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On (H_1, H_2)-Magic Generalized Total Composition

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Abstract. Let H_1 and H_2 be two non-isomorphic graphs. A graph G is said to admit an (H_1, H_2)-*covering* if every edge of G is contained in either a subgraph of G isomorphic to H_1 or to H_2. We say that a graph G admitting an (H_1, H_2)-covering is (H_1, H_2)-*magic* if there exists a total labeling $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ such that there exist magic constants c_1 and c_2 such that the weight of every subgraph H_i^* of G isomorphic to H_i equals to c_i, $i = 1, 2$. The weight of a subgraph H is defined as

$$w(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e).$$

Moreover, a graph G is called (H_1, H_2)-*supermagic* if the vertices are labeled with the numbers from 1 up to $|V(G)|$. In this paper, we present some constructions of (H_1, H_2)-magic graphs.

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1. Introduction

Let $G = (V, E)$ be a finite, simple, and undirected graph. For two integers $a < b$, let $[a, b] = \{k \in \mathbb{Z} \mid a \le k \le b\}$.

In 2005, Gutiérrez and Lladó [1] introduced a concept of H-(super)magic graphs. A graph G is said to admit an H-*covering* if every edge $e \in E(G)$ is contained in some

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subgraph of G isomorphic to H. Suppose that $G = (V, E)$ admits an H-covering. A bijective function $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ is called an *H-magic labeling* if there exists a positive integer $c \in \mathbb{N}$, called a magic constant, such that the weight $w(H^*) = \sum_{v \in V(H^*)} f(v) + \sum_{e \in E(H^*)} f(e) = c$ for every subgraph H^* of G isomorphic to H. In addition, if $\{f(v) \mid v \in V(G)\} = [1, |V(G)|]$ then f is called an *H-supermagic labeling*. A graph which admits an H-(super)magic labeling is called an H-(super)magic graph. To date, there are some results of H-supermagicness in planar graphs [2], grid graphs [3], polygonal snake graphs [4], edge coronation of graphs [5], and disjoint union of prisms [6].

It is known that for any graph H other than a K_2, we can always find several graphs which are not H-magic. One possible way to do this is by considering graphs which did not admit H-covering. Therefore, if we want to have similar magicness property for graphs which did not admit H-magic, we can consider a relaxation of H-magicness.

Recently, Ashari and Salman [7] introduced a notion of a generalization of H-(super)magic labeling, namely an (H_1, H_2)-(super)magic labeling. Let H_1 and H_2 be two non-isomorphic graphs. A graph G is said to admit an *(H_1, H_2)-covering* if every edge $e \in E(G)$ is contained in either a subgraph of G isomorphic to H_1 or a subgraph of G isomorphic to H_2. Let G admit an (H_1, H_2)-covering. A bijection $f : V(G) \cup E(G) \to [1, |V(G)| + |E(G)|]$ is called an *(H_1, H_2)-magic labeling* if there exist two positive integers c_1 and c_2, called magic constants, such that for every subgraph H_1^* of G isomorphic to H_1 holds

$$w(H_1) = \sum_{v \in V(H_1)} f(v) + \sum_{e \in E(H_1)} f(e) = c_1$$

and for every subgraph H_2^* of G isomorphic to H_2 holds

$$w(H_2) = \sum_{v \in V(H_2)} f(v) + \sum_{e \in E(H_2)} f(e) = c_2.$$

Moreover, an (H_1, H_2)-magic labeling f is called *(H_1, H_2)-supermagic* if the vertices are labeled with the smallest possible numbers, i.e., $\{f(v) \mid v \in V(G)\} = [1, |V(G)|]$. A graph G called is called *(H_1, H_2)-(super)magic* if G admits an (H_1, H_2)-(super)magic labeling. Some other variations of H-(super)magic valuations of graphs can be seen in [8–11]. For more insights about graph labeling, please see [12]. Furthermore, there are several recent applications of graph theory which can be seen in [13, 14].

In this paper, we present several new constructions of (H_1, H_2)-magic graphs and also results for P_h-(super)magicness of copies of paths.

2. The (k, θ)-balanced multisets

Maryati et al. [15] presented a characterization of mG being G-supermagic.

Theorem 1. *[15] Let m be a positive integer and let G be a graph such that all its components have at least 2 vertices. Then mG is G-magic if and only if $|V(G)| + |E(G)|$ is even or m is odd.*

The proof of this theorem is based on a technique called (k,θ)-balanced multisets, see [15, 16]. In this paper, we will also use this method; therefore, we begin by introducing several definitions and basic properties. Multiset is a generalization of a set, where multiple instances of each element in a set is allowed. The notion $\uplus$ combines multisets with counting repeated occurrences of elements from each multisets, i.e., $\{a\}\uplus\{a,b\}=\{a,a,b\}$.

Let $k=h\theta$ for some positive integers h and θ. Let Y be a multiset containing positive integers.

The multiset Y is said to be (k,θ)-*balanced* if there exist k submultisets of Y, namely Y_i for $i\in[1,k]$, and there exist θ distinct integers a_j for $j\in[1,\theta]$, such that

(i) $\uplus_{i=1}^{k} Y_i = Y$,

(ii) $|Y_i| = \frac{|Y|}{k}$ for every $i\in[1,k]$,

(iii) $\sum\limits_{b\in Y_{th+r}} b = a_{t+1}$ for $t\in[0,\theta-1]$ and $r\in[1,h]$.

