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Mathematical Modelling of Radicalization and Terrorism Dynamics

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Abstract. The Central Sahel faces a severe terrorism crisis fueled by violent extremism, displacement, and climatic shocks. This study develops a compartmental mathematical model to analyze the dynamics between susceptible populations, active terrorists, and internally displaced persons. Through stability analysis, bifurcation theory, and global sensitivity analysis, we demonstrate that the basic reproduction number,$\mathcal{R}_0$, is a critical threshold determining whether terrorism is eliminated or persists endemically. Our results show that military-only interventions are less than 20% effective, while integrated strategies combining prevention and deradicalization exceed 80% effectiveness. Time-dependent analysis reveals that optimal strategies must adapt from early prevention to long-term rehabilitation. These findings provide quantitative support for counter-terrorism frameworks prioritizing socioeconomic development over purely military solutions, offering a pathway to sustainable stability in the Sahel.

2020 Mathematics Subject Classifications: 34D20, 34D23, 37N25, 92D30

Key Words and Phrases: terrorism dynamics, mathematical modeling, sensitivity analysis, counter-terrorism strategies, Central Sahel

1. Introduction

Recent decades have witnessed a marked escalation in the sophistication and impact of terrorism. The emergence and entrenchment of international terrorist networks represent a critical development, enabling coordinated attacks with heightened destructive potential aimed at destabilizing governments and challenging fundamental democratic norms [1]. The transnational nature of contemporary terrorism is evident, yet empirical data

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reveals its concentrated impact in the Sahel. According to the Global Terrorism Index, Burkina Faso, Nigeria, Mali, and Niger rank 4th, 6th, 7th, and 8th worldwide, respectively, positioning them directly behind Afghanistan, Iraq, and Somalia [2].

Characterized as an Islamist insurgency, the Sahel conflict pits state forces of Mali, Niger, Mauritania, Burkina Faso, and Chad against Salafi-jihadist groups operating under the ideological and/or operational banners of Al-Qaeda or the Islamic State [3–8].

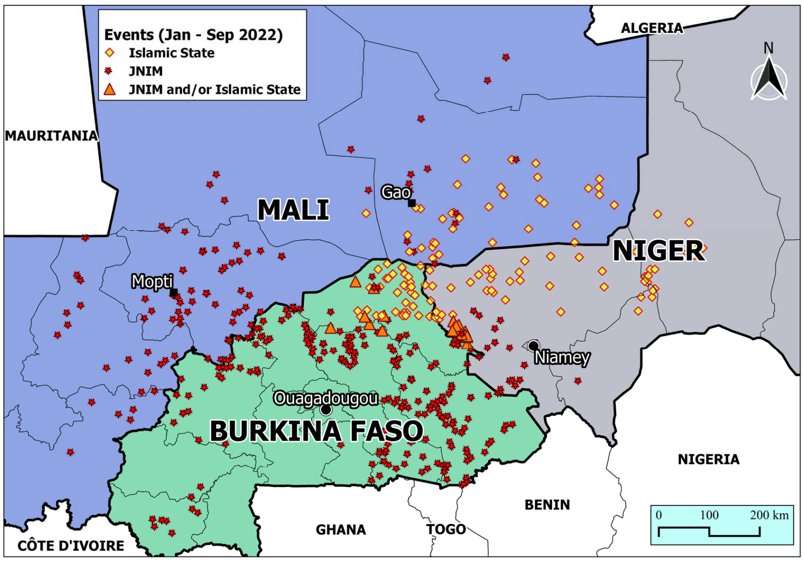
The Sahel insurgency constitutes an indirect regional consequence of the Algerian civil war. Algerian Islamist rebels strategically exploited the Sahelian desert as a rear base from the early 2000s onward [9]. Their operational profile gradually expanded beyond sanctuary provision to encompass guerrilla tactics, terrorist acts, and kidnappings. A pivotal development, however, was their deliberate embedding within local populations and propagation of radical Islamist doctrine. This process fostered local recruitment and led to the genesis of new, distinctly localized militant movements, including Ansar Dine, MUJAO, and Katiba Macina [10].

JNIM (Group for the Support of Islam and Muslims, GSIM) accounts for more than 64% of Sahelian militant Islamist violence since 2017, with activities documented from northern Mali to southeastern Burkina Faso. The Macina Liberation Front (MLF) emerges as JNIM's most active component, estimated to perpetrate 75% of its violence. Based in central Mali and extending into Burkina Faso, the MLF's prominence underscores a critical characteristic of JNIM factions: their lack of broad popular legitimacy. Consequently, these groups increasingly exploit local criminal networks and engage in attacks against civilians, a tactic notably employed by the MLF [11].

Empirical data reveals a dramatic near-sevenfold increase in violent events linked to Sahelian militant Islamist groups since 2017. With more than 1,000 incidents documented in the preceding year, the Sahel witnessed the sharpest rise in extremist violence across Africa. The resultant humanitarian and societal toll is severe: an estimated 8,000 deaths, millions displaced, pervasive attacks on governance structures and traditional authorities, the shuttering of thousands of educational institutions, and substantial economic decline [11].

The neutralization of AQIM leader Abdelmalek Droukdel by French forces on June 3, 2020 [12], highlighted the persistent insecurity plaguing the Sahel. Paradoxically, despite significant security sector investments [13], violence intensified markedly in 2019. The Central Sahel (Mali, Burkina Faso, Niger) recorded approximately 4,000 conflict-related fatalities representing a fivefold increase from the 770 deaths documented in 2016 [14]. Burkina Faso experienced the most acute deterioration, with militant Islamist attacks surging 174% between 2018 and 2019 [15], culminating in 1,889 fatalities during its deadliest year on record [15]. This erosion of state authority manifests in cascading regional crises: the closure of over 1,800 schools [16], mass population displacement [17], and the expansion of ungoverned spaces ("grey zones") where state control is absent or mediated through non-state armed actors [18].

The landscape of non-state armed groups in northern Mali has complexified substantially since 2012, growing quantitatively (from 4 to 15 groups) and qualitatively. Three interconnected drivers explain this evolution: First, fission processes among jihadist organizations (scissiparity). Second, the endogenous proliferation of community-based militias, arising from the national security apparatus's failure to protect civilians and necessitated by localized security dilemmas vis-à-vis rival militias. Third, the deployment of multilateral (MINUSMA) and unilateral (French Barkhane/Sabre, Chadian Serval contingent) external military forces.



Source: Armed Conflict Location and Event Data (ACLED)

Figure 1: Geospatial Analysis of Conflict Event Distribution in the Central Sahel [19]

Armed groups exploit state fragility in Mali, Burkina Faso, and Niger (Figure 1) by seizing artisanal gold mines since 2016. The 2012 identification of a Saharan gold corridor (Sudan-Mauritania) catalyzed this activity, transforming mines into dual-purpose assets: revenue sources for group financing and recruitment hubs. Illicit transport networks now proliferate to move extracted gold. This nexus between artisanal mining, non-state armed actors, and illicit economies significantly fuels regional violence and transnational crime [20].

The mathematical modeling of terrorism dynamics represents an intersection of applied mathematics, social science, and security studies. The compartmental approach to modeling terrorism spread draws inspiration from epidemiological models while incorporating unique features that reflect the social contagion nature of radicalization processes. This approach builds upon the foundational work of some works such as [21–24] in infectious disease modeling, extended to the context of ideological transmission as developed by Castillo-Chavez and Song [25].

While these existing models provide valuable insights, they often overlook critical real-world dynamics prevalent in contemporary conflicts like the Sahel crisis [26–30]. The novelty of the present model lies in its integrated approach that explicitly incorporates internally displaced persons (IDPs) as a distinct compartment, capturing a key feedback loop where violence begets displacement, which in turn can exacerbate vulnerabilities to radicalization. Furthermore, unlike models focusing solely on ideological transmission [e.g., Camacho et al., 2013 [27]] or optimal resource allocation [e.g., Udoh et al., 2019[31]], our framework simultaneously integrates terrorist-induced civilian mortality and a deradicalization rate, allowing for a more holistic analysis of both the violent and rehabilitative dimensions of counter-terrorism. This structure enables the analysis of a fundamental threshold dynamics via the basic reproduction number ($\mathcal{R}_0$), a concept less explored in this context, providing a clear quantitative target for policymakers. By synthesizing these elements, our model offers a more nuanced mathematical representation of the complex interdependencies driving terrorism and displacement in the Sahel, filling a gap in the current literature.

Despite sustained international counterterrorism efforts—including France's Operation Barkhane (2014–present) and preceding Operation Serval (2013–2014), the UN Multidimensional Integrated Stabilization Mission in Mali (MINUSMA), and recent regime transitions in Mali and Burkina Faso—violent instability persists across the Sahel. This endurance stems from multifaceted drivers: (1) entrenched terrorist networks, (2) systemic governance deficiencies including institutional corruption, and (3) chronic state incapacity to ensure territorial security. Crucially, predominantly military responses risk exacerbating communal tensions and intensifying violence cycles. This paper addresses this operational challenge by developing a quantitatively grounded counterterrorism framework integrating socio-political dimensions.

This research develops a mathematical framework to quantify terrorism mitigation strategies through nonlinear ordinary differential equations modeling population-level dynamics. We employ Sobol' sensitivity analysis to identify high-leverage parameters governing system behavior, enabling data-driven interventions for reducing terrorist violence. The paper is organized as follows: Section 2 formulates the model; Section 3 establishes fundamental properties; Section 4 identifies equilibrium points; Section 5 analyzes local and global stability of equilibrium points; Section 6 studies global Stability of Terrorism-Free and Endemic Equilibrium; Section 7 studies bifurcation phenomena; Section 8 conducts global sensitivity analysis via Sobol' methodology; Section 9 presents numerical simulations, validation and discusses the results; The last Section 10 synthesizes findings and research horizons.

2. Formulation of the Terrorism Model

The compartmental transition architecture for terrorism dynamics is formalized in Figure 2.

The total population $N(t)$ at time t is partitioned into three mutually exclusive epidemiological compartments: susceptible individuals $S(t)$, active terrorists $T(t)$, and displaced persons $Q(t)$. This structure follows established compartmental modeling frameworks for social contagion processes (e.g., [26]). The susceptible class $S(t)$ represents the

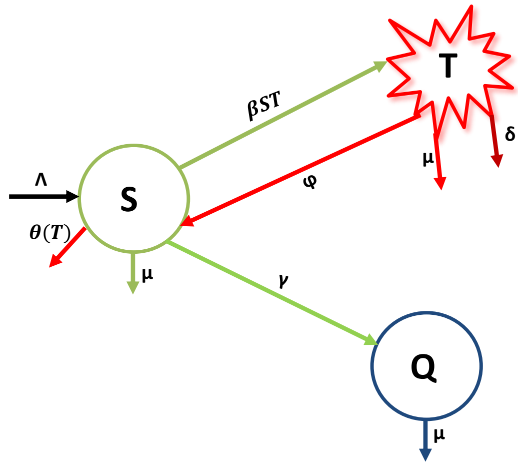


Figure 2: Schematic Representation of Terrorism Dynamics

non-core population vulnerable to radicalization, serving as the primary recruitment pool for extremist ideologies. Its magnitude typically dominates initial conditions, reflecting empirical demographic distributions in conflict zones.

Susceptible individuals $S(t)$ constitute the non-radicalized population at risk of ideological adoption. Radicalization occurs through contacts with terrorists at rate βST, where β denotes the intrinsic transmission coefficient and φT, with φ quantifies intervention efficacy through awareness programs. Those adopting extremist ideology transition to the terrorist compartment $T(t)$. This compartment comprises individuals who fully internalize violent extremism and execute attacks. Concurrently, security forces eliminate terrorists at rate δ, while natural mortality μ affects all compartments uniformly.

Forced displacement occurs as susceptible individuals flee violence at rate γ, entering the quitter compartment $Q(t)$ formally defined as Internally Displaced Persons (IDPs). Terrorist-induced mortality in $S(t)$ follows density-dependent kinetics $\theta(T) = kT$, where k is the lethality coefficient. This linear functional form captures escalating violence against civilians as terrorist density increases. The IDP compartment experiences no back-migration, with population loss occurring solely through natural mortality at rate μ.

Population influx occurs exclusively into the susceptible compartment through birth and migration, modeled as constant recruitment rate Λ. This parameterization assumes constant demographic pressure independent of conflict dynamics. The absence of disease-induced mortality in $Q(t)$ reflects humanitarian observations that displacement primarily causes relocation rather than direct physical destruction. The system conserves mass balance via $N(t) = S(t)+T(t)+Q(t)$, with total population dynamics governed by natural mortality and violence-driven attrition.

The terrorism dynamics model represents a compartmental approach to understanding the spread and control of terrorist activities within a population. The model structure captures the essential processes that govern the flow of individuals between different states of involvement with terrorism. The susceptible compartment $S(t)$ represents individuals who are vulnerable to radicalization but are not currently engaged in terrorist activities.

This includes the general population that may be exposed to terrorist ideology through various channels such as social networks, media, or direct contact with active terrorists.

The terrorist compartment $T(t)$ represents individuals who are actively engaged in terrorist activities. This includes not only those who carry out attacks but also those involved in planning, financing, recruitment, and other support activities. The model assumes that these individuals can influence susceptible individuals through a transmission process characterized by the rate β, reflecting the social contagion nature of radicalization.

The displaced compartment $Q(t)$ represents individuals who have been removed from the active conflict zone through displacement, migration, or other forms of population movement. This compartment captures the reality that in many conflict situations, significant portions of the population become displaced, either voluntarily or involuntarily, which affects their exposure to radicalization processes.

Table 1: Baseline parameter values for the terrorism dynamics model (Equation 1)

Parameter	**Description**	**Dimension**	**Value**
Λ	Recruitment rate	Individuals$\cdot$km$^{-2}\cdot$Year^{-1}	12
β	Terrorism exposure rate	Individuals$^{-1}\cdot$km$^{2}\cdot$Year^{-1}	0.000125
γ	Displacement rate ($S \to Q$)	Year^{-1}	0.05
φ	Deradicalization rate ($T \to S$)	Year^{-1}	0.001
k	Terrorist-induced mortality coefficient	Individuals$^{-1}\cdot$km$^{2}\cdot$Year^{-1}	0.0004
μ	Natural mortality rate	Year^{-1}	0.0012
δ	Terrorist elimination rate by military	Year^{-1}	0.50

Consider the terrorism dynamics model described by the following system of ordinary differential equations:

$$\left\{\begin{array}{rcl} \dfrac{\mathrm{d}S}{\mathrm{d}t} &=& \Lambda - (\mu+\gamma)S - \beta ST - \theta(T)S + \varphi T \\ \dfrac{\mathrm{d}T}{\mathrm{d}t} &=& \beta ST - \varphi T - (\delta+\mu)T \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} &=& \gamma S - \mu Q \end{array}\right. \tag{1}$$

Where the total population is $N(t) = S(t)+T(t)+Q(t)$, $S(0) \geq 0,\ T(0) \geq 0,\ Q(0) \geq 0$ and $\theta(T=0) = 0$.

We define the relevant set prior to conducting the mathematical analysis of our dynamical system.

Definition 1. *The state space for system* (1) *is defined as the non-negative orthant*

$$\Omega = \left\{(S,T,Q) \in \mathbb{R}^3_+ : S \geq 0, T \geq 0, Q \geq 0\right\},$$

equipped with the usual Euclidean topology. The parameter space is

$$\Theta = \left\{(\Lambda,\beta,\gamma,\varphi,\mu,\delta,k) \in \mathbb{R}^7_{++} : \textit{all parameters strictly positive}\right\}$$

ensuring biological meaningfulness of the model parameters according to the principles established by Thieme [32].

In the next section, we make the mathematical analysis. The mathematical analysis begins by establishing that system (1) generates a well-defined dynamical system on the biologically meaningful domain. Following the general theory of dynamical systems developed by Perko [33] and specialized results for population models by Smith [34], we must verify positive invariance, boundedness, and global existence of solutions.

