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Ricci Solitons and Their Associated Vector Fields in LRS Bianchi Type I Spacetime

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Abstract. This article presents a complete classification of Ricci solitons and their associated vector fields in the context of locally rotationally symmetric (LRS) Bianchi type I spacetime, a crucial model in cosmological studies. To systematically address the complexities inherent in the Ricci soliton equations, we adopt the Rif tree technique. The equations defining the Ricci soliton and its vector field are transformed into a reduced involutive form using a computational algorithm, which assists in dividing the integration process into a collection of cases organized in a tree-like structure. Each of these cases is governed by specific constraints on the metric functions, which facilitates the solution process. Definite expressions for the metric functions and the corresponding vector field of the Ricci soliton are obtained by efficiently solving the system of equations characterizing the soliton vector field through the application of these constraints. This powerful approach enables us to derive novel and exact solutions that previous methods have overlooked. Our results demonstrate that this spacetime admits Ricci solitons of shrinking, steady, and expanding natures, characterized by vector fields with up to 11 free parameters. Crucially, we conduct a thorough physical analysis of the resulting models, determining their matter content through the equation of state and testing their physical viability via the standard energy conditions. We find specific families of solutions that correspond to physically significant scenarios, such as a spacetime filled with vacuum energy (a cosmological constant). This work not only provides a comprehensive mathematical classification but also establishes a direct link between these geometric structures and potentially realistic cosmological models.

2020 Mathematics Subject Classifications: 83C15, 83F05, 53C50

Key Words and Phrases: Soliton vector fields, locally rotationally symmetric Bianchi type I spacetime, Rif tree, energy conditions

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1. Introduction

In the theory of general relativity (GR), a primary aim of researchers is to construct gravitational potentials that satisfy the Einstein field equations (EFEs). For this purpose, various symmetry assumptions are frequently imposed on the geometry of the spacetime, along with a choice of the particular matter distribution under consideration. These symmetries are expressed through vector fields that preserve different kinds of tensors. Among the various symmetries in the literature [1]-[2], some of the more significant ones are Killing, homothetic, and conformal vector fields, as well as Noether symmetries. In terms of physical importance, these symmetries are connected to conservation laws of energy and momentum, among others. Their role also extends to the study of singularities in GR, cosmological problems, and astrophysics. Apart from these, Noether symmetries are considered a crucial tool for classifying spacetime Lagrangians to obtain exact solutions of EFEs and to reduce the number of variables and the order of partial differential equations [3].

It is crucial to distinguish the aforementioned symmetries from the structure of a Ricci soliton. A Ricci soliton is defined as a pseudo-Riemannian manifold (M, g) that admits a smooth vector field V satisfying Equation (1). In contrast to a symmetry vector field, which generates a transformation preserving certain properties of a given metric, the vector field V is an intrinsic part of the soliton structure itself. The metric g and the vector field V together satisfy the soliton equation. In this paper, we study the LRS Bianchi-I spacetime as a Ricci soliton, and our objective is to solve Equation (1) to find the metric functions $p(t)$, $q(t)$ and the components of the associated vector field V.

The study of Ricci solitons on spacetimes has gained considerable momentum, with recent works providing important classifications regarding RSVFs. For instance, Ibrar et al. [4] classified Ricci solitons in plane symmetric static spacetimes, finding a six- or ten-dimensional Lie algebra for the soliton vectors. Similarly, Mahmood et al. [5] and Tahirullah et al. [6] examined LRS Bianchi type V and static spherically symmetric spacetimes, respectively, identifying solutions with expanding, steady, and shrinking natures. Ali and Khan [7] investigated Kantowski-Sachs spacetimes, concluding they only admit expanding solitons. Furthermore, the exploration has extended to modified theories of gravity; Uzma et al. [8] found all three types of solitons for Bianchi type I spacetimes in $f(T)$ gravity, while Siddiqi et al. [9, 10] have classified them in $f(R)$ and $f(R, T)$ theories.

