EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6739
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Invariant Subspace Problem for Norm Attaining Operators

Aissa Nasli Bakir[1,2], Ayyoub Fellag Ariouat[1], Abdelkader Benali[1], Ibrahim Alraddadi[3,*], Saad M. Almuaddi[4,5,*]

[1] *Department of Mathematics, Faculty of Exact Sciences and Informatics, Laboratory of Mathematics and Application LMA, Hassiba Benbouali University of Chlef, Algeria*
[2] *National Higher School of Cybersecurity, Sidi Abdellah, Algeria*
[3] *Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia*
[4] *Basic & Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam 31441, Saudi Arabia*
[5] *Mathematics Department, College of Science, Imam Abdulrahman Bin Faisal University, Dammam 31441, Saudi Arabia*

Abstract. Our aim is to characterize norm attaining and absolutely norm attaining quasi-$*$-paranormal operators and class Ω_n operators defined on a separable Hilbert space. We define invariant non trivial subspaces for the considered operators and we give a matrix representation under certain condition. Compactness, reducing subspaces and the normality of such operators are also established.

2020 Mathematics Subject Classifications: 47A30, 47B47, 47B20

Key Words and Phrases: Quasi-$*$-paranormal operators, invariant subspaces, norm attaining operators, absolutely norm attaining operators

1. Preliminaries and notations

Let H denote an infinite separable complex Hilbert space, and let $B(H)$ be the Banach algebra of all bounded linear operators on H. An operator $T \in B(H)$ is said to be norm attaining, if there exists a unit vector $u \in H$ satisfying $\|Tu\| = \|T\|$, [1], and T is said to be absolutely norm attaining, briefly, AN-operator, if its restriction on any closed subspace of H is norm attaining, [2]. Obviously, AN-operators are norm attaining. Many authors

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6739

Email addresses: a.nasli@univ-chlef.dz (A. Nasli Bakir), a.fellagariouat@univ-chlef.dz (A. Fellag Ariouat), benali4848@gmail.com (A. Benali), ialraddadi@iu.edu.sa (I. Alraddadi), smuaddi@iau.edu.sa (S. M. Almuaddi)

studied the structure of some classes of norm attaining non normal operators, see [3, 4] and [5].

Authors in [5, 6] analyzed the properties of norm attaining and absolutely norm attaining operators, and provide a representation for the class of $*$-paranormal operators under an orthogonal decomposition of H. Authors in [4] showed that compact (resp. isometric) operators are AN-operators since their restrictions on closed invariant subspaces remain compact (resp. isometric). Moreover, if T is AN-operator, then T^* may not be one, see [4, 7] where it's shown that an isometry $U\colon \ell_2 \to \ell_2$ onto a subspace with infinite codimension is AN-operator, whereas its adjoint U^* is not. However, G. Ramesh in [5] presented an additional condition for which T^* remains an AN-operator.

An operator $T \in B(H)$ is said to be non-negative and we shall write $T \geq 0$, if $\langle Tu, u\rangle \geq 0$ for all $u \in H$ and A is said to be normal if $T^*T = TT^*$, isometric if $T^*T = I$, where I is the identity operator on H. If T is isometric and onto, then T is said to be unitary. The operator $T \in B(H)$ is said to be $*$-paranormal if $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$ for each $\lambda > 0$, that is, $\|T^*u\|^2 \leq \|T^2u\|\|u\|$ for all $u \in H$ [8]. Also, T is said to be quasi-$*$-paranormal if $T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \geq 0$ for all $\lambda > 0$, [9]. Clearly, a $*$-paranormal operator is quasi-$*$-paranormal while the converse is in general false, see [10]. It is known that quasi-$*$-paranormal operators are normaloid, that is, $r(T) = \|T\|$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is the spectral radius of T. Ample properties of these classes can be found in [11] and [12]. The operator T in $B(H)$ is said to be quasi-normal of order n for certain integer n, and we shall write $T \in \Omega_n$, if $TT^{\star n}T^n = T^{\star n}T^{n+1}$. For more information on the class Ω_n, we refer the reader to [13, 14].

