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Geometric Characterizations of Imaginary Error Functions in Subclasses of Spirallike Analytic Functions

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Abstract. In this paper, we investigate the geometric behavior of the generalized normalized imaginary error function $\Upsilon i_k\,(z)$ and the associated convolution operator $\mathcal{I}i_k(z)$ within the framework of analytic function theory. Specifically, we establish necessary and sufficient conditions under which these functions belong to the subclasses $\mathcal{SPE}(\vartheta,\zeta)$ and $\mathcal{CSPE}(\vartheta,\zeta)$ of spirallike and convex spirallike analytic functions, respectively. Additionally, we derive sharp criteria for an integral operator involving $\Upsilon i_k\,(z)$ to be a member of these subclasses. These results extend and generalize several known findings and may inspire further applications of the imaginary error function in geometric function theory.

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1. Introduction and Preliminaries

Let $\mathcal{E}$ symbolize for the class of analytic functions of the form:

$$q(z) = z + \sum_{\epsilon=2}^{\infty} \beta_\epsilon z^\epsilon \ , \quad z \in \Gamma = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

Further, let $\mathcal{NE}$ be a subclass of $\mathcal{E}$ consisting of functions of the form:

$$q(z) = z - \sum_{\epsilon=2}^{\infty} \beta_\epsilon z^\epsilon \ , \quad \beta_\epsilon \geq 0, \ z \in \Gamma. \tag{2}$$

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A function $q \in \mathcal{E}$ is spirallike if

$$\Re\left(e^{-i\vartheta}\frac{zq'(z)}{q(z)}\right) > 0,\ |\vartheta| < \pi/2,\ z \in \Gamma.$$

Also, $q(z)$ is convex spirallike if $zq'(z)$ is spirallike.

Selvaraj and Geetha [1] introduced the subclasses of uniformly spirallike functions $\mathcal{SP}(\vartheta,\zeta)$ and $\mathcal{CSP}(\vartheta,\zeta)$, as given in the following definition.

Definition 1. *A function q of the form (1) is said to be in the subclass $\mathcal{SP}(\vartheta,\zeta)$, if it satisfies the following condition:*

$$\Re\left\{e^{-i\vartheta}\left(\frac{zq'(z)}{q(z)}\right)\right\} \geq \left|\frac{zq'(z)}{q'(z)} - 1\right| + \zeta \qquad (z \in \Gamma;\ \ |\vartheta| < \pi/2\ ; 0 \leq \zeta < 1)$$

and $q \in \mathcal{CSP}(\vartheta,\zeta)$ iff $zq'(z) \in \mathcal{SP}(\vartheta,\zeta)$, which is equivalent the following condition:

$$\Re\left\{e^{-i\vartheta}\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} \geq \left|\frac{zq''(z)}{q'(z)}\right| + \zeta \qquad (z \in \Gamma;\ \ |\vartheta| < \pi/2\ ; 0 \leq \zeta < 1).$$

We write

$$\mathcal{SPE}(\vartheta,\zeta) = \mathcal{SP}(\vartheta,\zeta) \cap \mathcal{NE}$$

and

$$\mathcal{CSPE}(\vartheta,\zeta) = \mathcal{CSP}(\vartheta,\zeta) \cap \mathcal{NE}.$$

We note that, for $\zeta = 0$, the subclasses of uniformly spirallike $\mathcal{SP}(\vartheta,0) = \mathcal{SP}(\vartheta)$ and uniformly convex spirallike $\mathcal{CSP}(\vartheta,0) = \mathcal{CSP}(\vartheta)$ introduced by Ravichandran et al. [2]. For $\vartheta = 0$, the subclasses $\mathcal{SP}(\vartheta) = \mathcal{SP}$ and $\mathcal{CSP}(\vartheta) = \mathcal{CSP}$ introduced and studied by Rønning [3]. For more intriguing discoveries of some related subclasses of consistently uniformly spirallike and uniformly convex spirallike, see the works of Al-Hawary et al. [4, 5], Bharati et al. [6], Frasin et al. [7], Goodman [8], Kanas and Wisniowska [9].