The exploration of solitons extends beyond Ricci solitons to other geometrically significant types. Recently, Ali et al. [11] conducted a comprehensive analysis of conformal Ricci solitons within the framework of Kenmotsu manifolds, uncovering conditions under which such solitons become Einstein and providing valuable insights into the interplay between curvature and soliton structures. In the realm of almost paracontact geometry, Naik et al. [12] investigated generalized Ricci solitons, establishing key theorems that determine when these solitons transition to a nearly para-Einstein form, thereby enriching the classification of solitons in this specific geometric setting. Furthermore, the study of solitons has been advanced through the lens of symmetry, as demonstrated by Raza et al. [13], who explored the relationship between Ricci solitons and curvature inheritance symmetry in Riemannian manifolds. Their work is particularly relevant as it bridges the study of solitons with other fundamental symmetry principles in geometry, suggesting a deeper underlying structure.

A common thread in these studies is the reliance on the direct integration approach to solve the highly complex, coupled non-linear system of Ricci soliton equations. While this method has yielded valuable results, it is inherently burdensome, time-consuming, and prone to human error. Most critically, this methodological limitation carries a significant risk of overlooking particular classes of metrics or special solutions that satisfy the governing equations only under specific, non-generic conditions. This means that prior classifications, while insightful, are likely not exhaustive and may have missed entire branches of solutions. This constitutes a fundamental gap in the literature, as an incomplete classification hinders a full understanding of the spacetime's geometric and physical properties. Furthermore, the absence of a systematic method makes it difficult to compare results across different studies or to ensure reproducibility.

Our work advances beyond these prior studies by implementing a systematic and exhaustive classification methodology. We employ the Rif tree algorithm [14, 15], a powerful computational differential elimination tool, to address the aforementioned limitations. This approach algorithmically simplifies the Ricci soliton equations into a reduced involutive form and partitions the entire solution space into a tree-like structure of mutually exclusive cases. Each branch of this Rif tree represents a distinct set of constraints on the metric functions $p(t)$ and $q(t)$. This ensures that no possible case is overlooked, thereby guaranteeing a complete classification-a feat difficult to achieve through manual computation alone.

The application of the Rif tree approach in GR has recently proven its value in classifying spacetimes by other symmetries [16–19], often uncovering new metrics missed by direct integration. We apply this innovative technique for the first time to the problem of classifying Ricci solitons in LRS Bianchi type I spacetime. This allows us to not only recover all solutions that might be found by direct methods but, more importantly, to *discover novel and previously unreported solutions* that exist under specific differential constraints. Our study therefore provides the first complete picture of Ricci solitons in this important cosmological spacetime.

In the theories of GR and Riemannian geometry, geometric flows are essential tools. Among these, the Ricci flow, initially proposed by Hamilton [20], is of fundamental importance. Hamilton also introduced self-similar solutions for the Ricci flow, defined as Ricci soliton vector fields (RSVFs) [21], given by:

$$\mathcal{L}_V g_{mn} + 2R_{mn} = 2\delta g_{mn}. \tag{1}$$

Here, $\mathcal{L}_V g_{mn}$ represents the Lie derivative of the metric tensor g_{mn} along the vector field V, R_{mn} denotes the Ricci tensor, and δ is a constant. Depending on the value of δ, an RSVF may be shrinking, expanding, or steady when $\delta > 0$, $\delta < 0$ or $\delta = 0$, respectively. RSVFs are not only considered a fruitful tool for the simplification of EFEs but are also helpful in understanding singularity formation. They establish a relationship between physical spacetime models and theoretical geometric flows, providing a framework for exploring fundamental questions in both mathematics and theoretical physics. Another important feature of RSVFs is that when they reduce to an Einstein tensor, they provide a solution to the EFEs with a cosmological constant, which makes them beneficial for investigating various cosmological models. Furthermore, Ricci solitons are important for examining how spacetime evolves. For example, they can help analyze how the Ricci flow affects the spacetime structure over time. They are also significant for observing how a manifold behaves under the flow and can offer insights into the flow's behavior near singularities.

Drawing inspiration from the existing literature and the identified gap in methodology, the objective of this paper is to present a complete and exhaustive classification of Ricci solitons in the context of LRS Bianchi type I spacetime within the framework of GR, by adopting the innovative Rif tree approach.

