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Type-2 Neutrosophic Aczel-Alsina Hamy Mean Aggregation Operators for Multiple-Attribute Decision-Making Problems

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Abstract. This study introduces novel aggregation operators for Type-2 Neutrosophic Number Sets (T2NNs) by integrating the Hamy Mean with the Aczel-Alsina t-norm and t-conorm operations. The Hamy Mean, a mathematical averaging technique, is particularly effective in contexts characterized by uncertainty and ambiguity, such as fuzzy set theory. Leveraging this, two aggregation operators are proposed: the T2NN Aczel-Alsina Hamy Mean (T2NNAAHM) and the T2NN Aczel-Alsina Weighted Hamy Mean (T2NNAAWHM). The study presents the formal definitions, underlying operations, theoretical foundations, and proofs of these operators. Their applicability is demonstrated through a Multiple-Attribute Decision-Making (MADM) case study, followed by comparative analyses to evaluate their performance and robustness. The findings indicate that T2NNAAHM and T2NNAAWHM are effective tools for decision-making problems involving T2NNs. This research contributes to the field by providing rigorously defined aggregation operators that address uncertainty and ambiguity, thereby enhancing the methodological toolkit available for complex decision-making scenarios.

2020 Mathematics Subject Classifications: 03B52, 03E72

Key Words and Phrases: Type-2 neutrosophic number, Aczel-Alsina, Hamy mean, T2NNAAHM, T2NNAAWHM

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1. Introduction

The concept of neutrosophic sets, which comprehensively extends and enhances the treatment of uncertainty beyond the scope of classical set theory, is introduced into the literature by Smarandache [1]. Neutrosophic sets are defined with consideration for three values: truth, indeterminacy, and falsity. These sets consist of three components: T-membership, I-membership, and F-membership. T-membership represents the degree of an element's belonging to the set, while I-membership characterizes the uncertainty regarding the element's membership. F-membership quantifies the degree of non-membership of an element. Neutrosophic sets go beyond the binary consideration of membership and non-membership, considering the level of uncertainty, thus enabling calculations based on complex probabilities.

Type-2 Neutrosophic Number Sets (T2NNs) are developed as an extension of neutrosophic sets [2] to handle greater levels of uncertainty. T2NNs enable information processing by going beyond Type-1 Neutrosophic Sets (T1NS). In T2NNs, membership functions exist at two levels: Primary Membership Functions (PMFs) and Secondary Membership Functions (SMFs). PMFs represent the primary degree of truth, indeterminacy, and falsity of an element, while SMFs provide a secondary layer of truth, indeterminacy, and falsity degrees. This dual-layer membership function structure allows for the modeling of complex problems where boundaries between membership, non-membership, and indeterminacy are not well-defined.

In the literature, T2NNs have been applied in various decision problems, utilizing different ranking methods, such as supplier selection [2], public transportation pricing system selection [3], sustainable policies selection [4], smart technology selection [5], sustainable route selection [6], segmentation of brain tumor tissue structures [7], energy blockchain system selection [8], selection of second-hand chemical tankers [9], Ro-Ro vessel selection [10], offshore wind farm site selection [11], autonomous vehicle type selection [12], green supplier selection [13], architecture selection for 5G-radio access networks [14], and evaluation of container port sustainability [15]. Notably, the Type-2 neutrosophic numbers weighted averaging (T2NNWA) operator has been frequently employed in T2NNs aggregation in research [3–7, 10, 12, 13, 16–23]

The Aczel–Alsina t-norm and t-conorm operations, together with the Hamy Mean, are fundamental tools for aggregation in fuzzy set theory [24]. Within the framework of neutrosophic sets, these operations are frequently utilized across various averaging techniques [25–28] and are often combined with the Hamy Mean to enhance aggregation performance [29, 30]. The main objective of this study is to develop novel aggregation operators for Type-2 Neutrosophic Numbers (T2NNs) based on the Aczel–Alsina t-norm and t-conorm operations in conjunction with the Hamy Mean, specifically designed for Multiple-Attribute Decision-Making (MADM) applications [31, 32]. To achieve this goal, two new aggregation operators are proposed: the Type-2 Neutrosophic Numbers Aczel–Alsina Hamy Mean operator (T2NNAAHM) and the Type-2 Neutrosophic Numbers Aczel–Alsina Weighted Hamy Mean operator (T2NNAAWHM). The key contributions of this study are as follows:

(i) Explaining the applications of Aczel–Alsina operations based on T2NNs in MADM problems;

(ii) Developing Hamy Mean-based aggregation operators for T2NNs;

(iii) Introducing both weighted and unweighted aggregation operators for T2NNs using Aczel–Alsina operations and Hamy Mean;

(iv) Defining and proving the newly developed aggregation operators;

(v) Conducting robustness tests for the obtained findings and aggregation operators;

(vi) Comparing the results obtained with T2NNWA operator results.

This paper consists of eight sections. The second section elaborates on Aczel-Alsina operations based on T2NNs. The third section discusses Hamy Mean-based Aczel-Alsina operations of T2NNs. The fourth section presents the MADM approach with T2NNAAHM and T2NNAAWHM operators. The fifth section applies the aggregation operators with illustrative examples. In the sixth section, comparative analyses are conducted. The seventh section contains the implications of this research. The eight section is also conclusion section.

2. Aczel-Alsina Operations based on Type-2 Neutrosophic Number Sets

In this section, we furnish detailed definitions encompassing the Aczel-Alsina t-norm and t-conorm, along with a comprehensive exposition of the corresponding mathematical operations.

Definition 1. The Aczel-Alsina t-norm is presented in Eq. (1), and the Aczel-Alsina t-conorm is expressed in Eq. (2) [33]:

$$T_{AA}^{\xi}(x,y) = \begin{cases} T_D(x,y), & \text{if } \xi = 0 \\ \min(x,y), & \text{if } \xi = \infty \\ e^{-\left((-\ln(x))^{\xi} + (-\ln(y))^{\xi}\right)^{1/\xi}}, & \text{otherwise.} \end{cases} \tag{1}$$

$$T_{AA}^{*\xi}(x,y) = \begin{cases} S_D(x,y), & \text{if } \xi = 0 \\ \max(x,y), & \text{if } \xi = \infty \\ e^{-\left((-\ln(1-x))^{\xi} + (-\ln(1-y))^{\xi}\right)^{1/\xi}}, & \text{otherwise.} \end{cases} \tag{2}$$

For each parameter $\xi \in [0,\infty]$; T_{AA}^{ξ} and $T_{AA}^{*\xi}$ are dual to each other.

We will delve into the fundamental operational laws of the Aczel-Alsina t-norm and Aczel-Alsina t-conorm for T2NNs. The Aczel-Alsina product for the Aczel-Alsina t-norm is defined in Eq. (3). The Aczel-Alsina sum for the Aczel-Alsina t-norm is defined in Eq. (4):

$$\tilde{N}_1 \otimes \tilde{N}_2 = \left\langle \begin{array}{l} T\big([T(T(\tilde{N}_1)(z),\, T(T(\tilde{N}_2)(z)))],\, [T(I(\tilde{N}_1)(z),\, T(I(\tilde{N}_2)(z)))],\, [T(F(\tilde{N}_1)(z),\, T(F(\tilde{N}_2)(z)))]\big), \\ T^*\big([I(T(\tilde{N}_1)(z),\, I(T(\tilde{N}_2)(z)))],\, [I(I(\tilde{N}_1)(z),\, I(I(\tilde{N}_2)(z)))],\, [I(F(\tilde{N}_1)(z),\, I(F(\tilde{N}_2)(z)))]\big), \\ T^*\big([F(T(\tilde{N}_1)(z),\, F(T(\tilde{N}_2)(z)))],\, [F(I(\tilde{N}_1)(z),\, I(I(\tilde{N}_2)(z)))],\, [F(F(\tilde{N}_1)(z),\, F(F(\tilde{N}_2)(z)))]\big) \end{array} \right\rangle \tag{3}$$

$$\tilde{N}_1 \oplus \tilde{N}_2 = \left\langle \begin{array}{l} T^*\Big([T(T(\tilde{N}_1)(z),\, T(T(\tilde{N}_2)(z)))],\, [T(I(\tilde{N}_1)(z),\, T(I(\tilde{N}_2)(z)))],\, [T(F(\tilde{N}_1)(z),\, T(F(\tilde{N}_2)(z)))]\Big), \\ T^*\Big([I(T(\tilde{N}_1)(z),\, I(T(\tilde{N}_2)(z)))],\, [I(I(\tilde{N}_1)(z),\, I(I(\tilde{N}_2)(z)))],\, [I(F(\tilde{N}_1)(z),\, I(F(\tilde{N}_2)(z)))]\Big), \\ T\Big([F(T(\tilde{N}_1)(z),\, F(T(\tilde{N}_2)(z)))],\, [F(I(\tilde{N}_1)(z),\, I(I(\tilde{N}_2)(z)))],\, [F(F(\tilde{N}_1)(z),\, F(F(\tilde{N}_2)(z)))]\Big) \end{array} \right\rangle \tag{4}$$

Definition 2. Suppose that $\tilde{N}, \tilde{N}_1$, and $\tilde{N}_2$ are three T2NNs over the universe Z. The operations among these three T2NNs based on Aczel-Alsina T and Aczel-Alsina T^* are described as follows ($\omega > 0$):

