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A Differential Game of Collision Coordination Between Two Robots

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Abstract. This paper explores a differential game involving two robots whose movements are described by linear differential equations with integral energy constraints, where the first robot possesses twice the energy of the resource of the second. We propose a novel multi-stage control strategy enabling the robots to execute position swaps while ensuring collision avoidance across three distinct phases, maintaining safe separation throughout. By employing time-specific control functions, we achieve precise coordination, culminating in a planned convergence at a shared location at predetermined terminal time. The admissibility of control strategies under the given constraints is rigorously verified and also, the timing sequence to achieve collision avoidance until critical endpoint is mathematically demonstrated. This work advances differential game theory by introducing a structured, multi-stage approach to balancing collision-free navigation and intentional terminal convergence.

Key Words and Phrases: Robot, collision, integral constraint, strategy

1. Introduction

Autonomous mobile robots operating in shared environments face significant challenges, among which collision avoidance and optimal path planning stand out as critical

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requirements. In multi-robot systems (MRS), each robot not only seeks its own target but simultaneously acts as a dynamic obstacle to other robots in the field. This mutual interaction, compounded by non-holonomic constraints and energy limitations, necessitates sophisticated strategies that ensure both safety and efficiency.

Differential game theory offers a powerful mathematical framework to address such problems. Early contributions by Isaacs [1] laid the foundations for differential games in conflict scenarios, leading to a wide array of applications in pursuit-evasion and cooperative control. Gu [2] applied a differential game approach to formation control, formulating it as a linear-quadratic Nash differential game and leveraging graph-theoretic concepts to coordinate multiple mobile robots. Setter and Egerstedt in [3] instantiated a problem in the context of energy-constrained rendezvous, where multiple mobile robots must determine an optimal meeting location and time, aiming to minimize duration while accounting for their differing battery levels. Further extensions by Li et al. [4] employed differential game concepts to enhance physical human-robot interaction, enabling robots to dynamically infer and respond to human control strategies.

In dynamic environments, differential games have also been used for real-time obstacle avoidance. Deshpande and Walambe [5] introduced a differential game model incorporating a safety parameter (SP) to adapt the circumvention behavior of two mobile robots, allowing them to maintain customizable minimum distances and avoid sharp turning angles during avoidance maneuvers. This extension demonstrated improved avoidance of close encounters without sacrificing optimal path efficiency, validated through MATLAB simulations.

In parallel, guaranteed pursuit time problems have attracted attention, especially in scenarios where one or more pursuers must intercept or constrain evaders under integral control constraints. Umar and Aihong [6] studied a pursuit-evasion game where the players' motions were governed by first and second order differential equations, deriving conditions under which pursuit is guaranteed within a prescribed distance l, thus generalizing classic l-catch concepts.

In the work of Lin et al. [7], a state-of-art path-planning algorithms of multi-robot decision making was introduced to provide an analysis of multi-robot decision making considering real-time performance.

This paper builds upon these foundational works by investigating a differential game involving two autonomous robots whose motions are described by differential equations subject to integral constraints. Unlike traditional pursuit-evasion games or purely formation tasks, our focus is on designing explicit multi-stage avoidance strategies. We show that with carefully chosen control functions, the robots can avoid collision at several distinct times, respecting their energy budgets, before ultimately converging to the same position at a specified final time. By incorporating ideas similar to safety parameter adjustments, we ensure that avoidance is not only optimal with respect to control energy but also maintains practical separations during maneuvers.

The remainder of this paper is organized as follows: Section 2 formalizes the problem, introducing the dynamic models and integral constraints. Section 3 derives the main results, including explicit control strategies and proofs of admissibility. Section 4 provides a detailed numerical example with visual trajectories to illustrate the theoretical results. Finally, Section 5 summarizes the findings and discusses possible future extensions to more general multi-robot scenarios.