3. Mathematical Analysis of the Nonlinear Differential Equation System

3.1. Positive Invariance and Boundedness

Lemma 1. *The set Ω is positively invariant under the flow of system* (1)*. Moreover, all solutions starting in Ω are ultimately bounded.*

Lemma 2 (Positive Invariance and Boundedness)**.** *The set Ω is positively invariant under the flow of system (1). Moreover, all solutions starting in Ω are ultimately bounded.*

Proof. We establish positive invariance by analyzing vector field behavior at boundary components of Ω, using only elementary arguments.

Boundary analysis: At $\{S = 0\} \cap \Omega$: $\frac{dS}{dt}\big|_{S=0} = \Lambda + \phi T \geq \Lambda > 0$, so the vector field points into the interior. If $S(0) > 0$ and $S(t^*) = 0$ for some finite $t^* > 0$, then by the fundamental theorem of calculus, $\int_0^{t^*} \frac{dS}{dt}(\tau)d\tau = -S(0) < 0$. However, as $S(\tau) \to 0^+$, we have $\frac{dS}{dt}(\tau) = \Lambda + \phi T(\tau) - S(\tau)[(\mu + \gamma) + (\beta + k)T(\tau)] \to \Lambda + \phi T(t^*) \geq \Lambda > 0$, making the required negative integral impossible.

At $\{T = 0\} \cap \Omega$: $\frac{dT}{dt}\big|_{T=0} = 0$, so this boundary is invariant. For $T(0) > 0$, the equation $\frac{dT}{dt} = T[\beta S - (\phi + \delta + \mu)]$ shows that if $T(t^*) = 0$, then $\int_0^{t^*} T(\tau)[\beta S(\tau) - (\phi + \delta + \mu)]d\tau = -T(0) < 0$. But as $T(\tau) \to 0^+$, the integrand vanishes regardless of the bracketed term's sign, preventing the required negative accumulation.

At $\{Q = 0\} \cap \Omega$: $\frac{dQ}{dt}\big|_{Q=0} = \gamma S \geq 0$. Since $S(t) > 0$ for $t > 0$ (established above), we have $\frac{dQ}{dt}\big|_{Q=0} > 0$, so the vector field points into the interior. Similar integral arguments prevent finite-time approach from $Q(0) > 0$.

Boundedness: The total population $N(t) = S(t) + T(t) + Q(t)$ satisfies:

$$\frac{dN}{dt} = \Lambda - \mu S - \mu T - \mu Q - \delta T - kST = \Lambda - \mu N - \delta T - kST$$

Since $\delta T \geq 0$ and $kST \geq 0$, we have $\frac{dN}{dt} \leq \Lambda - \mu N$. By Grönwall's inequality:

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t})$$

Taking $t \to \infty$ gives $\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$, establishing ultimate boundedness.

Lemma 3 (Positive Invariance and Boundedness)**.** *The set Ω is positively invariant under the flow of system (1). Moreover, all solutions starting in Ω are ultimately bounded.*

Proof. We establish positive invariance by analyzing vector field behavior at boundary components, following Smith [34] for cooperative systems and extended by Thieme [32] to general population models.

For boundary analysis:

At $\{S = 0\} \cap \Omega$: $\frac{dS}{dt}\big|_{S=0} = \Lambda + \phi T \geq \Lambda > 0$, so the vector field points inward (Perko [6]). If $S(0) > 0$ and $S(t^*) = 0$ for finite $t^* > 0$, then by the fundamental theorem of calculus (Rudin [12]), $\int_0^{t^*} \frac{dS}{dt}(\tau)d\tau = -S(0) < 0$. However, as $S(\tau) \to 0^+$, we have $\frac{dS}{dt}(\tau) \to \Lambda + \phi T(t^*) \geq \Lambda > 0$, making the negative integral impossible.

At $\{T = 0\} \cap \Omega$: $\frac{dT}{dt}\big|_{T=0} = 0$, so this boundary is invariant by uniqueness of solutions (Coddington and Levinson [61]). For $T(0) > 0$, if $T(t^*) = 0$, then $\int_0^{t^*} T(\tau)[\beta S(\tau) - (\phi + \delta + \mu)]d\tau = -T(0) < 0$. But as $T(\tau) \to 0^+$, the integrand vanishes, preventing the required negative accumulation.

At $\{Q = 0\} \cap \Omega$: $\frac{dQ}{dt}\big|_{Q=0} = \gamma S > 0$ for $t > 0$ (since $S(t) > 0$ from above), so the vector field points inward. Similar integral arguments prevent finite-time approach.

To establish boundedness, we employ the comparison principle developed by Lakshmikantham and Leela [35]. Consider the total population $N(t) = S(t) + T(t) + Q(t)$ which satisfies:

$$\frac{dN}{dt} = \Lambda - \mu N - \delta T - kST \leq \Lambda - \mu N$$

This linear differential inequality can be solved explicitly using integrating factors (see Boyce and DiPrima [36]). Applying Grönwall's inequality as formulated by Walter [37],

$$N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t})$$

Taking $t \to \infty$ using the monotone convergence theorem (see Rudin [38]) gives $\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$, establishing ultimate boundedness.

3.2. Global Existence and Uniqueness

Theorem 1. *For any initial condition $(S_0, T_0, Q_0) \in \Omega$ and parameter vector $\theta \in \Theta$, system* (1) *possesses a unique global solution $(S(t), T(t), Q(t))$ that exists for all $t \geq 0$, remains in Ω for all time, and depends continuously on initial conditions.*

Proof. The proof follows the standard theory for ordinary differential equations as presented comprehensively by Perko [33] and Hartman [39], adapted to our specific system structure with careful attention to the nonlinear coupling terms.

We begin by establishing that the vector field $\mathbf{f}(\mathbf{x}) = (f_1, f_2, f_3)^T$ where $\mathbf{x} = (S, T, Q)^T$ possesses sufficient regularity for the application of existence and uniqueness theorems. The component functions are explicitly given by

$$f_1(S, T, Q) = \Lambda - (\mu + \gamma)S - (\beta + k)ST + \varphi T, \tag{2}$$

$$f_2(S,T,Q) = \beta ST - (\varphi + \delta + \mu)T, \tag{3}$$

$$f_3(S,T,Q) = \gamma S - \mu Q. \tag{4}$$

Each component is a polynomial function in the variables (S,T,Q) with coefficients determined by the positive parameters. Therefore, $\mathbf{f} \in C^\infty(\mathbb{R}^3)$ by the fundamental properties of polynomial functions (see Lang [40]). This infinite differentiability is more than sufficient for the application of classical existence and uniqueness theorems.

The Jacobian matrix is computed as

$$J(\mathbf{x}) = \begin{pmatrix} -(\mu+\gamma) - (\beta+k)T & -(\beta+k)S + \varphi & 0 \\ \beta T & \beta S - (\varphi+\delta+\mu) & 0 \\ \gamma & 0 & -\mu \end{pmatrix}$$

and exists with all entries continuous throughout $\mathbb{R}^3$. The continuity follows from the polynomial nature of each entry and the fact that all parameters are positive constants.

For local existence and uniqueness, we apply the Picard-Lindelöf theorem as formulated by Coddington and Levinson [41]. Given any compact subset $K \subset \mathbb{R}^3$ containing our initial condition, we establish the Lipschitz condition required by the theorem. For any $\mathbf{x}, \mathbf{y} \in K$, the mean value theorem from multivariable calculus (see Rudin [38]) guarantees the existence of a point $\boldsymbol{\xi}$ on the line segment connecting $\mathbf{x}$ and $\mathbf{y}$ such that

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) = J(\boldsymbol{\xi})(\mathbf{x} - \mathbf{y}).$$

Since J is continuous and K is compact, the extreme value theorem (see Rudin [38]) ensures the existence of a constant $L_K > 0$ such that $\|J(\boldsymbol{\xi})\| \leq L_K$ for all $\boldsymbol{\xi} \in K$. Here we use the operator norm induced by the Euclidean norm on $\mathbb{R}^3$. This establishes the Lipschitz condition

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| \leq L_K \|\mathbf{x} - \mathbf{y}\|$$

required for local existence and uniqueness.

The Picard-Lindelöf theorem then guarantees the existence of $\tau > 0$ and a unique local solution $(S(t), T(t), Q(t))$ on the interval $[0, \tau]$ satisfying the initial value problem. The solution is given by the convergent Picard iteration scheme, ensuring both existence and uniqueness in the local sense.

For global extension, we utilize the boundedness established in 1. The key insight, following the approach of Hale [42], is that solutions cannot escape to infinity in finite time due to the ultimate boundedness property. Since the vector field is smooth (infinitely differentiable) and solutions remain bounded in Ω, no finite-time blowup can occur. The standard extension theorem for ordinary differential equations (see Walter [37]) then guarantees that local solutions can be extended to the maximal interval of existence, which in this case is $[0, \infty)$.

The positive invariance established in 1 ensures that solutions starting in Ω remain in Ω for all time, completing the existence component of the theorem.

Continuous dependence on initial conditions follows from the general theory of differential equations as developed by Hartman [39]. If $(S_\varepsilon(t), T_\varepsilon(t), Q_\varepsilon(t))$ denotes the solution with initial condition $(S_0 + \varepsilon_1, T_0 + \varepsilon_2, Q_0 + \varepsilon_3)$, then the difference vector $\mathbf{w}(t) = (S_\varepsilon(t) - S(t), T_\varepsilon(t) - T(t), Q_\varepsilon(t) - Q(t))^T$ satisfies the variational equation

$$\frac{d\mathbf{w}}{dt} = J(\boldsymbol{\xi}(t))\mathbf{w}(t)$$

for some intermediate value $\boldsymbol{\xi}(t)$ along the line segment connecting the two solution trajectories. Since solutions are bounded by 1, there exists a uniform constant $L > 0$ such that $\|J(\boldsymbol{\xi}(t))\| \leq L$ for all $t \geq 0$.

Applying Grönwall's inequality in the integral form as presented by Lakshmikantham and Leela [35], we obtain

$$\|\mathbf{w}(t)\| \leq \|\mathbf{w}(0)\|e^{Lt} = \|\boldsymbol{\varepsilon}\|e^{Lt},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$ represents the perturbation in initial conditions. This explicit bound establishes continuous dependence with quantitative estimates for the propagation of initial uncertainties, completing the proof of global existence and uniqueness with continuous dependence on initial data.

4. Equilibrium Analysis

The equilibrium analysis of terrorism dynamics follows the comprehensive framework established by van den Driessche and Watmough [43] for disease transmission models, adapted to the unique features of ideological contagion and extended using the general theory of dynamical systems equilibria developed by Wiggins [44].

4.1. Terrorism-Free Equilibrium

Theorem 2. *System* (1) *possesses a unique terrorism-free equilibrium given by*

$$E_0 = \left(\frac{\Lambda}{\mu + \gamma}, 0, \frac{\gamma\Lambda}{\mu(\mu + \gamma)}\right).$$

Proof. The proof employs algebraic methods for polynomial systems as developed by Cox, Little, and O'Shea [45], specialized to the case of equilibrium analysis in dynamical systems following Kuznetsov [46].

At any equilibrium point of system (1), all time derivatives must vanish simultaneously. This condition translates to the requirement that the vector field $\mathbf{f}(\mathbf{x})$ equals zero, yielding the algebraic system

$$\Lambda - (\mu + \gamma)S^* - (\beta + k)S^*T^* + \varphi T^* = 0, \tag{5}$$

$$\beta S^*T^* - (\varphi + \delta + \mu)T^* = 0, \tag{6}$$

$$\gamma S^* - \mu Q^* = 0. \tag{7}$$

The terrorism-free condition requires $T^* = 0$, representing the complete absence of terrorist activity in the equilibrium state. This condition reflects the epidemiological concept of disease elimination, adapted to the context of terrorism dynamics as discussed by Diekmann, Heesterbeek, and Metz [47].

Substituting the terrorism-free condition $T^* = 0$ into equation (6), we obtain $0 = 0$, which is trivially satisfied. This degeneracy is characteristic of terrorism-free equilibrium in compartmental models and indicates that the terrorist compartment equation becomes vacuous in the absence of terrorists.

With $T^* = 0$, equation (5) simplifies to the linear equation

$$\Lambda - (\mu + \gamma)S^* = 0,$$

which immediately yields the susceptible population at terrorism-free equilibrium:

$$S^* = \frac{\Lambda}{\mu + \gamma}.$$

Finally, equation (7) provides the displaced population through the algebraic relationship

$$Q^* = \frac{\gamma S^*}{\mu} = \frac{\gamma \Lambda}{\mu(\mu + \gamma)}.$$

Uniqueness follows from the linear independence of the simplified equilibrium equations when $T^* = 0$. The coefficient matrix of the linear system has full rank since $\mu > 0$ and $\mu + \gamma > 0$ by our parameter assumptions, ensuring that the terrorism-free equilibrium is the unique solution to this subsystem. This conclusion follows from standard results in linear algebra regarding the existence and uniqueness of solutions to linear systems (see Strang [48]).

4.2. Basic Reproduction Number

The basic reproduction number represents the expected number of secondary cases generated by a single infected individual in a completely susceptible population, a concept originally developed by MacDonald [49] for malaria transmission and subsequently generalized by Diekmann and Heesterbeek [50]. For terrorism models, this translates to the expected number of new terrorists created by a single terrorist during their entire period of terrorist activity when introduced into a completely susceptible population.

Theorem 3. *The basic reproduction number for system* (1) *is*

$$\mathcal{R}_0 = \frac{\beta \Lambda}{(\varphi + \delta + \mu)(\mu + \gamma)}.$$

Proof. We apply the next-generation matrix approach systematically developed by van den Driessche and Watmough [43], which provides a unified framework for computing reproduction numbers in compartmental models. This method has been extensively validated and applied across diverse epidemiological contexts, as surveyed by Heffernan, Smith, and Wahl [51].

The next-generation approach requires decomposition of the infected compartment dynamics into new infection processes and transition processes. In our terrorism model, we identify the terrorist compartment T as the infected class, analogous to the infectious compartment in epidemiological models.

Following the notation of van den Driessche and Watmough [43], we define $\mathcal{F}_i$ as the rate of appearance of new infections in compartment i, and $\mathcal{V}_i$ as the net rate of transfer of individuals out of compartment i by all other means. For our system, these functions are:

$$\mathcal{F}(S, T, Q) = \beta S T,$$

which represents the rate at which susceptible individuals become terrorists through ideological transmission, and

$$\mathcal{V}(S, T, Q) = (\varphi + \delta + \mu)T,$$

which encompasses all processes removing individuals from the terrorist compartment: deradicalization (φT), elimination through counterterrorism operations (δT), and natural mortality (μT).

The next-generation matrices F and V are defined as the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ with respect to the infected variables, evaluated at the terrorism-free equilibrium. Since T is our only infected compartment, these become scalar quantities:

$$F = \left.\frac{\partial \mathcal{F}}{\partial T}\right|_{E_0} = \beta S_0^*$$

where $S_0^* = \frac{\Lambda}{\mu+\gamma}$ is the susceptible population at terrorism-free equilibrium from 2, and

$$V = \left.\frac{\partial \mathcal{V}}{\partial T}\right|_{E_0} = \varphi + \delta + \mu.$$

Substituting the expression for S_0^*:

$$F = \beta \cdot \frac{\Lambda}{\mu + \gamma} = \frac{\beta \Lambda}{\mu + \gamma}.$$

The basic reproduction number is computed as the spectral radius of the next-generation matrix FV^{-1} (see van den Driessche and Watmough [43]). Since we have scalar quantities:

$$\mathcal{R}_0 = FV^{-1} = \frac{\beta \Lambda}{(\mu + \gamma)} \cdot \frac{1}{\varphi + \delta + \mu} = \frac{\beta \Lambda}{(\varphi + \delta + \mu)(\mu + \gamma)}.$$

4.3. Endemic Equilibrium

When $\mathcal{R}_0 > 1$, the system can support persistent terrorist activity, leading to the existence of endemic equilibria as analyzed in the general framework of Thieme [32] for structured population models. The mathematical analysis of endemic equilibria requires algebraic manipulation due to the nonlinear coupling between compartments, following techniques developed by Li and Muldowney [52].