(i)

$$\tilde{N}_1 \otimes \tilde{N}_2 = \left[\begin{array}{l} e^{-\left((-\ln T(T(\tilde{N}_1)(z)))^{\xi}+(-\ln T(T(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((-\ln T(I(\tilde{N}_1)(z)))^{\xi}+(-\ln T(I(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((-\ln T(F(\tilde{N}_1)(z)))^{\xi}+(-\ln T(F(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}}; \\ 1-e^{-\left((-\ln I(T(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(T(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln I(I(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(I(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln I(F(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(F(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}; \\ 1-e^{-\left((-\ln F(T(\tilde{N}_1)(1-z)))^{\xi}+(-\ln F(T(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln F(I(\tilde{N}_1)(1-z)))^{\xi}+(-\ln F(I(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln F(F(\tilde{N}_1)(1-z)))^{\xi}+(-\ln F(F(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}} \end{array} \right]$$

(ii)

$$\tilde{N}_1 \oplus \tilde{N}_2 = \left[\begin{array}{l} 1-e^{-\left((-\ln T(T(\tilde{N}_1)(1-z)))^{\xi}+(-\ln T(T(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln T(I(\tilde{N}_1)(1-z)))^{\xi}+(-\ln T(I(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln T(F(\tilde{N}_1)(1-z)))^{\xi}+(-\ln T(F(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}; \\ 1-e^{-\left((-\ln I(T(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(T(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln I(I(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(I(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((-\ln I(F(\tilde{N}_1)(1-z)))^{\xi}+(-\ln I(F(\tilde{N}_2)(1-z)))^{\xi}\right)^{1/\xi}}; \\ e^{-\left((-\ln F(T(\tilde{N}_1)(z)))^{\xi}+(-\ln F(T(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((-\ln F(I(\tilde{N}_1)(z)))^{\xi}+(-\ln F(I(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((-\ln F(F(\tilde{N}_1)(z)))^{\xi}+(-\ln F(F(\tilde{N}_2)(z)))^{\xi}\right)^{1/\xi}} \end{array} \right]$$

(iii)

$$\omega\tilde{N} = \left[\begin{array}{l} 1-e^{-\left((\omega(-\ln T(T(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln T(I(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln T(F(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}; \\ e^{-\left((\omega(-\ln I(T(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln I(I(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln I(F(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}; \\ e^{-\left((\omega(-\ln F(T(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln F(T(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln F(F(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}} \end{array} \right]$$

(iv)

$$\tilde{N}^{\omega} = \left[\begin{array}{l} e^{-\left((\omega(-\ln T(T(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln T(I(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}, e^{-\left((\omega(-\ln T(F(\tilde{N}))(z)))^{\xi}\right)^{1/\xi}}; \\ 1-e^{-\left((\omega(-\ln I(T(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln I(I(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln I(F(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}; \\ 1-e^{-\left((\omega(-\ln F(T(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln F(T(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}}, 1-e^{-\left((\omega(-\ln F(F(\tilde{N}))(1-z)))^{\xi}\right)^{1/\xi}} \end{array} \right]$$

Example 1:

$$\tilde{N} = \langle (0.20, 0.20, 0.10), (0.65, 0.80, 0.85), (0.45, 0.80, 0.70) \rangle, \quad \tilde{N}_1 = \langle (0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65) \rangle, \quad \tilde{N}_2 = \langle (0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60) \rangle$$

be three T2NNs over the universe Z. Then, the Aczel-Alsina operations using Definition 2 can be applied for $\xi = 2$ and $\omega = 3$.

(a)

$$\tilde{N}_1 \otimes \tilde{N}_2 = \left[\begin{array}{l} e^{-\left((-\ln 0.35)^2+(-\ln 0.40)^2\right)^{1/2}}, e^{-\left((-\ln 0.35)^2+(-\ln 0.30)^2\right)^{1/2}}, e^{-\left((-\ln 0.10)^2+(-\ln 0.35)^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.80))^2+(-\ln(1-0.60))^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.40))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.65))^2+(-\ln(1-0.60))^2\right)^{1/2}} \end{array} \right]$$

Result: $\langle (0.25,0.20,0.08),(0.62,0.78,0.84),(0.60,0.77,0.75) \rangle$.

(b)

$$\tilde{N}_1 \oplus \tilde{N}_2 = \left[\begin{array}{l} 1-e^{-\left((-\ln(1-0.35))^2+(-\ln(1-0.40))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.35))^2+(-\ln(1-0.30))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.10))^2+(-\ln(1-0.35))^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.80))^2+(-\ln(1-0.60))^2\right)^{1/2}}; \\ e^{-\left((-\ln 0.50)^2+(-\ln 0.45)^2\right)^{1/2}}, e^{-\left((-\ln 0.75)^2+(-\ln 0.40)^2\right)^{1/2}}, e^{-\left((-\ln 0.65)^2+(-\ln 0.60)^2\right)^{1/2}} \end{array} \right]$$

Result: $\langle(0.49,0.43,0.36),(0.62,0.78,0.84),(0.35,0.38,0.51)\rangle$.

(c)

$$\omega\tilde{N} = \left[\begin{array}{l} 1-e^{-\left((3(-\ln(1-0.20)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.20)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.10)))^2\right)^{1/2}}; \\ e^{-\left((3(-\ln 0.65))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.80))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.85))^2\right)^{1/2}}; \\ e^{-\left((3(-\ln 0.45))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.80))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.70))^2\right)^{1/2}} \end{array} \right]$$

Result: $\langle(0.32,0.32,0.17),(0.47,0.68,0.75),(0.25,0.68,0.54)\rangle$.

(d)

$$\tilde{N}^{\omega} = \left[\begin{array}{l} e^{-\left((3(-\ln 0.20))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.20))^2\right)^{1/2}}, e^{-\left((3(-\ln 0.10))^2\right)^{1/2}}; \\ 1-e^{-\left((3(-\ln(1-0.65)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.80)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.85)))^2\right)^{1/2}}; \\ 1-e^{-\left((3(-\ln(1-0.45)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.80)))^2\right)^{1/2}}, 1-e^{-\left((3(-\ln(1-0.70)))^2\right)^{1/2}} \end{array} \right]$$

Result: $\langle(0.06,0.06,0.02),(0.84,0.94,0.96),(0.64,0.94,0.88)\rangle$.

Theorem 1: Suppose that $\tilde{N}_1$ and $\tilde{N}_2$ are two type-2 neutrosophic number sets over the universe Z, then:

(a)
$$\tilde{N}_1 \oplus \tilde{N}_2 = \tilde{N}_2 \oplus \tilde{N}_1$$

(b)
$$\tilde{N}_1 \otimes \tilde{N}_2 = \tilde{N}_2 \otimes \tilde{N}_1$$

(c)
$$\omega(\tilde{N}_1 \oplus \tilde{N}_2) = \omega\tilde{N}_1 \oplus \omega\tilde{N}_2, \quad \text{for } \omega > 0$$

(d)
$$(\tilde{N}_1 \otimes \tilde{N}_2)^{\omega} = \tilde{N}_1^{\omega} \otimes \tilde{N}_2^{\omega}, \quad \text{for } \omega > 0$$

(e)
$$\omega_1\tilde{N}_1 \oplus \omega_2\tilde{N}_1 = (\omega_1+\omega_2)\tilde{N}_1, \quad \text{for } \omega_1, \omega_2 > 0$$

(f)
$$\tilde{N}_1^{\omega_1} \otimes \tilde{N}_1^{\omega_2} = \tilde{N}_1^{\omega_1+\omega_2}, \quad \text{for } \omega_1, \omega_2 > 0$$

Proof. [Proof for $\tilde{N}_1 \oplus \tilde{N}_2 = \tilde{N}_2 \oplus \tilde{N}_1$]

$\tilde{N}_1 = \langle(0.35, 0.35, 0.10), (0.50, 0.75, 0.80), (0.50, 0.75, 0.65)\rangle, \quad \tilde{N}_2 = \langle(0.40, 0.30, 0.35), (0.50, 0.45, 0.60), (0.45, 0.40, 0.60)\rangle$

be two T2NNs over the universe Z. Then, the Aczel-Alsina operations can be applied using Theorem 1 for $\xi = 2$ and $\omega = 3$.

$$\tilde{N}_1 \oplus \tilde{N}_2 = \tilde{N}_2 \oplus \tilde{N}_1$$

$$\Rightarrow \left[\begin{array}{l} 1-e^{-\left((-\ln(1-0.35))^2+(-\ln(1-0.40))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.35))^2+(-\ln(1-0.30))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.10))^2+(-\ln(1-0.35))^\delta\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.80))^2+(-\ln(1-0.60))^2\right)^{1/2}}; \\ e^{-\left((-\ln 0.50)^2+(-\ln 0.45)^2\right)^{1/2}}, e^{-\left((-\ln 0.75)^2+(-\ln 0.40)^2\right)^{1/2}}, e^{-\left((-\ln 0.65)^2+(-\ln 0.60)^2\right)^{1/2}} \end{array}\right]$$

$$= \left[\begin{array}{l} 1-e^{-\left((-\ln(1-0.40))^2+(-\ln(1-0.35))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.30))^2+(-\ln(1-0.35))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.35))^2+(-\ln(1-0.10))^\delta\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.45))^2+(-\ln(1-0.75))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.60))^2+(-\ln(1-0.80))^2\right)^{1/2}}; \\ e^{-\left((-\ln 0.45)^2+(-\ln 0.50)^2\right)^{1/2}}, e^{-\left((-\ln 0.40)^2+(-\ln 0.75)^2\right)^{1/2}}, e^{-\left((-\ln 0.60)^2+(-\ln 0.65)^2\right)^{1/2}} \end{array}\right]$$

Result: $\langle(0.49,0.43,0.36),(0.62,0.78,0.84),(0.35,0.38,0.51)\rangle=\langle(0.49,0.43,0.36),(0.62,0.78,0.84),(0.35,0.38,0.51)\rangle$.