2. Ricci Soliton Equations

The line element for LRS Bianchi type I spacetime is given as:

$$ds^2 = -dt^2 + p^2(t)\ dx^2 + q^2(t)(\ dy^2 + dz^2), \tag{2}$$

where $p \neq 0$ and $q \neq 0$. The set $\{\partial_x,\ \partial_y,\ \partial_z,\ y\partial_z - z\partial_y\}$ represents the minimal set of Killing vector fields (KVFs) possessed by the above spacetime. The following expressions demonstrate the non-vanishing components of R_{mn} :

$$\begin{aligned} R_{00} &= -\frac{1}{pq}\{p''q + 2q''p\}, \\ R_{11} &= \frac{p}{q}\{p''q + 2p'q'\}, \\ R_{22} &= \frac{1}{p}\{pqq'' + pq'^2 + qp'q'\}, \\ R_{33} &= R_{22}. \end{aligned} \tag{3}$$

Substituting the metric tensor from Eq. (2) along with the components of R_{mn} from Eq. (3) into Eq. (1) results in the following system of differential equations:

$$V^0_{,0} = \delta + R_{00}, \tag{4}$$

$$p^2V^1_{,0} - V^0_{,1} = 0, \tag{5}$$

$$q^2V^2_{,0} - V^0_{,2} = 0, \tag{6}$$

$$q^2V^3_{,0} - V^0_{,3} = 0, \tag{7}$$

$$V^1_{,1} + \frac{p'}{p}V^0 = \delta - \frac{1}{p^2}R_{11}, \tag{8}$$

$$p^2\ V^1_{,2} + q^2\ V^2_{,1} = 0, \tag{9}$$

$$p^2\ V^1_{,3} + q^2V^3_{,1} = 0, \tag{10}$$

$$V^2_{,2} + \frac{q'}{q}V^0 = \delta - \frac{1}{q^2}R_{22}, \tag{11}$$

$$V^2_{,3} + V^3_{,2} = 0, \tag{12}$$

$$V^3_{,3} + \frac{q'}{q}V^0 = \delta - \frac{1}{q^2}R_{22}. \tag{13}$$

We begin by analyzing the system of RSVF equations (Eqs. 4-13) using the Rif algorithm through Maple software, which produces the Rif tree shown in Fig. (1). The Rif algorithm not only simplifies the system but also identifies and prunes branches that lead to mathematical inconsistencies (e.g., a condition that implies $p' = 0$ while another condition in the same branch implies $p' \neq 0$). These inconsistent branches are shown in the tree (Fig. 1) as terminating without further subdivision (often marked as "$0 = 1$" or similar in Maple's output). The branches presented in the figure are the consistent, non-singular cases that require further analysis. The pivots leading to these dead ends are not listed in Equation (14) as they are automatically discarded by the algorithm. Each branch of the Rif tree includes nodes labeled $p1,\ p2,\ p3,\ ...,p10$, known as pivots, with their expressions listed in Eq. (14). These pivot expressions demonstrate the conditions on p and q under which the RSVF Eqs. (4)-(13) are required to be solved. The solution of Eqs. (4)-(13), based on the conditions from every branch, provides the precise form of the RSVFs. Moreover, in the figure, the indications " $=$ " and " <> " denote the vanishing and non-vanishing nature of the pivot, respectively.

$$\begin{aligned}
p1 &= p',\\
p2 &= qq'' - q'^2,\\
p3 &= q''p - p''q,\\
p4 &= pq' - p'q,\\
p5 &= p''q - p'q',\\
p6 &= p''p' - pp''',\\
p7 &= p''q^2 - pq'^2,\\
p8 &= q',\\
p9 &= p''p - p'^2,\\
p10 &= pq' + p'q,\\
p11 &= q'(pq' - p'q).
\end{aligned} \tag{14}$$

