wellness policies and rehabilitation interventions, ultimately

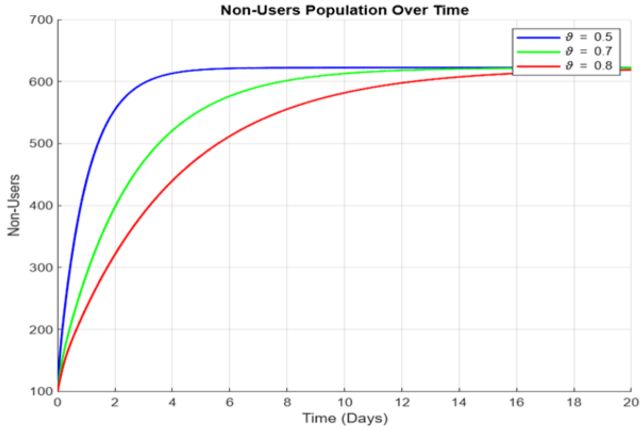


Figure 8: Simulation of population under Different Fractional Orders for non-users

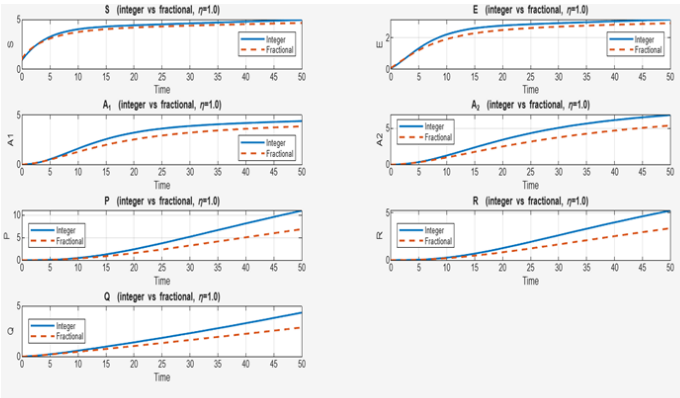

Figure 9: Comparison of integer-order ($\eta = 1$) and fractional-order ($0 < \eta < 1$) trajectories in the social media addiction model, showing the impact of memory effects on addiction and recovery dynamics

leading to healthier web use behaviours and better mental health in the digital age.

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## References


[1] Martina Drahošová and Peter Balco. The analysis of advantages and disadvantages of use of social media in european union. *Procedia Computer Science*, 109:1005–1009, 2017.

[2] Rdouan Faizi, Abdellatif El Afia, and Raddouane Chiheb. Exploring the potential benefits of using social media in education. *International Journal of Engineering Pedagogy (iJEP)*, 3(4):50–53, 2013.

[3] Dana Howard, W Glynn Mangold, and Tim Johnston. Managing your social campaign strategy using facebook, twitter, instagram, youtube & pinterest: An interview with dana howard, social media marketing manager. *Business Horizons*, 5(57):657–665, 2014.

[4] Linah Aburahmah, Hajar AlRawi, Yamamah Izz, and Liyakathunisa Syed. Online social gaming and social networking sites. *Procedia Computer Science*, 82:72–79, 2016.

[5] Hsuan-Ting Chen and Xueqing Li. The contribution of mobile social media to social capital and psychological well-being: Examining the role of communicative use, friending and self-disclosure. *Computers in Human Behavior*, 75:958–965, 2017.

[6] James A Roberts and Meredith E David. The social media party: Fear of missing out (fomo), social media intensity, connection, and well-being. *International Journal of Human–Computer Interaction*, 36(4):386–392, 2020.

[7] Davide Marengo, Matteo Angelo Fabris, Claudio Longobardi, and Michele Settanni. Smartphone and social media use contributed to individual tendencies towards social media addiction in italian adolescents during the covid-19 pandemic. *Addictive behaviors*, 126:107204, 2022.

[8] Sultan Alshamrani, Ahmed Abusnaina, Mohammed Abuhamad, Daehun Nyang, and David Mohaisen. Hate, obscenity, and insults: Measuring the exposure of children to inappropriate comments in youtube. In *Companion proceedings of the web conference 2021*, pages 508–515, 2021.

[9] S Malesev and M Cherry. Digital and social media marketing-growing market share for construction smes. construction economics and building, 21 (1), 65–82, 2021.

[10] Farai Nyabadza, John BH Njagarah, and Robert J Smith. Modelling the dynamics of crystal meth ('tik') abuse in the presence of drug-supply chains in south africa. *Bulletin of mathematical biology*, 75(1):24–48, 2013.

[11] J Mushanyu, F Nyabadza, and AGR Stewart. Modelling the trends of inpatient and outpatient rehabilitation for methamphetamine in the western cape province of south africa. *BMC research notes*, 8(1):797, 2015.

[12] Isaac Kwasi Adu, MAREN Osman, and Cuihong Yang. Mathematical model of drinking epidemic. *Br. J. Math. Computer Sci*, 22(5):1–10, 2017.

[13] Yubo Hou, Dan Xiong, Tonglin Jiang, Lily Song, and Qi Wang. Social media addiction: Its impact, mediation, and intervention. *Cyberpsychology: Journal of psychosocial research on cyberspace*, 13(1), 2019.

[14] Bouchaib Khajji, Abderrahim Labzai, Omar Balatif, and Mostafa Rachik. Mathematical modeling and analysis of an alcohol drinking model with the influence of

alcohol treatment centers. *International Journal of Mathematics and Mathematical Sciences*, 2020(1):4903168, 2020.