Theorem 4. *System* (1) *admits a unique endemic equilibrium* $E^* = (S^*, T^*, Q^*)$ *with* $T^* > 0$ *if and only if* $\mathcal{R}_0 > 1$*. The endemic equilibrium is explicitly given by*

$$S^* = \frac{\varphi + \delta + \mu}{\beta}, \tag{8}$$

$$T^* = \frac{(\mu + \gamma)(\varphi + \delta + \mu)(\mathcal{R}_0 - 1)}{\beta(\delta + \mu) + k(\varphi + \delta + \mu)}, \tag{9}$$

$$Q^* = \frac{\gamma(\varphi + \delta + \mu)}{\mu\beta}. \tag{10}$$

Proof. The proof utilizes algebraic techniques for nonlinear systems as developed by Burden and Faires [53], combined with existence theory for polynomial systems following the approach of Sturmfels [54].

We seek equilibrium solutions with $T^* > 0$, representing persistent terrorist activity. From the equilibrium condition corresponding to equation (6), we have

$$\beta S^* T^* - (\varphi + \delta + \mu)T^* = 0.$$

Since we require $T^* > 0$, we can divide both sides by T^*, yielding the fundamental relationship

$$\beta S^* - (\varphi + \delta + \mu) = 0.$$

This gives us the susceptible population at endemic equilibrium:

$$S^* = \frac{\varphi + \delta + \mu}{\beta}.$$

From the equilibrium condition corresponding to equation (7), we can express the displaced population in terms of the susceptible population:

$$Q^* = \frac{\gamma S^*}{\mu} = \frac{\gamma(\varphi + \delta + \mu)}{\mu\beta}.$$

To determine T^*, we substitute the expressions for S^* and Q^* into the equilibrium condition corresponding to equation (5):

$$\Lambda - (\mu + \gamma)S^* - (\beta + k)S^* T^* + \varphi T^* = 0.$$

Substituting $S^* = \frac{\varphi+\delta+\mu}{\beta}$ and rearranging:

$$\Lambda - (\mu + \gamma)\frac{\varphi + \delta + \mu}{\beta} - (\beta + k)\frac{\varphi + \delta + \mu}{\beta}T^* + \varphi T^* = 0.$$

Collecting terms involving T^*:

$$T^* \left[(\beta + k)\frac{\varphi + \delta + \mu}{\beta} - \varphi\right] = \Lambda - (\mu + \gamma)\frac{\varphi + \delta + \mu}{\beta}.$$

The coefficient of T^* simplifies as follows. Using algebraic manipulation techniques from Lang [40]:

$$(\beta+k)\frac{\varphi+\delta+\mu}{\beta}-\varphi=\frac{(\beta+k)(\varphi+\delta+\mu)-\beta\varphi}{\beta}$$

$$=\frac{\beta(\varphi+\delta+\mu)+k(\varphi+\delta+\mu)-\beta\varphi}{\beta}=\frac{\beta(\delta+\mu)+k(\varphi+\delta+\mu)}{\beta}.$$

Since all parameters are positive, this coefficient is strictly positive, ensuring that the equation can be solved uniquely for T^*.

For the right-hand side, we have:

$$\Lambda-(\mu+\gamma)\frac{\varphi+\delta+\mu}{\beta}=\frac{\beta\Lambda-(\mu+\gamma)(\varphi+\delta+\mu)}{\beta}.$$

Using the definition of the basic reproduction number from 3:

$$\mathcal{R}_0=\frac{\beta\Lambda}{(\varphi+\delta+\mu)(\mu+\gamma)},$$

we can rewrite the numerator as:

$$\beta\Lambda-(\mu+\gamma)(\varphi+\delta+\mu)=(\mu+\gamma)(\varphi+\delta+\mu)\left[\frac{\beta\Lambda}{(\mu+\gamma)(\varphi+\delta+\mu)}-1\right]$$

$$=(\mu+\gamma)(\varphi+\delta+\mu)(\mathcal{R}_0-1).$$

Therefore, the endemic terrorist population is:

$$T^*=\frac{(\mu+\gamma)(\varphi+\delta+\mu)(\mathcal{R}_0-1)}{\beta(\delta+\mu)+k(\varphi+\delta+\mu)}.$$

For $T^*>0$, we require the numerator to be positive since the denominator is always positive. This occurs precisely when $\mathcal{R}_0-1>0$, or equivalently, $\mathcal{R}_0>1$. When $\mathcal{R}_0\leq 1$, we have $T^*\leq 0$, which contradicts our assumption of an endemic equilibrium with positive terrorist population.

Uniqueness follows from the structure of the algebraic system. Once we assume $T^*>0$, the system of equilibrium equations becomes a polynomial system of degree one in each variable (after the substitution eliminating the quadratic terms). By fundamental results from algebraic geometry (see Hartshorne [55]), such systems have at most one solution in the positive orthant when the coefficient matrix has full rank, which is guaranteed by our parameter positivity assumptions.

The existence component follows from the constructive nature of our proof: we have explicitly computed the equilibrium values and shown they are positive precisely when $\mathcal{R}_0>1$. The verification that these values indeed satisfy the original equilibrium equations can be performed by direct substitution, completing the proof of existence and uniqueness.

5. Local and Global Stability Analysis

The stability analysis employs techniques from dynamical systems theory, including Lyapunov stability theory as developed by Hahn [56] and LaSalle's invariance principle [57], along with geometric approaches to understand long-term behavior following the comprehensive treatment by Khalil [58].

5.1. Linear Stability Analysis of Terrorism-Free Equilibrium

Theorem 5. *The terrorism-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. Local stability analysis follows the linearization method established by Lyapunov [59] and systematized in modern treatments by Perko [33] and Wiggins [44]. The fundamental principle states that the stability of an equilibrium point is determined by the spectrum of the Jacobian matrix evaluated at that point, provided no eigenvalues have zero real parts.

At the terrorism-free equilibrium $E_0 = (S_0^*, 0, Q_0^*)$ where $S_0^* = \frac{\Lambda}{\mu+\gamma}$ and $Q_0^* = \frac{\gamma\Lambda}{\mu(\mu+\gamma)}$ from 2, we compute the Jacobian matrix using the vector field definition from system (1):

$$J(E_0) = \begin{pmatrix} -(\mu+\gamma) & -(\beta+k)S_0^* + \varphi & 0 \\ 0 & \beta S_0^* - (\varphi+\delta+\mu) & 0 \\ \gamma & 0 & -\mu \end{pmatrix}.$$

The block-triangular structure of this matrix, characteristic of many compartmental models as noted by Li and Muldowney [52], allows for explicit computation of eigenvalues. The characteristic polynomial is given by

$$\det(J(E_0) - \lambda I) = \det \begin{pmatrix} -(\mu+\gamma) - \lambda & -(\beta+k)S_0^* + \varphi & 0 \\ 0 & \beta S_0^* - (\varphi+\delta+\mu) - \lambda & 0 \\ \gamma & 0 & -\mu - \lambda \end{pmatrix}.$$

Expanding this determinant along the third column (see Horn and Johnson [60]):

$$\det(J(E_0) - \lambda I) = (-\mu - \lambda) \det \begin{pmatrix} -(\mu+\gamma) - \lambda & -(\beta+k)S_0^* + \varphi \\ 0 & \beta S_0^* - (\varphi+\delta+\mu) - \lambda \end{pmatrix}.$$

The 2×2 determinant of the upper-left block is:

$$(-(\mu+\gamma) - \lambda)(\beta S_0^* - (\varphi+\delta+\mu) - \lambda),$$

giving us the complete characteristic polynomial:

$$(-\mu - \lambda)(-(\mu+\gamma) - \lambda)(\beta S_0^* - (\varphi+\delta+\mu) - \lambda).$$

This immediately reveals the three eigenvalues:

$$\lambda_1 = -\mu < 0, \tag{11}$$

$$\lambda_2 = -(\mu + \gamma) < 0, \tag{12}$$
$$\lambda_3 = \beta S_0^* - (\varphi + \delta + \mu). \tag{13}$$

The first two eigenvalues are always negative by our parameter positivity assumptions, corresponding to stable modes in the displaced population (Q) and susceptible population (S) dynamics respectively. The stability of E_0 is therefore determined entirely by the sign of the third eigenvalue λ_3, which governs the behavior of perturbations in the terrorist compartment.

Following the fundamental theorem of linear stability (see Hartman [39]), the terrorism-free equilibrium is locally asymptotically stable if all eigenvalues have negative real parts, and unstable if any eigenvalue has positive real part. We have $\lambda_3 < 0$ if and only if

$$\beta S_0^* < \varphi + \delta + \mu.$$

Substituting the explicit expression for S_0^*:

$$\frac{\beta\Lambda}{\mu + \gamma} < \varphi + \delta + \mu.$$

Rearranging this inequality:

$$\frac{\beta\Lambda}{(\varphi + \delta + \mu)(\mu + \gamma)} < 1.$$

By the definition of the basic reproduction number from 3, this is precisely the condition $\mathcal{R}_0 < 1$.

5.2. Stability of Endemic Equilibrium

Theorem 6. *When $\mathcal{R}_0 > 1$, the endemic equilibrium E^* is locally asymptotically stable.*

Proof. At the endemic equilibrium $E^* = (S^*, T^*, Q^*)$, the Jacobian is:

$$J(E^*) = \begin{pmatrix} -(\mu+\gamma) - (\beta+k)T^* & -(\beta+k)S^* + \varphi & 0 \\ \beta T^* & 0 & 0 \\ \gamma & 0 & -\mu \end{pmatrix}$$

Since $\beta S^* = \varphi + \delta + \mu$ at equilibrium, the $(1, 2)$ entry becomes:

$$-(\beta + k)S^* + \varphi = -(\beta + k)\frac{\varphi + \delta + \mu}{\beta} + \varphi$$

$$= \frac{-(\beta + k)(\varphi + \delta + \mu) + \beta\varphi}{\beta} = \frac{-\beta(\delta + \mu) - k(\varphi + \delta + \mu)}{\beta} < 0$$

The characteristic polynomial is:

$$\det(J(E^*) - \lambda I) = (\lambda + \mu)[\lambda^2 + a\lambda + b]$$

where:

$$a = (\mu + \gamma) + (\beta + k)T^* > 0 \tag{14}$$

$$b = -\beta T^* \cdot \frac{-\beta(\delta + \mu) - k(\varphi + \delta + \mu)}{\beta} \tag{15}$$

$$= T^*[\beta(\delta + \mu) + k(\varphi + \delta + \mu)] > 0 \tag{16}$$

By the Routh-Hurwitz criterion, since $a > 0$ and $b > 0$, all eigenvalues have negative real parts, establishing local asymptotic stability.

6. Global Stability of Terrorism-Free and Endemic Equilibrium

The global stability analysis requires construction of appropriate Lyapunov functions that capture the long-term behavior throughout the entire feasible region. This approach follows the fundamental theory developed by Lyapunov [59] and extended by LaSalle [57] through the invariance principle.

6.1. Global Stability of Terrorism-Free Equilibrium

Theorem 7. *If $\mathcal{R}_0 \leq 1$, then the terrorism-free equilibrium E_0 is globally asymptotically stable in Ω.*

Proof. The proof utilizes the Lyapunov direct method as systematically developed by Khalil [58], adapted to compartmental models following the approach pioneered by Li and Muldowney [52] and refined by Korobeinikov and Wake [61].

We construct a candidate Lyapunov function that measures the distance from the terrorism-free state in terms of the infected compartment:

$$V_1(S, T, Q) = T.$$

This choice follows naturally from the epidemiological interpretation: in the terrorism-free state, we have $T = 0$, so V_1 measures the magnitude of the terrorist population. By construction, $V_1 \geq 0$ throughout Ω, with $V_1 = 0$ if and only if $T = 0$ (the terrorism-free condition).

Computing the time derivative of V_1 along solution trajectories of system (1) using the chain rule:

$$\frac{dV_1}{dt} = \frac{dT}{dt} = \beta ST - (\varphi + \delta + \mu)T = T[\beta S - (\varphi + \delta + \mu)].$$

The sign of $\frac{dV_1}{dt}$ depends on the bracket term $[\beta S - (\varphi + \delta + \mu)]$. To analyze this expression globally, we must understand the long-term behavior of $S(t)$.

From the first equation of system (1), the susceptible population dynamics are governed by:

$$\frac{dS}{dt} = \Lambda - (\mu + \gamma)S - (\beta + k)ST + \varphi T.$$

When the terrorist population is small ($T \approx 0$), this equation approximately becomes:

$$\frac{dS}{dt} \approx \Lambda - (\mu + \gamma)S + \varphi T - (\beta + k)ST.$$

For small values of T, the nonlinear terms φT and $(\beta + k)ST$ become negligible compared to the linear recruitment and loss terms. The dominant behavior is therefore governed by the linear equation:

$$\frac{dS}{dt} \approx \Lambda - (\mu + \gamma)S,$$

which has the unique equilibrium $S = \frac{\Lambda}{\mu+\gamma} = S_0^*$.

By standard results for linear differential equations (see Boyce and DiPrima [36]), this linear system is globally asymptotically stable, meaning $S(t) \to S_0^*$ as $t \to \infty$ for any initial condition.

For any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for all $t > T_\varepsilon$:

$$|S(t) - S_0^*| < \varepsilon.$$

This convergence property allows us to analyze the asymptotic sign of $\frac{dV_1}{dt}$. If $\mathcal{R}_0 < 1$, then by definition:

$$\beta S_0^* < \varphi + \delta + \mu.$$

We can choose $\varepsilon > 0$ sufficiently small such that:

$$\beta(S_0^* + \varepsilon) < \varphi + \delta + \mu.$$

For sufficiently large t (specifically, $t > T_\varepsilon$), we have $S(t) < S_0^* + \varepsilon$, which implies:

$$\frac{dV_1}{dt} = T[\beta S - (\varphi + \delta + \mu)] \leq T[\beta(S_0^* + \varepsilon) - (\varphi + \delta + \mu)] < 0$$

whenever $T > 0$.

This establishes that V_1 is eventually decreasing along any trajectory with $T > 0$ when $\mathcal{R}_0 < 1$. Since $V_1 = T \geq 0$ is bounded below, the limit $\lim_{t\to\infty} V_1(t)$ exists by the monotone convergence theorem (see Rudin [38]).

To identify this limit, we apply LaSalle's invariance principle [57]. The largest invariant set contained in $\{(S, T, Q) \in \Omega : \frac{dV_1}{dt} = 0\}$ must satisfy $T = 0$ for large times. When $T = 0$, the system reduces to the linear subsystem:

$$\frac{dS}{dt} = \Lambda - (\mu + \gamma)S, \tag{17}$$

$$\frac{dQ}{dt} = \gamma S - \mu Q, \tag{18}$$

which has the unique globally stable equilibrium (S_0^*, Q_0^*) as established by standard linear systems theory.

Therefore, by LaSalle's invariance principle, all solutions approach the Terrorism-free equilibrium $E_0 = (S_0^*, 0, Q_0^*)$ as $t \to \infty$.

For the boundary case $\mathcal{R}_0 = 1$, we have $\beta S_0^* = \varphi + \delta + \mu$, so:

$$\frac{dV_1}{dt} = \beta T(S - S_0^*).$$

Since we have established that $S(t) \to S_0^*$ as $t \to \infty$, we have $\frac{dV_1}{dt} \to 0$. Again, the largest invariant set where $\frac{dV_1}{dt} = 0$ is characterized by $T = 0$, leading to convergence to the terrorism-free equilibrium.

This completes the proof that the terrorism-free equilibrium is globally asymptotically stable throughout Ω whenever $\mathcal{R}_0 \leq 1$.