Definition 2. [10] *A function $h \in \mathcal{E}$ is said to be in the class $G^{\tau}(C_1, C_2)$, $\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq C_2 < C_1 \leq 1$, if it satisfies the condition*

$$\left|\frac{q'(z) - 1}{(C_1 - C_2)\tau - C_2[q'(z) - 1]}\right| < 1, \qquad z \in \Gamma.$$

If we put $\tau = 1$, $C_1 = \varrho$ and $C_2 = -\varrho (0 < \varrho \leq 1)$, we get the class of functions $q \in \mathcal{E}$ satisfying the condition

$$\left|\frac{q'(z) - 1}{q'(z) + 1}\right| < \varrho, \qquad (z \in \Gamma,\ 0 < \varrho \leq 1)$$

which was studied by (among others) Caplinger and Causey [11].

It is commonly known that special functions are crucial to the theory of geometric functions, and that their use is not restricted to the theory of geometric functions; they are used in a wide variety of problems and in other areas of mathematics and the applied sciences, see [12–23].

The error function erq defined by Abramowitz and Stegun [24] as:

$$erq\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\epsilon=0}^{\infty} \frac{(-1)^{\epsilon}\, z^{2\epsilon+1}}{(2\epsilon+1)\,\epsilon!}, \qquad (z \in \mathbb{C}), \tag{3}$$

whereas the imaginary error function

$$erqi\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{\epsilon=0}^{\infty} \frac{z^{2\epsilon+1}}{(2\epsilon+1)\,\epsilon!}, \qquad (z \in \mathbb{C}). \tag{4}$$

The error function is widely used in statistics, probability theory, applied mathematics, and the physics of partial differential equations. In quantum physics, the error function is an essential tool for calculating the probability of observing a particle in a specific location. While Alzer [25] and Coman [26] demonstrated numerous features and inequalities of the error function, Elbert et al. [27] examined the characteristics of the complementary error function. Figure 1 below illustrates the behavior of the real and imaginary parts of $erq\,(z)$ in the complex plane. It reveals rich geometric structure, including symmetry and curvature, which motivates its role in the geometric characterization of subclasses of analytic functions.

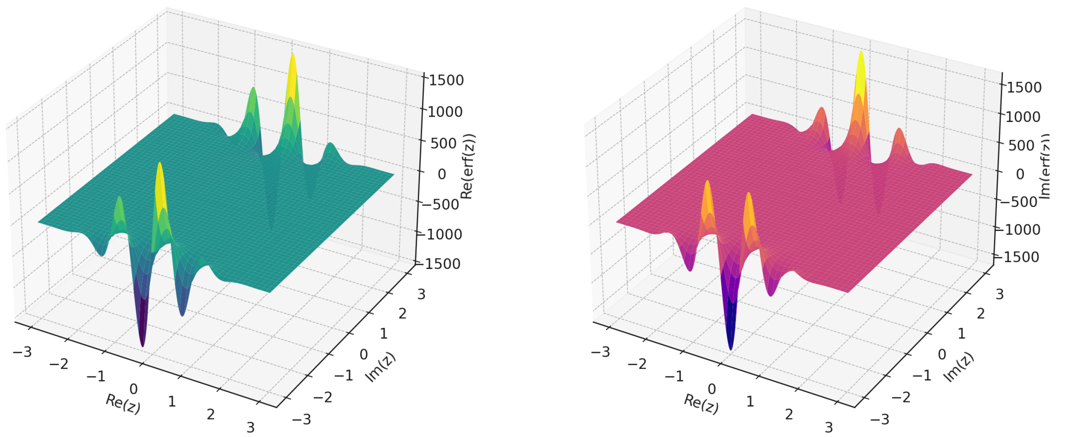

Figure 1: Real (left) and imaginary (right) parts of the error function $erq\,(z)$ over the complex plane.