[15] Shuang-Hong Ma, Hai-Feng Huo, and Xin-You Meng. Modelling alcoholism as a contagious disease: a mathematical model with awareness programs and time delay. *Discrete Dynamics in Nature and Society*, 2015(1):260195, 2015.

[16] Swarnali Sharma and GP Samanta. Drinking as an epidemic: a mathematical model with dynamic behaviour. *Journal of applied mathematics & informatics*, 31(1_2):1–25, 2013.

[17] Youming Guo and Tingting Li. Optimal control and stability analysis of an online game addiction model with two stages. *Mathematical Methods in the Applied Sciences*, 43(7):4391–4408, 2020.

[18] Tingting Li and Youming Guo. Stability and optimal control in a mathematical model of online game addiction. *Filomat*, 33(17):5691–5711, 2019.

[19] Adamu Ishaku, Bashir Saidu Musa, Ayuba Sanda, and Abubakar Muhammad Bakoji. Mathematical assessment of social media impact on academic performance of students in higher institution. *IOSR J. Math*, 14(1):72–79, 2018.

[20] Xun-Yang Wang, Hai-Feng Huo, Qing-Kai Kong, and Wei-Xuan Shi. Optimal control strategies in an alcoholism model. In *Abstract and Applied Analysis*, volume 2014, page 954069. Wiley Online Library, 2014.

[21] H. T. Alemneh and N. Y. Alemu. Mathematical modeling with optimal control analysis of social media addiction. *Infectious Disease Modelling*, 6:405–419, 2021.

[22] I. Slimane, G. Nazir, J. J. Nieto, and F. Yaqoob. Mathematical analysis of hepatitis c virus infection model in the framework of non-local and non-singular kernel fractional derivative. *International Journal of Biomathematics*, 16(1):2250064, 2023.

[23] I. Slimane, J. J. Nieto, and S. Ahmad. A fractional-order bovine babesiosis epidemic transmission model with non-singular mittag-leffler law. *Fractals*, 31(2):2340033, 2023.

[24] N. Özdemir and E. Uçar. Investigating of an immune system-cancer mathematical model with mittag-leffler kernel. *Chaos, Solitons & Fractals*, 2020. Unpublished preprint or conference abstract; full details may be incomplete.

[25] Z. Ahmad, S. A. El-Kafrawy, T. A. Alandijany, F. Giannino, A. A. Mirza, M. M. El-Daly, and E. I. Azhar. A global report on the dynamics of COVID-19 with quarantine and hospitalization: A fractional order model with non-local kernel. *Computational Biology and Chemistry*, 98:107645, 2022.

[26] S. Momani, R. P. Chauhan, S. Kumar, and S. Hadid. Analysis of social media addiction model with singular operator. *Fractals*, 31(10):2340097, 2023.

[27] M. Dasumani, B. S. Lassong, I. K. Adu, F. A. Wireko, and S. E. Moore. Fractional derivative technique for modeling the dynamics of social media impacts. *Discrete Dynamics in Nature and Society*, 2024, 2024.

[28] S. Rashid, R. Ashraf, and E. Bonyah. Nonlinear dynamics of the media addiction model using the fractal–fractional derivative technique. *Complexity*, 2022, 2022.

[29] S. Qureshi, A. Soomro, I. K. Argyros, K. Gdawiec, A. Akgül, and M. Alquran. Use of fractional calculus to avoid divergence in newton-like solver for solving one-

dimensional nonlinear polynomial-based models. *Communications in Nonlinear Science and Numerical Simulation*, 143:108631, 2025.

[30] C. W. Chang, S. Qureshi, I. K. Argyros, K. M. Saraz, and E. Hincal. A modified fractional newton's solver. *Axioms*, 13(10):689, 2024.

[31] D. Baleanu, S. Qureshi, A. Yusuf, A. Soomro, and M. S. Osman. Bi-modal COVID-19 transmission with caputo fractional derivative using statistical epidemic cases. *Partial Differential Equations in Applied Mathematics*, 10:100732, 2024.

[32] P. A. Naik, B. M. Yeolekar, S. Qureshi, M. Yeolekar, and A. Madzvamuse. Modeling and analysis of the fractional-order epidemic model to investigate mutual influence in HIV/HCV co-infection. *Nonlinear Dynamics*, 112(13):11679–11710, 2024.

[33] A. Padder, L. Almutairi, S. Qureshi, A. Soomro, A. Afroz, E. Hincal, and A. Tassaddiq. Dynamical analysis of generalized tumor model with caputo fractional-order derivative. *Fractal and Fractional*, 7(3):258, 2023.

[34] L. Zhang, E. Addai, J. Ackora-Prah, Y. D. Arthur, and J. K. K. Asamoah. Fractional-order ebola-malaria coinfection model with a focus on detection and treatment rate. *Computational and Mathematical Methods in Medicine*, 2022, 2022.

[35] N. Zhang, E. Addai, L. Zhang, M. Ngungu, E. Marinda, and J. K. K. Asamoah. Fractional modeling and numerical simulation for unfolding marburg–monkeypox virus co-infection transmission. *Fractals*, 31(7):2350086, 2023.

[36] A. Atangana. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos, Solitons & Fractals*, 102:396–406, 2017.

[37] S. W. Yao, A. Ahmad, M. Inc, M. Farman, A. Ghaffar, and A. Akgul. Analysis of fractional order diarrhea model using fractal fractional operator. *Fractals*, 30(5):2240173, 2022.