For $i\in[1,k]$, Y_i is called a *balanced submultiset* of Y. Particularly, a $(k,1)$-balanced multiset is exactly a k-balanced multiset. This method can also be applied to identify (H_1,H_2)-magic graphs. Additionally, some established results concerning (k,θ)-balanced multisets are presented below.

Lemma 1. *[15] Let x,y,z and k be non-negative integers and*

$$Y=[x+1,x+k]\uplus[y+1,y+k]\uplus[z+1,z+k]$$

be a multiset. Then

(1) for even $k\geq 2$, Y is $(k,2)$-balanced, with

$$\sum_{b\in Y_i} b = \begin{cases} x+y+z+\frac{3k}{2}+2, & \text{for } i\in[1,\frac{k}{2}],\\ x+y+z+\frac{3k}{2}+1, & \text{for } i\in[\frac{k}{2}+1,k],\end{cases}$$

(2) for odd $k\geq 3$, Y is k-balanced, with $\sum\limits_{b\in Y_i} b = x+y+z+\frac{3}{2}(k+1)$.

3. The P_h-supermagic graphs

Gutiérrez and Lladó [1] characterized the path-supermagicness of the path P_n on n vertices.

Theorem 2. *[1] The path P_n is P_h-supermagic for any integer $h\in[2,n]$.*

Moreover, Maryati et al. [17] showed that the odd copies of paths with at least 3 vertices is also P_h-supermagic with certain restrictions on h. By mG we denote the union of disjoint m copies of a graph G.

Theorem 3. *[17] Let m be odd and $n \geq 3$. Then mP_n is kP_h-supermagic for $k \in [1, m]$ and $h \in \left[\lceil \frac{n}{2} \rceil + 1, n\right]$.*

They also proposed that mP_n is kP_h-supermagic also for $h \in \left[2, \lceil \frac{n}{2} \rceil\right]$.

In the following theorem, we present a complete characterization when odd copies of paths is path-supermagic.

Theorem 4. *Let $n \geq 5$ be a positive integer, $m \geq 3$ be a positive odd integer and $h \in [3, n]$. Then, the disjoint union of paths mP_n is kP_h-supermagic for $k \in [1, m\lfloor \frac{n}{h} \rfloor - 1]$.*

Proof. Let mP_n be a graph with the vertex set and the edge set

$$V(mP_n) = \{v_{i,j} \mid i \in [1, m], j \in [1, n]\},$$
$$E(mP_n) = \{v_{i,j}v_{i,j+1} \mid i \in [1, m], j \in [1, n-1]\}.$$

Let $P_h^{(i,l)}, i \in [1, m], l \in [1, n-h+1]$ be the subgraph of mP_n such that

$$V(P_h^{(i,l)}) = \{v_{i,j} \mid j \in [l, l+h-1]\},$$
$$E(P_h^{(i,l)}) = \{v_{i,j}v_{i,j+1} \mid j \in [l, l+h-2]\}.$$

According to Theorem 2, there exists a P_h-supermagic labeling g of P_n which induces the magic constant c. For m odd, define a total labeling f of mP_n in the following way

$$f(v_{i,j}) = \begin{cases} m \cdot (g(v_{i,j}) - 1) + i + \frac{m+1}{2}, & \text{for } j \equiv 1 \pmod h,\ i \in [1, \frac{m-1}{2}], \\ m \cdot (g(v_{i,j}) - 1) + i - \frac{m-1}{2}, & \text{for } j \equiv 1 \pmod h,\ i \in [\frac{m+1}{2}, m], \\ m \cdot (g(v_{i,j}) - 1) + i, & \text{for } j \not\equiv 1 \pmod h,\ i \in [1, m], \end{cases}$$

$$f(v_{i,j}v_{i,j+1}) = \begin{cases} m \cdot g(v_{i,j}v_{i,j+1}) - 2i + 1, & \text{for } j \equiv 0 \pmod{h-1},\ i \in [1, \frac{m-1}{2}], \\ m \cdot (g(v_{i,j}v_{i,j+1}) + 1) - 2i + 1, & \text{for } j \equiv 0 \pmod{h-1},\ i \in [\frac{m+1}{2}, m], \\ m \cdot g(v_{i,j}v_{i,j+1}) - i + 1, & \text{for } j \not\equiv 0 \pmod{h-1},\ i \in [1, m]. \end{cases}$$

It is easy to see that f is a bijection and the vertices are labeled with numbers $1, 2, \ldots, mn$. Moreover, for the weight of a subgraph $P_h^{(i,l)}$, $i \in [1, m]$, $l \in [1, n-h+1]$ under the labeling f we have

$$w(P_h^{(i,l)}) = mc - h(m-1) + \tfrac{m-1}{2}.$$

Therefore, the weight of every subgraph of mP_n which is isomorphic to P_h is constant. This immediately implies that mP_n is kP_h-supermagic for $k \in [1, m\lfloor \frac{n}{h} \rfloor - 1]$. □

Remark 1. *It is also possible to show that mP_n is kP_h-supermagic for $k = m\lfloor \frac{n}{h} \rfloor$ whenever h does not divide n using the same labeling as in Theorem 2.*

Figure 1 illustrates the P_3-supermagic labeling of $5P_7$ obtained from the P_3-supermagic labeling of P_7 by the construction described in the proof of Theorem 2.