A generalization of the error function given by (3) is defined as:

$$\begin{aligned} erq_k\,(z) &= \frac{k!}{\sqrt{\pi}} \int_0^z e^{-t^k} dt, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\ &= \frac{k!}{\sqrt{\pi}} \sum_{\epsilon=0}^{\infty} \frac{(-1)^{\epsilon}\, z^{k\epsilon+1}}{(k\epsilon+1)\,\epsilon!}, \qquad (z \in \mathbb{C}). \end{aligned} \tag{5}$$

And a generalization of the imaginary error function given by (4) is defined by

$$\begin{aligned} erqi_k(z) &= \frac{k!}{\sqrt{\pi}} \int_0^z e^{t^k} dt, \quad k \in \mathbb{N}_0 \\ &= \frac{k!}{\sqrt{\pi}} \sum_{\epsilon=0}^{\infty} \frac{z^{k\epsilon+1}}{(k\epsilon+1)\,\epsilon!}, \qquad (z \in \mathbb{C}). \end{aligned} \tag{6}$$

From (5) and (6), we get

$$erq_0(z) = \frac{z}{e\sqrt{\pi}},\; erq_1(z) = \frac{1-e^z}{\sqrt{\pi}} = -erqi_1(z),\; erq_2(z) = erq(z) \text{ and } erqi_2(z) = erqi(z).$$

The functions $erq_k(z)$ and $erqi_k(z)$ are not in the class $\mathcal{E}$. Therefore, we will consider the following functions given by Al-Hawary et al. [28].

$$\varepsilon_k(z) = \frac{\sqrt{\pi}}{k!} z_k^{\left(1-\frac{1}{k}\right)} erq_k\left(z^{1/k}\right) = z + \sum_{\epsilon=2}^{\infty} \frac{(-1)^{\epsilon-1}}{((\epsilon-1)k+1)(\epsilon-1)!} z^{\epsilon}, \qquad (k \in \mathbb{N}), \tag{7}$$

and

$$\varepsilon i_k(z) = \frac{\sqrt{\pi}}{k!} z^{\left(1-\frac{1}{k}\right)} erqi_k\left(z^{1/k}\right) = z + \sum_{\epsilon=2}^{\infty} \frac{1}{((\epsilon-1)k+1)(\epsilon-1)!} z^{\epsilon}, \quad (k \in \mathbb{N}). \tag{8}$$

From (7) and (8), we get

$$\varepsilon_1(z) = \sqrt{\pi} erq_1(z) = 1 - e^z,\; \varepsilon i_1(z) = \sqrt{\pi} erqi_1(z) = e^z - 1$$

and

$$\varepsilon_2(z) = \frac{\sqrt{\pi z}}{2} erq_2\left(\sqrt{z}\right) \text{ and } \varepsilon i_1(z) = \frac{\sqrt{\pi z}}{2} erqi_2\left(\sqrt{z}\right).$$

Let the function $\Upsilon i_k(z)$ be defined as:

$$\Upsilon i_k(z) = 2z - \varepsilon i_k(z) = z - \sum_{\epsilon=2}^{\infty} \frac{1}{((\epsilon-1)k+1)(\epsilon-1)!} z^{\epsilon}, \qquad z \in \Gamma, \tag{9}$$

and the linear operator

$$\mathcal{I}i_k : \mathcal{E} \to \mathcal{E}$$

defined as:

$$\mathcal{I}i_k(z) = \varepsilon i_k(z) * q(z) = z + \sum_{\epsilon=2}^{\infty} \frac{1}{((\epsilon-1)k+1)(\epsilon-1)!} \beta_\epsilon z^{\epsilon}. \tag{10}$$

Inspired by the works of several researchers who have employed a variety of special functions to identify certain conditions to belong to subclasses of analytic functions (see, [29–36]), we will determine some conditions for the error functions $\Upsilon i_k(z)$ and $\mathcal{I}i_k(z)$, and an integral operator to belong to the subclasses $\mathcal{SPE}(\vartheta, \zeta)$ and $\mathcal{CSPE}(\vartheta, \zeta)$.