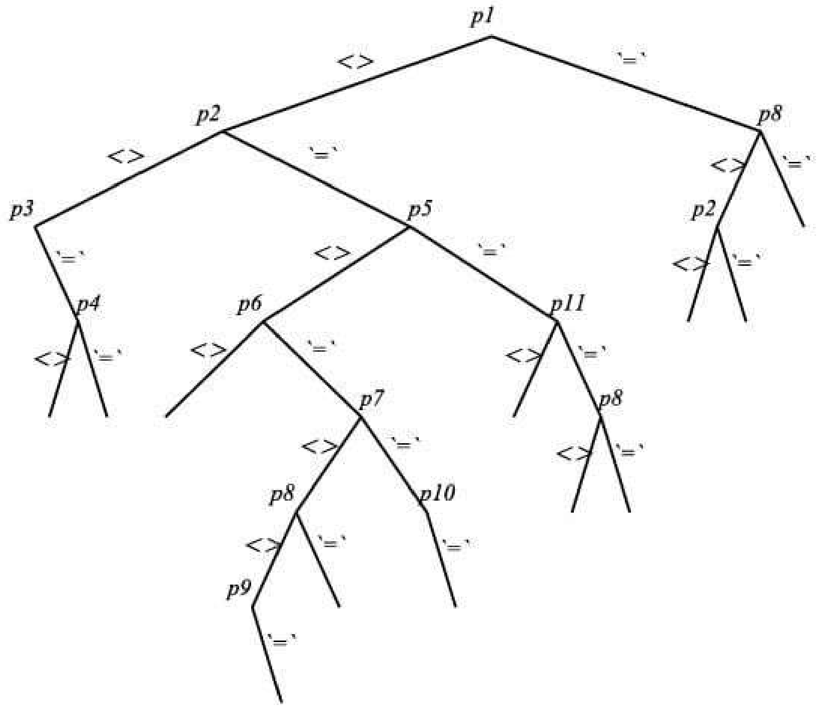

Figure 1: Rif tree structure for the system of RSVF equations. Branches that terminate prematurely were pruned by the algorithm as they led to mathematical inconsistencies (e.g., $p1 = p' = 0$ and $p1 \neq 0$ arising in the same branch). The displayed branches represent all consistent, solvable cases..

The general solution for the vector field V in branches 1 and 2 contains 8 and 9 independent arbitrary constants $(e_1, e_2, ...)$ respectively, while δ is arbitrary. In both branches,

Branch No.	Metric Functions	Components of V
1	$p \neq q,\ p' \neq 0,$ $qq'' - q'^2 \neq 0,$ $p''q - q''p = 0.$	$V^0 = \delta t - \int \frac{p''}{p} dt - 2\int \frac{q''}{q} dt + e_1,$ $V^1 = e_2 x + e_3,$ $V^2 = e_4 z + e_5 y + e_6,$ $V^3 = -e_4 y + e_5 z + e_7.$ Where p and q must satisfy the following expressions: $e_2 + \frac{p'}{p}\{\delta t - \int \frac{p''}{p} dt - 2\int \frac{q''}{q} dt + e_1\} + \frac{qp'' + 2p'q'}{pq} = \delta.$ $e_5 + \frac{q'}{q}\{\delta t - \int \frac{p''}{p} dt - 2\int \frac{q''}{q} dt + e_1\} + \frac{q''}{q} + \frac{q'^2}{q^2} + \frac{p'q'}{pq} = \delta.$
2	$p = q,\ p' \neq 0,$ $pp'' - p'^2 \neq 0.$	$V^0 = \delta t - 3\int \frac{p''}{p} dt + e_1,$ $V^1 = e_2 x - e_3 y - e_4 z + e_5,$ $V^2 = e_6 z + e_2 y + e_3 x + e_7,$ $V^3 = e_2 z - e_6 y + e_4 x + e_8.$ Where p satisfies the non-linear equation given by: $e_2 + \frac{p'}{p}\{\delta t - 3\int \frac{p''}{p} dt + e_1\} + \frac{pp'' + 2p'^2}{p^2} = \delta.$
3	$p = \tan t,$ $q = e^{kt},$ $k \neq 0.$	$V^0 = -2\tan t,$ $V^1 = e_1,$ $V^2 = e_2 z + e_3,$ $V^3 = -e_2 y + e_4.$
4	$p = e^{k_1 t},$ $q = e^{k_2 t},$ $k_1 \neq k_2,$ and $\delta = k_1^2 + 2k_2^2.$	$V^0 = e_1,$ $V^1 = 2k_2(k_2 - k_1)x - e_1 k_1 x + e_2,$ $V^2 = e_3 z + k_1(k_1 - k_2)y - e_1 k_2 y + e_4,$ $V^3 = k_1(k_1 - k_2)z - e_3 y - e_1 k_2 z + e_5.$
5	$p = k_2 e^{kt} + k_3 e^{-kt},$ $q = k_1 \neq 0,$ $k \neq 0, \delta = k^2,$ $\alpha = 2k\sqrt{k_2 k_3}.$	$V^0 = \frac{1}{\alpha}\{e_1 \sin \alpha x - e_2 \cos \alpha x\},$ $V^1 = \frac{k(k_2 e^{kt} - k_3 e^{-kt})}{\alpha^2 (k_2 e^{kt} + k_3 e^{-kt})}\{e_1 \cos \alpha x + e_2 \sin \alpha x\} + e_3,$ $V^2 = e_4 z + k^2 y + e_5,$ $V^3 = -e_4 y + k^2 z + e_6.$
6	$p = e^{-k_1 t},$ $q = e^{k_1 t}$ where $k_1 \neq 0$ and $\delta = 3k_1^2.$	$V^0 = 4k_1^2 - e_1,$ $V^1 = \{4k_1^2 - k_1(4k_1^2 - e_1)\}x + e_2,$ $V^2 = e_3 z + e_1 y + e_4,$ $V^3 = -e_3 y + e_1 z + e_5.$
7	$p = \frac{1}{k_1} e^{k_1 t},$ $q = e^{k_1 t}$ where $k_1 \neq 0, 1$ and $\delta = 3k_1^2.$	$V^0 = \frac{1}{k_1}\{e_1 z - e_2 y - e_3 x - e_4\},$ $V^1 = \frac{e_3}{2}(x^2 + y^2 + z^2 + \frac{e^{-2k_1 t}}{k_1^2}) + (e_2 x - e_5)y$ $+(e_6 - e_1 x)z + e_4 x + e_7,$ $V^2 = \frac{e_2}{2}(2yz + \frac{e^{-2k_1 t}}{k_1^2} + x^2) + \frac{e_1}{2}(z^2 - y^2)$ $+e_8 z + (e_3 x + e_4)y + e_5 x + e_9,$ $V^3 = -\frac{e_1}{2}(2yz + \frac{e^{-2k_1 t}}{k_1^2} - x^2) + \frac{e_2}{2}(z^2 - y^2)$ $-e_8 y + (e_3 x + e_4)z + e_6 x + e_{10}.$