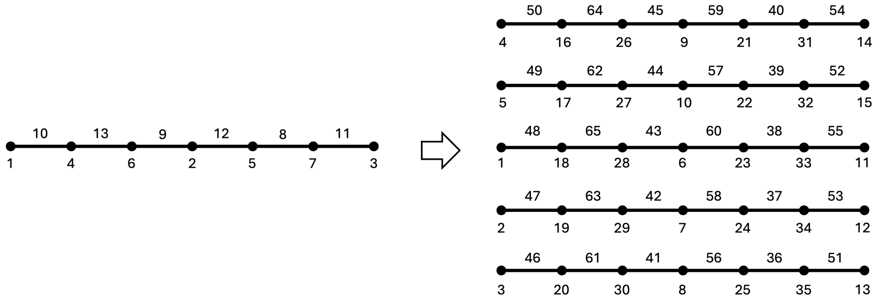

Figure 1: A P_3-supermagic labeling of P_7 and the corresponding P_3-supermagic labeling of $5P_7$.

4. The (H_1, H_2)-magic graphs

In this section, we study several graph operations and related results on the magicness of the obtained graphs. The first operation is a generalized total composition graph. Let F and G be two graphs. Let H_i, $i \in [1, n]$, be a graph which contains $2F$ as a subgraph. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$. A *generalized total composition* $G[F; \mathcal{H}]$ is a graph obtained from the graph G by replacing each vertex $v \in V(G)$ with the graph F, and every edge $e \in E(G)$ with any graph from $\mathcal{H}$. Note that there are many non-isomorphic graphs $G[F; \mathcal{H}]$. For convenience, if $\mathcal{H} = \{H_1, H_2\}$, then $G[F; \mathcal{H}] = G[F; H_1, H_2]$. For any graph Γ, let $t_\Gamma = |V(\Gamma)| + |E(\Gamma)|$.

In the next theorem we give a sufficient condition when a generalized total composition $G[F; H_1, H_2]$ is (H_1, H_2)-magic.

Theorem 5. *Let G and F be connected nontrivial graphs. Let H_1 and H_2 be non-isomorphic connected graphs which have size at least $2|E(F)| + 2$ and contain $2F$ as a subgraph. Let s_i be the number of subgraphs of $G[F; H_1, H_2]$ isomorphic to H_i, $i = 1, 2$. Let $s_1 + s_2 = |E(G)|$. Let $|V(F)| + |E(F)|$ be even or $|V(G)|$ be odd. If both $t_{H_1}(s_1 - 1)$ and $t_{H_2}(s_2 - 1)$ are even then the graph $G[F; H_1, H_2]$ is (H_1, H_2)-magic.*

Proof. Let G be a connected graph of order n. The idea of the proof is to label the vertices and edges of $G[F; H_1, H_2]$ such that its every subgraph isomorphic to F (which is obtained by replacing a vertex of G) has constant sum of labels, and then ensure that every pair of vertices or edges in the same 'component' have the constant sums.

Suppose that either t_F is even or n is odd. Then nF is F-magic due to Theorem 1. Let $g : V(nF) \cup E(nF) \to [1, nt_F]$ be a F-magic labeling of nF with the magic constant c, i.e., the weights of subgraphs F corresponding to the vertices of G are $w_g(F) = c$.

Let s_i denote the number of subgraphs of $G[F; H_1, H_2]$ isomorphic to H_i, $i = 1, 2$, and let $s_1 + s_2 = |E(G)|$. Now suppose that $t_{H_1}(s_1 - 1)$ is even. Consider the following two cases.

Case 1. When t_{H_1} is even. Let $X^1 = [nt_F + 1, s_1(t_{H_1} - 2t_F) + nt_F]$. Create a partition of X^1 into 2-sets, X^1_j for $j \in [1, \frac{1}{2}(s_1(t_{H_1} - 2t_F))]$ such that

$$\sum_{a \in X^1_j} a = s_1(t_{H_1} - 2t_F) + 2nt_F + 1$$

and put $s_1(t_{H_1} - 2t_F) + 2nt_F + 1 = \overline{X^1}$.

Case 2. When both s_1 and t_{H_1} are odd. Let $Y^1 = [nt_F + 1, s_1(t_{H_1} - 2t_F - 3) + nt_F]$. Similarly, create a partition of Y^1 into 2-sets, Y_k^1 for $k \in [1, \frac{1}{2}(s_1(t_{H_1} - 2t_F - 3))]$ such that

$$\sum_{a \in Y_k^1} a = s_1(t_{H_1} - 2t_F - 3) + 2nt_F + 1,$$

and denote $s_1(t_{H_1} - 2t_F - 3) + 2nt_F + 1 = \overline{Y^1}$. Next, let

$$Z^1 = [s_1(t_{H_1} - 2t_F - 3) + nt_F + 1, s_1(t_{H_1} - 2t_F) + nt_F]$$

and consider $x = s_1(t_{H_1} - 2t_F - 3) + nt_F$, $y = x + s_1$, $z = x + 2s_1$. By Lemma 1, Z^1 is s_1-balanced. Let Z_l be a balanced multisets of Z^1 for $l \in [1, s_1]$. Hence

$$\sum_{a \in Z_l^1} a = 3(s_1(t_{H_1} - 2t_F - 2) + nt_F + \tfrac{1}{2}(s_1 + 1)),$$

let $3(s_1(t_{H_1} - 2t_F - 2) + nt_F + \frac{1}{2}(s_1 + 1)) = \overline{Z^1}$.