2. Statement of the problem

Consider the motion of two robots described by the following equations:

$$\begin{cases} R_1 : \ \dot{\lambda}(t) = (\theta - t)u(t), \quad \lambda(0) = \lambda_0, , \\ \\ R_2 : \dot{\eta}\ (t) = (\theta - t)v(t), \ \eta(0) = \eta_0, \end{cases} \tag{1}$$

where $\lambda, \lambda_0, u, \eta, \eta_0, v \in \mathbb{R}^n$, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$ is a control function of the first robot R_1, and $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$ is that of the second robot R_2.
The solutions to dynamic equations (1) are given by

$$\begin{cases} \lambda(\theta) = \lambda_0 + \displaystyle\int_0^{\theta-\epsilon} (\theta - s)u(s)ds \\ \\ \eta(\theta) = \eta_0 + \displaystyle\int_0^{\theta-\epsilon} (\theta - s)v(s)ds, \end{cases} \tag{2}$$

$\epsilon > 0 \in \mathbb{R}$

Definition 1. *An admissible control of the first robot R_1 is a measurable function $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))$; $t \geq 0$, such that*

$$\int_0^t |u(s)|^2 \, ds \leq \rho^2, \tag{3}$$

where ρ is positive number representing the energy resource of the first robot R_1.

Definition 2. *An admissible control of the second robot R_2 is a measurable function $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$; $t \geq 0$, such that*

$$\int_0^t |v(s)|^2 \, ds \leq \sigma^2, \tag{4}$$

where σ is positive number representing the energy resource of the second robot R_2.

3. Main Results

Lemma 1.

$$\left(\int_{\tau_{i-1}}^{\tau_i - \epsilon} \frac{1}{(\tau_i - t)^2} dt \right)^{\frac{1}{2}} < \sqrt{\tau_i} \tag{5}$$

where τ_i is any given time, $\epsilon > 0$ is a very small real number.

Proof. Direct integration yield to

$$\int_{\tau_{i-1}}^{\tau_i-\epsilon} \frac{1}{(\tau_i-t)^2}dt = \int_{\tau_{i-1}}^{\tau_i-\epsilon} (\tau_i-t)^{-2}dt$$

Let $u=(\tau_i-t)$ implies $du=-dt$. Then

$$\begin{aligned}\int_{\tau_{i-1}}^{\tau_i-\epsilon} \frac{1}{(\tau_i-t)^2}dt = \int_{\tau_{i-1}}^{\tau_i-\epsilon} -u^{-2}du =&(\tau_i-\tau_i+\epsilon)^{-1}-(\tau_i-\tau_{i-1})^{-1}\\ =&\frac{1}{\epsilon}-\frac{1}{(\tau_i-\tau_{i-1})}\\ =&\frac{\tau_i-\tau_{i-1}-\epsilon}{\epsilon(\tau_i-\tau_{i-1})}\\ <&\tau_i-\tau_{i-1}\\ <&\tau_i.\end{aligned}$$

Hence $\left(\int_{\tau_{i-1}}^{\tau_i}\frac{1}{(\tau_i-t)^2}dt\right)^{\frac{1}{2}} < \sqrt{\tau_i}$.

Theorem 1. *If the energy resource of the first robot ρ is twice that of the second robot σ that is $\rho=2\sigma$, and $\sigma>1$ then at time τ_1,τ_2,τ_3 avoidance of collision is possible where $\tau_1<\tau_2<\tau_3<\theta$ and collision occur at time θ. These times are defined by*

$$\tau_1 = \left(\frac{\rho}{2|\eta_0-\lambda_0|+1}\right)^2 \tag{6}$$

$$\tau_2 = \left(\frac{\sigma}{|\eta_0-\lambda_0|}\right)^2 \tag{7}$$

$$\tau_3 = \left(\frac{-2\sigma}{|\eta_0-\lambda_0|}\right)^2 \tag{8}$$

$$\theta = (-2\sigma)^2 \tag{9}$$

Proof. In order to escape collision of these two robots, we construct our strategies in such away that when the first robot moves, the second robot will immediately replace its position, as it is going to be shown below step by step.
The strategies to be use by each robot is given below.