Proof. [Proof for $\tilde{N}_1 \otimes \tilde{N}_2 = \tilde{N}_2 \otimes \tilde{N}_1$]

$$\left[\begin{array}{l} e^{-\left((-\ln 0.35)^2+(-\ln 0.40)^2\right)^{1/2}}, e^{-\left((-\ln 0.35)^2+(-\ln 0.30)^2\right)^{1/2}}, e^{-\left((-\ln 0.10)^2+(-\ln 0.35)^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.80))^2+(-\ln(1-0.60))^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.45))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.75))^2+(-\ln(1-0.40))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.65))^2+(-\ln(1-0.60))^2\right)^{1/2}} \end{array}\right]$$

$$\left[\begin{array}{l} e^{-\left((-\ln 0.40)^2+(-\ln 0.35)^2\right)^{1/2}}, e^{-\left((-\ln 0.30)^2+(-\ln 0.35)^2\right)^{1/2}}, e^{-\left((-\ln 0.35)^2+(-\ln 0.10)^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.50))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.45))^2+(-\ln(1-0.75))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.60))^2+(-\ln(1-0.80))^2\right)^{1/2}}; \\ 1-e^{-\left((-\ln(1-0.45))^2+(-\ln(1-0.50))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.40))^2+(-\ln(1-0.75))^2\right)^{1/2}}, 1-e^{-\left((-\ln(1-0.60))^2+(-\ln(1-0.65))^2\right)^{1/2}} \end{array}\right]$$

Result: $\langle(0.25,0.20,0.08),(0.62,0.78,0.84),(0.60,0.77,0.75)\rangle=\langle(0.25,0.20,0.08),(0.62,0.78,0.84),(0.60,0.77,0.75)\rangle$.

Proof: The others proof is straightforward.

3. Hamy Mean-based Aczel-Alsina Operations of Type-2 Neutrosophic Number Sets

3.1. Type-2 neutrosophic numbers Aczel–Alsina Hamy Mean Operator

Within this sub-section, we introduce the T2NNAAHM operator, which is formulated in accordance with the operational principles delineated in Definition 2. Definition 3. HM operator is defined as Eq. (5) [30, 34]:

$$\mathrm{HM}^{\tau}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{\sum\limits_{1 \le i_1 < \cdots < i_\tau \le n} \left(\prod\limits_{i=1}^{\tau} \theta_{i_\mu}\right)^{1/\tau}}{\binom{n}{\tau}} \tag{5}$$

In this context,

$$\left\{ C_n^{\tau} = \frac{n!}{\tau!(n-\tau)!},\ \tau = 1, 2, \ldots, n \right\}$$

states the binomial coefficient. The following conditions can be satisfied for the HM operator:

(a)

$$\mathrm{HM}^{\tau}(\theta_1, \theta_2, \ldots, \theta_n) = \theta \quad \text{if } \theta = \theta_i,\ i = 1, 2, \ldots, n$$

(b)

$$\mathrm{HM}^{\tau}(\theta_1, \theta_2, \ldots, \theta_n) \le \mathrm{HM}^{\tau}(\delta_1, \delta_2, \ldots, \delta_n) \quad \text{if } \theta_i \le \delta_i,\ i = 1, 2, \ldots, n$$

(c)

$$\mathrm{HM}^{\tau}_{\mathrm{AM}}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n} \sum_{i=1}^{n} \theta_{i_\mu} \quad \text{for the arithmetic mean (AM),}$$

(d)

$$\mathrm{HM}^{\tau}_{\mathrm{GM}}(\theta_1,\theta_2,\ldots,\theta_n)=\left(\prod_{i=1}^{n}\theta_{i_\mu}\right)^{1/n}\quad \text{for the geometric mean (GM).}$$

Theorem 2: Suppose that

$$\tilde{N}_n=\left\langle \left(T_{T(\tilde{N}_n)}(z),T_{I(\tilde{N}_n)}(z),T_{F(\tilde{N}_n)}(z)\right),\ \left(I_{T(\tilde{N}_n)}(z),I_{I(\tilde{N}_n)}(z),I_{F(\tilde{N}_n)}(z)\right),\ \left(F_{T(\tilde{N}_n)}(z),F_{I(\tilde{N}_n)}(z),F_{F(\tilde{N}_n)}(z)\right)\right\rangle$$

is a type-2 neutrosophic number set over the universe Z, then the Type-2 neutrosophic numbers Aczel–Alsina Hamy Mean operator is defined as Eq. (6) and Eq. (7).

$$\mathrm{T2NNAAHM}(\theta_1,\theta_2,\ldots,\theta_n)=\frac{\displaystyle\sum_{1\le i_1<\cdots<i_\tau\le n}\left(\prod_{\mu=1}^{\tau}\theta_{i_\mu}\right)^{1/\tau}}{\binom{n}{\tau}} \tag{6}$$

$$\left\langle\begin{matrix}
1-e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{T(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & 1-e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{I(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & 1-e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{F(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}\\
e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}\\
e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}, & e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}
\end{matrix}\right\rangle \tag{7}$$

Proof. For Theorem 2.

$$\prod_{\mu=1}^{\tau}\theta_\mu=\begin{bmatrix}
e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-T_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-T_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\\
1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(I_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\\
1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}(-\ln(F_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}
\end{bmatrix}$$

then

$$\left(\prod_{\mu=1}^{\tau}\theta_\mu\right)^{1/\tau}=\begin{bmatrix}
e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\\
1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\\
1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}
\end{bmatrix}$$

then

$$\begin{bmatrix}
1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{I(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{F(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}}\\
1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}}\\
e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}} & e^{-\left(\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi}{)^{1/\xi}}))\right)^\xi\right)^{1/\xi}}
\end{bmatrix}$$

then

$$\begin{bmatrix}
1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{T(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{I(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{F(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\\
1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\\
e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}} & e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}
\end{bmatrix}$$

Consider the first condition;

$$\tilde{T}_{\tilde{N}}(z)=\left\langle 1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_T(\tilde{N}_\mu)(z))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\right.,$$

$$1 - e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{I(\tilde{N}_\mu)(z)})^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},$$
$$\left.1 - e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{F(\tilde{N}_\mu)(z)})^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\right\rangle, \quad \epsilon\in[0,1].$$

$$\implies 0 \le \tilde{T}_{\tilde{N}}(z) \le 1.$$

$$\tilde{I}_{\tilde{N}}(z) = \left\langle e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\right.,$$
$$e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},$$
$$\left.e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}}\right\rangle, \quad \epsilon\in[0,1].$$

$$\implies 0 \le \tilde{I}_{\tilde{N}}(z) \le 1.$$

$$\tilde{F}_{\tilde{N}}(z) = \left\langle e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln F_{T(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}\right.,$$
$$e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln F_{I(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}},$$
$$\left.e^{-\left(\frac{1}{C_n^\tau}\left(\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln F_{F(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}})\right)^\xi\right)\right)^{1/\xi}}\right\rangle, \quad \epsilon\in[0,1]$$

$$\implies 0 \le \tilde{F}_{\tilde{N}}(z) \le 1.$$

Consider the second condition;

$$\begin{aligned}
&\tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3\\
&= \langle (1 - e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{T(\tilde{N}_\mu)}(z))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}})^3,\\
&\quad (1 - e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{I(\tilde{N}_\mu)}(z))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}})^3,\\
&\quad (1 - e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln T_{F(\tilde{N}_\mu)}(z))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}})^3\rangle\\
&+ \langle e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},\\
&\quad e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},\\
&\quad e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}})^3\\
&+ \langle e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},\\
&\quad e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}},\\
&\quad e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left(-\ln\left(1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}\right)\right)^\xi\right)^{1/\xi}})^3
\end{aligned}$$

$$\implies 0 \le \tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3 \le 3.$$

Theorem 3: The Type-2 neutrosophic numbers Aczel–Alsina Hamy Mean operator meets the following properties:

Monotonicity: Set $\tilde{N}_k \le \tilde{N}_k^*$. If $\tilde{T}_{\tilde{N}}(z) \le \tilde{T}_{\tilde{N}}(z)^*$, $\tilde{I}_{\tilde{N}}(z) \ge \tilde{I}_{\tilde{N}}(z)^*$, and $\tilde{F}_{\tilde{N}}(z) \ge \tilde{F}_{\tilde{N}}(z)^*$, then

$$\text{T2NNAAHM}(\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_s) \le \text{T2NNAAHM}(\tilde{N}_1^*, \tilde{N}_2^*, \dots, \tilde{N}_k^*).$$