Furthermore, let

$$\begin{aligned}
X^2 &= [s_1(t_{H_1} - 2t_F) + nt_F + 1, s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F) + nt_F], \\
Y^2 &= [s_1(t_{H_1} - 2t_F) + nt_F + 1, s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F - 3) + nt_F], \\
Z^2 &= [s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F - 3) + nt_F + 1, s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F) + nt_F].
\end{aligned}$$

By a similar approach, when $t_{H_2}(s_2 - 1)$ is even, we obtain balanced multisets X_j^2, Y_k^2 and Z_l^2 such that

$$\begin{aligned}
\sum_{a \in X_j} a &= 2s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F) + 2nt_F + 1 = \overline{X^2}, \\
\sum_{a \in Y_k} a &= 2s_1(t_{H_1} - 2t_F) + s_2(t_{H_2} - 2t_F - 3) + 2nt_F + 1 = \overline{Y^2}, \\
\sum_{a \in Z_l} a &= 2s_1(t_{H_1} - 2t_F) + s_2(2t_{H_2} - 4t_F - 3) + 2nt_F + 1 = \overline{Z^2}.
\end{aligned}$$

Now, define a total labeling f of $G[F; H_1, H_2]$ as follows.

- For $v \in V(nF)$ put $f(v) = g(v)$.
- Assign the elements of X_j^1, Y_k^1 or Z_l^1 to label unlabeled vertices and unlabeled edges of a subgraph isomorphic to H_1.
- Likewise, use the elements of X_j^2, Y_k^2, Z_l^2 to label unlabeled vertices and unlabeled edges of subgraph isomorphic to H_2.

It is a routine to check that f is a bijection. To prove that $G[F; H_1, H_2]$ is (H_1, H_2)-magic, consider a subgraph H_1^* of $G[F; H_1, H_2]$ isomorphic to H_1. We get that

$$w_f(H_1^*) = \begin{cases} 2c + \frac{1}{2}\overline{X^1}(t_{H_1} - 2t_F), & \text{if } s_1 \text{ is even,} \\ 2c + \frac{1}{2}\overline{Y^1}(t_{H_1} - 2t_F - 3) + \overline{Z^1}, & \text{if } s_1 \text{ and } t_{H_1} \text{ are odd.} \end{cases}$$

Moreover, for a subgraph H_2^* of $G[F; H_1, H_2]$ isomorphic to H_2, we have

$$w_f(H_2^*) = \begin{cases} 2c + \frac{1}{2}\overline{X^2}(t_{H_2} - 2t_F), & \text{if } s_2 \text{ is even,} \\ 2c + \frac{1}{2}\overline{Y^2}(t_{H_2} - 2t_F - 3) + \overline{Z^2}, & \text{if } s_2 \text{ and } t_{H_2} \text{ are odd.} \end{cases}$$

Thus it may be concluded that $G[F; H_1, H_2]$ is (H_1, H_2)-magic. □

An illustration of a construction described in the proof of Theorem 5 is given in Figure 2.

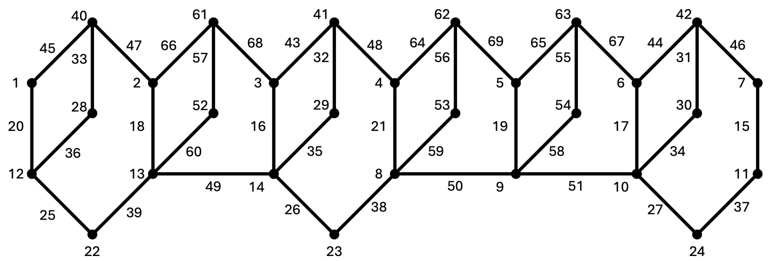

Figure 2: The $P_7[K_2; H_1, H_2]$ is (H_1, H_2)-magic, where H_1 is a cycle on 6 vertices with a subdivided chord and H_2 is a cycle on 5 vertices with a subdivided chord.

In the next part we present another method of generating (H_1, H_2)-magic graphs. Let F_1 and F_2 be finite graphs containing a graph A as a subgraph. We call A as a connector. An A-amalgamation of graphs F_1 and F_2, denoted by $Amal(F_1, F_2; A)$, is a graph obtained by taking F_1 and F_2 and identifying their connectors A. Let H_1 be a connected graph which contains $Amal(F_1, F_2; A)$ as a proper subgraph and let H_2 be a connected graph which contains $F_1 \cup F_2$ as a proper subgraph. The graph $P_n[H_1; H_2]$ is constructed as an alternating sequence of $\lceil n/2 \rceil$ copies of the graph H_1 and $\lfloor n/2 \rfloor$ copies of the graph H_2. For $i = 1, 2, \ldots, \lceil n/2 \rceil - 1$, the ith copy of H_1 is connected to the ith copy of H_2 by identifying the connector F_2, and the ith copy of H_2 is connected to the $(i+1)$th copy of H_1 by identifying the connector F_1. To aid visualization, a diagram of $P_3[H_1; H_2]$ is provided in Figure 3.

Let H be a subgraph of G. For the sake of clarity we use the notation $t_{G-H} = t_G - t_H$.