$$u(t) = \begin{cases} \frac{2(\eta_0-\lambda_0)}{(\tau_1-t)\delta_1}, & \tau_0 \le t < \tau_1 \\ \frac{e}{(\tau_2-t)\delta_2}, & \tau_1 \le t < \tau_2 \\ \frac{2e}{(\tau_3-t)\delta_3}, & \tau_2 \le t < \tau_3 \\ v(t)+\frac{\eta_0-\lambda_0}{(\theta-t)\delta_4}, & \tau_3 \le t < \theta \end{cases}$$

$$v(t) = \begin{cases} \frac{\lambda_0-\eta_0}{(\tau_1-t)\delta_1}, & \tau_0 \leq t < \tau_1 \\ \frac{\eta_0-\lambda_0}{(\tau_2-t)\delta_2}, & \tau_1 \leq t < \tau_2 \\ \frac{\lambda_0-\eta_0+e}{(\tau_3-t)\delta_3}, & \tau_2 \leq t < \tau_3 \\ v(t), & \tau_3 \leq t < \theta \end{cases}$$

Where $u(t)$ and $v(t)$ are the control functions of the first robot R_1 and second robot R_2 respectively, $\delta_i = (\tau_i - \epsilon) - \tau_{i-1}$, $\tau_0 = 0$ and e is a unit vector.

Step 1: Let the initial position of the first robot (R_1) and second robot (R_2) be given by λ_0 and η_0 respectively.
At time $t \in [\tau_0, \tau_1)$, R_1 and R_2 uses the following strategies $u(t) = \frac{2(\eta_0-\lambda_0)}{(\tau_1-t)\delta_1}$ and $v(t) = \frac{(\eta_0-\lambda_0)}{(\tau_1-t)\delta_1}$ respectively. After the construction of strategies one may not just go on and apply it, because it must satisfy some constraint for its admissibility as shown below.

$$\begin{aligned} \left(\int_{\tau_0}^{\tau_1-\epsilon} |u(t)|^2 dt\right)^{\frac{1}{2}} &= \left(\int_{\tau_0}^{\tau_1-\epsilon} \left|\frac{2(\eta_0-\lambda_0)}{(\tau_1-t)\delta_1}\right|^2 dt\right)^{\frac{1}{2}} \\ &= \frac{2|\eta_0-\lambda_0|}{\delta_1} \left(\int_{\tau_0}^{\tau_1-\epsilon} \left|\frac{1}{(\tau_1-t)}\right|^2 dt\right)^{\frac{1}{2}} \end{aligned}$$

using **Lemma** 1

$$\begin{aligned} \left(\int_{\tau_0}^{\tau_1-\epsilon} |u(t)|^2 dt\right)^{\frac{1}{2}} &\leq \frac{2|\eta_0-\lambda_0|}{\delta_1} \times \sqrt{\tau_1} \\ &< 2|\eta_0-\lambda_0| \times \sqrt{\tau_1} \\ &\leq 2|\eta_0-\lambda_0| \times \sqrt{\left(\frac{\rho}{2|\eta_0-\lambda_0|+1}\right)^2} \\ &< \rho. \end{aligned}$$

Therefore $\left(\int_{\tau_0}^{\tau_1-\epsilon} |u(t)|^2 dt\right)^{\frac{1}{2}} < \rho$. Hence the strategy of R_1 is admissible.

For R_2 it is easy to see that $\left(\int_{\tau_0}^{\tau_1-\epsilon} |v(t)|^2 dt\right)^{\frac{1}{2}} < \sigma$, using the same method used in showing R_1 admissibility. Hence the strategy of R_2 is admissible.