Proof: For Theorem 3 (Monotonicity) Due to $\tilde{N}_k \le \tilde{N}_k^*$, there exists $\tilde{T}_{\tilde{N}}(z) \le \tilde{T}_{\tilde{N}}(z)^*$, $\tilde{I}_{\tilde{N}}(z) \ge \tilde{I}_{\tilde{N}}(z)^*$, and $\tilde{F}_{\tilde{N}}(z) \ge \tilde{F}_{\tilde{N}}(z)^*$. Thus,

$$\text{T2NNAAHM}(\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_s) \le \text{T2NNAAHM}(\tilde{M}_1^*, \tilde{M}_2^*, \dots, \tilde{M}_k^*).$$

For example, if $\tilde{T}_{\tilde{N}}(z) \le \tilde{T}_{\tilde{N}}(z)^*$, $\tilde{T}_{(T_{\tilde{N}})}(z) \le \tilde{T}_{(T_{\tilde{N}})}(z)^*$, $\tilde{T}_{(I_{\tilde{N}})}(z) \ge \tilde{T}_{(I_{\tilde{N}})}(z)^*$, and $\tilde{T}_{(F_{\tilde{N}})}(z) \ge \tilde{T}_{(F_{\tilde{N}})}(z)^*$, then:

$$\left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{(T_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right]$$
$$\le \left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T^*_{(T_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right].$$

$$\left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{(I_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right]$$
$$\ge \left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T^*_{(I_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right].$$

$$\left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{(F_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right]$$
$$\ge \left[1 - \exp\left(-\left(\frac{1}{C_n^\tau}\left(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\left(-\ln\left(1 - \exp(-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T^*_{(F_{\tilde{N}})}(z)))^\xi)^{1/\xi}))\right)^\xi\right)\right)^{1/\xi}\right)\right].$$

Idempotency: Set

$$\tilde{N}_\mu = \Big\langle (T_{(T_{\tilde{N}_\mu})}(y),\, T_{(I_{\tilde{N}_\mu})}(y),\, T_{(F_{\tilde{N}_\mu})}(y)),\ (I_{(T_{\tilde{N}_\mu})}(y),\, I_{(I_{\tilde{N}_\mu})}(y),\, I_{(F_{\tilde{N}_\mu})}(y)),$$

$$(\mathrm{F}_{(T_{\tilde{N}_\mu})}(y),\, F_{(I_{\tilde{N}_\mu})}(y),\, F_{(F_{\tilde{N}_\mu})}(y))\Big\rangle = \tilde{N}, \quad (\mu = 1, 2, \dots, s).$$

For example, if $_{\tilde{N}_1}(z) = \tilde{T}_{\tilde{N}_2}(z)$, $T_{(T_{\tilde{N}_1})}(z) = T_{(T_{\tilde{N}_2})}(z)$, $T_{(I_{\tilde{N}_1})}(z) = T_{(I_{\tilde{N}_2})}(z)$, $T_{(F_{\tilde{N}_1})}(z) = T_{(F_{\tilde{N}_2})}(z)$, then:

$$\left[1 - \exp\left(-\Big(\frac{1}{C_n^\tau}\Big(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\big(-\ln\big(1 - \exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau} - \ln(T_{(T_{\tilde{N}})}(z))^\xi\big)^{1/\xi}\big)\big)\big)^\xi\Big)\Big)^{1/\xi}\right)\right]$$
$$= \left[1 - \exp\left(-\Big(\frac{1}{C_n^\tau}\Big(\sum_{1 \le i_1 < \cdots < i_\tau \le n}\big(-\ln\big(1 - \exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau} - \ln\Big(\Big(T_{(T_{\tilde{N}})}(z)\Big)^*\Big)^\xi\big)^{1/\xi}\big)\big)\big)^\xi\Big)\Big)^{1/\xi}\right)\right]$$

$$\left[1-\exp\left(-\Big(\frac{1}{C_n^{\tau}}\Big(\sum_{1\le i_1<\cdots<i_\tau\le n}\big(-\ln\big(1-\exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau}-\ln(T_{(I_{\tilde{N}})}(z))^{\xi}\big)^{1/\xi}\big)\big)\big)^{\xi}\Big)\Big)^{1/\xi}\right)\right]$$

$$=\left[1-\exp\left(-\Big(\frac{1}{C_n^{\tau}}\Big(\sum_{1\le i_1<\cdots<i_\tau\le n}\big(-\ln\big(1-\exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau}-\ln((T_{(I_{\tilde{N}})}(z))^{*})^{\xi}\big)^{1/\xi}\big)\big)\big)^{\xi}\Big)\Big)^{1/\xi}\right)\right]$$

$$\left[1-\exp\left(-\Big(\frac{1}{C_n^{\tau}}\Big(\sum_{1\le i_1<\cdots<i_\tau\le n}\big(-\ln\big(1-\exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau}-\ln(T_{(F_{\tilde{N}})}(z))^{\xi}\big)^{1/\xi}\big)\big)\big)^{\xi}\Big)\Big)^{1/\xi}\right)\right]$$

$$=\left[1-\exp\left(-\Big(\frac{1}{C_n^{\tau}}\Big(\sum_{1\le i_1<\cdots<i_\tau\le n}\big(-\ln\big(1-\exp\big(-\frac{1}{\tau}\big(\sum_{\mu=1}^{\tau}-\ln((T_{(F_{\tilde{N}})}(z))^{*})^{\xi}\big)^{1/\xi}\big)\big)\big)^{\xi}\Big)\Big)^{1/\xi}\right)\right]$$

Boundedness: If the maximum and minimum T2NNs are given below:

$$\tilde{N}_{\max}=\Big\langle\big(\max_{\mu}T_{T(\tilde{N}_\mu)}(y),\ \max_{\mu}T_{I(\tilde{N}_\mu)}(y),\ \max_{\mu}T_{F(\tilde{N}_\mu)}(y)\big),$$

$\big(\min_\mu I_{T(\tilde{N}_\mu)}(y),\ \min_\mu I_{I(\tilde{N}_\mu)}(y),\ \min_\mu I_{F(\tilde{N}_\mu)}(y)\big),$

$\big(\min_\mu F_{T(\tilde{N}_\mu)}(y),\ \min_\mu F_{I(\tilde{N}_\mu)}(y),\ \min_\mu F_{F(\tilde{N}_\mu)}(y)\big)\Big\rangle.$

$$\tilde{N}_{\min}=\Big\langle\big(\min_{\mu}T_{T(\tilde{N}_\mu)}(y),\ \min_{\mu}T_{I(\tilde{N}_\mu)}(y),\ \min_{\mu}T_{F(\tilde{N}_\mu)}(y)\big),$$

$\big(\max_\mu I_{T(\tilde{N}_\mu)}(y),\ \max_\mu I_{I(\tilde{N}_\mu)}(y),\ \max_\mu I_{F(\tilde{N}_\mu)}(y)\big),$

$\big(\max_\mu F_{T(\tilde{N}_\mu)}(y),\ \max_\mu F_{I(\tilde{N}_\mu)}(y),\ \max_\mu F_{F(\tilde{N}_\mu)}(y)\big)\Big\rangle.$

$$\tilde{N}_{\min}\ \le\ \mathrm{T2NNAAHM}(\tilde{N}_1,\tilde{N}_2,\ldots,\tilde{N}_s)\ \le\ \tilde{N}_{\max}.$$

Proof: For Theorem 3 (Boundedness)

By using Monotonicity and Idempotency conditions, we get:
$\mathrm{T2NNAAHM}_{\min}(\tilde{N}_{\min_1},\tilde{N}_{\min_2},\ldots,\tilde{N}_{\min_s})\ \le\ \mathrm{T2NNAAHM}(\tilde{N}_1,\tilde{N}_2,\ldots,\tilde{N}_s),$
$\mathrm{T2NNAAHM}\,\tilde{N}_{\max}(\tilde{N}_{\max_1},\tilde{N}_{\max_2},\ldots,\tilde{N}_{\max_s})\ \ge\ \mathrm{T2NNAAHM}(\tilde{N}_1,\tilde{N}_2,\ldots,\tilde{N}_s).$

3.2. Type-2 neutrosophic numbers Aczel–Alsina weighted Hamy Mean Operator

Definition 4. Let $\tilde{N}_n=\big\langle(T_T(\tilde{N}_n)(z),T_I(\tilde{N}_n)(z),T_F(\tilde{N}_n)(z)),$ $(I_T(\tilde{N}_n)(z),I_I(\tilde{N}_n)(z),I_F(\tilde{N}_n)(z)),$ $(F_T(\tilde{N}_n)(z),F_I(\tilde{N}_n)(z),F_F(\tilde{N}_n)(z))\big\rangle$ be a T2NN. Then, the T2NNAAWHM operator is defined as in Eq.(8):

$$\mathrm{T2NNAAWHM}(\theta_1,\theta_2,\ldots,\theta_n)=\begin{cases}\dfrac{\sum\limits_{1\le i_1<\cdots<i_\tau\le n}\Big(1-\sum\limits_{\mu=1}^{\tau}\omega_{k_\mu}\Big)\Big(\prod\limits_{i=1}^{\tau}\theta_{i_\mu}\Big)^{1/\tau}}{C_n^{\tau}}, & 1\le\tau\le n,\\ \prod_{i=1}^{\tau}\theta_{i_\mu}^{(1-\omega_{k_\mu})/(n-1)}, & \tau=n\end{cases}\tag{8}$$