Theorem 6. *Let F_1 and F_2 be connected graphs containing a graph A as a proper subgraph and let $F^* = Amal(F_1, F_2; A)$ contain exactly one subgraph isomorphic to F_i, $i = 1, 2$.*

Let H_1 be a connected graph containing F^* as a proper subgraph and let H_2 be a connected graph containing $F_1 \cup F_2$ as a proper subgraph. Let each of $t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)}$, $t_{H_2-(F_1\cup F_2)}$, and $t_{F_1\cap F_2} + t_{H_1-F^*}$ be an even number. If there are exactly $n+1-i$ subgraphs in $P_n[H_1; H_2]$ isomorphic to H_i, $i = 1, 2$, then the graph $P_n[H_1; H_2]$ is (H_1, H_2)-magic for $n \geq 2$.

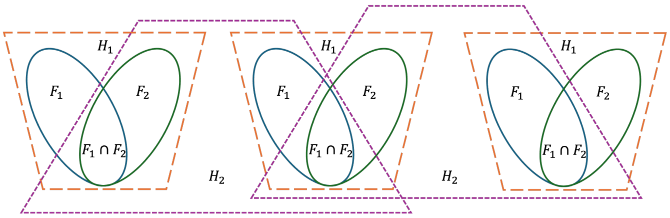

Figure 3: Visualization of $P_3[H_1; H_2]$.

Proof. The idea of the proof is similar to the proof of Theorem 5. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ in a natural way. For every $\Gamma \in \{F_1, F_2, F_1 \cap F_2, H_1 - F^*\}$, where $F^* = Amal(F_1, F_2; A)$, let $\Gamma^{(i)}$ be a subgraph isomorphic to Γ which is originated from $v_i \in V(P_n)$.

First, let $Z = [n(t_{H_1}) + 1, n(t_{H_1} + t_{H_2-(F_1 \cup F_2)})]$. Create a partition of Z into 2-sets, Z_k for $k \in [1, \frac{1}{2}n(t_{H_2-(F_1\cup F_2)})]$ such that

$$\sum_{a \in Z_k} a = n(2t_{H_1} + t_{H_2-(F_1\cup F_2)}) + 1,$$

and put $n(2t_{H_1} + t_{H_2-(F_1\cup F_2)}) + 1 = \overline{Z}$. Now, we consider several cases based on the parity of t_{F_1} and t_{F_2}. Since $t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)}$ is even, then both of them have the same parity.

Case 1.1. When $t_{F_1-(F_1\cap F_2)}$ and $t_{F_2-(F_1\cap F_2)}$ are even. Let $X^1 = [1, n(t_{F_1-(F_1\cap F_2)})]$. Create a partition X^1 into 2-sets, X_i^1 for $i \in [1, \frac{1}{2}n(t_{F_1-(F_1\cap F_2)})]$ such that

$$\sum_{a \in X_i^1} a = n(t_{F_1-(F_1\cap F_2)}) + 1,$$

and denote $n(t_{F_1-(F_1\cap F_2)}) + 1 = \overline{X^1}$. Likewise, let

$$Y^1 = [n(t_{F_1-(F_1\cap F_2)}) + 1, n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)})]$$

and create a partition of Y^1 into 2-sets, Y_i^1 for $i \in [1, \frac{1}{2}n(t_{F_2-(F_1\cap F_2)})]$ such that

$$\sum_{a \in Y_i^1} a = n(2(t_{F_1-(F_1\cap F_2)}) + t_{F_2-(F_1\cap F_2)}) + 1,$$

and put $n(2(t_{F_1-(F_1\cap F_2)}) + t_{F_2-(F_1\cap F_2)}) + 1 = \overline{Y^1}$.

Case 1.2. When $t_{F_1-(F_1\cap F_2)}$ and $t_{F_2-(F_1\cap F_2)}$ are odd. Let $X^2 = [1, n(t_{F_1-(F_1\cap F_2)} - 1)]$. Create a partition X^2 into 2-sets, X_i^2 for $i \in [1, \frac{1}{2}n(t_{F_1-(F_1\cap F_2)} - 1)]$ such that

$$\sum_{a \in X_i^2} a = n(t_{F_1-(F_1\cap F_2)} - 1) + 1,$$

and let $n(t_{F_1-(F_1\cap F_2)} - 1) + 1 = \overline{X^2}$. Again, let

$$Y^2 = [n(t_{F_1-(F_1\cap F_2)}) + 1, n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)} - 1)]$$

and create a partition of Y^2 into 2-sets, Y_i^2 for $i \in [1, \frac{1}{2}n(t_{F_2-(F_1\cap F_2)} - 1)]$ such that

$$\sum_{a\in Y_i^2} a = n(2(t_{F_1-(F_1\cap F_2)}) + t_{F_2-(F_1\cap F_2)} - 1) + 1,$$

and denote $n(2(t_{F_1-(F_1\cap F_2)}) + t_{F_2-(F_1\cap F_2)} - 1) + 1 = \overline{Y^2}$.

Next, we consider the parity of $t_{F_1\cap F_2}$ and $t_{H_1-F^*}$ in a similar manner.