Now Using the solution in (2) If R_2 uses the above admissible strategy, then its position will be

$$\begin{aligned} \eta(t) &= \eta_0 + \int_0^t (t-s)v(s)ds \\ \eta(\tau_1) &= \eta_0 + \int_0^{\tau_1-\epsilon} (\tau_1 - s)\frac{(\lambda_0-\eta_0)}{(\tau_1-s)\delta_1} ds \end{aligned}$$

$$=\lambda_0$$

which is the initial position of R_1. This shows that R_2 will be at same position left by R_1 to avoid been in contact during their work. After this, R_1 will not stand steady at one position because they are applying their strategy simultaneously, so if R_1 make its first move using the above strategy, it will now be at the following position

$$\begin{aligned}\lambda(t) =&\lambda_0 + \int_0^t (t-s)u(s)ds\\ \lambda(\tau_1) =&\lambda_0 + \int_{\tau_0}^{\tau_1-\epsilon} (\tau_1 - s)\frac{2(\eta_0-\lambda_0)}{(\tau_1-s)\delta_1}ds\\ =&2\eta_0 - \lambda_0.\end{aligned}$$

It is obvious that at time $t \in [\tau_0, \tau_1)$ R_2 moved to the initial position of R_1 that is λ_0, and R_1 moved to another point $2\eta_0 - \lambda_0$. This indicates that at time $t \in [\tau_0, \tau_1)$ the two robots avoids colliding with one another.

Step 2:
At time $t \in [\tau_1, \tau_2)$, R_1 and R_2 uses the following strategies $u(t) = \frac{e}{(\tau_2-t)\delta_2}$ and $v(t) = \frac{\eta_0-\lambda_0}{(\tau_2-t)\delta_2}$ respectively. One can verify that these strategies are admissibility with $\tau_2 = \left(\frac{\sigma}{|\eta_0-\lambda_0|}\right)^2$. That is

$$\left(\int_{\tau_1}^{\tau_2-\epsilon} |u(t)|^2 dt\right)^{\frac{1}{2}} < \rho \;\; and \;\; \left(\int_{\tau_1}^{\tau_2-\epsilon} |v(t)|^2 dt\right)^{\frac{1}{2}} < \sigma.$$

Now if these robots uses the above strategies, their positions will again change. Starting with R_2 we have.

$$\begin{aligned}\eta(t) =&\eta_0 + \int_0^t (t-s)v(s)ds\\ \eta(\tau_2) =&\eta_0 + \int_{\tau_1}^{\tau_2-\epsilon} (\tau_2 - s)\frac{\eta_0-\lambda_0}{(\tau_2-s)\,\delta_2}ds\\ =&2\eta_0 - \lambda_0\end{aligned}$$

which is the position of R_1 when it made first step. Now the second move by R_1 using the strategy will be as follows

$$\begin{aligned}\lambda(t) =&\lambda_0 + \int_0^t (\tau_2-s)u(s)ds\\ \lambda(\tau_2) =&\lambda_0 + \int_{\tau_1}^{\tau_2-\epsilon} (\tau_2 - s)\frac{e}{(\tau_2-s)\,\delta_2}ds\\ =&\lambda_0 + e.\end{aligned}$$

Therefore at time $t \in [\tau_1, \tau_2)$ R_2 moved to the previous position of the first robot R_1 that is $2\eta_0 - \lambda_0$, and R_1 moved to another point $\lambda_0 + e$. This clearly shows that at this time interval, the two robots avoids collision.

Step 3: At this step, the time t is in the set $[\tau_2, \tau_3)$, R_1 and R_2 will use the following strategies $u(t) = \frac{2e}{(\tau_3 - t)\delta_3}$ and $v(t) = \frac{\lambda_0 - \eta_0 + e}{(\tau_3 - t)\delta_3}$ respectively, $\tau_3 = \left(\frac{-2\sigma}{|\eta_0 - \lambda_0|}\right)^2$. Obviously these strategies are admissible, one may verify.
If R_2 uses the above strategy it position will become,

$$\begin{aligned}\eta(t) =& \eta_0 + \int_0^t (t-s)v(s)ds \\ \eta(\tau_3) =& \eta_0 + \int_{\tau_2}^{\tau_3 - \epsilon} (\tau_3 - s)\frac{\lambda_0 - \eta_0 + e}{(\tau_3 - s)\,\delta_3}ds \\ =& \lambda_0 + e\end{aligned}$$