For an operator $T \in B(H)$, the range of T, the null space and the modulus of T will be denoted by $R(T)$, $N(T)$ and $|T| = \sqrt{T^*T}$ respectively. If $T \in B(H)$, then $T = U|T|$ is the polar decomposition of T, where U is a partial isometry, that is, $U\big|_{N(A)^\perp}$ is an isometry, $R(U) = R(|T|)$. According to [15], U is a partial isometry if and only if $UU^*U = U$. The sets $\sigma(T), \sigma_p(T)$ denote respectively, the spectrum and the set of eigenvalues of T. For a self-adjoint operator $T \in B(H)$, that is, $T^* = T$, the discrete spectrum of T is the set $\sigma_d(T) = \{\lambda \in \sigma_p(T)\colon \lambda \text{ is isolated and has a finite multiplicity}\}$. The set $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$ is said to be the essential spectrum of T, [16]. If $\dim \mathcal{H} < +\infty$, then $\sigma_{ess}(T) = \emptyset$.

The positive real number $m(T) = \inf\{\|Tu\|\colon u \in\ H \text{ and } \|u\| = 1\}$ is said to be the minimum modulus of $T \in B(H)$, and the quantity $m_e(T) = \inf\{\lambda\colon \lambda \in \sigma(|T|)\}$ is said to be the essential minimum modulus of T. For more details, reader is referred to [4, 17] and [16].

In [18], authors gave a characterization of norm attaining and absolutely norm attaining $*$-paranormal operators and defined a closed non trivial invariant subspace for this class of operators. In this article, we generalize these results for a large class of quasi-$*$-paranormal operators. We show several spectral properties. We also provide invariant subspaces for both of classes of quasi-$*$-paranormal operators and class Ω_n operators, and we show that the given subspaces become reducing under certain conditions. Other properties related to the compactness, the normality and the matrix representation are also established.

2. Norm attaining quasi-$*$-paranormal operators

Definition 1. *[19, 20] An operator $T \in B(H)$ is said to be quasi-$*$-paranormal if*

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \geq 0$$

for all $\lambda > 0$.
The given definition implies that

$$\|T^*Tu\|^2 \leq \|T^3u\|\|Tu\|$$

for all $u \in H$.

Example 1. *[11] Let $\mu = (\mu_n)_{n\geq 1}$ be a positive real sequence. Define the weighted shift $\mathcal{S}_\mu$ on the Hilbert space ℓ_2 by*

$$\mathcal{S}_\mu e_n = \mu_n e_{n+1},\ n \geq 1$$

where $(e_n)_{n\geq 1}$ is the standard basis of ℓ_2. Then, $\mathcal{S}_\mu$ is quasi-$*$-paranormal if and only if the inequality $\mu_n^2 \leq \mu_{n+1}\mu_{n+2}$ holds for any n, $n \geq 1$.

[20] The restriction of a quasi-$*$-paranormal operator on a closed invariant subspace is also quasi-$*$-paranormal. [10, Lemma 3.4] For any quasi-$*$-paranormal operator $T \in B(H)$, and each non-zero complex scalar λ, we've $N(T - \lambda I) \subset N(T - \lambda I)^*$.

Remark 1. *Lemma 2 is in general not true for $\lambda = 0$. A counter-example can be found in [9].*

Definition 2. *[4] An operator $T \in B(H)$ is said to be norm attaining (or achieving the norm) if there exists a unit vector $u \in H$ for which $\|Tu\| = \|T\|$.*

Example 2. *[21] Let $\theta = (\theta_n)_{n\geq 1}$ be a real strictly increasing sequence. The operator T_θ defined on the usual Hilbert space ℓ_2 by*

$$T_\theta x = (\theta_n x_n)_{n\geq 1},\ x = (x_n)_{n\geq 1} \in \ell_2$$

is not norm attaining.

Example 3. *[18] The usual Hilbert space ℓ_2 is equipped with its standard orthonormal basis $(e_n)_{n\geq 1}$, and $\mathcal{S}$ is the unilateral left shift on ℓ_2 defined by*

$$\mathcal{S}e_n = e_{n-1},\ n \geq 2 \quad \text{and } \mathcal{S}e_1 = 0$$

Then, $\mathcal{S}$ is norm attaining since $\|\mathcal{S}e_2\| = \|\mathcal{S}\| = 1$.

Recall that a closed subspace $M \subset H$ is said to be invariant for an operator $T \in B(H)$, if $Tu \in M$ for each $u \in M$, and M is said to be reducing for T if M is invariant for both T and T^*.