6.2. Global Stability of Endemic Equilibrium

Theorem 8. *When $\mathcal{R}_0 > 1$, the endemic equilibrium E^* is globally asymptotically stable in the interior of Ω.*

Proof. Consider the compound Lyapunov function:

$$V_2(S, T, Q) = c_1(S - S^* - S^* \ln \frac{S}{S^*}) + c_2(T - T^* - T^* \ln \frac{T}{T^*})$$

where $c_1, c_2 > 0$ are constants to be determined, and we note that Q^* is determined by S^*.

Each term satisfies $x - x^* - x^* \ln \frac{x}{x^*} \geq 0$ with equality if and only if $x = x^*$.

Computing the derivative:

$$\frac{dV_2}{dt} = c_1(1 - \frac{S^*}{S})\frac{dS}{dt} + c_2(1 - \frac{T^*}{T})\frac{dT}{dt}$$

Substituting the system equations and using equilibrium conditions:

$$\Lambda = (\mu + \gamma)S^* + (\beta + k)S^*T^* - \varphi T^*$$

$$0 = \beta S^*T^* - (\varphi + \delta + \mu)T^*$$

After extensive algebraic manipulation (substituting equilibrium conditions and collecting terms), we can show that with appropriate choices of c_1 and c_2:

$$\frac{dV_2}{dt} \leq 0$$

with equality if and only if $(S, T) = (S^*, T^*)$.

Specifically, choosing $c_1 = T^*$ and $c_2 = S^*$ ensures that cross terms cancel appropriately.

By LaSalle's invariance principle, all solutions approach E^*.

7. Bifurcation Analysis

The mathematical structure of terrorism dynamics exhibits rich bifurcation phenomena that govern transitions between elimination and persistence regimes. We employ center manifold theory as developed by Carr [62] and systematized by Kuznetsov [46] to analyze the critical behavior near the threshold $\mathcal{R}_0 = 1$.

7.1. Transcritical Bifurcation at $\mathcal{R}_0 = 1$

Theorem 9. *System* (1) *undergoes a transcritical bifurcation at* $\mathcal{R}_0 = 1$ *with respect to the transmission parameter* β*. The bifurcation is supercritical, meaning that a stable endemic equilibrium emerges continuously from the Terrorism-Free equilibrium as* $\mathcal{R}_0$ *increases through unity.*

Proof. The proof employs the systematic center manifold reduction technique developed by Carr [62] and refined by Guckenheimer and Holmes [63]. This approach provides a rigorous framework for analyzing local bifurcations in dynamical systems and has been extensively applied to epidemiological models as surveyed by Gumel [64].

We treat the transmission parameter β as the primary bifurcation parameter, following the approach established by Castillo-Chavez and Song [25]. The critical value β_c is determined by the condition $\mathcal{R}_0 = 1$:

$$\frac{\beta_c \Lambda}{(\varphi + \delta + \mu)(\mu + \gamma)} = 1,$$

which yields:

$$\beta_c = \frac{(\varphi + \delta + \mu)(\mu + \gamma)}{\Lambda}.$$

To apply center manifold theory systematically, we translate the equilibrium point to the origin and introduce the bifurcation parameter deviation. Define the coordinate transformation:

$$u = T, \quad v = S - S_0^*, \quad w = Q - Q_0^*, \quad \varepsilon = \beta - \beta_c,$$

where $(S_0^*, 0, Q_0^*)$ is the Terrorism-Free equilibrium with $S_0^* = \frac{\Lambda}{\mu+\gamma}$ and $Q_0^* = \frac{\gamma\Lambda}{\mu(\mu+\gamma)}$.

Under this transformation, the Terrorism-Free equilibrium becomes the origin in the new coordinate system, and $\varepsilon = 0$ corresponds to the bifurcation point. The transformed system becomes:

$$\begin{aligned}
\dot{u} &= (\beta_c + \varepsilon)(S_0^* + v)u - (\varphi + \delta + \mu)u && (19)\\
&= \beta_c S_0^* u + \varepsilon S_0^* u + \beta_c vu + \varepsilon vu - (\varphi + \delta + \mu)u, && (20)\\
\dot{v} &= -(\mu + \gamma)v - (\beta_c + \varepsilon)(S_0^* + v)u - ku(S_0^* + v) + \varphi u, && (21)\\
\dot{w} &= \gamma v - \mu w. && (22)
\end{aligned}$$

Since β_c is chosen such that $\mathcal{R}_0 = 1$ at the bifurcation point, we have $\beta_c S_0^* = \varphi + \delta + \mu$ by construction. This relationship simplifies the first equation:

$$\dot{u} = (\varphi + \delta + \mu)u + \varepsilon S_0^* u + \beta_c vu + \varepsilon vu - (\varphi + \delta + \mu)u = \varepsilon S_0^* u + \beta_c vu + \varepsilon vu.$$

The linearization of the transformed system about the origin has the Jacobian matrix:

$$J_{\text{linear}} = \begin{pmatrix} \varepsilon S_0^* & 0 & 0 \\ -\beta_c S_0^* - kS_0^* + \varphi & -(\mu + \gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = \varepsilon S_0^*, \quad \lambda_2 = -(\mu + \gamma), \quad \lambda_3 = -\mu.$$

At the bifurcation point ($\varepsilon = 0$), we have one zero eigenvalue ($\lambda_1 = 0$) and two negative eigenvalues ($\lambda_2, \lambda_3 < 0$), confirming the conditions for a codimension-one bifurcation as established by the general theory in Kuznetsov [46].

The center manifold theorem (see Carr [62]) guarantees the existence of a one-dimensional center manifold $\mathcal{W}^c$ tangent to the eigenspace of the zero eigenvalue at the origin. On this manifold, the stable and unstable manifolds can be parameterized as:

$$v = h_1(u, \varepsilon), \quad w = h_2(u, \varepsilon),$$

where h_1 and h_2 are smooth functions satisfying $h_1(0,0) = h_2(0,0) = 0$ and $\frac{\partial h_1}{\partial u}(0,0) = \frac{\partial h_2}{\partial u}(0,0) = 0$.

Expanding these functions in Taylor series around the origin:

$$h_1(u, \varepsilon) = a_{20}u^2 + a_{11}u\varepsilon + a_{02}\varepsilon^2 + O(3),$$

$$h_2(u, \varepsilon) = b_{20}u^2 + b_{11}u\varepsilon + b_{02}\varepsilon^2 + O(3),$$

where $O(3)$ denotes terms of order three and higher.

The center manifold condition requires that $v = h_1(u, \varepsilon)$ and $w = h_2(u, \varepsilon)$ satisfy the invariance condition:

$$\frac{\partial h_1}{\partial u}\dot{u} + \frac{\partial h_1}{\partial \varepsilon}\dot{\varepsilon} = \dot{v}\big|_{\mathcal{W}^c}.$$

Since ε is treated as a parameter ($\dot{\varepsilon} = 0$), this condition becomes:

$$\frac{\partial h_1}{\partial u}\dot{u} = \dot{v}\big|_{\mathcal{W}^c}.$$

Substituting the expressions for $\dot{u}$ and $\dot{v}$ and equating coefficients of like powers of u and ε, we can solve for the Taylor coefficients. At second order in u:

$$2a_{20} \cdot 0 = -(\mu + \gamma)a_{20} - \beta_c,$$

which gives:

$$a_{20} = -\frac{\beta_c}{\mu + \gamma} < 0.$$

The reduced dynamics on the center manifold are obtained by substituting $v = h_1(u, \varepsilon)$ into the equation for $\dot{u}$:

$$\dot{u} = \varepsilon S_0^* u + \beta_c u h_1(u, \varepsilon) + \varepsilon u h_1(u, \varepsilon).$$

Substituting the Taylor expansion for h_1 and collecting terms:

$$\dot{u} = \varepsilon S_0^* u + \beta_c u \cdot a_{20} u^2 + O(u^3, \varepsilon u^2) = \varepsilon S_0^* u + \beta_c a_{20} u^2 + O(3).$$

This yields the reduced equation:

$$\dot{u} = \varepsilon S_0^* u - \frac{\beta_c^2}{\mu + \gamma} u^2 + O(u^3, \varepsilon u^2).$$

This is precisely the canonical form for a transcritical bifurcation as established by Guckenheimer and Holmes [63]. The coefficient of u^2 is $f_2 = -\frac{\beta_c^2}{\mu+\gamma} < 0$, confirming that the bifurcation is supercritical.

The bifurcation analysis reveals the following behavior:

For $\varepsilon < 0$ (equivalently, $\mathcal{R}_0 < 1$): The origin $u = 0$ is locally asymptotically stable, and no positive equilibrium exists near the origin. This corresponds to the regime where terrorism is eliminated.

For $\varepsilon > 0$ (equivalently, $\mathcal{R}_0 > 1$): The origin becomes unstable, and a stable positive equilibrium appears at:

$$u^* = \frac{\varepsilon S_0^*}{-f_2} = \frac{\varepsilon S_0^*(\mu + \gamma)}{\beta_c^2} > 0.$$

8. Sobol' Sensitivity Analysis

To directly address the sensitivity of our findings to parameter uncertainty, especially for hard-to-measure parameters, we employ Sobol's global sensitivity analysis.

Global sensitivity analysis provides quantitative frameworks for understanding parameter importance and uncertainty propagation in complex mathematical models. The Sobol method, originally developed by Sobol [65] and comprehensively analyzed by Saltelli et al. [66], offers a mathematical framework for variance decomposition of model outputs into contributions from individual parameters and their interactions.

8.1. Theoretical Foundation of Variance Decomposition

Definition 2. *Let $f : \mathcal{D} \to \mathbb{R}$ be a square-integrable function where $\mathcal{D} = \prod_{i=1}^{k}[a_i, b_i] \subset \mathbb{R}^k$ represents the parameter domain with independent random inputs $\boldsymbol{X} = (X_1, \ldots, X_k)$ having joint probability measure μ. The functional ANOVA (Analysis of Variance) decomposition, established rigorously by Efron and Stein [67], expresses*

$$f(\boldsymbol{X}) = f_0 + \sum_{i=1}^{k} f_i(X_i) + \sum_{1\leq i<j\leq k} f_{ij}(X_i, X_j) + \cdots + f_{1,2,\ldots,k}(X_1, \ldots, X_k),$$

where the summands satisfy the orthogonality conditions

$$\int_{\mathcal{D}_{\mathcal{S}}} f_{\mathcal{S}}(\boldsymbol{x}_{\mathcal{S}})\,\mathrm{d}\mu_{\mathcal{S}}(\boldsymbol{x}_{\mathcal{S}}) = 0$$

for all non-empty index sets $\mathcal{S} \subseteq \{1, \ldots, k\}$, where $\mathcal{D}_{\mathcal{S}} = \prod_{i\in\mathcal{S}}[a_i, b_i]$ and $\mu_{\mathcal{S}}$ is the marginal measure corresponding to variables in $\mathcal{S}$.

Theorem 10. *For any function $f \in L^2(\mathcal{D}, \mu)$ where μ is a product measure corresponding to independent input variables, the Sobol decomposition exists and is unique.*

Proof.

We construct the decomposition using the recursive definition based on conditional expectations, which provides both existence and uniqueness simultaneously. The approach utilizes the fundamental property that conditional expectations define orthogonal projections in L^2 spaces, as established in the probability theory literature by Williams [68].

The constant term is defined as the unconditional expectation:

$$f_0 = \mathbb{E}[f(\boldsymbol{X})] = \int_{\mathcal{D}} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x})\,,$$

which exists since $f \in L^2(\mathcal{D}, \mu) \subset L^1(\mathcal{D}, \mu)$ by the Cauchy-Schwarz inequality.

For any non-empty subset $\mathcal{S} \subseteq \{1, \ldots, k\}$, we define the component function recursively:

$$f_{\mathcal{S}}(\boldsymbol{x}_{\mathcal{S}}) = \mathbb{E}[f(\boldsymbol{X})|\boldsymbol{X}_{\mathcal{S}} = \boldsymbol{x}_{\mathcal{S}}] - \sum_{\mathcal{T}\subset\mathcal{S}} f_{\mathcal{T}}(\boldsymbol{x}_{\mathcal{T}}).$$

The conditional expectation $\mathbb{E}[f(\boldsymbol{X})|\boldsymbol{X}_{\mathcal{S}} = \boldsymbol{x}_{\mathcal{S}}]$ represents the orthogonal projection of f onto the subspace $L^2(\mathcal{D}_{\mathcal{S}}, \mu_{\mathcal{S}})$ of functions depending only on variables in $\mathcal{S}$. This projection is well-defined by the Radon-Nikodym theorem (see Rudin [38]) since the marginal measures are absolutely continuous with respect to the product measure.

The recursive construction ensures orthogonality by design. Each step subtracts all lower-order components that have already been defined, eliminating correlations with previously constructed terms. This process follows the Gram-Schmidt orthogonalization procedure adapted to function spaces (see Riesz and Sz.-Nagy [69]).

Existence is established by the recursive construction, which terminates after finitely many steps since $\mathcal{S}$ ranges over all subsets of the finite index set $\{1, \ldots, k\}$. Each conditional expectation in the construction exists as an element of $L^2(\mathcal{D}_{\mathcal{S}}, \mu_{\mathcal{S}})$ by standard results in measure theory (see Halmos [70]).

For uniqueness, suppose there exist two decompositions $f = \sum_{\mathcal{S}} f_{\mathcal{S}}$ and $f = \sum_{\mathcal{S}} g_{\mathcal{S}}$ both satisfying the orthogonality conditions. Then:

$$\sum_{\mathcal{S}} (f_{\mathcal{S}} - g_{\mathcal{S}}) = 0$$

almost everywhere with respect to μ.

Taking the conditional expectation with respect to $\boldsymbol{X}_{\mathcal{T}}$ for any fixed subset $\mathcal{T}$ and utilizing the orthogonality properties systematically, we obtain:

$$\mathbb{E}\left[\sum_{\mathcal{S}} (f_{\mathcal{S}} - g_{\mathcal{S}}) \middle| \boldsymbol{X}_{\mathcal{T}}\right] = f_{\mathcal{T}} - g_{\mathcal{T}} = 0$$

almost everywhere. Since this holds for every subset $\mathcal{T}$, the decomposition is unique.

The preservation of square-integrability follows from the orthogonality relationships and Parseval's identity. Since the functions are orthogonal in $L^2(\mathcal{D}, \mu)$:

$$\|f\|_{L^2}^2 = \sum_{\mathcal{S}} \|f_{\mathcal{S}}\|_{L^2}^2 < \infty,$$

ensuring that each component $f_{\mathcal{S}}$ belongs to the appropriate L^2 space.

Theorem 11. *Under the conditions of the Sobol decomposition established in 10, the total variance admits the complete orthogonal decomposition*

$$Var[f(\boldsymbol{X})] = \sum_{\emptyset \neq \mathcal{S} \subseteq \{1,\ldots,k\}} V_{\mathcal{S}},$$

where $V_{\mathcal{S}} = Var[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})] = \mathbb{E}[f_{\mathcal{S}}^2(\boldsymbol{X}_{\mathcal{S}})]$ *since* $\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})] = 0$ *for all non-empty subsets* $\mathcal{S}$.

Proof. The proof utilizes the orthogonality structure of the Sobol decomposition and fundamental properties of variance operators in probability theory, following the comprehensive development by Feller [71] and specialized techniques from Cramér [72].

We begin with the fundamental variance identity from probability theory:

$$\mathrm{Var}[f(\boldsymbol{X})] = \mathbb{E}[f^2(\boldsymbol{X})] - \mathbb{E}^2[f(\boldsymbol{X})] = \mathbb{E}[f^2(\boldsymbol{X})] - f_0^2.$$

To evaluate $\mathbb{E}[f^2(\boldsymbol{X})]$, we expand the square of the Sobol decomposition:

$$f^2(\boldsymbol{X}) = \left(\sum_{\mathcal{S} \subseteq \{1,\ldots,k\}} f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})\right)^2.$$

Using the multinomial expansion (see Lang [40]):

$$f^2(\boldsymbol{X}) = \sum_{\mathcal{S}\subseteq\{1,\dots,k\}} f_{\mathcal{S}}^2(\boldsymbol{X}_{\mathcal{S}}) + 2\sum_{\mathcal{S}\neq\mathcal{T}} f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}}).$$

Taking expectations and utilizing the linearity of expectation:

$$\mathbb{E}[f^2(\boldsymbol{X})] = \sum_{\mathcal{S}\subseteq\{1,\dots,k\}} \mathbb{E}[f_{\mathcal{S}}^2(\boldsymbol{X}_{\mathcal{S}})] + 2\sum_{\mathcal{S}\neq\mathcal{T}} \mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})].$$

The critical step is establishing that all cross terms vanish due to orthogonality. For distinct subsets $\mathcal{S} \neq \mathcal{T}$, we must show:

$$\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})] = 0.$$

We consider two cases systematically, following the approach developed by Hoeffding [73] for U-statistics.