The lemmas listed below will be useful in deriving our main findings.

Lemma 1. *(see* [1]*) (i) A sufficient condition for a function q of the form (1) to be in the subclass* $\mathcal{SP}(\vartheta, \zeta)$ *is*

$$\sum_{\epsilon=2}^{\infty}(2\epsilon - \zeta - \cos\vartheta)\,|\beta_\epsilon| \leq \cos\vartheta - \zeta \qquad (|\vartheta| < \pi/2\ ; 0 \leq \zeta < 1), \tag{11}$$

and a necessary and sufficient condition for a function q of the form (2) to be in the subclass $\mathcal{SPE}(\vartheta, \zeta)$ *is that the condition (11) is satisfied. In particular, when* $\zeta = 0$, *we obtain a sufficient condition for a function q of the form (1) to be in the subclass* $\mathcal{SP}(\vartheta)$ *is*

$$\sum_{\epsilon=2}^{\infty}(2\epsilon - \cos\vartheta)\,|\beta_\epsilon| \leq \cos\vartheta \qquad (|\vartheta| < \pi/2), \tag{12}$$

and a necessary and sufficient condition for a function q of the form (2) to be in the subclass $\mathcal{SPE}(\vartheta)$*is that the condition (12) is satisfied.*

(ii) A sufficient condition for a function q of the form (1) to be in the subclass $\mathcal{CSP}(\vartheta, \zeta)$*is*

$$\sum_{\epsilon=2}^{\infty}\epsilon(2\epsilon - \zeta - \cos\vartheta)\,|\beta_\epsilon| \leq \cos\vartheta - \zeta \qquad (|\vartheta| < \pi/2\ ; 0 \leq \zeta < 1) \tag{13}$$

and a necessary and sufficient condition for a function q of the form (2) to be in the subclass $\mathcal{CSPE}(\vartheta, \zeta)$ *is that the condition (13) is satisfied. In particular, when* $\zeta = 0$, *we obtain a sufficient condition for a function q of the form (1) to be in the subclass* $\mathcal{CSP}(\vartheta)$ *is that*

$$\sum_{\epsilon=2}^{\infty}\epsilon(2\epsilon - \cos\vartheta)\,|\beta_\epsilon| \leq \cos\vartheta \qquad (|\vartheta| < \pi/2) \tag{14}$$

and a necessary and sufficient condition for a function q of the form (2) to be in the subclass $\mathcal{CSPE}(\vartheta)$*is that the condition (14) is satisfied.*

Lemma 2. [10] *If q of the form (1) and* $q \in G^{\tau}(C_1, C_2)$, *then*

$$|\beta_\epsilon| \leq \frac{(C_1 - C_2)\,|\tau|}{\epsilon}, \qquad \epsilon \in \mathbb{N} - \{1\}. \tag{15}$$

The result is sharp for the function $q(z)$ given by

$$q(z) = \int_0^z \left(1 + \frac{(C_1 - C_2)\tau t^{\epsilon-1}}{1 + C_2 t^{\epsilon-1}}\right) dt \quad (\ z \in \Gamma,\ \epsilon \geq 2). \tag{16}$$

We need the following well-known series sums to prove our main findings.

$$\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon - 1)\,2^{\epsilon}} = \frac{1}{2}\ln 2 \tag{17}$$

and

$$\sum_{\epsilon=3}^{\infty} \frac{1}{2^{\epsilon}(\epsilon-1)} = \frac{1}{2}\ln 2 - \frac{1}{4}. \tag{18}$$

Note that

$$\sum_{\epsilon=d}^{\infty} \frac{1}{(\epsilon-1)\,2^{\epsilon}} = \frac{1}{2}\ln 2 - \sum_{\epsilon=2}^{d-1} \frac{1}{(\epsilon-1)\,2^{\epsilon}},\ d = 3, 4, \cdots . \tag{19}$$

The following inequalities are also required

$$(\epsilon-1)k + 1 > (\epsilon-1)k \quad (\epsilon, k \in \mathbb{N}) \tag{20}$$

and

$$\epsilon! \geq 2^{\epsilon-1} \quad (\epsilon \in \mathbb{N}). \tag{21}$$

2. Necessary and sufficient conditions for the function Υi_k

In this section, we find some necessary and sufficient conditions for the function Υi_k to be in the subclasses $\mathcal{SPE}(\vartheta,\zeta)$ and $\mathcal{CSPE}(\vartheta,\zeta)$.