As an extension of results given in [18] and [5], where are provided invariant non trivial subspaces for both of norm achieving $*$-paranormal and norm achieving paranormal operators respectively, we shall define in the following, a non trivial subspace for norm attaining quasi-$*$-paranormal operators as a positive answer to the problem of invariant subspaces for operators on Hilbert spaces that asks if any bounded linear operator acting on a Hilbert space admits at least, a non trivial invariant subspace.

Theorem 1. *Let $T \in B(H)$ be a quasi-$*$-paranormal operator that achieves the norm. Then, the subspace $M = \{u \in H : \|Tu\| = \|T\|\|u\|\}$ is invariant for T.*

Proof. Since T is norm attaining, $M = N(T^*T - \|T\|^2 I)$ is a non-zero closed subspace of H. Next, as T is quasi-$*$-paranormal,

$$\begin{aligned}\|T\|^2\|u\|^2 = \|Tu\|^2 = \langle T^*Tu, u\rangle \leq \|T^*Tu\|\|u\| &\leq \sqrt{\|T^3u\|\|Tu\|}\|u\| \\ &\leq \sqrt{\|T^2\|\|Tu\|^2}\|u\| \\ &\leq \sqrt{\|T\|^2\|Tu\|^2}\|u\| \\ &\leq \|T\|\|Tu\|\|u\| \\ &\leq \|T\|^2\|u\|^2\end{aligned}$$

for each $u \in M$. Hence,

$$\|T\|^2\|u\|^2 = \|T^*Tu\|\|u\| = \|Tu\|^2$$

i.e.,

$$\|T\|\|Tu\| = \|T\|^2\|u\| = \|T^*Tu\|, \; u \in M \tag{1}$$

Using equality (1), the fact that $\|T^*T\| = \|T\|^2$ and by Cauchy-Schwarz's inequality, we get for each vector u in M,

$$\|T^2u\|^2 = \langle T^*T^2u, Tu\rangle \leq \|T^*T\|\|Tu\|^2 = \|T\|^2\|Tu\|^2 = \|T^*Tu\|^2$$

That is,

$$\|T^2u\| \leq \|T^*Tu\| \tag{2}$$

for all $u \in M$. On another hand,

$$\|T^*Tu\|^2 \leq \|T^3u\|\|Tu\| \leq \|T^2u\|\|T\|\|Tu\| = \|T^2u\|\|T^*Tu\|$$

Hence, for all $u \in M$,

$$\|T^*Tu\| \leq \|T^2u\| \tag{3}$$

By (2) and (3),

$$\|T^*Tu\| = \|T\|\|Tu\| = \|T^2u\|$$

for each $u \in M$. This shows that M is invariant for T.

By a similar way as in [5, Lemma 3.1], we can easily prove the following result For an operator $T \in B(H)$, $M = N(\|T\|^2 I - TT^*) = N(|T^*| - \|T\|I)$. Furthermore, if T^* is norm attaining, then $M \neq \{0\}$. [7] Let $T \in B(H)$. The following statements are equivalent

A. T achieves the norm.

b. T^* achieves the norm.

c. $|T|$ achieves the norm.

d. $|T^*|$ achieves the norm.

e. $\|T\|$ is an eigenvalue of T.

f. $\|T\|$ is an eigenvalue of T^*.

Corollary 1. *Let $T \in B(H)$ be a quasi-$*$-paranormal operator achieving the norm. Then $\|T\|$ is an eigenvalue of $|T|$.*

Proof. The operator $|T^*| - \|T\|I$ is not one-to-one acoording to Corollary 1 and Lemma 2. Then, the result holds by Lemma 2.

Corollary 2. *If both of T and T^* are quasi-$*$-paranormal in $B(H)$, then the subspace $M = N(\|T\|^2 I - TT^*)$ reduces T.*

Proof. By Corollary 1, M is invariant for T, and $M \subset N(\|T\|^2 I - T^*T)$ according to the proof of Theorem 1. Since T^* is also quasi-$*$-paranormal, $N(\|T\|^2 I - T^*T) \subset M$ is an invariant subspace for T^*. Thus, $M = N(\|T\|^2 I - T^*T)$ is a reducing subspace for T.