Case 2.1. When $t_{F_1\cap F_2}$ and $t_{H_1-F^*}$ are even. Let

$$U^1 = [n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)}) + 1, n(t_{F^*})].$$

Create a partition of U^1 into 2-sets, U_j^1 for $j \in [1, \frac{1}{2}n(t_{F_1\cap F_2})]$ such that

$$\sum_{a\in U_j^1} a = n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)} + t_{F^*}) + 1 = \overline{U^1}.$$

Similarly, let $W^1 = [n(t_{F^*}) + 1, n(t_{H_1})]$ and create a partition of W^1 into 2-sets, W_j^1 for $j \in [1, \frac{1}{2}n(t_{H_1-F^*})]$ such that

$$\sum_{a\in W_j^1} a = n(t_{H_1} + t_{F^*}) + 1 = \overline{W^1}.$$

Case 2.2. When $t_{F_1\cap F_2}$ and $t_{H_1-F^*}$ are odd. Let

$$U^2 = [n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)}) + 1, n(t_{F^*} - 1)].$$

Create a partition of U^2 into 2-sets, U_j^2 for $j \in [1, \frac{1}{2}n(t_{F_1\cap F_2} - 1)]$ such that

$$\sum_{a\in U_j^2} a = n(t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)} + t_{F^*} - 1) + 1 = \overline{U^2}.$$

Similarly, let $W^2 = [n(t_{F^*}) + 1, n(t_{H_1} - 1)]$ and create a partition of W^2 into 2-sets, W_j^2 for $j \in [1, \frac{1}{2}n(t_{H_1-F^*} - 1)]$ such that

$$\sum_{a\in W_j^2} a = n(t_{H_1} + t_{F^*} - 1) + 1 = \overline{W^2}.$$

Now construct a total labeling f of $P_n[H_1; H_2]$ as follows.

- According to the parity of $t_{F_1\cap F_2}$ use either the elements of U_j^1 or the elements of U_j^2 to label vertices and edges of $(F_1 \cap F_2)^{(j)}$. Moreover, if $t_{F_1\cap F_2}$ is odd, label the unlabeled vertex or unlabeled edge in $(F_1 \cap F_2)^{(j)}$ with $n(t_{F^*} - 1) + j$.
- According to the parity of $t_{H_1-F^*}$ use either the elements of W_j^1 or W_j^2 to label vertices and edges of $(H_1 - F^*)^{(j)}$. Moreover, if $t_{H_1-F^*}$ is odd, label the unlabeled vertex or unlabeled edge in $(H_1 - F^*)^{(j)}$ with $n(t_{H_1}) - j + 1$.

- According to the parity of $t_{F_1-(F_1\cap F_2)}$ assign the elements of X_i^1 or X_i^2 to unlabeled vertices and unlabeled edges of $F_1^{(i)}$. In addition, if $t_{F_1-(F_1\cap F_2)}$ is odd, label the unlabeled vertex or unlabeled edge in $F_1^{(i)}$ with $n(t_{F_1-(F_1\cap F_2)}-1)+i$.
- According to the parity of $t_{F_2-(F_1\cap F_2)}$ assign the elements of Y_i^1 or Y_i^2 to unlabeled vertices and unlabeled edges of $F_2^{(i)}$. Similarly, if $t_{F_2-(F_1\cap F_2)}$ is odd, label the unlabeled vertex or unlabeled edge in $F_2^{(i)}$ with $n(t_{F_1-(F_1\cap F_2)}+t_{F_2-(F_1\cap F_2)})-i+1$.
- Lastly, use the elements of Z to label unlabeled vertices and unlabeled edges of a subgraph isomorphic to H_2.

It can be shown that f is a bijection. To show that $P_n[H_1;H_2]$ is (H_1,H_2)-magic, consider a subgraph $H_1^{(i)}$ of $P_n[H_1;H_2]$ isomorphic to H_1. If $t_{F_1-(F_1\cap F_2)}$ and $t_{F_1\cap F_2}$ are even we get that

$$w(H_1^{(i)})=\tfrac{1}{2}\overline{X^1}t_{F_1-(F_1\cap F_2)}+\tfrac{1}{2}\overline{Y^1}t_{F_2-(F_1\cap F_2)}+\tfrac{1}{2}\overline{U^1}t_{F_1\cap F_2}+\tfrac{1}{2}\overline{W^1}t_{H_1-F^*}.$$

If $t_{F_1-(F_1\cap F_2)}$ is odd but $t_{F_1\cap F_2}$ is even, then

$$\begin{aligned}w(H_1^{(i)})&=\tfrac{1}{2}\overline{X^2}(t_{F_1-(F_1\cap F_2)}-1)+\tfrac{1}{2}\overline{Y^2}(t_{F_2-(F_1\cap F_2)}-1)+\tfrac{1}{2}\overline{U^1}t_{F_1\cap F_2}+\tfrac{1}{2}\overline{W^1}t_{H_1-F^*}\\&\quad+n(2t_{F_1-(F_1\cap F_2)}+t_{F_2-(F_1\cap F_2)}-1)+1.\end{aligned}$$

If $t_{F_1-(F_1\cap F_2)}$ is even but $t_{F_1\cap F_2}$ is odd, we have

$$\begin{aligned}w(H_1^{(i)})&=\tfrac{1}{2}\overline{X^1}t_{F_1-(F_1\cap F_2)}+\tfrac{1}{2}\overline{Y^1}t_{F_2-(F_1\cap F_2)}+\tfrac{1}{2}\overline{U^2}(t_{F_1\cap F_2}-1)+\tfrac{1}{2}\overline{W^2}(t_{H_1-F^*}-1)\\&\quad+n(2t_{F^*}+t_{H_1-F^*}-1)+1.\end{aligned}$$