Table 1: Metrics admitting RSVFs

the nature of the RSVFs can be shrinking, expanding, or steady. In the case of a steady

Branch No.	Metric Functions	Components of V
8	$p = e^{k_1 t}$, $q = e^{k_1 t}$ where $k_1 \neq 0$ and $\delta = 3k_1^2$.	$V^0 = \frac{1}{k_1}\{e_1 z - e_2 y - e_3 x - e_4\}$, $V^1 = \frac{e_3}{2}(x^2 + y^2 + z^2 + \frac{e^{-2k_1 t}}{k_1^2}) + (e_2 x - e_5)y$ $+(e_6 - e_1 x)z + e_4 x + e_7$, $V^2 = \frac{e_2}{2}(2yz + \frac{e^{-2k_1 t}}{k_1^2} + x^2) + \frac{e_1}{2}(z^2 - y^2)$ $+e_8 z + (e_3 x + e_4)y + e_5 x + e_9$, $V^3 = -\frac{e_1}{2}(2yz + \frac{e^{-2k_1 t}}{k_1^2} - x^2) + \frac{e_2}{2}(z^2 - y^2)$ $-e_8 y + (e_3 x + e_4)z + e_6 x + e_{10}$.
9	$p = k_1 t + k_2$, $q = k_3 \neq 0$, where $k_1 \neq 0, k_2 \neq 0$ $\delta = \frac{k_1}{k_2} e_1$.	$V^0 = \delta t + e_1$, $V^1 = e_2 x + e_3$, $V^2 = \delta y + e_4 z + e_5$, $V^3 = \delta z - e_4 y + e_6$.
10	$p = k_1$, $q' \neq 0$, where $qq'' - q'^2 \neq 0$.	$V^0 = \delta t + e_1 - 2\int \frac{q''}{q} dt$, $V^1 = \delta x + e_2$, $V^2 = e_3 z + e_4 y + e_5$, $V^3 = -e_3 y + e_4 z + e_6$. where q satisfies the non-linear expression given by $e_4 + \frac{q'}{q}\{\delta t - 3\int \frac{q''}{q} dt + e_1\} + \frac{qq'' + 2q'^2}{q^2} = \delta$.
11	$p = k_1 \neq 0$, $q = e^{k_2 t}$, where $k_2 \neq 0, \delta = 2k_2^2$.	$V^0 = e_1$, $V^1 = \delta x + e_2$, $V^2 = e_3 z - e_1 k_2 y + e_4$, $V^3 = -e_3 y - e_1 k_2 z + e_5$.
12	$p = k_1 \neq 0$, $q = k_2 \neq 0$.	$V^0 = \delta t + k_2^2(e_1 y + e_2 z) + k_1^2 e_3 t + e_4$, $V^1 = \delta x - \frac{k_2^2}{k_1^2}(e_5 y + e_6 z) + e_3 x + e_7$, $V^2 = \delta y + e_8 z + e_5 x + e_1 t + a_9$, $V^3 = \delta z - e_8 y + e_6 x + e_2 t + a_{10}$.