Theorem 2. *Let $T \in B(H)$ be a norm attaining quasi-$*$-paranormal operator. If $M = N(T^*T - \|T\|^2 I)$ is finite dimensional, then*

a. M is a reducing subspace for T.

b. The restriction $T\big|_{M^\perp}$ of T on M is also quasi-$*$-paranormal.

Proof. a. By Theorem 1, M is an invariant subspace for T. Since M is of finite dimension, the isometry $\frac{T^*}{\|T\|}$ is unitary on M. We can then write

$$T = \begin{pmatrix} T\big|_M & S \\ 0 & R \end{pmatrix}$$

under the decomposition $H = M \oplus M^{\perp}$, where $S \in B(M^{\perp}, M)$ and $R \in B(M^{\perp}, M^{\perp})$. Since T is quasi-$*$-paranormal,

$$\begin{aligned} 0 &\leq T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T \\ &= \begin{pmatrix} (T\big|_M)^* X T\big|_M & (T\big|_M)^*(XS + YR) \\ (S^*X + R^*Y^*)T\big|_M & (S^*X + R^*Y^*)S + (S^*Y + R^*Z)R \end{pmatrix} \end{aligned}$$

for all $\lambda > 0$, where

$$\begin{aligned} X &= (T\big|_M)^{*2}(T\big|_M)^2 - 2\lambda(T\big|_M(T\big|_M)^* + SS^* + \lambda^2 I \\ Y &= (T\big|_M)^{*2}(T\big|_M S + SR) - 2\lambda SR^* \\ Z &= (T\big|_M S + SR)^*(T\big|_M S + SR) + R^{*2}R^2 - 2\lambda RR^* + \lambda^2 I \end{aligned}$$

By [22, Theorem 6], we get

$$(T\big|_M)^* X T\big|_M \geq 0$$

and

$$(S^*X + R^*Y^*)S + (S^*Y + R^*Z)R \geq 0$$

Hence,

$$(T\big|_M)^{*2}\left(T\big|_M\right)^2 - 2\lambda(T\big|_M(T\big|_M)^* + SS^*) + \lambda^2 I \geq 0$$

for all $\lambda > 0$. As the operator $\frac{1}{\|T\|}(T\big|_M)^*$ is unitary, we get for $\lambda = 1$ that $SS^* \leq 0$. Hence, $S = 0$. This shows that

$$T = \begin{pmatrix} T\big|_M & 0 \\ 0 & R \end{pmatrix}$$

Thus, the subspace M reduces T.

b. The operator $R^*ZR = R^*(R^{*2}R^2 - 2\lambda RR^* + \lambda^2 I)R$ is non-negative for all $\lambda > 0$. Thus, the restriction $R = T\big|_{M^{\perp}}$ is also quasi-$*$-paranormal.

Corollary 3. *Let $T \in B(H)$ be a compact quasi-$*$-paranormal operator. Then, the subspace $M = N(\|T\|^2 I - TT^*)$ is reducing for T.*

Proof. By the hypothesis, the operator TT^* is also compact. Then, M is a nonzero finite dimensional subspace by Fredholm Alternative. The desired result follows then by Theorem 2.

3. Absolutely norm attaining quasi-*-paranormal operators

In the sequel, we present certain structure results on the absolutely norm attaining quasi-$*$-paranormal operators as an extension of certain results given in [3] and [18].

Definition 3. *[4] An operator $T \in B(H)$ is said to be absolutely norm attaining, briefly AN-operator, if the restriction $T\big|_V$ is norm attaining for any closed subspace $V \subset H$, that is, there exists a unit vector $u \in V$ for which*

$$\|T\big|_V u\| = \|Tu\| = \|T\big|_V\|$$

Example 4. *[7] The operator S defined on the Hilbert space ℓ_2 by*

$$Se_1 = \tfrac{1}{2}e_1 \text{ and } Se_n = e_n, \ (n \geq 2)$$

is absolutely norm attaining.

Example 5. *In [3], author showed that the operator $A \in \mathscr{L}(\ell_2 \oplus \ell_2)$ defined by*

$$A(x,y) = \Big((y_1, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ..., \frac{x_n}{n}, ...), (y_2, y_3, ..., y_n, y_{n+1}, ...)\Big)$$

for all $x = (x_k)_{k\geq 1}, y = (y_k)_{k\geq 1} \in \ell_2$, is not absolutely norm attaining on $\ell_2 \oplus \ell_2$.