If $t_{F_1-(F_1\cap F_2)}$ and $t_{F_1\cap F_2}$ are odd, it follows that

$$\begin{aligned}w(H_1^{(i)})&=\tfrac{1}{2}\overline{X^2}(t_{F_1-(F_1\cap F_2)}-1)+\tfrac{1}{2}\overline{Y^2}(t_{F_2-(F_1\cap F_2)}-1)+\tfrac{1}{2}\overline{U^2}(t_{F_1\cap F_2}-1)\\&\quad+\tfrac{1}{2}\overline{W^2}(t_{H_1-F^*}-1)+n(2t_{F_1-(F_1\cap F_2)}+t_{F_2-(F_1\cap F_2)}-1)\\&\quad+n(2t_{F^*}+t_{H_1-F^*}-1)+2.\end{aligned}$$

Furthermore, consider $H_2^{(i)}\cong H_2$. If $t_{F_1-(F_1\cap F_2)}$ and $t_{F_1\cap F_2}$ are even, then

$$w(H_2^{(i)})=\tfrac{1}{2}\overline{Z}t_{H_2-(F_1\cup F_2)}+\tfrac{1}{2}\overline{X^1}t_{F_1-(F_1\cap F_2)}+\tfrac{1}{2}\overline{Y^1}t_{F_2-(F_1\cap F_2)}+\overline{U^1}t_{F_1\cap F_2}.$$

If $t_{F_1-(F_1\cap F_2)}$ is odd but $t_{F_1\cap F_2}$ is even, we have

$$\begin{aligned}w(H_2^{(i)})&=\tfrac{1}{2}\overline{Z}t_{H_2-(F_1\cup F_2)}+\tfrac{1}{2}\overline{X^2}(t_{F_1-(F_1\cap F_2)}-1)+\tfrac{1}{2}\overline{Y^2}(t_{F_2-(F_1\cap F_2)}-1)+\overline{U^1}t_{F_1\cap F_2}\\&\quad+n(2t_{F_1-(F_1\cap F_2)}+t_{F_2-(F_1\cap F_2)}-1)+1.\end{aligned}$$

If $t_{F_1-(F_1\cap F_2)}$ is even but $t_{F_1\cap F_2}$ is odd, it follows that

$$\begin{aligned} w(H_2^{(i)}) &= \tfrac{1}{2}\overline{Z}t_{H_2-(F_1\cup F_2)} + \tfrac{1}{2}\overline{X^1}t_{F_1-(F_1\cap F_2)} + \tfrac{1}{2}\overline{Y^1}t_{F_2-(F_1\cap F_2)} + \overline{U^2}(t_{F_1\cap F_2}-1) \\ &\quad + n(2t_{F^*} + t_{H_1-F^*} - 1) + 1. \end{aligned}$$

If $t_{F_1-(F_1\cap F_2)}$ and $t_{F_1\cap F_2}$ are odd, it holds that

$$\begin{aligned} w(H_2^{(i)}) &= \tfrac{1}{2}\overline{Z}t_{H_2-(F_1\cup F_2)} + \tfrac{1}{2}\overline{X^2}(t_{F_1-(F_1\cap F_2)}-1) + \tfrac{1}{2}\overline{Y^2}(t_{F_2-(F_1\cap F_2)}-1) \\ &\quad + \overline{U^2}(t_{F_1\cap F_2}-1) + n(2t_{F_1-(F_1\cap F_2)} + t_{F_2-(F_1\cap F_2)} - 1) \\ &\quad + n(2t_{F^*} + t_{H_1-F^*} - 1) + 2. \end{aligned}$$

Therefore, $P_n[H_1; H_2]$ is (H_1, H_2)-magic. □

For instance, we present an example of $P_3[H_1; H_2]$ which is (H_1, H_2)-magic, see Figure 4.

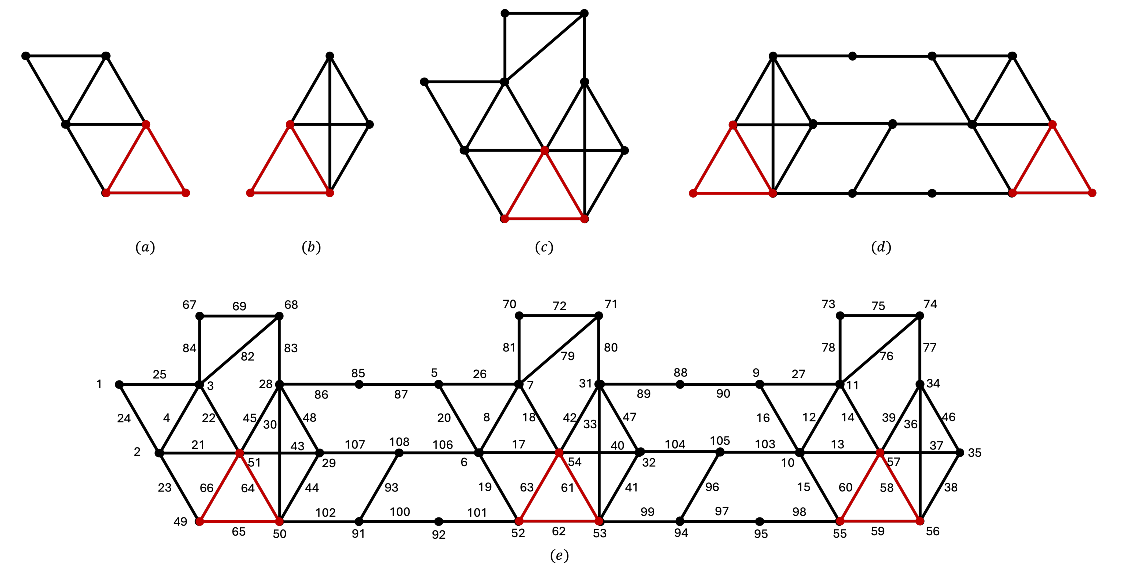

Figure 4: The graphs (a) F_1, (b) F_2, (c) H_1, (d) H_2, (e) (H_1, H_2)-magic labeling of $P_3[H_1; H_2]$. In addition, vertices and edges of $F_1 \cap F_2$ are outlined in red.