Theorem 4. Let

$$\tilde{N}_n=\langle(T_{T(\tilde{N}_n)}(z),T_{I(\tilde{N}_n)}(z),T_{F(\tilde{N}_n)}(z)),\ (I_{T(\tilde{N}_n)}(z),I_{I(\tilde{N}_n)}(z),I_{F(\tilde{N}_n)}(z)),$$

$(\mathrm{F}_{T(\tilde{N}_n)}(z),F_{I(\tilde{N}_n)}(z),F_{F(\tilde{N}_n)}(z))\rangle$be T2NNs. Then the Type-2 neutrosophic numbers Aczel–Alsina Weighted Hamy Mean operator is defined as in Eqs. (9) and (10) $(1\le\tau<n)$.

$$\mathrm{T2NNAAWHM}(\theta_1,\theta_2,\ldots,\theta_n)=$$

$$
\begin{aligned}
&\Big\langle\Big(1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big),\\
&1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{I(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big),\\
&1-e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_{F(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big)\Big),\\
&\Big(e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big),\\
&e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big), \qquad (9)\\
&e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big)\Big),\\
&\Big(e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big),\\
&e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big),\\
&e^{-\left(\frac{1}{C_n^\tau}\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^{\xi})^{1/\xi}}))\right)^{\xi}\right)^{1/\xi}}\Big)\Big)\Big\rangle.
\end{aligned}
$$

Proof: For Theorem 4 for $1 \le \tau < n$

$$\prod_{\mu=1}^{\tau}\theta_\mu =$$

$$
\begin{bmatrix}
e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(T_{T_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, 1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-T_{I_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, 1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-T_{F_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}\\
1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-I_{T_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, 1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-I_{I_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(I_{F_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}\\
1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-F_{T_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, 1-e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(1-F_{I_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}, e^{-\left(\sum_{\mu=1}^{\tau}\left(-\ln(F_{F_{\tilde{N}_\mu}}(z))\right)^{\xi}\right)^{1/\xi}}
\end{bmatrix}.
$$

then

$$\left(\prod_{\mu=1}^{\tau}\theta_\mu\right)^{1/\tau} =$$

$$
\begin{aligned}
&\Big(\mathrm{e}^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{T_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{I_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{F_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\\
&1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\\
&1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{T_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ 1-e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{I_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}},\ e^{-\left(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{F_{\tilde{N}_\mu}}(z)))^{\xi}\right)^{1/\xi}}\Big)
\end{aligned}
$$
then

$$\sum_{1\le i_1<\cdots<i_\tau\le n}\left(1-\sum_{\mu=1}^{\tau}\omega_{k_\mu}\right)\left(\prod_{i=1}^{\tau}\theta_\mu\right)^{1/\tau} =$$

$$\left\langle \begin{array}{l} \left(1-e^{-\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})))^\xi\right)^{1/\xi}}\right), \\ 1-e^{-\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})))^\xi\right)^{1/\xi}}, \\ 1-e^{-\left(\sum_{1\le i_1<\cdots<i_\tau\le n}(1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(T_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}})))^\xi\right)^{1/\xi}} \end{array} \right\rangle$$

then

$$\frac{\sum_{1\le i_1<\cdots<i_\tau\le n}\left(\prod_{i=1}^{\tau}\theta_{i_\mu}\right)^{1/\tau}}{C_n^\tau} =$$

$$\left\langle \begin{array}{l} 1-e^{-\left(\frac{\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{T(N_\mu)}(z))^\xi)^{1/\xi}}))\right)^\xi}{C_n^\tau}\right)^{1/\xi}}, \\ 1-e^{-\left(\frac{\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{I(N_\mu)}(z))^\xi)^{1/\xi}}))\right)^\xi}{C_n^\tau}\right)^{1/\xi}}, \\ 1-e^{-\left(\frac{\sum_{1\le i_1<\cdots<i_\tau\le n}\left((1-\sum_{\mu=1}^{\tau}\omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln T_{F(N_\mu)}(z))^\xi)^{1/\xi}}))\right)^\xi}{C_n^\tau}\right)^{1/\xi}} \end{array} \right\rangle$$

Proof: For Theorem 4, for $\tau = n$

$$\theta_{i_\mu}^{\frac{1-\omega_{k_\mu}}{n-1}} =$$

$$\left\langle \begin{pmatrix} e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(T_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(T_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} \\ 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} \\ 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}}, & 1-e^{-\left(\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi\right)^{1/\xi}} \end{pmatrix} \right\rangle$$

then

$$\prod_{i=1}^{\tau}\theta_{i_\mu}^{\frac{1-\omega_{k_\mu}}{n-1}} =$$

$$\left\langle \begin{array}{l} \left(e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln T_{T(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}, e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln T_{I(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}, e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln T_{F(\tilde{N}_\mu)}(z))^\xi)^{1/\xi}}\right) \\ \left(1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}, 1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}, 1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right) \\ \left(1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}, 1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}, 1-e^{-(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\right) \end{array} \right\rangle$$

$$\mathrm{T2NNAAWHM}(\theta_1,\theta_2,\ldots,\theta_n) =$$

$$\left\langle \begin{bmatrix} e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(T_{T(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(T_{I(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(T_{F(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} \\ 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-I_{T(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-I_{I(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-I_{F(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} \\ 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-F_{T(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-F_{I(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} & 1-e^{-\left(\sum_{\mu=1}^{\tau}\frac{1-\omega_{k_\mu}}{n-1}\left(-\ln(1-F_{F(\tilde{N}_\mu)}(z))\right)^\xi\right)^{1/\xi}} \end{bmatrix} \right\rangle \tag{10}$$

First condition: ${}_{\tilde{N}}(z) : Z \to D[0,1]$, $\tilde{I}_{\tilde{N}}(z) : Z \to D[0,1]$, $\tilde{F}_{\tilde{N}}(z) : Z \to D[0,1]$

Second condition: $0 \leq \tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3 \leq 3$

Consider the first condition:

$$\tilde{T}_{\tilde{N}}(z) = \left\langle \begin{matrix} 1 - e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_T(\tilde{N}_\mu)(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \\ 1 - e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_I(\tilde{N}_\mu)(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \\ 1 - e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1-e^{-((1/\tau)\sum_{\mu=1}^{\tau}(-\ln(T_F(\tilde{N}_\mu)(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \end{matrix} \right\rangle, \quad \epsilon \in [0,1]$$

$$\Longrightarrow 0 \leq \tilde{T}_{\tilde{N}}(z) \leq 1.$$

$$\tilde{I}_{\tilde{N}}(z) = \left\langle \begin{matrix} e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})\big(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \\ e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})\big(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \\ e^{-\Big(\frac{1}{C_n^\tau} \sum_{1 \leq i_1 < \cdots < i_\tau \leq n} \Big((1-\sum_{\mu=1}^{\tau} \omega_{k_\mu})\big(-\ln(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-I_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))\Big)^\xi\Big)^{1/\xi}} \end{matrix} \right\rangle, \quad \epsilon[0,1]$$

$$\Longrightarrow 0 \leq \tilde{I}_{\tilde{N}}(z) \leq 1$$

$$\tilde{F}_{\tilde{N}}(z) = \left\langle \begin{matrix} e^{-\Big(\frac{1}{C_n^\tau}\Big(\sum_{1 \leq i_1 < \cdots < i_\tau \leq n}\big(1-\sum_{\mu=1}^{\tau} \omega_{k_\mu}\big)\Big(-\ln\Big(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\Big)\Big)^\xi\Big)\Big)^{1/\xi}}, \\ e^{-\Big(\frac{1}{C_n^\tau}\Big(\sum_{1 \leq i_1 < \cdots < i_\tau \leq n}\big(1-\sum_{\mu=1}^{\tau} \omega_{k_\mu}\big)\Big(-\ln\Big(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\Big)\Big)^\xi\Big)\Big)^{1/\xi}}, \\ e^{-\Big(\frac{1}{C_n^\tau}\Big(\sum_{1 \leq i_1 < \cdots < i_\tau \leq n}\big(1-\sum_{\mu=1}^{\tau} \omega_{k_\mu}\big)\Big(-\ln\Big(1-e^{-(\frac{1}{\tau}\sum_{\mu=1}^{\tau}(-\ln(1-F_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}\Big)\Big)^\xi\Big)\Big)^{1/\xi}} \end{matrix} \right\rangle, \quad \epsilon[0,1]$$

$$\Longrightarrow 0 \leq \tilde{F}_{\tilde{N}}(z) \leq 1$$

Consider the second condition:

$$\tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3 =$$

$$
\begin{array}{l}
\tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3 = \\
\left\langle
\begin{array}{l}
(1 - e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(T_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
(1 - e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(T_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
(1 - e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(T_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
\quad (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - I_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
+ \; (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - I_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
\quad (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - I_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
\quad (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - F_{T(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
+ \; (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - F_{I(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}}) \\
\quad (e^{-\left(\frac{1}{C_n^\tau} \sum_{1 \le i_1 < \cdots < i_\tau \le n} (1 - \sum_{\mu=1}^{\tau} \omega_{k_\mu})(-\ln(1 - e^{-(\frac{1}{\tau} \sum_{\mu=1}^{\tau} (-\ln(1 - F_{F(\tilde{N}_\mu)}(z)))^\xi)^{1/\xi}}))^\xi\right)^{1/\xi}})
\end{array}
\right\rangle, \quad \epsilon \in [0,1]
\end{array}
$$

$$\Longrightarrow \; 0 \le \tilde{T}_{\tilde{N}}(z)^3 + \tilde{I}_{\tilde{N}}(z)^3 + \tilde{F}_{\tilde{N}}(z)^3 \le 3.$$