which is the position of R_1 when it made second move. Now the third move by the first robot (R_1) using the strategy above, will be as follows

$$\begin{aligned}\lambda(t) =& \lambda_0 + \int_0^t (\tau_3 - s)u(s)ds \\ \lambda(\tau_3) =& \lambda_0 + \int_{\tau_2}^{\tau_3 - \epsilon} (\tau_3 - s)\frac{2e}{(\tau_3 - s)\,\delta_3}ds \\ =& \lambda_0 + 2e.\end{aligned}$$

Step 4

At the time $t \in [\tau_3, \theta)$ which is the final stage it is expected that both the two robots will stop on the same position. Here we only require one strategy of either robot that will satisfy the condition for what ever strategy of the other one.
Let us take R_1, with the strategy:

$$u(s) = v(s) + \frac{\eta_0 - \lambda_0}{(\theta - s)\,\delta_4}$$

It is easy to see that $\left(\int_{\tau_3}^{\theta-\epsilon} |u(s)|^2 ds\right)^{\frac{1}{2}} < \rho$. Therefore if the first robot use this admissible strategy, then we have

$$\begin{aligned}\lambda(t) =& \lambda_0 + \int_{\tau_3}^t (\theta - s)u(s)ds \\ \lambda(\theta) =& \lambda_0 + \int_{\tau_3}^{\theta-\epsilon} (\theta - s)\left(v(s) + \frac{\eta_0 - \lambda_0}{(\theta - s)\,\delta_4}\right)ds \\ =& \lambda_0 + \int_{\tau_3}^{\theta-\epsilon} (\theta - s)v(s)ds + \int_{\tau_3}^{\theta-\epsilon} (\theta - s)\left(\frac{\eta_0 - \lambda_0}{(\theta - s)\,\delta_4}\right)ds\end{aligned}$$

$$
\begin{aligned}
&= \eta_0 + \int_0^{\theta-\epsilon} (\theta - s) v(s) ds \\
&= \eta(\theta),
\end{aligned}
$$

indicating that both robots are in the same position as t approach time θ

Looking at these steps, one may notice that at time τ_1, τ_2, τ_3 avoidance of collision is possible where $\tau_1 < \tau_2 < \tau_3 < \theta$ and collision occur at time θ which proved the theorem.

4. Numerical Example

Consider a motion of two robots described by (1) in $\mathbb{R}^2$. Let the initial potions of the first and second robot be $\lambda_0 = (1, 0)$ and $\eta_0 = (0, 1)$ respectively, let $e = (0, 1)$, $\rho = 4$, $\sigma = 2$. Now we can compute for the times defined in **Theorem 1** as follows

$$
\begin{aligned}
\tau_1 &= \left(\frac{\rho}{2|\eta_0 - \lambda_0| + 1} \right)^2 = \left(\frac{4}{2|(-1,1)| + 1} \right)^2 = 0.7 \\
\tau_2 &= \left(\frac{\sigma}{|\eta_0 - \lambda_0|} \right)^2 = \left(\frac{2}{\sqrt{2}} \right)^2 = 2 \\
\tau_3 &= \left(\frac{-2\sigma}{|\eta_0 - \lambda_0|} \right) = 8 \\
\theta &= (-2\sigma)^2 = 16.
\end{aligned}
$$