Table 2: Metrics admitting RSVFs

nature ($\delta = 0$), the number of independent arbitrary constants in the solution for V is reduced by one for both branches.

Branch 3 shows that the spacetime under study possesses RSVFs of a steady nature ($\delta = 0$ implied by the solution) with a expanding, and the solution for V is characterized by 3 independent parameters.

Our analysis shows that the constant δ is positive for both branches 4 and 6, indicating that the RSVFs found are naturally expanding with a expanding, and the solution for V is characterized by 5 independent parameters.

The dimension of the RSVFs is 7 under the conditions of branches 5 and 10. For branch 5, $\delta = k^2$, which reveals that the nature of the RSVFs is only expanding. In branch 10, the nature of the RSVFs may be expanding, shrinking, or steady, corresponding to δ being positive, negative, or zero, respectively.

The solution of Eqs.(4)-(13), under the limitations of branching scenarios 7 and 8, provides a solution for V containing 10 independent parameters, which are only expanding by nature ($\delta = 3k_1^2 > 0$).

Investigation of RSVFs for LRS Bianchi type I spacetime, within the limits of the branching constraints of 9 and 11, gives a solution for V characterized by 6 independent parameters.. In branch 9, δ may take any value (as it is proportional to e_1), which indicates that the nature of the found RSVFs may be steady, shrinking, or expanding. In contrast, δ turns out to be positive in branch 11 ($\delta = 2k_2^2$), showing that the RSVFs are expanding.

The solution of Eqs.(4)-(13) leads to the precise form of a solution for V characterized by 11 independent parameters. under the conditions imposed by branch 12. Here the constant δ is arbitrary, and the RSVFs can be of any nature among the three mentioned.

3. Physical Implications

To ensure the physical validity of the obtained metrics, it is necessary to find the non-vanishing components of the energy-momentum tensor (EMT), denoted by T_{mn}, for the specified spacetime. Equation (15) gives these terms for the spacetime (2).

$$\begin{aligned} T_{00} &= \frac{2p'q'}{pq} + \frac{q'^2}{q^2}, \\ T_{11} &= -\frac{p^2}{q^2}\left(2qq'' + q'^2\right), \\ T_{22} &= -\frac{p}{q}\left(pq'' + qp'' + p'q'\right), \\ T_{33} &= T_{22}. \end{aligned} \tag{15}$$

Based on the type of matter it describes, the EMT takes distinct forms. The form $T_{mn} = (\rho + P)u_m u_n + P g_{mn}$ indicates a perfect fluid, while the form $T_{mn} = (\rho + P_\parallel)u_m u_n + (P_\parallel - P_\perp)v_m v_n + P_\perp g_{mn}$ illustrates an anisotropic fluid. Here the symbols $\rho,\ P,\ P_\parallel,\ P_\perp$ stand for energy density, isotropic pressure, pressure parallel to the preferred spatial direction, and pressure perpendicular to it, respectively. Moreover, u_m and v_m indicate the four-velocity vector and a space-like unit vector, respectively, such that $u_m u^m = -1$, $v_m v^m = 1$ and $u_m v^m = 0$ [22]. The terms $P_\parallel$ and $P_\perp$ become equal when the matter is a perfect fluid.

The equation of state (EoS) for a perfect fluid has the form $P = w\rho$, where w is the EoS parameter. To ensure that the system preserves mechanical stability and adheres to causality, the condition $-1 \leq w \leq 1$ should hold. Different values of w indicate various matter sources. For $w = -1$, the matter source is a cosmological constant, also called vacuum energy. Taking $w = 0$, the matter source is dust, also referred to as non-relativistic matter. The matter source will be stiff matter (Zel'dovich matter) when $w = 1$ [23].

For branches 7 and 8, we obtained the physical terms as $\rho = 3k_1^2$ and $P = -3k_1^2$, for which the EoS parameter turns out to be $w = -1$, identifying that the source of matter in the branches under study corresponds to the cosmological constant, or vacuum energy.

This result reveals that these branches represent a constant energy density that drives cosmic acceleration. In a similar way, one can study all metrics obtained in all branches with respect to the EoS.