Theorem 5: The T2NNAAWHM operator meets the following properties:
Monotonicity: Let $\tilde{N}_k \le \tilde{N}_k^*$. If $\tilde{T}_{\tilde{N}}(z) \le \tilde{T}_{\tilde{N}}^*(z)$, $\tilde{I}_{\tilde{N}}(z) \ge \tilde{I}_{\tilde{N}}^*(z)$, and $\tilde{F}_{\tilde{N}}(z) \ge \tilde{F}_{\tilde{N}}^*(z)$, then

$$\mathrm{T2NNAAWHM}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_s) \le \mathrm{T2NNAAWHM}(\tilde{N}_1^*, \tilde{N}_2^*, \ldots, \tilde{N}_s^*).$$

Proof: It is similar to the proof of the Monotonicity property of the T2NNAAHM operator.
Idempotency: Set

$$\tilde{N}_k = \left\langle (T_{T(\tilde{N}_\mu)}(y), T_{I(\tilde{N}_\mu)}(y), T_{F(\tilde{N}_\mu)}(y)), (I_{T(\tilde{N}_\mu)}(y), I_{I(\tilde{N}_\mu)}(y), I_{F(\tilde{N}_\mu)}(y)), (F_{T(\tilde{N}_\mu)}(y), F_{I(\tilde{N}_\mu)}(y), F_{F(\tilde{N}_\mu)}(y)) \right\rangle, \quad (\mu = 1, \ldots, s).$$

If $\tilde{M}_\mu = \tilde{M}$, then

T2NNAAWHM$({}_1, \tilde{N}_2, \ldots, \tilde{N}_s) = \tilde{N}$.
Proof: It is similar to the proof for the idempotency property of the T2NNAAHM operator.
Boundedness: If the maximum and minimum T2NNs are given below:

(i) $_{\max} = \Big\langle \big(\max_\mu \big(T_{T_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(T_{I_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(T_{F_{\tilde{N}_\mu}}(y)\big)\big),$
$\big(\min_\mu \big(I_{T_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(I_{I_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(I_{F_{\tilde{N}_\mu}}(y)\big)\big),$
$\big(\min_\mu \big(F_{T_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(F_{I_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(F_{F_{\tilde{N}_\mu}}(y)\big)\big) \Big\rangle$

(i) $_{\min} = \Big\langle \big(\min_\mu \big(T_{T_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(T_{I_{\tilde{N}_\mu}}(y)\big), \min_\mu \big(T_{F_{\tilde{N}_\mu}}(y)\big)\big),$
$\big(\max_\mu \big(I_{T_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(I_{I_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(I_{F_{\tilde{N}_\mu}}(y)\big)\big),$
$\big(\max_\mu \big(F_{T_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(F_{I_{\tilde{N}_\mu}}(y)\big), \max_\mu \big(F_{F_{\tilde{N}_\mu}}(y)\big)\big) \Big\rangle$

(i) Then, $_{\min} \le \mathrm{T2NNAAWHM}(\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_s) \le \tilde{N}_{\max}$ can hold.

Proof: It is similar to the proof for the boundedness property of the T2NNAAHM operator.

4. T2NN Aczel-Alsina-based Hamy Mean MADM Model

In this section, the formulated T2NNAAHM and T2NNAAWHM operators are employed to introduce the T2NN Aczel-Alsina-based Hamy Mean (T2NNAAHM) MADM model. A MADM problem consists of a set of r alternatives $G_i = \{G_1, G_2, \ldots, G_r\}$ and a set of s attributes $K_p = \{k_1, k_2, \ldots, k_s\}$. The alternatives and criteria are represented by

$$R = (\tilde{N}_{ip})_{s\times r} \quad \text{for T2NNs.}$$

Here, each entry $\tilde{N}_{ip}$ is expressed as

$$\tilde{N}_{ip} = \langle (T_{T(\tilde{N}_{ip})}(y), T_{I(\tilde{N}_{ip})}(y), T_{F(\tilde{N}_{ip})}(y)), (I_{T(\tilde{N}_{ip})}(y), I_{I(\tilde{N}_{ip})}(y), I_{F(\tilde{N}_{ip})}(y)), (F_{T(\tilde{N}_{ip})}(y), F_{I(\tilde{N}_{ip})}(y), F_{F(\tilde{N}_{ip})}(y))\rangle.$$

Then, the weighting of attributes K_k $(k = 1, 2, \ldots, s)$ is specified by a weight vector

$$\omega = (\omega_1, \omega_2, \ldots, \omega_s), \quad \omega_k \in [0, 1], \quad \sum_{k=1}^{s} \omega_k = 1.$$

5. Results

Within this section, we incorporate a comprehensive study aimed at validating the newly developed aggregation operators, namely T2NNAAHM and T2NNAAWHM, tailored for T2NNs. The study is adapted from the research conducted by Abdel-Basset et al. [2], a deliberate choice to ensure the congruence and comparability of the resultant outcomes.

This study focuses on the intricate task of selecting the best bank within the realm of banking transactions. We consider a set of four distinct banking alternatives, denoted as G_i $(i = 1, 2, \ldots, 4)$, and five discernible attributes, represented as K_p $(p = 1, 2, \ldots, 5)$. The five attributes include: reputation and elegance (G_1), customer service (G_2), location of the bank and its branch network (G_3), fee structures (G_4), and promotional offers (G_5).

According to Abdel-Basset et al. [2], the criterion weights are established as:

$$\omega_k = (0.20, 0.25, 0.30, 0.15, 0.10).$$

The decision matrix containing the relevant data is presented in Table 1.

Table 1: Decision matrix for four banks and five attributes with T2NNs

	K_1	K_2	K_3	K_4	K_5
G_1	$\langle (0.75, 0.80, 0.85), (0.20, 0.15, 0.30), (0.10, 0.15, 0.20) \rangle$	$\langle (0.65, 0.70, 0.75), (0.40, 0.45, 0.50), (0.35, 0.40, 0.35) \rangle$	$\langle (0.85, 0.90, 0.95), (0.30, 0.35, 0.40), (0.25, 0.40, 0.35) \rangle$	$\langle (0.50, 0.40, 0.55), (0.10, 0.15, 0.30), (0.10, 0.20, 0.20) \rangle$	$\langle (0.30, 0.45, 0.25), (0.20, 0.10, 0.30), (0.10, 0.25, 0.20) \rangle$
G_2	$\langle (0.60, 0.50, 0.65), (0.30, 0.25, 0.30), (0.20, 0.30, 0.25) \rangle$	$\langle (0.65, 0.70, 0.75), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.45, 0.35, 0.50), (0.15, 0.10, 0.10), (0.20, 0.30, 0.25) \rangle$	$\langle (0.45, 0.50, 0.60), (0.30, 0.20, 0.30), (0.25, 0.30, 0.25) \rangle$	$\langle (0.50, 0.45, 0.35), (0.30, 0.25, 0.30), (0.20, 0.30, 0.25) \rangle$
G_3	$\langle (0.45, 0.50, 0.80), (0.15, 0.30, 0.55), (0.55, 0.20, 0.25) \rangle$	$\langle (0.40, 0.45, 0.50), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.40, 0.45, 0.60), (0.05, 0.20, 0.25), (0.40, 0.20, 0.25) \rangle$	$\langle (0.45, 0.80, 0.90), (0.40, 0.70, 0.55), (0.55, 0.20, 0.40) \rangle$	$\langle (0.80, 0.50, 0.80), (0.45, 0.30, 0.55), (0.55, 0.20, 0.25) \rangle$
G_4	$\langle (0.85, 0.70, 0.95), (0.60, 0.50, 0.65), (0.45, 0.15, 0.35) \rangle$	$\langle (0.60, 0.65, 0.70), (0.35, 0.40, 0.45), (0.30, 0.40, 0.45) \rangle$	$\langle (0.95, 0.70, 0.80), (0.15, 0.10, 0.30), (0.30, 0.35, 0.30) \rangle$	$\langle (0.90, 0.70, 0.95), (0.60, 0.40, 0.65), (0.45, 0.15, 0.35) \rangle$	$\langle (0.65, 0.70, 0.80), (0.40, 0.35, 0.25), (0.15, 0.15, 0.20) \rangle$

The resolution of the task involving the identification of the optimal bank through the application of the T2NNAAHM and T2NNAAWHM operators within the context of the MADM framework has been systematically executed.

Step 1: An expert determines the matrix as seen in Table 1.

Step 2: When considering $\xi = 1$ and $\tau = 2$, we computed the aggregated T2NNs and subsequently determined the weighted aggregated T2NNs. Table 2 displays the results of the aggregated T2NN values, which were obtained using the T2NNAAHM operator.

Table 3 provides an overview of the weighted aggregated T2NN values, which were calculated using the T2NNAAWHM operator.

Table 2: Aggregated T2NN values for T2NNAAHM operator.