Using the above defined strategies, at time τ_1 the position of R_1 and R_2 are

$$
\begin{aligned}
\lambda(\tau_1) &= \lambda_0 + \int_{\tau_0}^{\tau_1-\epsilon} (\tau_1 - s) \frac{2(\eta_0 - \lambda_0)}{(\tau_1 - s)\delta_1} ds \\
\lambda(0.7) &= (1, 0) + \int_0^{0.7-\epsilon} (0.7 - s) \frac{2\left((0,1) - (1,0)\right)}{(0.7 - s)(0.7 - \epsilon)} ds \\
&= (-1, 2)
\end{aligned}
$$

and

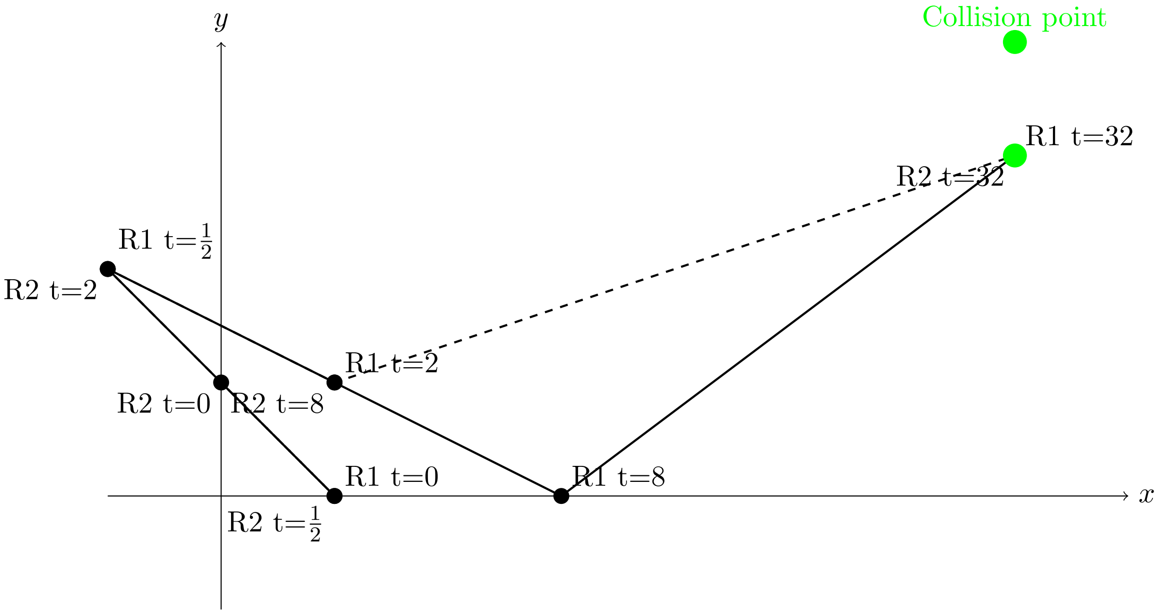
$$
\begin{aligned}
\eta(t) &= \eta_0 + \int_0^t (t - s) v(s) ds \\
\eta(0.7) &= (0, 1) + \int_0^{0.7-\epsilon} (0.7 - s) \frac{(1, -1)}{(0.7 - s)(0.7 - \epsilon)} ds \\
&= (1, 0)
\end{aligned}
$$

respectively. Continuing in the same tract, we can generate the following table

From this table, we can illustrate the motion of these robots below

time	R_1 position	R_2 *position*
τ_0	$(1,0)$	$(0,1)$
τ_1	$(-1,2)$	$(1,0)$
τ_2	$(1,1)$	$(-1,2)$
τ_3	$(3,0)$	$(1,1)$
θ	$(7,3)$	$(7,3)$

Table 1: Positions of robots R_1 and R_2 at different times



5. Conclusion

This time, we studied a pursuit-avoidance differential game with two robots whose dynamics are described by second-order differential equations subjected to integral constraints. We developed explicit admissible control strategies whereby the robots alternately position themselves at distinct intermediate times while avoiding collision, overactive gracefully despite the first robot having double the energy of the second. Ultimately, collision is unavoidable at a specified terminal time due to designed trajectory convergence. These findings presents a structured multi-stage approach to balancing collision-free navigation and intentional terminal convergence under resource limitations. The proposed strategies have broader implications for multi-agent systems requiring guaranteed no-collision conditions within finite windows and energy constraints. For instance, in swarm robotics, this framework could optimize coordinated exploration or search-and-rescue missions where agents must avoid interference while converging on a target.

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