For an anisotropic fluid, once the values of density and pressure are determined, the following energy conditions are evaluated in detail to analyze the physical viability of the obtained models [22]:
1. Null energy condition (NEC) requires that $(\rho + P_{\|} \geq 0, \quad \rho + P_{\perp} \geq 0)$.
2. Weak energy condition (WEC) requires $(\rho \geq 0)$ in addition to the NEC.
3. Strong energy condition (SEC), in addition to the NEC, requires $(\rho + P_{\|} + 2P_{\perp} \geq 0)$.
4. Dominant energy condition (DEC) demands $(\rho \geq 0, \quad \rho \geq \left|P_{\|}\right|, \quad \rho \geq |P_{\perp}|)$.

In branch 4, we get $\rho = 2k_1k_2 + k_2^2$, $P_{\|} = -3k_2^2$ and $P_{\perp} = -e^{2(k_1-k_2)t}(k_1^2 + k_2^2 + k_1k_2)$. The NEC holds true when $k_1 > k_2$ and $2k_1k_2 + k_2^2 - e^{2(k_1-k_2)t}(k_1^2 + k_2^2 + k_1k_2) \geq 0$. Additionally, if $2k_1k_2 + k_2^2 \geq 0$, then the WEC is also satisfied. Furthermore, $2k_1k_2 - 2k_2^2 - e^{2(k_1-k_2)t}(k_1^2 + k_2^2 + k_1k_2) \geq 0$ reveals that the SEC is fulfilled. The restrictions $k_1 > k_2$ and $2k_1k_2 + k_2^2 \geq e^{2(k_1-k_2)t}(k_1^2 + k_2^2 + k_1k_2)$ confirm the satisfaction of the DEC.

For branch 5, the physical terms are obtained as $\rho = P_{\|} = 0$ and $P_{\perp} = -\frac{k^2}{k_1^2}(k_2e^{kt} + k_3e^{-kt})$. Under the condition $\frac{k^2}{k_1^2}(k_2e^{kt} + k_3e^{-kt}) < 0$ (i.e., $P_{\perp} > 0$), the NEC, WEC, and SEC hold true, while the DEC does not hold (as $\rho = 0$ is not greater than $|P_{\perp}| > 0$).

For the metric of branch 9, the values of ρ, $P_{\|}$, and $P_{\perp}$ turn out to be zero, which demonstrates the presence of a vacuum space, and all the energy conditions are satisfied trivially.

In the case of branch 10, $\rho = \frac{q'^2}{q^2}$, $P_{\|} = -\frac{1}{q^2}(2qq'' + q'^2)$, and $P_{\perp} = -k_1^2\frac{q''}{q^3}$. If $q'' < 0$ (making $P_{\perp} > 0$) and $qq'^2 - k_1q'' \geq 0$ (ensuring $\rho + P_{\perp} \geq 0$), then the NEC, WEC, and SEC hold true. Additionally, the DEC requires $q'^2 \geq k_1^2\left|\frac{q''}{q}\right|$ (since $P_{\perp}$ is positive under $q'' < 0$, this becomes $q'^2 \geq -k_1^2\frac{q''}{q}$).

By adopting the same procedure, one can easily examine the above-mentioned energy conditions for the remaining metrics.

3.1. Implications for Spacetime Evolution and Stability

The discovery of Ricci soliton solutions within the LRS Bianchi-I framework has profound implications for understanding the possible evolution and stability of such spacetimes. A Ricci soliton metric is, by definition, a fixed point (or equilibrium solution) of the Ricci flow $\frac{\partial g_{mn}}{\partial \lambda} = -2R_{mn}$ under a combination of the flow and diffeomorphisms generated by the vector field V [20]. This connection allows us to interpret our results in a dynamical context.

The nature of the soliton-shrinking $(\delta > 0)$, steady $(\delta = 0)$, or expanding $(\delta < 0)$-provides a direct classification of the spacetime's behavior under the Ricci flow. For instance, the **expanding solitons** found in Branches 4, 5, 7, 8, and 11 suggest a spacetime that, under the flow, exhibits inflationary-like expansion. This is not a physical expansion in time t, but an expansion in the flow parameter λ, which can be interpreted as a coarsegrained or averaged evolution of the geometry. The prevalence of expanding solutions in our classification indicates that the LRS Bianchi-I geometry naturally tends towards less curved, more voluminous states under this geometric flow.