Theorem 1. *If $k \in \mathbb{N}$, then $\Upsilon i_k(z)$ is in the subclass $\mathcal{SPE}(\vartheta,\zeta)$ if and only if*

$$2(6-\zeta-\cos\vartheta)\ln 2 \leq k(\cos\vartheta-\zeta). \tag{22}$$

Proof. Since

$$\Upsilon i_k(z) = z - \sum_{\epsilon=2}^{\infty} \frac{1}{((\epsilon-1)k+1)(\epsilon-1)!} z^{\epsilon}, \tag{23}$$

by virtue of (11) it suffices to show that $L_1(\vartheta,\zeta) \leq \cos\vartheta - \zeta$, where

$$L_1(\vartheta,\zeta) = \sum_{\epsilon=2}^{\infty} [2\epsilon - \zeta - \cos\vartheta] \; \frac{1}{((\epsilon-1)k+1)(\epsilon-1)!}.$$

Writing $\epsilon = (\epsilon-1)+1$, we get

$$\begin{aligned} L_1(\vartheta,\zeta) &= \sum_{\epsilon=2}^{\infty} \frac{2(\epsilon-1)}{((\epsilon-1)k+1)(\epsilon-1)!} + \sum_{\epsilon=2}^{\infty} \frac{2-\zeta-\cos\vartheta}{((\epsilon-1)k+1)(\epsilon-1)!} \\ &= \sum_{\epsilon=2}^{\infty} \frac{2}{((\epsilon-1)k+1)(\epsilon-2)!} + \sum_{\epsilon=2}^{\infty} \frac{2-\zeta-\cos\vartheta}{((\epsilon-1)k+1)(\epsilon-1)!}. \end{aligned}$$

By (20), we get

$$L_1(\vartheta,\zeta) \leq \frac{2}{k} \sum_{\epsilon=2}^{\infty} \frac{1}{(\epsilon-1)(\epsilon-2)!} + \frac{2-\zeta-\cos\vartheta}{k} \sum_{\epsilon=2}^{\infty} \frac{1}{(\epsilon-1)(\epsilon-1)!}.$$

By (21), we get

$$L_1(\vartheta,\zeta) \le \frac{16}{k}\sum_{\epsilon=3}^{\infty}\frac{1}{(\epsilon-1)\,2^{\epsilon}} + \frac{4\,(2-\zeta-\cos\vartheta)}{k}\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)\,2^{\epsilon}}.$$

Using the series sums (17), we get

$$L_1(\vartheta,\zeta) \le \frac{8}{k}\,(\ln 2) + \frac{2\,(2-\zeta-\cos\vartheta)}{k}(\ln 2) = \frac{2(6-\zeta-\cos\vartheta)}{k}\ln 2.$$

However, if and only if (22) holds, the last expression is bounded above by $\cos\vartheta-\zeta$.

Theorem 2. *If $k\in\mathbb{N}$, then $\Upsilon i_k\,(z)$ is in the subclass $\mathcal{CSPE}(\vartheta,\zeta)$ if and only if*

$$2(22-3\zeta-3\cos\vartheta)\ln 2 \le 8 + k(\cos\vartheta-\zeta). \tag{24}$$

Proof. Since $\Upsilon i_k\,(z)$ is given by (23) and by virtue (13), it suffices to show that $L_2(\lambda_1,\lambda_2)\le\cos\vartheta-\zeta$, where

$$\begin{aligned} L_2(\vartheta,\zeta) &= \sum_{\epsilon=2}^{\infty}\epsilon\,[2\epsilon-\zeta-\cos\vartheta]\,\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \\ &= \sum_{\epsilon=2}^{\infty}\left[2\epsilon^2-(\zeta+\cos\vartheta)\,\epsilon\right]\,\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!}. \end{aligned}$$