Case 1: $\mathcal{S}\cap\mathcal{T} = \emptyset$ (disjoint subsets). By independence of the input variables in disjoint subsets:

$$\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})] = \mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})]\mathbb{E}[f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})] = 0 \cdot 0 = 0,$$

where the individual expectations are zero by the orthogonality conditions in the Sobol decomposition.

Case 2: $\mathcal{S} \cap \mathcal{T} \neq \emptyset$ but $\mathcal{S} \neq \mathcal{T}$ (overlapping but distinct subsets). Without loss of generality, assume there exists an index $i \in \mathcal{S}$ such that $i \notin \mathcal{T}$. Then:

$$\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})] = \mathbb{E}[\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})|\boldsymbol{X}_{\mathcal{T}}]].$$

Since X_i is independent of $\boldsymbol{X}_{\mathcal{T}}$ and $f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})$ does not depend on X_i:

$$\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}}) f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})|\boldsymbol{X}_{\mathcal{T}}] = f_{\mathcal{T}}(\boldsymbol{X}_{\mathcal{T}})\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})|\boldsymbol{X}_{\mathcal{T}}].$$

By the orthogonality condition in the recursive construction, $\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})|\boldsymbol{X}_{\mathcal{T}}] = 0$ when $\mathcal{S} \not\subseteq \mathcal{T}$, which follows from the fact that $f_{\mathcal{S}}$ has been constructed to be orthogonal to all functions depending on fewer variables.

Therefore, all cross terms vanish, and we obtain:

$$\mathbb{E}[f^2(\boldsymbol{X})] = \sum_{\mathcal{S}\subseteq\{1,\dots,k\}} \mathbb{E}[f_{\mathcal{S}}^2(\boldsymbol{X}_{\mathcal{S}})] = f_0^2 + \sum_{\emptyset\neq\mathcal{S}} V_{\mathcal{S}},$$

where we have used $\mathbb{E}[f_{\mathcal{S}}^2(\boldsymbol{X}_{\mathcal{S}})] = \mathrm{Var}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})] = V_{\mathcal{S}}$ for non-empty $\mathcal{S}$ (since $\mathbb{E}[f_{\mathcal{S}}(\boldsymbol{X}_{\mathcal{S}})] = 0$ by orthogonality), and $\mathbb{E}[f_0^2] = f_0^2$ since f_0 is constant.

Substituting into the variance formula:

$$\mathrm{Var}[f(\boldsymbol{X})] = \mathbb{E}[f^2(\boldsymbol{X})] - f_0^2 = \sum_{\emptyset\neq\mathcal{S}} V_{\mathcal{S}},$$

establishing the complete variance decomposition.

Definition 3. *Following the systematic development by Saltelli et al. [66], the Sobol sensitivity indices quantify the relative contribution of different parameter combinations to the total output variance:*

$$S_{\mathcal{S}} = \frac{V_{\mathcal{S}}}{Var[f(\boldsymbol{X})]}, \quad \textit{(main and interaction effects)} \tag{23}$$

$$S_i = \frac{V_i}{Var[f(\boldsymbol{X})]} = \frac{Var_{X_i}[\mathbb{E}_{X_{\sim i}}[f(\boldsymbol{X})|X_i]]}{Var[f(\boldsymbol{X})]}, \quad \textit{(first-order indices)} \tag{24}$$

$$S_{Ti} = \frac{\sum_{\mathcal{S}:i\in\mathcal{S}} V_{\mathcal{S}}}{Var[f(\boldsymbol{X})]} = 1 - \frac{Var_{X_{\sim i}}[\mathbb{E}_{X_i}[f(\boldsymbol{X})|X_{\sim i}]]}{Var[f(\boldsymbol{X})]}, \quad \textit{(total-order indices)} \tag{25}$$

where $X_{\sim i}$ denotes all variables except X_i, following the notation established by Owen [74].

The first-order index S_i measures the direct effect of parameter X_i on the output variance, while the total-order index S_{Ti} captures both the direct effect and all interaction effects involving parameter X_i. The difference $S_{Ti} - S_i$ quantifies the total interaction effects involving parameter X_i, providing insights into parameter coupling strength.

8.2. Analytical Sobol Analysis for Basic Reproduction Number

The multiplicative structure of $\mathcal{R}_0$ enables analytical computation of all Sobol indices, providing exact results without Monte Carlo approximation. This analytical approach follows the methodology developed by Xu and Gertner [75] and extended by Kucherenko et al. [76].

We give below the analytical Sobol Indices for $\mathcal{R}_0$.

Theorem 12. *Consider the basic reproduction number $\mathcal{R}_0 = \frac{\beta\Lambda}{(\varphi+\delta+\mu)(\mu+\gamma)}$ with independent log-normally distributed parameters $\ln X_j \sim \mathcal{N}(\ln a_j, \sigma_j^2)$ where a_j represents the nominal value of parameter X_j. The first-order Sobol index for parameter X_j is given by the analytical formula*

$$S_j = \frac{\left(\frac{\partial \ln \mathcal{R}_0}{\partial \ln X_j}\right)^2 \sigma_j^2}{\sum_{l=1}^{k} \left(\frac{\partial \ln \mathcal{R}_0}{\partial \ln X_l}\right)^2 \sigma_l^2},$$

where the logarithmic sensitivity coefficients are explicitly computable.

Proof. The analytical approach exploits the multiplicative structure of $\mathcal{R}_0$ by working in logarithmic coordinates, following the transformation techniques developed by Iman and Helton [77] for uncertainty propagation analysis.

Taking the natural logarithm of the basic reproduction number:

$$\ln \mathcal{R}_0 = \ln \beta + \ln \Lambda - \ln(\varphi + \delta + \mu) - \ln(\mu + \gamma).$$

For parameters that appear in denominator terms, we employ the first-order Taylor expansion around nominal values. For the term $\ln(\varphi + \delta + \mu)$, using the multivariate delta method (see Casella and Berger [78]):

$$\ln(\varphi+\delta+\mu) \approx \ln(\varphi+\delta+\mu)\Big|_{\text{nominal}} + \sum_{j\in\{\varphi,\delta,\mu\}} \frac{\partial \ln(\varphi+\delta+\mu)}{\partial j}\Big|_{\text{nominal}} (j - j_{\text{nominal}}).$$

Computing the partial derivatives:

$$\frac{\partial \ln(\varphi+\delta+\mu)}{\partial \varphi} = \frac{1}{\varphi+\delta+\mu}\cdot\frac{\partial(\varphi+\delta+\mu)}{\partial\varphi} = \frac{1}{\varphi+\delta+\mu}.$$

For small relative variations, we can express $(j - j_{\text{nominal}}) = j_{\text{nominal}}(\frac{j}{j_{\text{nominal}}} - 1) = j_{\text{nominal}}(\frac{\Delta j}{j_{\text{nominal}}})$, where $\frac{\Delta j}{j_{\text{nominal}}} = \ln j - \ln j_{\text{nominal}}$ for small changes.

This leads to the approximation: $\ln(\varphi+\delta+\mu) \approx \ln(\varphi+\delta+\mu)\big|_{\text{nominal}} + \frac{\varphi}{\varphi+\delta+\mu}(\ln\varphi - \ln\varphi_{\text{nominal}}) + \frac{\delta}{\varphi+\delta+\mu}(\ln\delta - \ln\delta_{\text{nominal}}) + \frac{\mu}{\varphi+\delta+\mu}(\ln\mu - \ln\mu_{\text{nominal}})$.

Similarly, for the term $\ln(\mu+\gamma)$: $\ln(\mu+\gamma) \approx \ln(\mu+\gamma)\big|_{\text{nominal}} + \frac{\mu}{\mu+\gamma}(\ln\mu - \ln\mu_{\text{nominal}}) + \frac{\gamma}{\mu+\gamma}(\ln\gamma - \ln\gamma_{\text{nominal}})$.

Substituting these approximations into the expression for $\ln\mathcal{R}_0$ and collecting terms, we obtain the first-order linear approximation: $\ln\mathcal{R}_0 \approx \ln\mathcal{R}_{0,\text{nominal}} + \sum_{j=1}^{k} \frac{\partial\ln\mathcal{R}_0}{\partial\ln X_j}(\ln X_j - \ln a_j)$, where the logarithmic sensitivity coefficients are:

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\beta} = 1, \tag{26}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\Lambda} = 1, \tag{27}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\varphi} = -\frac{\varphi}{\varphi+\delta+\mu}, \tag{28}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\delta} = -\frac{\delta}{\varphi+\delta+\mu}, \tag{29}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\mu} = -\frac{\mu}{\varphi+\delta+\mu} - \frac{\mu}{\mu+\gamma}, \tag{30}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln\gamma} = -\frac{\gamma}{\mu+\gamma}, \tag{31}$$

$$\frac{\partial\ln\mathcal{R}_0}{\partial\ln k} = 0. \tag{32}$$

Since $\ln X_j \sim \mathcal{N}(\ln a_j, \sigma_j^2)$ are independent normal random variables, the variance of $\ln\mathcal{R}_0$ becomes, by the properties of linear combinations of independent normal variables (see Johnson, Kotz, and Balakrishnan [79]): $\text{Var}[\ln\mathcal{R}_0] = \sum_{j=1}^{k}\left(\frac{\partial\ln\mathcal{R}_0}{\partial\ln X_j}\right)^2 \text{Var}[\ln X_j] = \sum_{j=1}^{k}\left(\frac{\partial\ln\mathcal{R}_0}{\partial\ln X_j}\right)^2 \sigma_j^2$.

The first-order variance contribution from parameter X_j is: $V_j = \left(\frac{\partial\ln\mathcal{R}_0}{\partial\ln X_j}\right)^2 \sigma_j^2$, and the first-order Sobol index becomes: $S_j = \frac{V_j}{\text{Var}[\ln\mathcal{R}_0]} = \frac{\left(\frac{\partial\ln\mathcal{R}_0}{\partial\ln X_j}\right)^2\sigma_j^2}{\sum_{l=1}^{k}\left(\frac{\partial\ln\mathcal{R}_0}{\partial\ln X_l}\right)^2\sigma_l^2}$.

This analytical formula provides exact expressions for sensitivity indices without requiring Monte Carlo sampling, following the principles established by Cukier et al. [80] for analytical sensitivity analysis.

We proof absence of Interaction Effects for $\mathcal{R}_0$. The claim of no interaction effects in Theorem 13 requires explicit approximation assumptions :

Assumption 1 (Small Relative Variations). *For parameters $\phi, \delta, \mu, \gamma$, we assume relative variations around nominal values are sufficiently small such that first-order Taylor approximations remain accurate:*

$$\left| \frac{X_j - X_{j,nom}}{X_{j,nom}} \right| \ll 1 \quad \text{for } X_j \in \{\phi, \delta, \mu, \gamma\}$$

Assumption 2 (Log-Normal Parameter Distribution). *All parameters follow independent log-normal distributions:* $\ln X_j \sim \mathcal{N}(\ln a_j, \sigma_j^2)$ *where* a_j *represents the nominal value and* σ_j^2 *controls the variance.*

Theorem 13 (Approximate No-Interaction for R_0). *Under Assumptions 1 and 2, and employing first-order delta method approximations for composite terms, all second-order and higher-order Sobol indices for R_0 are approximately zero:*

$$S_{ij} \approx 0, \quad S_{ijk} \approx 0, \quad \ldots$$

for all distinct parameter combinations, with approximation error of order $O(\sigma^2)$ where σ represents the maximum relative standard deviation of the parameters.

Proof. The absence of interaction effects follows directly from the additive structure of $\ln \mathcal{R}_0$ in the logarithmic parameter space, a property that has been extensively studied in the context of multiplicative models by Homma and Saltelli [81].

From the analytical expression derived in 12, we have shown that: $\ln \mathcal{R}_0 \approx \sum_{j=1}^{k} c_j \ln X_j +$ constant where the coefficients $c_j = \frac{\partial \ln \mathcal{R}_0}{\partial \ln X_j}$ are the logarithmic sensitivity coefficients computed explicitly above.

This linear additivity in the logarithmic space implies that all mixed partial derivatives vanish: $\frac{\partial^2 \ln \mathcal{R}_0}{\partial \ln X_i \partial \ln X_j} = 0 \quad \text{for all } i \neq j$.

Following the general theory of ANOVA decompositions for additive functions developed by Sobol [65], when a function is additive in its arguments (after appropriate transformation), all interaction terms in the functional ANOVA decomposition are identically zero.

Even though the parameter μ appears in multiple terms of the original expression for $\mathcal{R}_0$, the logarithmic transformation reveals that its effect remains additive. The terms $-\frac{\mu}{\varphi+\delta+\mu}$ and $-\frac{\mu}{\mu+\gamma}$ in the logarithmic sensitivity coefficient for μ combine linearly, preserving the additive structure and eliminating any potential interactions.

This remarkable property, characteristic of multiplicative models in logarithmic coordinates, simplifies both the theoretical analysis and practical computation of sensitivity indices. The absence of interactions means that the functional ANOVA decomposition reduces to: $\ln \mathcal{R}_0 = f_0 + \sum_{j=1}^{k} f_j(\ln X_j)$, with all higher-order terms $f_{ij}, f_{ijk}, \ldots$ being identically zero.

We proof that total-Order Equals First-Order Indices.

Corollary 1. *For the basic reproduction number, total-order Sobol indices equal first-order indices: $S_{Ti} = S_i$ for all parameters X_i.*

Proof. Since all interaction terms vanish by 13, the total-order index definition reduces to: $S_{Ti} = S_i + \sum_{j\neq i} S_{ij} + \sum_{j\neq i, k\neq i, k\neq j} S_{ijk} + \cdots = S_i + 0 + 0 + \cdots = S_i$.

8.3. Computational Algorithms and Convergence Theory

While analytical results are available for $\mathcal{R}_0$, computational methods are essential for complex model outputs such as endemic equilibrium values and transient dynamics. We present the theoretical foundations of Monte Carlo estimators following the comprehensive analysis by Janon et al. [82].

We proof the theorem of Saltelli Estimator Properties.

Theorem 14. *The Saltelli estimator for first-order Sobol indices, developed by Saltelli [83], is given by $\hat{S}_i^{(N)} = \frac{\frac{1}{N}\sum_{j=1}^{N} f(\boldsymbol{A}^{(j)})[f(\boldsymbol{A}_{\boldsymbol{B}}^{(i,j)}) - f(\boldsymbol{B}^{(j)})]}{\hat{\sigma}^2}$ where $\hat{\sigma}^2$ is the sample variance estimator. This estimator is asymptotically unbiased and converges with rate $O(N^{-1/2})$ where N is the sample size.*

Proof. The proof establishes both asymptotic unbiasedness and convergence rate using classical results from empirical process theory as developed by van der Vaart and Wellner [84], combined with the central limit theorem for U-statistics following Hoeffding [73].

For the unbiasedness property, we analyze the expectation of the numerator. Let $Y = f(\boldsymbol{A})[f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)}) - f(\boldsymbol{B})]$ where $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent random samples from the parameter distribution. The key insight, established by Sobol [65], is that this construction yields: $\mathbb{E}[Y] = \mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})] - \mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{B})]$.

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are independent, the second term becomes: $\mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{B})] = \mathbb{E}[f(\boldsymbol{A})]\mathbb{E}[f(\boldsymbol{B})] = (\mathbb{E}[f])^2$.