Theorem 3. *Let $T \in B(H)$ be an absolutely norm attaining quasi-$*$-paranormal operator. If $\sigma_{ess}(|T|) = \{\|T\|\}$, then*

$$T = \begin{pmatrix} \|T\|S & B \\ 0 & C \end{pmatrix}$$

under the orthogonal decomposition $H = M \oplus M^\perp$, where

1. *$S \in B(M)$ is an isometry.*
2. *$B^*S = 0$.*

Proof. 1. Let $T = U|T|$ be the polar decomposition of T. Then, for all $u \in M$, we get

$$Tu = U|T|u = \|T\|Uu$$

That is,

$$T\big|_M = \|T\|U\big|_M = \|T\|S$$

Since $U\big|_M$ is an isometry, the operator S so is.

2. The subspace M is invariant for T. Hence, on $H = M \oplus M^\perp$,

$$T = \begin{pmatrix} \|T\|S & B \\ 0 & C \end{pmatrix}$$

Since T is quasi-$*$-paranormal, we get for all $\lambda > 0$,

$$T^*(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \geq 0$$

where

$$X = \|T\|^2 S^*(\|T\|^4 + \lambda^2 - 2\lambda\|T\|^2 SS^* - 2\lambda BB^*)S \geq 0$$

and for some bounded linear operators Y, Z, W. Then, for $\lambda = \|T\|^2$, and since S is an isometry, $S^*BB^*S = (B^*S)^*(B^*S) \leq 0$. Thus, $B^*S = 0$ since B^*S is a positive operator.

4. Norm attaining class Ω_n operators

Definition 4. *[13] An operator $T \in B(H)$ is said to be quasi-normal of order n for some integer n, or a class Ω_n operator if $TT^{\star n}T^n = T^{\star n}T^{n+1}$.*

Example 6. *[13] Matrices $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are quasi-normal operators of order 2, i.e., $B, S \in \Omega_2$.*

In the following, we provide invaraiant subspaces for operators belonging to class Ω_n.

Theorem 4. *Let $T \in B(H)$ be quasi-normal operator of order 2 such that T^2 achieves the norm. Then, the subspace $V = \{u \in \mathcal{H} : \|T^2u\| = \|T^2\|\|u\|\}$ is invariant for T^2.*

Proof. Since $T \in \Omega_2$, $TT^{\star 2}T^2 = T^{\star 2}T^3$. Then, $(T^{*2}T^2)^2 = T^{*4}T^4$. Hence, for all $u \in H$, $\|T^{*2}T^2u\| = \|T^4u\|$. By Cauchy-Schwarz's inequality,

$$\begin{aligned} \|T^2u\|^2 = \langle T^2u, T^2u\rangle = \langle T^{*2}T^2u, u\rangle \le \|T^{*2}T^2u\|\|u\| &\le \|T^4u\|\|u\| \\ &\le \|T^2(T^2u)\|\|u\| \\ &\le \|T^2\|\|T^2u\|\|u\| \end{aligned}$$

That is for all $u \in V$,

$$\|T^2\|^2\|u\| \le \|T^4u\| \le \|T^2\|^2\|u\|$$

since $\|T^2u\| = \|T^2\|\|u\|,\ u \in V$. Thus,

$$\|T^4u\| = \|T^2(T^2u)\| = \|T^2\|^2\|u\| = \|T^2\|\|T^2u\|$$

for each $u \in V$. This achieves the proof.

Corollary 4. *Let $T \in B(H)$ a class Ω_2 operator. If T^{*2} achieves the norm, then the subspace $V_* = \{u \in H : \|T^{*2}u\| = \|T^2\|\|u\|\}$ is invariant under T^{*2}.*

5. Conclusion

Structures of norm attaining quasi-$*$-paranormal operators and class Ω_n are established in the present manuscript. It's shown that elements of these classes of operators admit at least an invariant non trivial subspace. Some results depending on compactness, and finite dimension are given too. As perspective works, we ask if a such structure can be provided for large classes of norm achieving k-quasi-$*$-paranormal operators, and class $\Omega_{n,k}$ operators defined for certain integer k as $T^{*k}(TT^{\star n}T^n - T^{\star n}T^{n+1})T^k = 0$.