5. Conclusion

In this paper, we have presented a path-magic family of disjoint union of paths and two general constructions of (H_1, H_2)-magic graphs. Our findings partially address the gaps in the existing knowledge on this emerging topic.

A potential application of (H_1, H_2)-magic graphs is in the foundational design of structured networks, where they help enforce a hidden uniformity. By ensuring that specific, critical patterns within the larger system all share an identical cumulative property, these graphs enable a built-in balance and symmetry. This inherent harmony simplifies system-wide management, promotes fault tolerance by making key components interchangeable, and provides a robust mathematical framework.

The next research direction in this topic is to systematically investigate the (H_1, H_2)-magic properties of graphs formed by a graph operation, such as the Cartesian product or strong product. While some specific cases are not that hard to be determined, a general theory is lacking. Establishing necessary and sufficient conditions when a graph is (H_1, H_2)-magic would represent a significant advancement.

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References

[1] A. Gutiérrez and A. Lladó. Magic covering. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 55:43–56, 2005.

[2] B. Yang, M. A. Rashid, S. Ahmad, M. F. Nadeem, and M. K. Siddiqui. Cycle super magic labeling of planar graphs. *International Journal of Applied Mathematics*, 32(6):945–957, 2019.

[3] M. Asif, G. Ali, M. Numan, and A. Semaničová-Feňovčíková. Cycle-supermagic labeling for some families of graphs. *Utilitas Mathematica*, 103:51–59, 2017.

[4] T. Öner, M. Hussain, and S. Baranas. C_n-supermagic labeling of polygonal snake graphs. *Journal of Mathematics and Computer Science*, 20(3):189–195, 2019.

[5] H. Sandariria and Y. Susanti. H-supermagic labeling on edge coronation of some graphs with a cycle. In *AIP Conference Proceedings*, volume 2192, page 040014, 2019.

[6] K. Ali, S. T. R. Rizvi, and A. Semaničová-Feňovčíková. C_4-supermagic labelings of disjoint union of prisms. *Mathematical Reports*, 18(3):315–320, 2016.

[7] Y. F. Ashari and A. N. M. Salman. On (H_1, H_2)-supermagic labeling of some graph operation. *Electronic Journal of Graph Theory and Applications*, 2025. submitted.

[8] M. Bača, P. Jeyanthi, N. T. Muthuraja, P. N. Selvagopal, and A. Semaničová-Feňovčíková. Ladders and fan graphs are cycle-antimagic. *Hacettepe Journal of Mathematics and Statistics*, 49(3):1093–1106, 2020.

[9] C. Chithra, G. Marimuthu, and G. Kumar. C_m-E-supermagic labelings of graphs, journal = AKCE International Journal of Graphs and Combinatorics. 17(1):510–518, 2020.

[10] X. Ma, M. A. Umar, S. Nazeer, Y. Chu, and Y. Liu. Stacked book graphs are cycle-antimagic. *AIMS Mathematics*, 5(6):6043–6050, 2020.

[11] T. K. Maryati, F. F. Hadiputra, and A. N. M. Salman. Forbidden family of P_h-magic graphs. *Electronic Journal of Graph Theory and Applications*, 12(1):43–54, 2024.

[12] J. A. Gallian. A dynamic survey of graph labelings, 2024.

[13] D. I. Lanlege, S. E. Fadugba, N. Ali, A. L. Ozioko, N. Alam, S. Ahmad, N. Jeeva, and M. Z. Sayed-Ahmed. Mathematical model of the social pathogen of HIV/AIDS stigma. *Communications in Mathematical Biology and Neuroscience*, page Article ID 6, 2025.

[14] S. N. Saleh, M. K. Naseer, N. Ali, Ü. Karabiyik, M. S. Zakir, and M. Arshad. Graph-theoretical approaches to entropy in Cu_2O crystalline structures: Implications for biomedical and energy applications. *Communications in Mathematical Biology and Neuroscience*, page Article ID 64, 2025.

[15] T. K. Maryati, A. N. M. Salman, and E. T. Baskoro. Supermagic coverings of the disjoint union of graphs and amalgamations. *Discrete Mathematics*, 313:397–405, 2013.

[16] T. K. Maryati, A. N. M. Salman, E. T. Baskoro, J. Ryan, and M. Miller. On H-supermagic labelings for certain shackles and amalgamations of a connected graph. *Utilitas Mathematica*, 83:333–342, 2010.

[17] T. K. Maryati, A. N. M. Salman, E. T. Baskoro, and Irawati. On P_h-supermagic labelings of cP_n, booktitle = Proceedings of the 14th National Conference of Mathematics, pages = 281–285, year = 2009.