Writing

$$\epsilon = (\epsilon-1)+1, \tag{25}$$

and

$$\epsilon^2 = (\epsilon-1)(\epsilon-2)+3(\epsilon-1)+1, \tag{26}$$

we get

$$\begin{aligned} L_2(\vartheta,\zeta) = 2&\sum_{\epsilon=2}^{\infty}\frac{(\epsilon-1)(\epsilon-2)}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \\ &+ (6-\zeta-\cos\vartheta)\sum_{\epsilon=2}^{\infty}\frac{\epsilon-1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \\ &+ (2-\zeta-\cos\vartheta)\sum_{\epsilon=2}^{\infty}\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \end{aligned}$$

$$= 2\sum_{\epsilon=3}^{\infty}\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-3)!} + (6-\zeta-\cos\vartheta)\sum_{\epsilon=2}^{\infty}\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-2)!}$$
$$+\;(2-\zeta-\cos\vartheta)\sum_{\epsilon=2}^{\infty}\frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!}.$$

By (20), we get

$$L_2(\vartheta,\zeta) \le \frac{2}{k}\sum_{\epsilon=3}^{\infty}\frac{1}{(\epsilon-1)(\epsilon-3)!} + \frac{(6-\zeta-\cos\vartheta)}{k}\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)(\epsilon-2)!} + \frac{(2-\zeta-\cos\vartheta)}{k}\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)(\epsilon-1)!}.$$

By (21), we get

$$L_2(\vartheta,\zeta) \le \frac{32}{k}\sum_{\epsilon=3}^{\infty}\frac{1}{(\epsilon-1)2^{\epsilon}} + \frac{8(6-\zeta-\cos\vartheta)}{k}\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)2^{\epsilon}} + \frac{4(2-\zeta-\cos\vartheta)}{k}\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)2^{\epsilon}}.$$

Using the series sums (17) and (18), we get

$$\begin{aligned} L_2(\vartheta,\zeta) &\le \frac{1}{k}(16\ln 2-8) + \frac{6-\zeta-\cos\vartheta}{k}(4\ln 2) + \frac{2-\zeta-\cos\vartheta}{k}(2\ln 2) \\ &= \frac{2(22-3\zeta-3\cos\vartheta)}{k}\ln 2 - \frac{8}{k}. \end{aligned}$$

However, if and only if (24) holds, the last expression is bounded above by $\cos\vartheta-\zeta$.

3. Necessary and sufficient conditions for the convolution operator $\mathcal{I}i_k(z)$

In this section, we find sufficient conditions for the convolution operator $\mathcal{I}i_k(z)$ to be in the subclasses $\mathcal{SPE}(\vartheta,\zeta)$ and $\mathcal{CSPE}(\vartheta,\zeta)$.

Theorem 3. *Let $k \in \mathbb{N}$. If $q \in G^{\tau}(C_1,C_2)$, then $\mathcal{I}i_k(z)$ is in the subclass $\mathcal{SPE}(\vartheta,\zeta)$ if*

$$(C_1-C_2)|\tau|(4-\zeta-\cos\vartheta)\ln 2 \le k(\cos\vartheta-\zeta). \tag{27}$$

Proof. In view of (11), it suffices to show that

$$M_1(\vartheta,\zeta) = \sum_{\epsilon=2}^{\infty}[2\epsilon-\zeta-\cos\vartheta]\frac{1}{((\epsilon-1)k+1)(\epsilon-1)!}|\beta_\epsilon| \le \cos\vartheta-\zeta.$$