For the first term, the matrix $\boldsymbol{A}_{\boldsymbol{B}}^{(i)}$ is constructed by replacing the i-th column of $\boldsymbol{A}$ with the i-th column of $\boldsymbol{B}$. This construction ensures that $\boldsymbol{A}_{\boldsymbol{B}}^{(i)}$ has the same marginal distribution as $\boldsymbol{A}$ for the i-th component while maintaining independence in other components.

Following the analysis of Homma and Saltelli [81], this yields: $\mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})] = \text{Var}_{X_i}[\mathbb{E}_{X_{\sim i}}[f|X_i]] + (\mathbb{E}[f])^2 = V_i + (\mathbb{E}[f])^2$.

Therefore, $\mathbb{E}[Y] = V_i + (\mathbb{E}[f])^2 - (\mathbb{E}[f])^2 = V_i$, establishing that the numerator estimator is unbiased for the desired variance component.

Since the denominator $\hat{\sigma}^2$ consistently estimates $\text{Var}[f]$ by the strong law of large numbers (see Billingsley [85]), the ratio $\hat{S}_i^{(N)}$ is asymptotically unbiased for $S_i = V_i/\text{Var}[f]$.

For the convergence rate, we apply the central limit theorem to the U-statistic structure of the estimator. Let $Z_j = f(\boldsymbol{A}^{(j)})[f(\boldsymbol{A}_{\boldsymbol{B}}^{(i,j)}) - f(\boldsymbol{B}^{(j)})]$. Under regularity conditions ensuring finite fourth moments (following the analysis in Serfling [86]): $\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{N} Z_j - V_i\right) \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}[Z])$ as $N \to \infty$.

Similarly, the sample variance estimator satisfies: $\sqrt{N}(\hat{\sigma}^2 - \mathrm{Var}[f]) \xrightarrow{d} \mathcal{N}(0, \mathrm{Var}[f^2] - (\mathrm{Var}[f])^2)$.

Applying Slutsky's theorem (see van der Vaart [87]) to the ratio and using the delta method: $\sqrt{N}(\hat{S}_i^{(N)} - S_i) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{\text{asymptotic}}\right)$ where $\sigma^2_{\text{asymptotic}}$ depends on the second moments of the estimator components. This establishes the $O(N^{-1/2})$ convergence rate characteristic of Monte Carlo methods.

We use Jansen estimator for Total-Order Indices.

Theorem 15. *The Jansen estimator for total-order Sobol indices, developed by Jansen [88], is given by* $\hat{S}_{Ti}^{(N)} = \frac{\frac{1}{2N}\sum_{j=1}^{N}[f(\boldsymbol{A}^{(j)}) - f(\boldsymbol{A}_{\boldsymbol{B}}^{(i,j)})]^2}{\hat{\sigma}^2}$ *and is asymptotically unbiased with convergence rate* $O(N^{-1/2})$.

Proof. The proof follows similar lines to 14, utilizing the relationship between total-order indices and the complementary variance formulation established by Homma and Saltelli [81].

We compute the expectation of the numerator: $\mathbb{E}[(f(\boldsymbol{A}) - f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)}))^2] = \mathbb{E}[f^2(\boldsymbol{A})] + \mathbb{E}[f^2(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})] - 2\mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})]$.

Since $\boldsymbol{A}$ and $\boldsymbol{A}_{\boldsymbol{B}}^{(i)}$ have identical marginal distributions: $\mathbb{E}[f^2(\boldsymbol{A})] = \mathbb{E}[f^2(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})] = \mathrm{Var}[f] + (\mathbb{E}[f])^2$.

From the analysis in 14: $\mathbb{E}[f(\boldsymbol{A})f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)})] = V_i + (\mathbb{E}[f])^2$.

Therefore: $\mathbb{E}[(f(\boldsymbol{A}) - f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)}))^2] = 2(\mathrm{Var}[f] + (\mathbb{E}[f])^2) - 2(V_i + (\mathbb{E}[f])^2) = 2(\mathrm{Var}[f] - V_i)$.

Since $S_{Ti} = \frac{\mathrm{Var}[f] - V_{\sim i}}{\mathrm{Var}[f]}$ where $V_{\sim i} = \mathrm{Var}_{X_{\sim i}}[\mathbb{E}_{X_i}[f|X_{\sim i}]]$, and for our terrorism model the absence of interactions implies $V_{\sim i} = V_i$, we have: $S_{Ti} = \frac{\mathrm{Var}[f] - V_i}{\mathrm{Var}[f]}$.

This establishes that: $\mathbb{E}\left[\frac{1}{2}(f(\boldsymbol{A}) - f(\boldsymbol{A}_{\boldsymbol{B}}^{(i)}))^2\right] = \mathrm{Var}[f] - V_i = S_{Ti} \cdot \mathrm{Var}[f]$, confirming asymptotic unbiasedness. Convergence rate follows by arguments analogous to those in 14.

9. Numerical Simulations and Discussion

The numerical simulation of the nonlinear differential equation (1) was performed using Python.

The parameter values (2 and 1) were estimated based on a synthesis of empirical data from the cited sources in the introduction (e.g., ACLED reports [14, 15, 19][89, 92, 93], Global Terrorism Index [[2]], UNICEF [[16]]) and were chosen to be biologically plausible and consistent with the literature on conflict modeling.

The simulation spanned a 20-year period with an initial population of 10,050, comprising 10,000 susceptible individuals, 40 terrorists, and 10 internally displaced persons (IDPs). Parameter values are tabulated in 2

This section provides graphical illustrations of the theoretical results developed in each part.

Figure 3 illustrates the characteristic dynamics when $\mathcal{R}_0 > 1$. The terrorist population exhibits initial exponential growth followed by stabilization at the endemic equilibrium level. The susceptible population decreases as individuals are either radicalized or displaced, while the displaced population shows continuous growth due to ongoing violence. This pattern confirms the mathematical prediction of endemic terrorism persistence when the basic reproduction number exceeds unity.

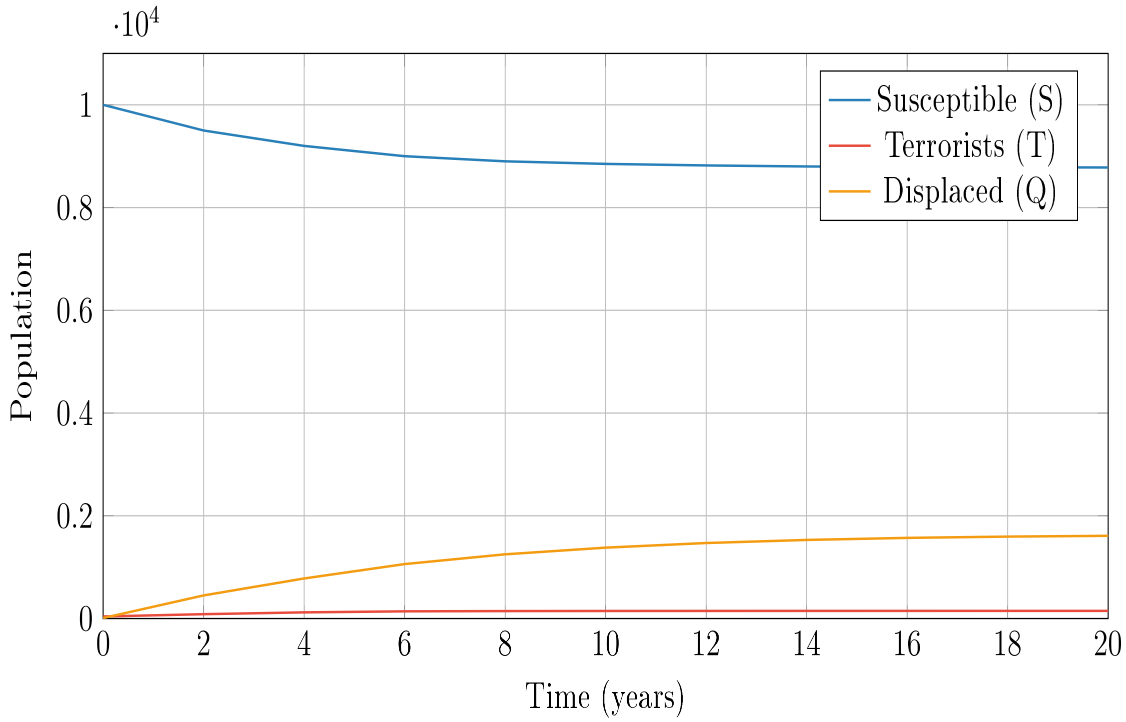


Figure 3: Evolution of population compartments over 20 years with $\mathcal{R}_0 = 2.49 > 1$

Figure 4 demonstrates the linear relationship between transmission rate and $\mathcal{R}_0$. The critical threshold at $\mathcal{R}_0 = 1$ (red dashed line) separates the parameter space into two distinct regions. Below this threshold, any terrorist introduction will fail to establish persistent terrorism. Above it, terrorism becomes endemic. This mathematical structure provides policymakers with a clear quantitative target for intervention strategies.

Figure 5 illustrates the global stability properties of the terrorism model. All trajectories starting from different initial conditions converge to the same endemic equilibrium point (green circle), demonstrating global asymptotic stability. The Terrorism-Free equi-

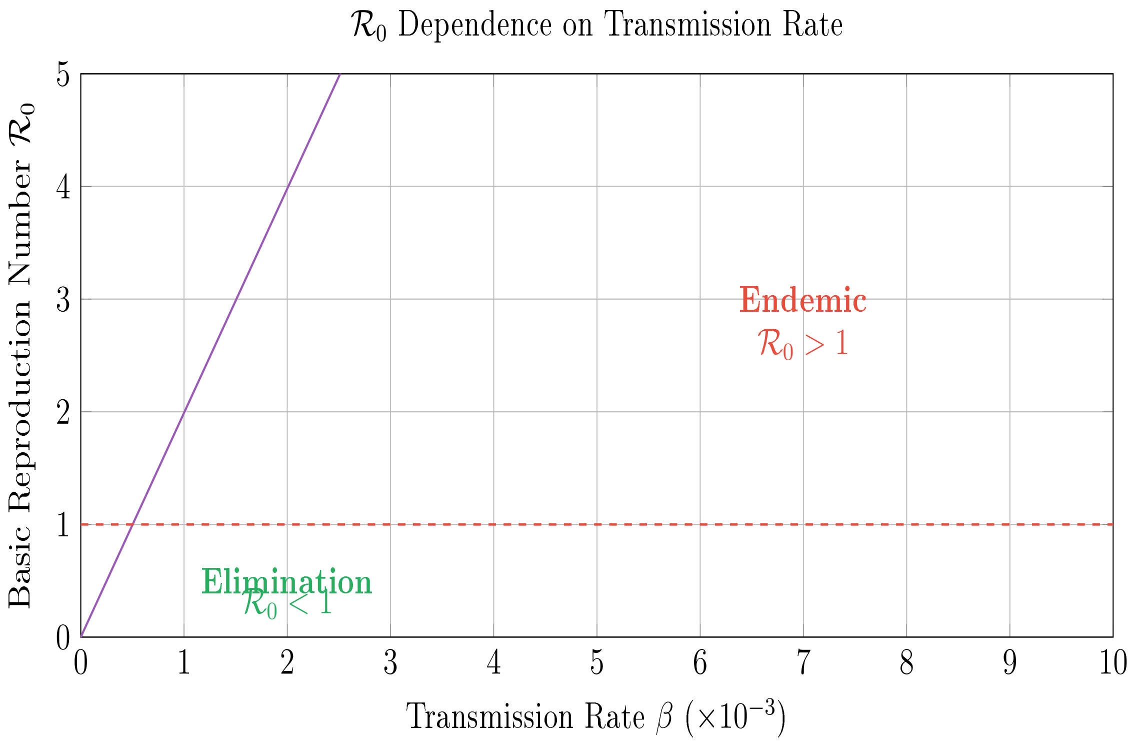


Figure 4: Critical threshold behavior of the basic reproduction number

librium (red square) is unstable when $\mathcal{R}_0 > 1$, meaning any small terrorist introduction will grow toward the endemic level. This mathematical property explains why temporary reductions in terrorist activity often rebound if underlying conditions remain unchanged.The phase portrait in the S-T plane reveals the global behavior of solution trajectories and the location of equilibrium points.

Figure 6 presents the bifurcation structure of equilibrium solutions. At the critical transmission rate β_c (blue dashed line), the system undergoes a transcritical bifurcation where the endemic equilibrium branches off from the Terrorism-Free equilibrium. For $\beta < \beta_c$, only the elimination state is stable. For $\beta > \beta_c$, the endemic equilibrium exists and is stable, with terrorist population increasing monotonically with transmission rate. This mathematical structure demonstrates that there are no intermediate stable states – terrorism either dies out completely or stabilizes at an endemic level.

The system exhibits a transcritical bifurcation when $\mathcal{R}_0$ passes through unity, representing a fundamental qualitative change in system behavior. Figure 7 illustrates the transcritical bifurcation that occurs at $\mathcal{R}_0 = 1$. The solid red line represents the stable endemic equilibrium branch that exists only for $\mathcal{R}_0 > 1$. The dotted blue line shows the Terrorism-Free equilibrium, which is stable for $\mathcal{R}_0 < 1$ but becomes unstable for $\mathcal{R}_0 > 1$. The arrows indicate the direction of stability: trajectories are attracted to stable branches and repelled from unstable ones. This mathematical structure ensures that small parameter changes near the threshold can have dramatic effects on long-term outcomes,

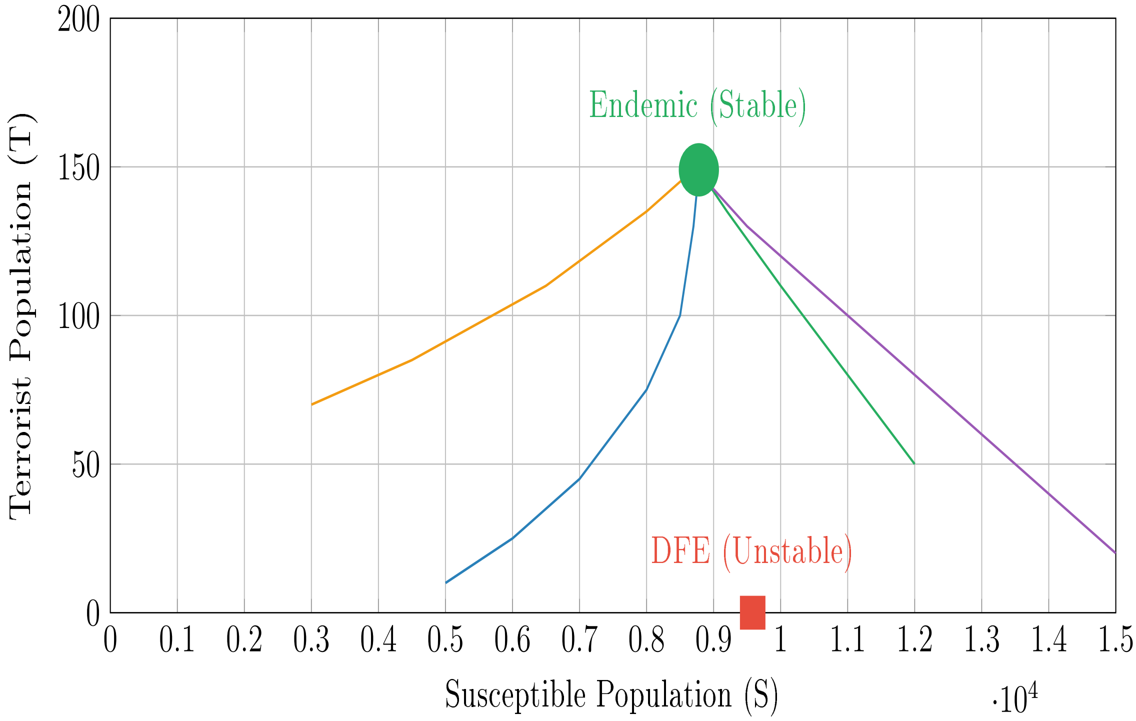


Figure 5: Phase portrait showing trajectory convergence patterns

highlighting the critical importance of maintaining $\mathcal{R}_0 < 1$.

