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Integral Formulas for the Noncentral Tanny-Dowling Polynomials

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Abstract. In this paper, the authors established some integral formulas for the noncentral Tanny-Dowling polynomials. These formulas are shown to be generalizations of some known results on the classical geometric polynomials.

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1. Introduction

Let $\left\{ n \atop k \right\}$ denote the *Stirling numbers of the second kind*, see [1]. In the classical distribution problems, $\left\{ n \atop k \right\}$ count the number of ways to distribute n distinct objects into k identical boxes such that no box is empty, see page 47 of [2]. These numbers also appear as coefficients in the expansion of

$$x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} (x)_k, \tag{1}$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ is the *Pochhammer symbol*, see [3]. It is easy to see that $k!\left\{ n \atop k \right\}$ i.e., the Stirling numbers of the second kind multiplied by $k!$, counts the number of ways to distribute n distinct objects to k distinct boxes such that no box is empty.

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The *geometric polynomials*, also known as *Fubini Polynomials*, see [4], are defined by

$$w_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k. \tag{2}$$

These polynomials are known to satisfy the exponential generating function given by [5, Eq. (3.14)]

$$\sum_{n=0}^{\infty} w_n(x) \frac{z^n}{n!} = \frac{1}{1 - x(e^z - 1)}. \tag{3}$$

These polynomials have strong links to combinatorics, exponential generating functions, and classical sequences such as the Bernoulli numbers.

The case when $x = 1$ given by

$$w_n := w_n(1) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} \tag{4}$$

is called *geometric numbers* or *Fubini numbers*. These count all the possible set partitions of an n element set such that the order of the blocks matters. The exponential generating function of w_n can be easily by setting $x = 1$ in (3). That is,

$$\sum_{n=0}^{\infty} w_n(1) \frac{z^n}{n!} := \sum_{n=0}^{\infty} w_n \frac{z^n}{n!} = \frac{1}{2 - e^z}. \tag{5}$$

The study of geometric polynomials has remained a thrend for among mathematicians to this date. For instance, Kellner [6] established several identities involving the polynomials $w_n(x)$. Among these identities is the integral identity over the interval $[-1, 0]$ given by

$$\int_{-1}^{0} w_n(x) dx = B_n. \tag{6}$$

Here, B_n denotes the n^{th} *Bernoulli number* defined by the exponential generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^k}{n!} = \frac{x}{e^x - 1}. \tag{7}$$

The proof of (6) uses Worpitzky's identity [7, pg. 215 (36)] given by

$$B_n = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{j^n}{k+1} \tag{8}$$

and its equivalent form

$$B_n = \sum_{k=1}^{n} (-1)^k \frac{k!}{k+1} \left\{ {n \atop k} \right\}. \tag{9}$$

Boyadzhiev [5] established transformation formulas for the geometric polynomials. In his paper, given the *exponential polynomials* or *Bell polynomials* $\phi_n(x)$ defined by

$$\phi_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k, \tag{10}$$

Boyadzhiev [5] expressed the geometric polynomials $w_n(x)$ in terms of the exponential polynomials, as follows

$$w_n(x) = \int_0^\infty \phi_n(x\lambda) e^{-\lambda} d\lambda. \tag{11}$$

This was used to derive more properties for $w_n(x)$ including the exponential generating function [5, Eq. (3.13)]

$$\int_0^\infty e^{-\lambda(1-x(e^x-1))} d\lambda = \sum_{n=0}^{\infty} w_n(x) \frac{z^n}{n!}. \tag{12}$$

Additional important works are due to Kargın [8], Dil and Kurt [9], Boyadzhiev and Dil [10], Kargın and Çekim [11], Ramírez and Cesarano [12], among others.

In 2016, Mangontarum et al. [13] introduced the *noncentral Tanny-Dowling polynomials* $\widetilde{F}_{m,a}(n;x)$ defined by

$$\widetilde{F}_{m,a}(n;x) = \sum_{k=0}^{n} k! \widetilde{W}_{m,a}(n,k) x^k \tag{13}$$

and satisfying the exponential generating function

$$\sum_{n=k}^{\infty} \widetilde{F}_{m,a}(n;x) \frac{z^n}{n!} = \frac{me^{-az}}{m - x(e^{mz} - 1)}, \tag{14}$$

where the numbers $\widetilde{W}_{m,a}(n,k)$ are the *noncentral Whitney numbers of the second kind*, a generalization of $\left\{ {n \atop k} \right\}$.