Conversely, the existence of **steady solitons** (e.g., when $\delta = 0$ in Branches 1, 2, 9, 10, and 12) represents equilibrium configurations. These spacetimes are stable under the combined action of the Ricci flow and the diffeomorphism generated by their associated vector field V. A prime example is Branch 9, where we found a vacuum solution ($\rho = P_{\|} = P_{\perp} = 0$) satisfying all energy conditions. This steady soliton can be viewed as a stable, fixed-point geometry within this class of models.

The **shrinking solitons** (possible in Branches 1, 2, 9, 10, and 12 for $\delta > 0$) are perhaps the most significant from a gravitational perspective. In the Ricci flow, shrinking solitons evolve towards a singularity (a curvature blow-up) in finite flow time. This mathematical behavior is analogous to the formation of a spacetime singularity in GR, such as those inside black holes or in cosmological scenarios like the Big Bang. Therefore, our identification of these shrinking soliton solutions provides concrete examples of LRS Bianchi-I spacetimes whose intrinsic geometry is predisposed to evolve into a singular state, offering a geometric flow perspective on singularity formation.

Finally, the high dimensionality of the soliton vector fields (e.g., up to 11 parameters in Branch 12) indicates a significant degree of **geometric rigidity**. The spacetime admits a large family of deformations (generated by V) that preserve the soliton structure. This suggests a form of stability: the spacetime is not a isolated solution but part of a larger family of solutions with similar geometric properties, making it a more robust and physically plausible model.

In conclusion, our classification does more than list solutions; it reveals the dynamical tendencies of LRS Bianchi-I spacetimes. The expanding, steady, and shrinking solitons map onto possible evolutionary pathways-inflationary expansion, stable equilibrium, or gravitational collapse-linking the algebraic structure of the Ricci soliton equation to profound physical phenomena in cosmology and gravitation.

4. Conclusion

This work presents a detailed theoretical study of Ricci solitons in LRS Bianchi-I spacetime, culminating in a complete classification. The investigation was carried out in multiple, comprehensive stages: First, the highly complex, coupled system of partial differential equations defining the Ricci soliton and its vector field was derived. Second, the Rif tree algorithm was employed to perform a systematic and exhaustive analysis of this system, leading to the identification of twelve distinct branches of solutions, each governed by specific geometric constraints on the metric functions $p(t)$ and $q(t)$. Third, the resulting equations were solved to determine the exact form of the soliton vector field V for each case, revealing solutions characterized by a number of free parameters ranging from 3 to 11. Finally, a thorough physical analysis was conducted, determining the matter content via the equation of state and testing the physical viability of the resulting models through the standard energy conditions.

The main results and findings reported in this manuscript are summarized as follows:

(i) The Rif tree algorithm proved to be a powerful and essential tool for this classification, successfully managing the high complexity of the PDE system and guaranteeing that no possible solution branch was overlooked. This represents a significant methodological advancement over the direct integration approach used in prior studies.

(ii) The LRS Bianchi-I spacetime admits a rich variety of Ricci soliton solutions, including **shrinking** ($\delta > 0$), steady ($\delta = 0$), and expanding ($\delta < 0$) types. Their existence depends critically on the specific differential constraints satisfied by the metric functions $p(t)$ and $q(t)$.

(iii) The dimensionality of the associated soliton vector fields V varies significantly across the different branches, ranging from 3 to 11 independent parameters. This diversity indicates varying degrees of geometric rigidity within the solution space.

(iv) The physical analysis, based on the equation of state and energy conditions, confirmed that several of the obtained solutions correspond to physically meaningful scenarios. Notably, the solutions in Branches 7 and 8 were found to perfectly describe a spacetime filled with a cosmological constant ($w = -1$), while the solution in Branch 9 describes a vacuum.

In conclusion, our classification does more than list solutions; it reveals the dynamical tendencies and stability properties of LRS Bianchi-I spacetimes. The expanding, steady, and shrinking solitons map onto possible evolutionary pathways-inflationary expansion, stable equilibrium, or gravitational collapse-linking the algebraic structure of the Ricci soliton equation to profound physical phenomena in cosmology and gravitation. The cases involving non-linear constraints on $p(t)$ and $q(t)$ present promising avenues for future work, potentially requiring numerical analysis to fully explore their dynamics and stability.

Declarations

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