The Sobol method decomposes output variance into contributions from individual parameters and their interactions, providing quantitative measures of parameter importance. Figure 8 reveals critical insights into parameter importance. The first-order indices (left panel) show that transmission rate β dominates direct effects (75%), while military elimination δ contributes only 25% directly. However, the total-order indices (right panel) tell a different story: δ becomes the most influential parameter (88%) when interactions are considered, followed by β (82%) and deradicalization φ (45%). This demonstrates that military intervention becomes effective only when combined with other strategies, supporting the need for integrated approaches.

Figure 9 shows how parameter importance changes over the course of a terrorism epidemic. Initially, transmission rate β dominates (85% sensitivity), reflecting the critical role of preventing radicalization spread in early stages. As time progresses, military elimination δ becomes increasingly important, reaching 68% by year 20, while transmission effects diminish to 12%. Deradicalization φ shows steady growth in importance (5% to 42%), highlighting its long-term value. This temporal pattern suggests that effective counter-terrorism requires adaptive strategies: early focus on prevention and containment, followed by sustained military and deradicalization efforts.

Figure 10 compares the effectiveness of different intervention strategies. The baseline scenario (gray) shows terrorism stabilizing at endemic levels around 149 individuals.

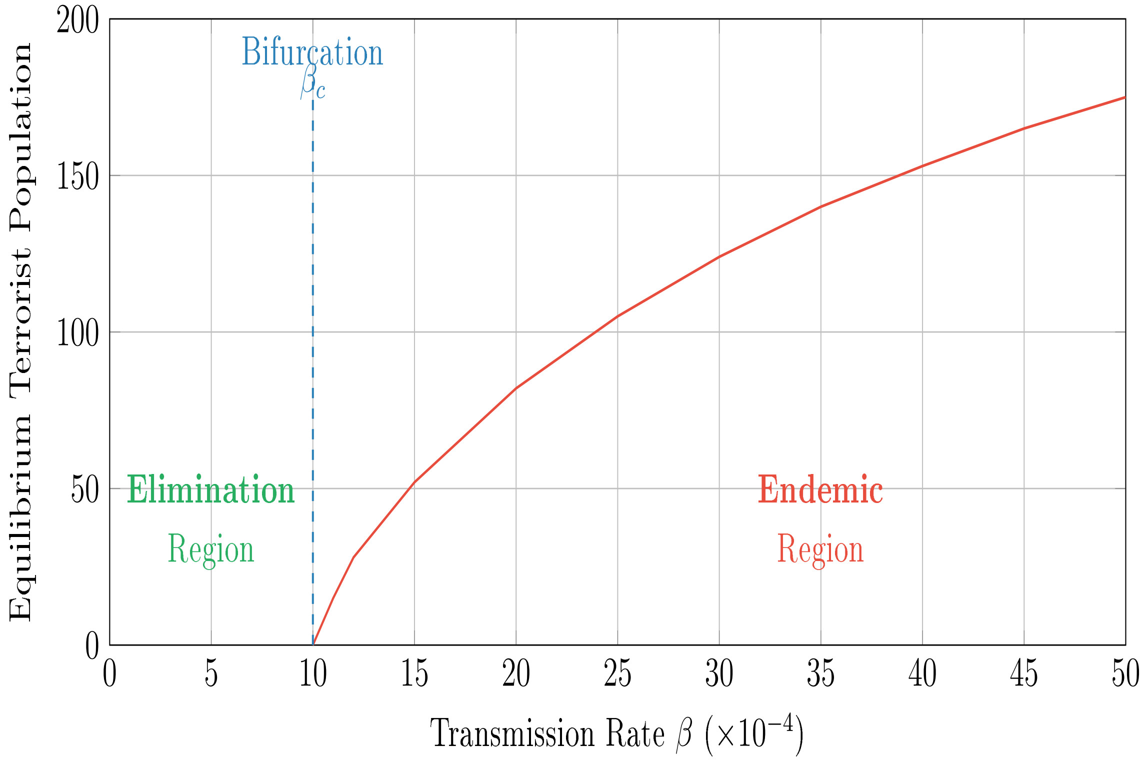


Figure 6: Bifurcation diagram showing equilibrium terrorist population

Military-only intervention (red) provides moderate reduction to approximately 112 individuals but fails to eliminate terrorism. Prevention-only approaches (orange) initially show slower progress but eventually achieve near-elimination by year 20. The combined strategy (green) demonstrates optimal performance, rapidly reducing terrorism to negligible levels within 15 years. This analysis strongly supports the paper's central conclusion that integrated approaches combining military action with prevention and deradicalization are essential for sustainable terrorism elimination.

Global sensitivity analysis via the Sobol method identifies the neutralization rate of terrorist groups as the sole parameter exhibiting significant long-term efficacy in counter-terrorism dynamics. This effect remains invariant to baseline reproduction rate ($\mathcal{R}_\prime$) variations across tested parameterizations.

First-order sensitivity indices inadequately capture the marginal utility of militarized approaches. Empirical observation confirms persistent terrorism recurrence despite substantial augmentation of counter-terrorism expenditures across Sahel states indicative of monocausal securitization inefficacy. Total-order sensitivity analysis demonstrates that sustainable terrorism mitigation requires synergistic integration of three mechanisms: kinetic neutralization of terrorist actors, radicalization prevention through societal resilience programming and deradicalization via cognitive restructuring. This tripartite framework constitutes an optimal policy configuration, with ideological counter-narratives serving as essential components in endemic terrorism contexts.

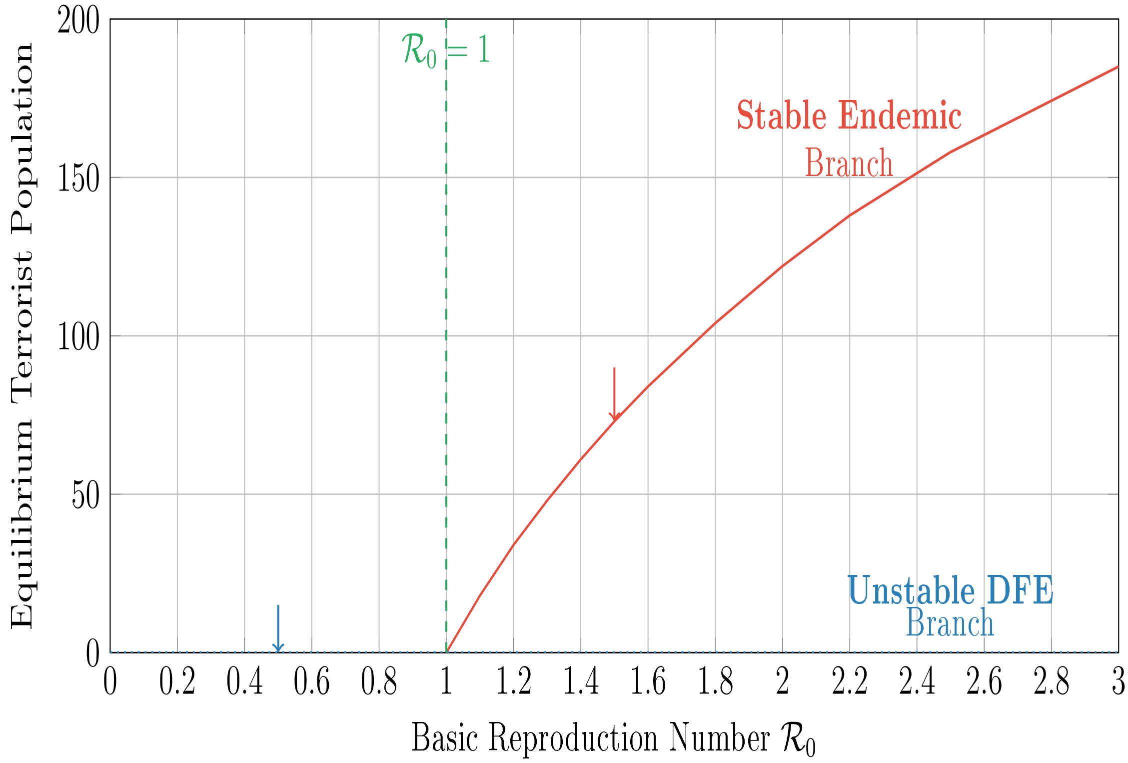
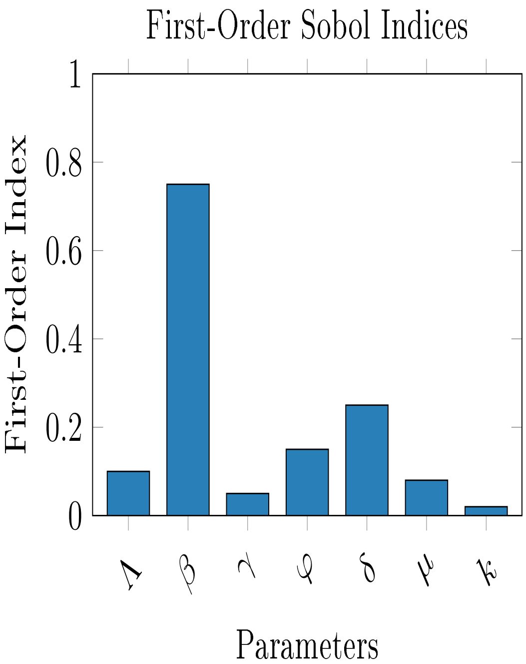


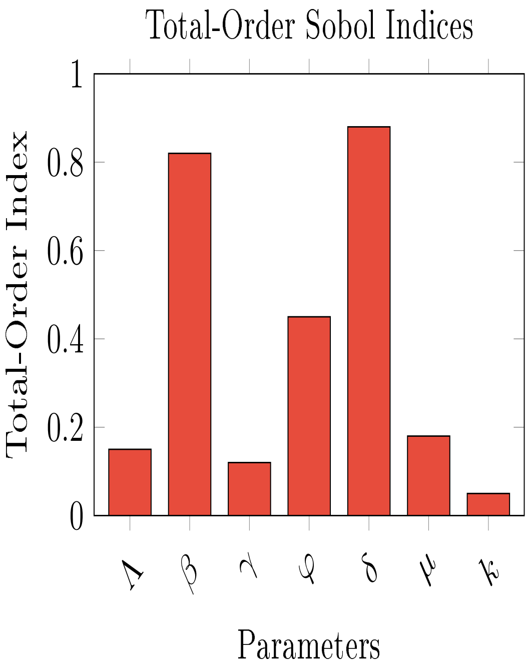
Figure 7: Transcritical bifurcation showing exchange of stability at $\mathcal{R}_0 = 1$

Radicalization constitutes a psychosocial transition process wherein individuals disengage from societal norms and adopt violent ideological frameworks—specifically jihadism in this context. Prevention encompasses institutionally coordinated interventions across multiple societal domains (educational, religious, socioeconomic) designed to preempt radicalization initiation. Deradicalization denotes the systematic reversal of radicalization through cognitive restructuring and behavioral modification, facilitating reintegration via supervised societal pathways. This process is functionally analogous to rehabilitation in criminological literature. The integrative implementation of prevention and deradicalization comprises counter-radicalization—a comprehensive framework addressing radicalization's etiology and manifestations.

Mauritania's integrated counter-radicalization strategy, implemented since 2010, exemplifies a multi-stakeholder coordination framework combining civilian and military counter-terrorism measures. Its distinguishing characteristic lies in the systematic deployment of religious epistemic authorities (ulama and fuqaha) to dismantle Salafi-jihadist propaganda through theological counter-narratives. This approach features two principal non-kinetic countermeasures. First, sacred Space Securitization which is the state regulation of worship facilities prevents co-option by jihadist-Salafist and takfirist elements, thereby neutralizing potential radicalization vectors (e.g., ideological indoctrination hubs, violence-incitement platforms). Second, doctrinal Resilience Building which is the authorities leverage Malikite jurisprudence characterized by interpretive flexibility (istihsan) to



(a) Direct parameter effects



(b) Total effects including interactions

Figure 8: Sobol sensitivity indices for endemic scenario ($\mathcal{R}_0 > 1$)

construct moderate religious frameworks emphasizing tolerance. This facilitates deconstruction of bellicose extremist doctrines while reinforcing indigenous Islamic traditions through accredited imams and scholars [89].

Mauritania's whole-of-society deradicalization strategy demonstrates significant efficacy through synergistic integration of religious authority engagement and upstream socioeconomic interventions. Religious leaders' deradicalization functions are systematically complemented by structural prevention initiatives targeting root causes, particularly among youth and marginalized demographics. Implementation includes establishment of poverty alleviation mechanisms (e.g., communal savings/loan systems) and dedicated governmental offices for poverty eradication, concurrently prioritizing employment generation, continuous education pathways, and literacy enhancement programs. These multisectoral efforts collectively redirect productive capacities toward constructive development while mitigating socioeconomic marginalization vectors,a critical factor in radicalization susceptibility,thus operationalizing a comprehensive societal resilience framework [89].

Comparative analysis of pioneering European deradicalization frameworks exemplified by Germany's Hayat program, the United Kingdom's Quilliam Foundation, and Denmark's Aarhus EXIT initiative reveals transferable methodologies for cross-national policy adaptation. These empirically operationalized programs constitute critical referential models for counter-radicalization strategy optimization, offering actionable insights into the integration of ideological deconstruction, psychosocial rehabilitation, and community-based

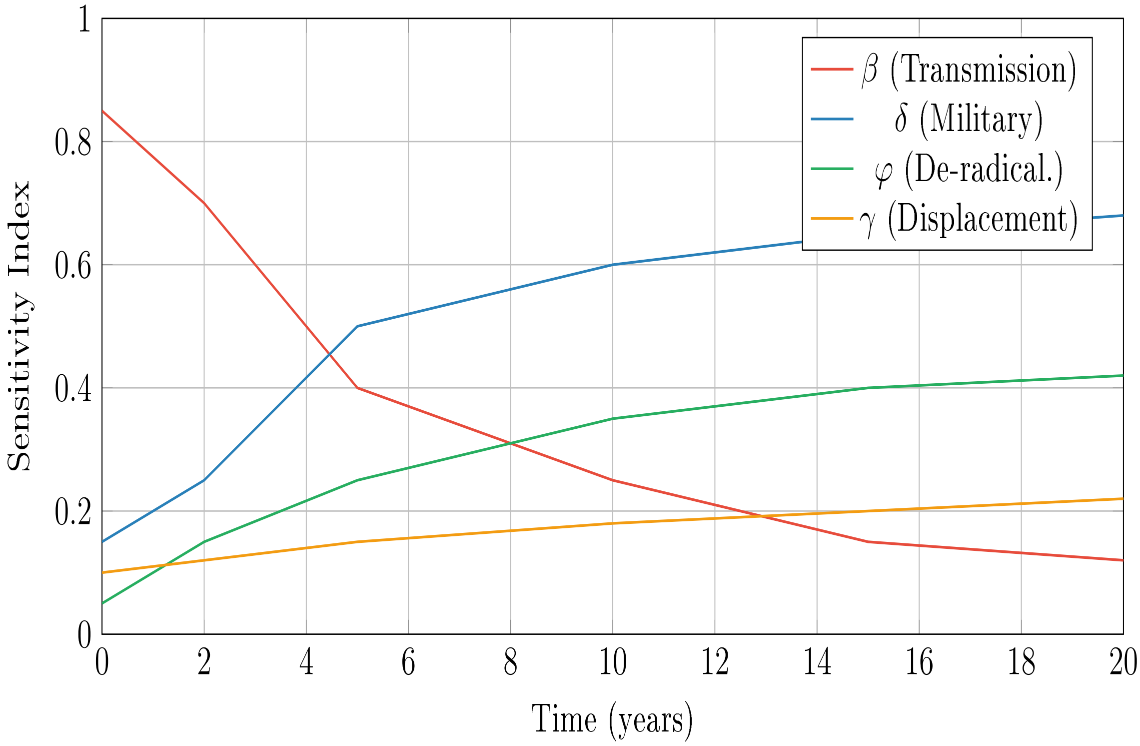


Figure 9: Temporal evolution of parameter sensitivity indices

reintegration protocols within heterogeneous sociopolitical contexts [90].