The parameters (m,a) deform the classical structure:

$$\widetilde{F}_{1,0}(n;x) = w_n(x).$$

Further, in a recent paper by Mangontarum and Madid [14], a number of identities for $\widetilde{F}_{m,a}(n;x)$ are established. Such identities are shown to generalize some known results on the geometric polynomials, including the ones in the paper of Kargın [8].

In the present paper, the authors establish integral formulas for and involving the noncentral Tanny-Dowling polynomials. In particulay, we will derive a generalization of Kellner's [6] integral formula relating the noncentral Tanny-Dowling polynomials with the Bernoulli polynomials, obtain generalizations of the Worpitzky's [7] explicit formulas in terms of noncentral Whitney numbers of the second kind, and derive a generalizations of Boyadzhiev's [5] identities for the noncentral Tanny-Dowling polynomials.

2. Results and Discussions

The first theorem establishes a relationship between the noncentral Tanny-Dowling polynomials and the Bernoulli polynomials, and extends the result of Kellner [6] presented in (6).

Theorem 1. *For any real number a and positive integer m, the following integral formula over the interval $[-1, 0]$ holds:*

$$\int_{-1}^{0} \widetilde{F}_{m,a}(n; mx)dx = m^n B_n\left(\frac{-a}{m}\right). \tag{15}$$

Proof. Note that from (14), we have

$$\sum_{n=0}^{\infty} \int_{-1}^{0} \widetilde{F}_{m,a}(n; mx)\frac{z^n}{n!}dx = \int_{-1}^{0} \frac{e^{-az}}{1 - xe^{mz} + x}dx. \tag{16}$$

Evaluating the integral and by (7), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \int_{-1}^{0} \widetilde{F}_{m,a}(n; mx)\frac{z^n}{n!}dx &= \frac{-e^{-az}}{e^{mz} - 1} \ln|1 - xe^{mz} + x| \Big|_{-1}^{0} \\ &= \frac{-e^{-az}}{e^{mz} - 1}\left(-\ln|e^{mz}|\right) \\ &= \frac{mze^{-az}}{e^{mz} - 1} \\ &= \sum_{n=0}^{\infty} m^n B_n\left(\frac{-a}{m}\right)\frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ completes the proof.

Remark 1. Since $\widetilde{F}_{1,0}(n; x) = w_n(x)$, then by setting $m = 1$ and $a = 0$ in Theorem 1, we get the integral

$$\int_{-1}^{0} \widetilde{F}_{1,0}(n; x)dx = B_n(0)$$

which is Kellner's [6] identity in (6).

Now, observe that from (13),

$$\begin{aligned}m^n B_n\left(\frac{-a}{m}\right) &= \int_{-1}^{0} \left(\sum_{k=0}^{n} m^k k! \widetilde{W}_{m,a}(n, k)x^k\right)dx \\ &= \sum_{k=0}^{n} m^k k! \widetilde{W}_{m,a}(n, k)\int_{-1}^{0} x^k dx\end{aligned}$$

$$= \sum_{k=0}^{n} m^k k! \widetilde{W}_{m,a}(n,k) \frac{(-1)^k}{k+1}.$$

Thus, we have the following corollary:

Corollary 1. *The* n^{th} *Bernoulli polynomial* $B_n\left(\frac{-a}{m}\right)$ *satisfies the following explicit formula:*

$$B_n\left(\frac{-a}{m}\right) = \sum_{k=0}^{n} k! \widetilde{W}_{m,a}(n,k) \frac{(-1)^k}{m^{n-k}(k+1)}. \tag{17}$$

Remark 2. Since $\widetilde{W}_{1,0}(n,k) = \left\{ {n \atop k} \right\}$, then when $m = 1$ and $a = 0$ in Corollary 1, we recover the Bernoulli formula in (9). Moreover, using the explicit formula of $\widetilde{W}_{m,a}(n,k)$ [13, Eq. (38)] given by

$$\widetilde{W}_{m,a}(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (mj-a)^n,$$

equation (17) can be written as

$$B_n\left(\frac{-a}{m}\right) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} (mj-a)^n \frac{(-1)^j}{m^n(k+1)}. \tag{18}$$

This is a generalization of Worpitzky's [7] identity in (8).