Author Contributions

A. Nasli Bakir: Conceptualization, methodology, writing—original draft, supervision. A. Fellag Ariouat: Investigation, data analysis, writing—original draft. A. Benali: Formal analysis, validation, writing—review and editing. I. Alraddadi: Software, visualization, project administration, funding acquisition. S.M. Almuaddi: Resources,formal analysis, data curation, writing—review and editing. All authors have read and agreed to the published version of the manuscript

References

[1] W. Schachermayer. Norm attaining operators on some classical Banach spaces. *Pac. J. Math.*, 105:427–438, 1983.

[2] G. Ramesh and D. V. Naido. On absolutely norm attaining operators. *Math. Anal. Appl.*, 465:547–556, 2018. Preprint, https://arxiv.org/abs/1801.02432.2016.

[3] N. Bala. Representation and normality of $*$-paranormal absolutely norm attaining operators. *Acta Scient. Math.*, 89:167–181, 2023.

[4] X. Carvajal and W. Neves. Operators that achieve the norm. *Int. Equa. Oper. Theory*, 72:179–195, 2012.

[5] G. Ramesh. Absolutely norm attaining paranormal operators. *J. Math. Anal. Appl.*, 465:547–556, 2018.

[6] N. Bala and G. Ramesh. A representation of hyponormal absolutely norm attaining operators. *Bull. Sci. Math.*, 171:1–15, 2021.

[7] S. K. Pandey and V. I. Paulsen. A spectral characterization of $\mathcal{AN}$ operators. *J. Aust. Math. Soc.*, 102:369–391, 2017. MR3650963.

[8] K. Tanahashi and A. Uchiyama. A note on $*$-paranormal operators and related classes of operators. *Bull. Korean Math. Soc.*, 51:357–371, 2014.

[9] J. L. Shen and C. Alatancang. The spectrum properties of quasi-$*$-paranaormal operators. *Chinese Ann. Math.*, 34:663–670, 2013. in Chinese.

[10] F. Zuo and H. Zuo. Structural and spectral properties of k-quasi-$*$-paranormal operators. *Kor. J. Math.*, 23:249–257, 2015.

[11] I. Hoxha, N. L. Braha, and A. Tato. Riesz idempotent and Weyl's theorem for k-quasi-$*$-paranormal operators. *Appl. Math. E-notes*, 19:80–100, 2019.

[12] F. Zuo and J. Shen. Polaroid and k-quasi-$*$-paranormal operators. *Filomat*, 30:313–319, 2016.

[13] A. Fellag Ariouat, A. Nasli Bakir, and A. Benali. Bishop's property, Weyl's theorem and Riesz idempotent. *J. Comp. Anal. Appl.*, 34:323–340, 2025.

[14] M. Maadani, A. Benali, and A. Nasli Bakir. On quasi totally m-class A_k^* operators. *J. Comp. Anal. Appl.*, 35:104–119, 2025.

[15] J. B. Conway. *A course in functional analysis.* Springer Verlag, second edition, 1990.

[16] V. Müller. *Spectral theory of linear operators and spectral systems in Banach algebras.* Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, second edition, 2007.

[17] J. Ganesh, G. Ramesh, and D. Sukumar. On the structure of absolutely minimum attaining operators. *J. Math. Anal. Appl.*, 428:457–470, 2015.

[18] S. Mecheri and A. Nasli Bakir. Norm attaining and absolutely norm attaining of $*$-paranormal operators. *Filomat*, 38:2381–2386, 2024.

[19] Y. M. Han and A. H. Kim. A note on $*$-paranormal operators. *Int. Equa. Oper. Theo.*, 49:435–444, 2004.

[20] I. Hoxha and N. L. Braha. A note on k-quasi-$*$-paranormal operators. *J. Ineq. Appl.*, 2013(350):1–7, 2013.

[21] J. I. Lee. On the norm attaining operators. *Kor. J. Math.*, 20:485–491, 2012.

[22] M. Dehghani, S. M. S. Modarres, and M. S. Moslehian. Positive block matrices on Hilbert and Krein C^*-modules. *Surveys in Math. Appl.*, 8:23–34, 2013. ISSN 1842-6298 (electronic), 1843-7265 (print).