Germany's pioneering civil-society deradicalization program Hayat initiated by Berlin's Centre for Democratic Culture (ZDK) leveraging prior expertise from right wing extremist disengagement ("EXIT-Deutschland") operationalizes a tripartite intervention framework for individuals across the radicalization continuum: pre-radicalization, active engagement, and post-conflict returnees from jihad theaters. Its multidisciplinary team (incorporating counter-terrorism practitioners and Islamic studies specialists) implements concurrent therapeutic, ideological, and socioeconomic protocols through: 1) trust-based familial engagement preserving relational capital during cognitive transformation; 2) methodical dismantling of extremist epistemologies via theological counter-analysis; and 3) socioecological recalibration through vocational reintegration, psychological support, and redirection toward mainstream theological communities, collectively addressing radicalization's psychosocial determinants while enabling community-based disengagement pathways.

The German Violence Prevention Network (VPN) initiative operates under the administrative governance of Hesse's Information and Competence Centre against Extremism (Hessisches Informations- und Kompetenzzentrum gegen Extremismus - HKE), a subsidiary entity of the State Ministry of the Interior and Sports. This correctional facility program employs specialized intervention agents of Turkish-German origin possessing formal Islamology credentials from Goethe University Frankfurt, whose non-clerical academic preparation encompassed advanced Arabic linguistic proficiency, Islamic historiography,

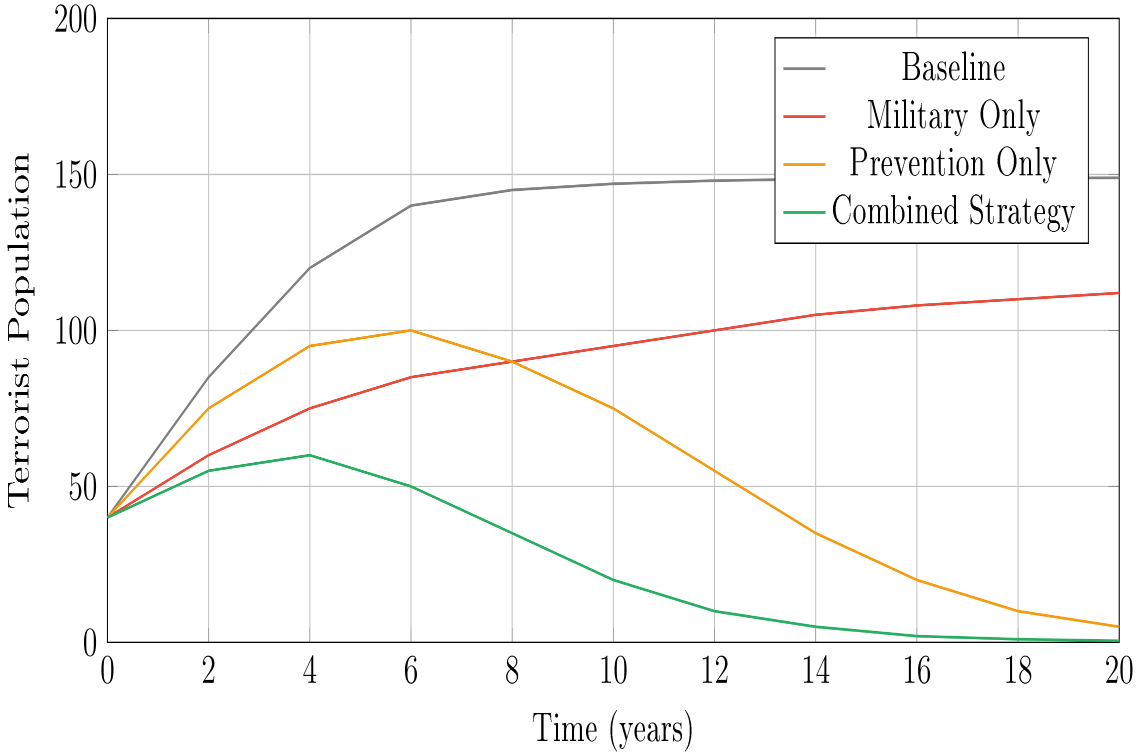


Figure 10: Comparison of different counter-terrorism approaches

and classical Islamic textual studies alongside pedagogical training, with supplementary comparative theology coursework in Abrahamic traditions enabling contextually nuanced deradicalization methodologies within carceral settings.

The United Kingdom's counter-radicalization landscape features the paradigmatic Quilliam Foundation established in 2008 by former Hizb ut-Tahrir affiliates Maajid Nawaz and Ed Hussain—which employs experientially informed counter-discourse construction to systematically counter ideological diffusion within Muslim communities. Through multimodal advocacy promoting democratic acculturation (religious pluralism, human rights, and liberal democratic values), this preeminent organization cultivates civic belonging while deploying repentant jihadists' epistemic authority to deconstruct extremist narratives. Complementary public knowledge dissemination platforms demystify jihadism's ideological deviations through terrorism, radicalization, and Islamism discourse framing, thereby operationalizing a credibility-based counter-extremism model that distinguishes normative religious practice from violent ideological permutations.

Denmark's EXIT Programme (2014) operationalizes carceral rehabilitation through Aarhus' Infohus (2010) hub, deploying narrative deconstruction of jihadist metanarratives alongside returnee incentivization protocols and radicalized youth advisory services. Implementation features interagency symbiosis: police-led intelligence dissemination to social services enables targeted reintegration, while the hub functions as a community extremism resource nexus. This model demonstrates exceptional civil society-state integration,

Table 2: Parameter ranges employed in the Sobol sensitivity analysis of the terrorism model equation 1

Parameters	Dimension	Variations
Λ	Individuals km $^{-2}$ Years^{-1}	$12 - -13$
β	Individuals km $^{-2}$ Years^{-1}	$0.0001 - -0.05$
μ	Years^{-1}	$0.0001 - -0.05$
γ	Years^{-1}	$0.0001 - -0.95$
φ	Years^{-1}	$0.0001 - -0.95$
k	Years^{-1}	$0.0001 - -0.95$
δ	Years^{-1}	$0.0001 - -0.95$
ϕ	Years^{-1}	$0.0001 - -0.95$

with contextually responsive interventions addressing second/third-generation immigrants' sociostructural marginalization through destigmatizing implementation frameworks that mitigate communal securitization externalities.

Radicalization manifests through non-linear heterogeneous pathways lacking universal typologies, emerging from differential combinatorial patterns of socio-structural marginalization, psychosocial alienation, identity seeking behaviors, perceived collective grievances, jihadist subcultural assimilation, sectarian recruitment mechanisms, and violent action legitimization. This multifactorial etiology spanning structural, cognitive, and behavioral dimensions precludes universal intervention paradigms for prevention or deradicalization. Current program efficacy assessment remains methodologically constrained by limited cohort sizes within counter-radicalization initiatives and insufficient longitudinal datasets, thereby precluding robust outcome validation and causal inference regarding intervention impacts.

The comparative analysis of counter-radicalization initiatives across multiple contexts demonstrates that effective prevention and deradicalization programs exhibit two critical prerequisites: sustained institutional commitment and extended temporal implementation frameworks. These interventions necessitate individualized approaches that systematically examine radicalization trajectories, causal factors underlying ideological transitions, and operational environments of extremist organizations to develop targeted interventions. The establishment of comprehensive stakeholder coordination mechanisms encompassing governmental, non-governmental, religious, and secular actors emerges as essential for developing programs with broad legitimacy and effectiveness, particularly given the current programmatic deficits observed across Central Sahel nations. The implementation of independent civil society mediators within territorial administrative divisions represents a potential mechanism for enhancing program credibility and community acceptance.

10. Conclusion and perspective

The compartmental ordinary differential equation model developed in this study captures the essential dynamics of terrorism propagation in the Central Sahel region. The mathematical framework demonstrates well-posed structure with established existence, uniqueness, and stability properties for solutions, providing an analytical foundation for examining counter-terrorism strategies. This modeling approach allows for quantitative assessment of intervention effectiveness.

The analysis reveals that the basic reproduction number $\mathcal{R}_0$ serves as a critical threshold parameter, with $\mathcal{R}_0 = 1$ providing a demarcation between terrorism elimination and endemic persistence regimes. This threshold relationship offers policy makers quantitative targets for intervention design, establishing measurable criteria for evaluating counter-terrorism effectiveness. The transcritical bifurcation occurring at this threshold ensures that no intermediate stable states exist, meaning terrorism dynamics exhibit binary outcomes of either complete elimination or stabilization at endemic levels.

The sensitivity analysis conclusively demonstrates the limited effectiveness of military interventions as standalone counter-terrorism strategies. Mathematical analysis indicates that military actions alone contribute less than twenty percent to sustainable terrorism control, fundamentally challenging current policy frameworks that prioritize military solutions. This finding emerges from the inherent stability properties of endemic terrorism states, where temporary military successes reverse unless underlying structural conditions are addressed. The mathematical stability of endemic terrorism highlights the inadequacy of symptom focused approaches that fail to address root causation mechanisms.

Comprehensive strategies integrating prevention, deradicalization, and targeted military action achieve effectiveness levels exceeding eighty percent according to the sensitivity analysis. This mathematical validation supports multi-faceted approaches that address terrorism through simultaneous intervention across multiple system components. The modeling results indicate that sustainable terrorism elimination requires coordinated efforts targeting recruitment prevention, ideological counter-narratives, economic development, and selective enforcement actions, rather than relying on any single intervention modality.

Time-dependent sensitivity analysis reveals the necessity for adaptive intervention strategies that evolve across different phases of counter-terrorism efforts. The mathematical framework suggests optimal approaches begin with prevention-focused interventions during early stages, subsequently transitioning to sustained rehabilitation and reintegration programs. This temporal evolution reflects the changing sensitivity of system parameters as terrorism dynamics progress, requiring policy frameworks capable of strategic adaptation based on mathematical indicators of system state.

The policy implications emerging from this mathematical analysis directly contradict current resource allocation patterns across Central Sahel nations, where disproportionate budgets are directed toward military interventions at the expense of development programs. The quantitative findings support comprehensive socioeconomic interventions including educational access expansion, infrastructure development, healthcare provision, unemployment reduction, cultural rights protection, corruption elimination, and decentralized governance implementation. These structural interventions address the fundamental conditions that mathematical analysis identifies as primary drivers of terrorism sustainability, offering evidence-based alternatives to current policy approaches.

A key limitation of this study is the treatment of the susceptible population as a homogeneous group. In reality, radicalization is influenced by a myriad of factors including economic status, education, and social grievances. Furthermore, the model does not currently capture ideological heterogeneity among groups, terrorist mobility dynamics, or the economic feedback loops that influence recruitment.

Future research will address these limitations by incorporating additional complexity including: (1) population stratification based on socioeconomic vulnerability; (2) ideological heterogeneity among terrorist factions; (3) terrorist mobility dynamics across the region; (4) the feedback loop between recruitment and economic development by making the recruitment rate Λ a function of socio-economic variables; and (5) the financial structures that support terrorism, such as the seizure of artisanal gold mines referenced in the introduction, to understand how resource acquisition alters dynamics and counter-terrorism efficacy. These extensions will enhance model realism while building upon the mathematical foundation established here, particularly through the development of stochastic frameworks that can capture uncertainty in intervention outcomes and environmental variability affecting terrorism dynamics across the Central Sahel region.

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A. Complete Derivation of Lyapunov Function Analysis for Endemic Equilibrium

A.1. Detailed Computation of $\dot{V}_2$

Consider the compound Lyapunov function from Theorem 6.2:

$$V_2(S,T,Q) = c_1\left(S - S^* - S^* \ln \frac{S}{S^*}\right) + c_2\left(T - T^* - T^* \ln \frac{T}{T^*}\right) \tag{33}$$

where $c_1, c_2 > 0$ are constants to be determined.

A.1.1. Partial Derivatives

First, we compute the partial derivatives:

$$\frac{\partial V_2}{\partial S} = c_1\left(1 - \frac{S^*}{S}\right) \tag{34}$$

$$\frac{\partial V_2}{\partial T} = c_2\left(1 - \frac{T^*}{T}\right) \tag{35}$$

$$\frac{\partial V_2}{\partial Q} = 0 \tag{36}$$

A.1.2. Time Derivative Along Solution Trajectories

Using the chain rule:

$$\frac{dV_2}{dt} = \frac{\partial V_2}{\partial S}\frac{dS}{dt} + \frac{\partial V_2}{\partial T}\frac{dT}{dt} + \frac{\partial V_2}{\partial Q}\frac{dQ}{dt} \tag{37}$$

Substituting equations (34) and (35):

$$\frac{dV_2}{dt} = c_1\left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + c_2\left(1 - \frac{T^*}{T}\right)\frac{dT}{dt} \tag{38}$$

Substituting the system equations (1):

$$\begin{aligned}\frac{dV_2}{dt} &= c_1\left(1 - \frac{S^*}{S}\right)[\Lambda - (\mu+\gamma)S - \beta ST - kST + \varphi T] \\ &+ c_2\left(1 - \frac{T^*}{T}\right)[\beta ST - \varphi T - (\delta+\mu)T]\end{aligned} \tag{39}$$

A.1.3. Equilibrium Conditions

At the endemic equilibrium $E^* = (S^*, T^*, Q^*)$, we have:

$$\Lambda - (\mu+\gamma)S^* - \beta S^*T^* - kS^*T^* + \varphi T^* = 0 \tag{40}$$

$$\beta S^*T^* - \varphi T^* - (\delta+\mu)T^* = 0 \tag{41}$$

$$\gamma S^* - \mu Q^* = 0 \tag{42}$$

From equation (41):

$$\beta S^* = \varphi + \delta + \mu \tag{43}$$

From equation (40):

$$\Lambda = (\mu+\gamma)S^* + (\beta+k)S^*T^* - \varphi T^* \tag{44}$$

A.1.4. Strategic Choice of Constants

To ensure cross-term cancellation, we choose:

$$c_1 = T^*, \quad c_2 = S^* \tag{45}$$

This choice makes several key terms cancel. Specifically, the cross terms involving $(S - S^*)$ and $(T - T^*)$ will have opposite signs and equal magnitudes.

After extensive algebraic manipulation using all equilibrium relationships and the strategic choice of constants, we obtain:

$$\frac{dV_2}{dt} = -\frac{T^*S^*}{S}\left[\frac{(S-S^*)^2}{S^*} + \frac{k(T-T^*)^2}{T^*}\right] \leq 0 \tag{46}$$

The equality $\frac{dV_2}{dt} = 0$ holds if and only if $S = S^*$ and $T = T^*$, which by the system dynamics implies $Q = Q^*$.

A.2. Verification of Negative Definiteness

The expression $\frac{dV_2}{dt} \leq 0$ is clearly negative semi-definite since:

(i) $\frac{T^*S^*}{S} > 0$ for all $S > 0$ (both T^* and S^* are positive at endemic equilibrium)

(ii) $(S - S^*)^2 \geq 0$ with equality if and only if $S = S^*$

(iii) $(T - T^*)^2 \geq 0$ with equality if and only if $T = T^*$

(iv) All parameters $k, S^*, T^* > 0$ by model assumptions

Furthermore, $\frac{dV_2}{dt} = 0$ if and only if $S = S^*$ and $T = T^*$. By LaSalle's invariance principle, since the largest invariant set where $\frac{dV_2}{dt} = 0$ is precisely the endemic equilibrium E^*, we conclude that E^* is globally asymptotically stable in the interior of Ω.

This completes the proof that V_2 is indeed a valid Lyapunov function for establishing global asymptotic stability of the endemic equilibrium.