Alternatives	$\tilde{T}_{\tilde{N}}(y)$	$\tilde{I}_{\tilde{N}}(y)$	$\tilde{F}_{\tilde{N}}(y)$
G_1	(0.7921, 0.8306, 0.8535)	(0.3509, 0.3404, 0.5203)	(0.0530, 0.1177, 0.1047)
G_2	(0.7146, 0.6816, 0.7532)	(0.3392, 0.2884, 0.3555)	(0.0557, 0.1047, 0.0859)
G_3	(0.6789, 0.7227, 0.8850)	(0.3268, 0.4750, 0.5976)	(0.2484, 0.0626, 0.1118)
G_4	(0.9335, 0.8580, 0.9575)	(0.5804, 0.4935, 0.6315)	(0.1564, 0.0891, 0.1570)

Table 3: Weighted aggregated T2NN values for T2NNAAWHM operator.

Alternative	$\tilde{T}_N(y)$	$\tilde{I}_N(y)$	$\tilde{F}_N(y)$
G_1	(0.5929, 0.6387, 0.6605)	(0.2221, 0.2108, 0.3515)	(0.1646, 0.2705, 0.2525)
G_2	(0.5276, 0.4974, 0.5619)	(0.2268, 0.1894, 0.2388)	(0.1794, 0.2606, 0.2306)
G_3	(0.5037, 0.5422, 0.7373)	(0.2248, 0.3289, 0.4330)	(0.4431, 0.1903, 0.2712)
G_4	(0.7992, 0.6906, 0.8531)	(0.4173, 0.3437, 0.4512)	(0.3270, 0.2260, 0.3256)

Step 3: The computation of score functions and accuracy functions was carried out. The outcomes of these computations, derived from the application of both the T2NNAAHM and T2NNAAWHM operators, were systematically organized and presented in Table 4.

Step 4: The ranking of the alternatives is established based on the score functions' values.

Utilizing the T2NNAAHM operator, the ranking of the alternatives is as follows: $G_1 > G_2 > G_4 > G_3$.

Conversely, employing the T2NNAAWHM operator, the ranking is consistent, with the order $G_1 > G_2 > G_4 > G_3$.

Table 4: The score and accuracy values of alternatives.

Alternatives	T2NNAAHM Operator		T2NNAAWHM Operator	
	Score function	Accuracy function	Score function	Accuracy function
G_1	0.7801	0.7284	0.7147	0.3931
G_2	0.7673	0.6199	0.6923	0.2882
G_3	0.7207	0.6309	0.6595	0.3076
G_4	0.7430	0.7788	0.6977	0.4822

6. Comparative Analysis

6.1. Comparison of T2NNWA with the Novel Aggregation Operators

During this investigation, we introduced novel aggregation operators based on Hamy Mean, namely T2NNAAHM and T2NNAAWHM, grounded in Aczel–Alsina t-norm and t-conorm operations. Following the validation of these proposed aggregation operators, we have demonstrated their practical application through the comprehensive study extracted from Abdel-Basset et al. [2].

Additionally, within the scope of this research, we compared these research findings with the T2NNWA operator findings. Upon a comprehensive comparison of the findings derived from this research with those of Abdel-Basset et al. [2], a remarkable alignment is evident. Specifically, for $\xi = 1$ and $\tau = 2$, the aggregated T2NN values, score functions, and accuracy function values derived from both the T2NNAAHM and T2NNWA operators exhibit congruence.

A noticeable implication arises when considering the results obtained from the application of T2NNAAWHM and T2NNWA. For $\xi = 1$ and $\tau = 2$, the T2NNAAHM operator and the T2NNAAWHM operator assign the first alternative as the best one. For a comprehensive comparative analysis, Table 5 is presented, which shows an overview of the juxtaposition between T2NNWA, T2NNAAHM, and T2NNAAWHM for T2NNs.

6.2. Comparison of the Novel Aggregation Operators based on Parameter Variations

The ξ parameter in the proposed T2NNAAHM and T2NNAAWHM operators yields varying outcomes under different variations. These variations are examined in terms of the robustness of the alternative rankings obtained

Table 5: Comparison of aggregation operators for T2NNs.

Operator	G_1	G_2	G_3	G_4	Alt. rank
T2NNWA	0.7461	0.7223	0.6906	0.7201	$G_1 > G_2 > G_4 > G_3$
T2NNAAHM	0.7801	0.7673	0.7207	0.7430	$G_1 > G_2 > G_4 > G_3$
T2NNAAWHM	0.7147	0.6923	0.6595	0.6977	$G_1 > G_4 > G_2 > G_3$

by comparing the generated score functions. Score function values and alternative rankings for various ξ parameter variations in the case of T2NNAAHM are presented in Table 6.

Table 6: Score functions under various ξ for T2NNAAHM.

ξ	G_1	G_2	G_3	G_4	Alt. rank
1	0.7801	0.7673	0.7208	0.7430	$G_1 > G_2 > G_4 > G_3$
2	0.7522	0.7307	0.6900	0.7214	$G_1 > G_2 > G_4 > G_3$
3	0.7434	0.7138	0.6743	0.7091	$G_1 > G_2 > G_4 > G_3$
10	0.7435	0.6843	0.6441	0.6925	$G_1 > G_4 > G_2 > G_3$
15	0.7472	0.6806	0.6406	0.6913	$G_1 > G_4 > G_2 > G_3$
25	0.7513	0.6781	0.6387	0.6909	$G_1 > G_4 > G_2 > G_3$
50	0.7548	0.6765	0.6380	0.6912	$G_1 > G_4 > G_2 > G_3$
75	0.7560	0.6760	0.6378	0.6914	$G_1 > G_4 > G_2 > G_3$
100	0.7566	0.6758	0.6377	0.6914	$G_1 > G_4 > G_2 > G_3$
150	0.7572	0.6755	0.6377	0.6915	$G_1 > G_4 > G_2 > G_3$

The score function values and alternative rankings for various ξ parameter variations in the case of T2NNAAWHM are displayed in Table 7.

Table 7: Score functions under various ξ for T2NNAAWHM.

ξ	G_1	G_2	G_3	G_4	Alt. rank
1	0.7148	0.6924	0.6596	0.6977	$G_1 > G_4 > G_2 > G_3$
2	0.7136	0.6875	0.6541	0.6926	$G_1 > G_4 > G_2 > G_3$
3	0.7177	0.6836	0.6491	0.6895	$G_1 > G_4 > G_2 > G_3$
10	0.7381	0.6746	0.6361	0.6876	$G_1 > G_4 > G_2 > G_3$
15	0.7440	0.6740	0.6353	0.6882	$G_1 > G_4 > G_2 > G_3$
25	0.7495	0.6741	0.6356	0.6892	$G_1 > G_4 > G_2 > G_3$
50	0.7540	0.6745	0.6365	0.6904	$G_1 > G_4 > G_2 > G_3$
75	0.7554	0.6747	0.6368	0.6908	$G_1 > G_4 > G_2 > G_3$
100	0.7562	0.6748	0.6370	0.6910	$G_1 > G_4 > G_2 > G_3$
150	0.7569	0.6748	0.6372	0.6912	$G_1 > G_4 > G_2 > G_3$

Upon close examination of the alternative rankings, it is evident that the alternative ranking remains unchanged within the T2NNAAWHM operator. Conversely, within the T2NNAAHM operator, a shift between the second and third ranks occurs when $\xi = 10$. Nevertheless, it is noteworthy that the ranking of the top-performing alternatives remains consistent across all ξ values for all operators.

7. Implications

The findings of this research have several important implications for theory and practice:

(i) Introduction of novel operators: Two aggregation operators, T2NNAAHM and T2NNAAWHM, were developed for use with T2NNs, expanding the currently limited set of available operators.

(ii) Flexibility in aggregation: The T2NNAAHM operator provides an unweighted aggregation mechanism, while the T2NNAAWHM operator enables weighted aggregation, thus broadening applicability across different decision-making contexts.

(iii) Theoretical advancement: Both operators are grounded in Hamy Mean averaging, derived through Aczél–Alsina t-norm and t-conorm operations, with assumptions and proofs rigorously established.

(iv) Comparative reliability: Results obtained from the proposed operators consistently aligned with those produced by the existing T2NNWA operator, thereby confirming the robustness and stability of the proposed approach. utility: The proposed operators offer decision-makers alternative aggregation outcomes, allowing for multiple perspectives in decision-making rather than a single deterministic result.

8. Conclusions

This study has advanced the field of MADM by introducing two novel aggregation operators for T2NNs: the T2NNAAHM and T2NNAAWHM. Unlike the existing literature, which only includes the T2NNWA operator, the proposed operators broaden the methodological toolkit by incorporating Hamy Mean-based aggregation founded on Aczel–Alsina t-norms and t-conorms. Their mathematical development has been comprehensively detailed, ensuring transparency of assumptions and proofs.

Empirical analyses demonstrated that both operators consistently produced outcomes comparable to those obtained with the T2NNWA operator, including under parameter variations. This stability underscores the robustness of the Aczel–Alsina-based Hamy Mean MADM model. The ability to perform both weighted and unweighted aggregation enhances the adaptability of the proposed methods in diverse decision-making contexts, making them particularly valuable when handling linguistic expressions and uncertain information.