Since $q \in G^{\tau}(C_1,C_2)$, then by virtue (15), we have

$$M_1(\vartheta,\zeta) \le (C_1-C_2)|\tau|\left(\sum_{\epsilon=2}^{\infty}\frac{2}{((\epsilon-1)k+1)(\epsilon-1)!} - \sum_{\epsilon=2}^{\infty}\frac{\zeta+\cos\vartheta}{((\epsilon-1)k+1)\epsilon!}\right).$$

By (20) and (21), we get

$$M_1(\vartheta,\zeta) \le \frac{(C_1-C_2)|\tau|}{k}\left(8\sum_{\epsilon=2}^{\infty}\frac{1}{(\epsilon-1)2^{\epsilon}} - 2\sum_{\epsilon=2}^{\infty}\frac{\zeta+\cos\vartheta}{(\epsilon-1)2^{\epsilon}}\right).$$

By (17), we get

$$M_1(\vartheta,\zeta) \leq \frac{(C_1-C_2)|\tau|}{k}(4-\zeta-\cos\vartheta)\ln 2.$$

However, the last expression is bounded above by $\cos\vartheta-\zeta$ if (27) holds.

Theorem 4. *Let $k\in\mathbb{N}$. If $q\in G^{\tau}(C_1,C_2)$, then $\mathcal{I}i_k(z)$ is in the subclass $\mathcal{CSPE}(\vartheta,\zeta)$ if*

$$2(C_1-C_2)|\tau|\left(6-\zeta-\cos\vartheta\right)\ln 2 \leq k\left(\cos\vartheta-\zeta\right). \tag{28}$$

Proof. By view of (13), it suffices to show that

$$M_2(\vartheta,\zeta)=\sum_{\epsilon=2}^{\infty}\epsilon\left[2\epsilon-\zeta-\cos\vartheta\right]\frac{1}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}\left|\beta_\epsilon\right| \leq \cos\vartheta-\zeta.$$

Since $q\in G^{\tau}(C_1,C_2)$, then by virtue (15), we have

$$M_2(\vartheta,\zeta) \leq (C_1-C_2)|\tau|\left(\sum_{\epsilon=2}^{\infty}\frac{2\epsilon}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}-\sum_{\epsilon=2}^{\infty}\frac{\zeta+\cos\vartheta}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}\right).$$

By (25), we get

$$\begin{aligned}M_2(\vartheta,\zeta) &\leq (C_1-C_2)|\tau|\left(\sum_{\epsilon=2}^{\infty}\frac{2(\epsilon-1)}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}+\sum_{\epsilon=2}^{\infty}\frac{2-\zeta-\cos\vartheta}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}\right)\\ &=(C_1-C_2)|\tau|\left(\sum_{\epsilon=2}^{\infty}\frac{2}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-2\right)!}+\sum_{\epsilon=2}^{\infty}\frac{2-\zeta-\cos\vartheta}{\left(\left(\epsilon-1\right)k+1\right)\left(\epsilon-1\right)!}\right).\end{aligned}$$

By (20) and (21), we get

$$M_2(\vartheta,\zeta) \leq \frac{(C_1-C_2)|\tau|}{k}\left(16\sum_{\epsilon=2}^{\infty}\frac{1}{\left(\epsilon-1\right)2^{\epsilon}}+4\sum_{\epsilon=2}^{\infty}\frac{2-\zeta-\cos\vartheta}{\left(\epsilon-1\right)2^{\epsilon}}\right).$$

By (17), we get

$$M_2(\vartheta,\zeta) \leq \frac{2(C_1-C_2)|\tau|}{k}\left(6-\zeta-\cos\vartheta\right)\ln 2.$$

However, the last expression is bounded above by $\cos\vartheta-\zeta$ if (28) holds.

4. Necessary and sufficient conditions for the integral operator $Li_k(z)$

In this section, we find necessary and sufficient conditions of the integral operator

$$Li_k(z) := \int_0^z \frac{\Upsilon i_k(t)}{t} dt, \ z \in \Gamma, \tag{29}$$

to be in the subclasses $\mathcal{SPE}(\vartheta, \zeta)$ and $\mathcal{CSPE}(\vartheta, \zeta)$.