Before proceeding, note that by induction on k, it is easy to show that

$$\int_0^\infty x^k e^{-x} dx = k!. \tag{19}$$

Also, the *noncentral Dowling polynomials* [13, Eq. (89)] defined by

$$\widetilde{D}_{m,a}(n;x) = \sum_{k=0}^{n} \widetilde{W}_{m,a}(n,k) x^k$$

satisfies the exponential generating function [13, Eq. (91)]

$$\sum_{n=0}^{\infty} \widetilde{D}_{m,a}(n;x) \frac{z^n}{n!} = e^{-az+(emz-a)(x/m)}. \tag{20}$$

The next theorem provides an integral representation of the noncentral Tanny-Dowling polynomials in terms of noncentral Dowling polynomials. This extends Boyadzhiev's [5] identity for geometric polynomials in (11).

Theorem 2. *The noncentral Tanny-Dowling polynomials satisfy the following relation:*

$$\widetilde{F}_{m,a}(n;x) = \int_0^\infty \widetilde{D}_{m,a}(n;x\lambda)e^{-\lambda}\ d\lambda. \tag{21}$$

Proof. By definition,

$$\begin{aligned}\int_0^\infty \widetilde{D}_{m,a}(n;x\lambda)e^{-\lambda}d\lambda &= \int_0^\infty \left[\sum_{k=0}^{n} \widetilde{W}_{m,a}(n,k)x^k\lambda^k\right] e^{-\lambda}d\lambda \\ &= \sum_{k=0}^{n} \widetilde{W}_{m,a}(n,k)x^k \int_0^\infty \lambda^k e^{-\lambda}d\lambda.\end{aligned}$$

Using (19) and then (13) yield

$$\begin{aligned}\int_0^\infty \widetilde{D}_{m,a}(n;x\lambda)e^{-\lambda}d\lambda &= \sum_{k=0}^{n} k!\widetilde{W}_{m,a}(n,k)x^k \\ &= \widetilde{F}_{m,a}(n;x)\end{aligned}$$

which is the desired result.

Remark 3. Since $\widetilde{D}_{1,0}(n;x\lambda) = \phi_n(x\lambda)$, then when $m = 1$ and $a = 0$, the following relation

$$\int_0^\infty \widetilde{D}_{1,0}(n;x\lambda)e^{-\lambda}\ d\lambda = \widetilde{F}_{1,0}(n;x) \tag{22}$$

is precisely Boyadzhiev's [5] formula in (11).

Finally, the next theorem presents another form of exponential generating function for the polynomials $\widetilde{F}_{m,a}(n;x)$.

Theorem 3. *The exponential generating function of the noncentral Tanny-Dowling polynomial satisfies the following integral formula:*

$$\sum_{n=0}^{\infty} \widetilde{F}_{m,a}(n;x)\frac{z^n}{n!} = \int_0^\infty exp\left[-az - \lambda\left(1 - \frac{x}{m}\left(e^{mz}-1\right)\right)\right] d\lambda. \tag{23}$$

Proof. Multiplying both sides of (21) by $\frac{z^n}{n!}$ and summing over n gives

$$\sum_{n=0}^{\infty} \widetilde{F}_{m,a}(n;x)\frac{z^n}{n!} = \int_0^\infty \left(e^{-\lambda}\sum_{n=0}^{\infty} \widetilde{D}_{m,a}(n;x\lambda)\frac{z^n}{n!}\right) d\lambda.$$

By apply (20) in the right-hand side,

$$\sum_{n=0}^{\infty} \widetilde{F}_{m,a}(n;x)\frac{z^n}{n!} = \int_0^\infty e^{-az-\lambda(1-\frac{x}{m}(e^{mz}-1))}d\lambda.$$

Remark 4. When $m = 1$ and $a = 0$, we obtain

$$\int_0^\infty e^{-(0)z-\lambda(1-\frac{x}{1}(e^{(1)z}-1)}d\lambda = \sum_{n=0}^{\infty} \widetilde{F}_{1,0}(n;x)\frac{z^n}{n!}, \tag{24}$$

an equivalent representation of Boyadzhiev's [5] exponential generating function in (12).

3. Conclusion

The results of this study demonstrate the relationship between the noncentral Tanny-Dowling polynomials, a natural generalization of the Bell polynomials, and the Bernoulli polynomials. It is interesting to explore similar connections between the noncentral Tanny-Dowling polynomials and other families of special polynomials discussed in [12], such as the *Apostol-Bernoulli*, *Apostol-Euler*, and *Apostol-Genocchi polynomials*. The work of Mangontarum [15] on the *r-Dowling polynomials* may offer valuable insights for establishing these extensions.

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