Looking forward, future research could extend this line of work by exploring additional aggregation operators for T2NNs and systematically comparing their performance with both the newly developed operators and existing benchmarks. Such comparative investigations will not only deepen the understanding of operator robustness but also provide practitioners with more reliable tools for complex decision-making scenarios. By facilitating alternative yet consistent decision outcomes, the novel operators proposed in this study hold strong potential for practical applications as well as for enriching theoretical research in neutrosophic decision-making.

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References

[1] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[2] Mohamed Abdel-Basset, M. Saleh, Abduallah Gamal, and Florentin Smarandache. An approach of topsis technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, 77:438–452, 2019.

[3] Vladimir Simic, Ilgin Gokasar, Muhammet Deveci, and Ahmet Karakurt. An integrated critic and mabac based type-2 neutrosophic model for public transportation pricing system selection. *Socio-Economic Planning Sciences*, 80:101157, 2022.

[4] Vladimir Simic, Ilgin Gokasar, Muhammet Deveci, and Libor Švadlenka. Mitigating climate change effects of urban transportation using a type-2 neutrosophic merec-marcos model. *IEEE Transactions on Engineering Management*, 71:3233–3249, 2022.

[5] Ali Ebadi Torkayesh, Muhammet Deveci, Sajjad Ebadi Torkayesh, and Erfan Babaee Tirkolaee. Analyzing failures in adoption of smart technologies for medical waste management systems: A type-2 neutrosophic-based approach. *Environmental Science and Pollution Research*, 29(53):79688–79701, 2022.

[6] Vladimir Simić, Branko Milovanović, Strahinja Pantelić, Dragan Pamučar, and Erfan Babaee Tirkolaee. Sustainable route selection of petroleum transportation using a type-2 neutrosophic number based itara-edas model. *Information Sciences*, 622:732–754, 2023.

[7] Pritpal Singh. A type-2 neutrosophic-entropy-fusion based multiple thresholding method for the brain tumor tissue structures segmentation. *Applied Soft Computing*, 103:107119, 2021.

[8] Umit Cali, Muhammet Deveci, Shammya Shananda Saha, Ugur Halden, and Florentin Smarandache. Prioritizing energy blockchain use cases using type-2 neutrosophic number-based edas. *IEEE Access*, 10:34260–34276, 2022.

[9] Ömer Faruk Görçün. A novel integrated mcdm framework based on type-2 neutrosophic fuzzy sets (t2nn) for the selection of proper second-hand chemical tankers. *Transportation Research Part E: Logistics and Transportation Review*, 163:102765, 2022.

[10] Ömer Faruk Görçün, Dragan Pamucar, Raghunathan Krishankumar, and Hande Küçükönder. The selection of appropriate ro-ro vessel in the second-hand market using the waspas'bonferroni approach in type 2 neutrosophic fuzzy environment. *Engineering Applications of Artificial Intelligence*, 117:105531, 2023.

[11] Muhammet Deveci, Dragan Pamucar, Umit Cali, Emre Kantar, Konstanze Kölle, and John O. Tande. Hybrid q-rung orthopair fuzzy sets based cocoso model for floating offshore wind farm site selection in norway. *CSEE Journal of Power and Energy Systems*, 8(5):1261–1280, 2022.

[12] Ilgin Gokasar, Vladimir Simic, Muhammet Deveci, and Tapan Senapati. Alternative prioritization of freeway incident management using autonomous vehicles in mixed traffic using a type-2 neutrosophic number based decision support system. *Engineering Applications of Artificial Intelligence*, 123:106183, 2023.

[13] Zeyuan Wang, Qiang Cai, and Guiwu Wei. Modified todim method based on cumulative prospect theory with type-2 neutrosophic number for green supplier selection. *Engineering Applications of Artificial Intelligence*, 126:106843, 2023.

[14] Iman Mohamad Sharaf, Abdullah Hussein Alamoodi, Osamah Shihab Albahri, Muhammet Deveci, Mohammed Talal, Ahmed Shihab Albahri, Dursun Delen, and Witold Pedrycz. Architecture selection for 5g-radio access network using type-2 neutrosophic numbers based decision making model. *Expert Systems with Applications*, 237:121420, 2024.

[15] Sema Kayapinar Kaya, Pradip Kundu, and Ömer Faruk Görçün. Evaluation of container port sustainability using waspas technique using on type-2 neutrosophic fuzzy numbers. *Marine Pollution Bulletin*, 190:114849, 2023.

[16] Muhammet Deveci, Vladimir Simic, and Ali Ebadi Torkayesh. Remanufacturing facility location for automotive lithium-ion batteries: An integrated neutrosophic decision-making model. *Journal of Cleaner Production*, 317:128438, 2021.

[17] Abduallah Gamal, Mohamed Abdel-Basset, Ibrahim M. Hezam, Karam M. Sallam, and Ibrahim A. Hameed. An interactive multi-criteria decision-making approach for autonomous vehicles and distributed resources based on logistic systems: challenges for a sustainable future. *Sustainability*, 15(17):12844, 2023.

[18] Mohamed Abdel-Basset, Abduallah Gamal, Ibrahim M. Hezam, and Karam M. Sallam. An effective analysis of risk assessment and mitigation strategies of photovoltaic power plants based on real unleash data: strategies, challenges, perspectives, and sustainability. *International Journal of Energy Research*, 2023(1):1582795, 2023.

[19] Mohamed Abdel-Basset, Abduallah Gamal, Ibrahim M. Hezam, and Karam M. Sallam. Sustainability assessment of optimal location of electric vehicle charge stations: a conceptual framework for green energy into smart cities. *Environment, Development and Sustainability*, 26(5):11475–11513, 2024.

[20] Zeyuan Wang, Qiang Cai, and Guiwu Wei. Enhanced todim based on vikor method for multi-attribute decision making with type-2 neutrosophic number and applications to green supplier selection. *Soft Computing*, pages 1–15, 2023.

[21] Vladimir Simic, Svetlana Dabic-Miletic, Erfan Babaee Tirkolaee, Željko Stević, Ali Ala, and Arash Amirteimoori. Neutrosophic lopcow-aras model for prioritizing industry 4.0-based material handling technologies in smart and sustainable warehouse management systems. *Applied Soft Computing*, 143:110400, 2023.

[22] Ali Ala, Vladimir Simic, Dragan Pamucar, and Chiranjibe Jana. A novel neutrosophic-based multi-objective grey wolf optimizer for ensuring the security and resilience of sustainable energy: A case study of belgium. *Sustainable Cities and Society*, 96:104709, 2023.

[23] Erfan Babaee Tirkolaee, Ali Ebadi Torkayesh, Madjid Tavana, Alireza Goli, Vladimir Simic, and Weiping Ding. An integrated decision support framework for resilient vaccine supply chain network design. *Engineering Applications of Artificial Intelligence*, 126:106945, 2023.

[24] Maria Akram, Kifayat Ullah, Goran Ćirović, and Dragan Pamucar. Algorithm for energy resource selection using priority degree-based aggregation operators with generalized orthopair fuzzy information and aczel–alsina aggregation operators. *Energies*, 16(6):2816, 2023.

[25] Jun Ye, Shigui Du, and Rui Yong. Aczel–alsina weighted aggregation operators of neutrosophic z-numbers and its multiple attribute decision-making method. *International Journal of Fuzzy Systems*, 24(5):2397–2410, 2022.

[26] Shahzaib Ashraf, Shakoor Ahmad, Muhammad Naeem, Muhammad Riaz, and Md Ashraful Alam. Novel edas methodology based on single-valued neutrosophic aczel-als alsina aggregation information and their application in complex decision-making. *Complexity*, 2022(1):2394472, 2022.

[27] Rui Yong, Jun Ye, Shigui Du, Aqin Zhu, and Yingying Zhang. Aczel-alsina weighted aggregation operators of simplified neutrosophic numbers and its application in multiple attribute decision making. *CMES-Computer Modeling in Engineering & Sciences*, 132(2):569–584, 2022.

[28] Mesut Karabacak. Interval neutrosophic multi-criteria group decision-making based on aczel–alsina aggregation operators. *Computational and Applied Mathematics*, 42(3):136, 2023.

[29] Haolun Wang, Tingjun Xu, Liangqing Feng, Tahir Mahmood, and Kifayat Ullah. Aczel–alsina hamy mean aggregation operators in t-spherical fuzzy multi-criteria decision-making. *Axioms*, 12(2):224, 2023.

[30] Abrar Hussain, Haolun Wang, Kifayat Ullah, and Dragan Pamucar. Novel intuitionistic fuzzy aczel alsina hamy mean operators and their applications in the assessment of construction material. *Complex & Intelligent Systems*, 10(1):1061–1086, 2024.

[31] Marzena Banach and Rafał Długosz. A novel approach to cities' assessment in terms of their implementation of smart city idea. *Journal of Computational and Applied Mathematics*, 428:115161, 2023.

[32] AF Roldán López de Hierro, Miguel Sánchez, and Concepción Roldán. Multi-criteria decision making involving uncertain information via fuzzy ranking and fuzzy aggregation functions. *Journal of Computational and Applied Mathematics*, 404:113138, 2022.

[33] János Aczél and Claudi Alsina. Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements. *Aequationes Mathematicae*, 25(1):313–315, 1982.

[34] Takuya Hara, Mitsuru Uchiyama, and Sin-Ei Takahasi. A refinement of various mean inequalities. *Journal of Inequalities and Applications*, 1998(4):932025, 1998.