Theorem 5. *Let $k \in \mathbb{N}$. The integral operator $Li_k(z)$ is in the subclass $\mathcal{SPE}(\vartheta, \zeta)$ if and only if the inequality*

$$(4 - \zeta - \cos\vartheta) \ln 2 \leq k(\cos\vartheta - \zeta) \tag{30}$$

holds.

Proof. According to (9) it follows that

$$Li_k(z) = z - \sum_{\epsilon=2}^{\infty} \frac{1}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \frac{z^\epsilon}{\epsilon}, \ z \in \Gamma. \tag{31}$$

From (11), the integral operator $Li_k(z)$ belongs to $\mathcal{SPE}(\vartheta, \zeta)$ if and only if

$$\begin{aligned} &\sum_{\epsilon=2}^{\infty} [2\epsilon - \zeta - \cos\vartheta] \ \frac{1}{\epsilon((\epsilon-1)\,k+1)\,(\epsilon-1)!} \\ = \ &\sum_{\epsilon=2}^{\infty} \frac{2}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} - \sum_{\epsilon=2}^{\infty} \frac{\zeta + \cos\vartheta}{((\epsilon-1)\,k+1)\epsilon!} \leq \cos\vartheta - \zeta. \end{aligned}$$

By a similar proof of Theorem 3, we get that $Li_k(z) \in \mathcal{SPE}(\vartheta, \zeta)$ if and only if (30) holds.

Theorem 6. *Let $k \in \mathbb{N}$. The integral operator $Li_k(z)$ is in the subclass $\mathcal{CSPE}(\vartheta, \zeta)$ if and only if the inequality (22) holds.*

Proof. Since $Li_k(z)$ is given by (31) and in view (13), the integral operator $Li_k(z)$ belongs to $\mathcal{CSPE}(\vartheta, \zeta)$ if and only if

$$\begin{aligned} &\sum_{\epsilon=2}^{\infty} \epsilon\,[2\epsilon - \zeta - \cos\vartheta] \ \frac{1}{\epsilon((\epsilon-1)\,k+1)\,(\epsilon-1)!} \\ &= \sum_{\epsilon=2}^{\infty} \frac{2\epsilon}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} - \sum_{\epsilon=2}^{\infty} \frac{\zeta + \cos\vartheta}{((\epsilon-1)\,k+1)\,(\epsilon-1)!} \leq \cos\vartheta - \zeta. \end{aligned}$$

By a similar proof of Theorem 4, we get that $Li_k(z) \in \mathcal{CSPE}(\vartheta, \zeta)$ if and only if (22) holds.

Remark 1. *Particularization of the parameters ϑ and ζ in our theorems, we get several subresults related to the subclasses $\mathcal{SPE}(\vartheta, \zeta)$ and $\mathcal{CSPE}(\vartheta, \zeta)$. For example, if $\zeta = 0$ or $\vartheta = 0$, we get many subresults for the subclasses $\mathcal{SPE}(\vartheta)$, $\mathcal{CSPE}(\vartheta)$, $\mathcal{SPE}(\zeta)$ and $\mathcal{CSPE}(\vartheta)$.*

5. Conclusion

In this paper, we have established several geometric criteria for the generalized normalized imaginary error function Υi_k, its associated convolution operator $\mathcal{I}i_k(z)$, and an integral operator involving Υi_k to belong to the subclasses $\mathcal{SPE}(\vartheta, \zeta)$ and $\mathcal{CSPE}(\vartheta, \zeta)$ of spirallike and convex spirallike analytic functions defined in the open unit disk Γ.

By employing a combination of analytic techniques and coefficient-based inequalities, we derived sharp necessary and sufficient conditions in terms of the parameters k, ϑ, and ζ, thereby extending known results in the theory of geometric function classes. Our findings highlight the analytical richness of the imaginary error function and its potential for characterizing function spaces through generalized transformations. This study may encourage researchers to include the generalized normalized imaginary error function in other classes of analytic functions defined on Γ and creating new necessary and sufficient conditions.

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