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# Decision-Making under Uncertainty with Bipolar Complex n,m-Rung Orthopair Fuzzy Sets: A Water Crisis Application

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**Abstract.** Decision-making in complex and uncertain environments often involves handling multidimensional, conflicting, and partially contradictory information. While existing fuzzy frameworks—such as bipolar, n,m-rung orthopair, and complex fuzzy sets—address specific aspects of uncertainty, none fully capture bipolarity, complex-valued membership, and flexible n,m-rung representation simultaneously. To address this gap, this study introduces the bipolar complex n,m-rung orthopair fuzzy set (BCn,m-ROFS), a unified framework capable of representing positive and negative evaluations alongside complex-valued uncertainties with adjustable $n$ and $m$ parameters. Within this framework, two novel aggregation operators—BCn,m-ROF weighted averaging (BCn,m-ROFWA) and weighted geometric (BCn,m-ROFWG)—are developed to integrate multidimensional attribute information efficiently, while maintaining discriminative power and computational feasibility. The proposed approach is applied to multi-attribute decision-making problems, illustrating its capability to rank alternatives consistently and interpretably under varying conditions. Comparative analyses with traditional fuzzy models demonstrate that BCn,m-ROFS-based operators offer superior stability, ranking discrimination, and adaptability in uncertain decision environments. Sensitivity studies further confirm the robustness of the approach, highlighting practical considerations for extreme parameter settings. Overall, the BCn,m-ROFS framework provides a flexible, theoretically grounded, and computationally practical methodology for decision support, enabling more informed and balanced choices in scenarios characterized by complex, bipolar, and uncertain information.

**2020 Mathematics Subject Classifications**: 03E72, 90B50

**Key Words and Phrases**: Bipolar complex n,m-rung orthopair fuzzy set, aggregation operators, decision support

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## 1. Introduction

Multi-attribute decision making (MADM) provides a structured framework for evaluating and selecting the most suitable alternative among several options based on multiple, often conflicting, attributes. It is widely used in fields such as operations research, engineering, management, and economics to address complex problems involving trade-offs between factors like cost, quality, and performance. Traditional MADM techniques, however, depend on precise numerical data, which is often unrealistic in real-world scenarios where information may be vague, incomplete, or qualitative in nature. To manage such uncertainty, Zadeh [1] introduced the concept of fuzzy sets (FSs), represented by membership functions assigning values between 0 and 1. Later, Atanassov [2] extended this concept to intuitionistic fuzzy sets (IFSs), incorporating both membership and non-membership degrees, along with a hesitation margin. Yager [3] further generalized this to Pythagorean fuzzy sets (PFSs), expanding the permissible range of uncertainty. To overcome the inherent limitations of PFSs, Yager [4] proposed the q-rung orthopair fuzzy set (q-ROFS), which introduces a flexible parameter q to better represent uncertainty. Senapati and Yager [5] reformulated these as Fermatean fuzzy sets (FFSs), enhancing their expressiveness. Recently, Ibrahim and Alshammari [6] introduced the n,m-rung orthopair fuzzy set (n,m-ROFS), where the sum of the nth power of membership and the mth power of non-membership values does not exceed one. This generalization offers greater flexibility and representational power, enabling more accurate modeling of complex decision-making environments.

Over the years, numerous studies on decision-making have demonstrated that although existing approaches can manage uncertain information, they often struggle to account for its temporal variability. To address this limitation, Ramot et al. [7] extended the concept of fuzzy sets by allowing membership degrees to take values within the entire unit disc of the complex plane, introducing the complex fuzzy set (CFS). Building on this foundation, Alkouri and Salleh [8] proposed the complex intuitionistic fuzzy set (CIFS), enhancing the ability to represent uncertainty more precisely. Later, Ullah et al. [9] developed the complex Pythagorean fuzzy set (CPFS) to better handle uncertainty associated with both amplitude and phase terms. Liu et al. [10] further generalized this concept by introducing the complex q-rung orthopair fuzzy set (Cq-ROFS), where the sum of the qth powers of the membership and non-membership degrees (including their real and imaginary components) does not exceed one. Recently, Ibrahim [11] advanced this field by formulating the complex n,m-rung orthopair fuzzy set (Cn,m-ROFS), which provides a broader and more flexible structure capable of capturing higher levels of uncertainty and allowing the combined membership and non-membership values to exceed one when necessary.

Bipolarity reflects the human ability to evaluate situations using both positive and negative perspectives. Positive information represents what is desirable, permissible, or advantageous, while negative information conveys what is unacceptable, prohibited, or undesirable. In decision-making, positive tendencies guide preferences by highlighting favorable options, whereas negative tendencies act as constraints, identifying alternatives that should be excluded. The concept of bipolar fuzzy sets (BFSs) was first introduced and later refined by Zhang [12, 13] to represent such dual-valued reasoning. Building on this foundation, Ezhilmaran and Sankar [14] developed bipolar intuitionistic fuzzy sets (BIFSs) and their relational structures, exploring their key properties. Later, Mohana and Jansi [15] proposed bipolar Pythagorean fuzzy sets (BPFSs) to extend the representational capacity of bipolar modeling. More recently, Ibrahim [16] generalized the n,m-rung orthopair fuzzy framework to introduce bipolar n,m-rung orthopair fuzzy sets (Bn,m-ROFSs), addressing existing limitations and enhancing their applicability in decision-making scenarios. Expanding these ideas into the complex domain led to the development of bipolar complex fuzzy sets (BCFSs), which integrate the dual membership structure of BFSs with complex-valued representations. In BCFSs, both positive and negative membership degrees are expressed through real and imaginary components, enabling a richer modeling of uncertainty. Mahmood and Rehman [17] applied BCFSs to generalized similarity measures, highlighting their effectiveness in pattern recognition, decision analysis, and system optimization. Subsequently, the concept evolved into bipolar complex intuitionistic fuzzy sets (BCIFSs), which represent both membership and non-membership degrees as complex numbers, allowing for finer representation of dual-aspect uncertainties. Mahmood et al. [18] formally established the theoretical foundation of BCIFSs, while later studies, such as those by Ibrahim and Alqahtani [19], demonstrated their successful application in areas like environmental impact evaluation, sustainable waste-to-energy decision-making, and multi-attribute analysis. These advancements provide a powerful and flexible framework for modeling multidimensional uncertainty in complex real-world decision-support systems.

Fuzzy sets and their extensions have become essential tools for modeling uncertainty and imprecision across a wide range of applications, including control systems, decision-making, artificial intelligence, healthcare, robotics, image processing, environmental management, and education. Classical FSs provide a framework for representing vague information, while IFSs extend this by incorporating both membership and non-membership degrees, enhancing their descriptive power in areas such as medical diagnosis, pattern recognition, and multi-attribute decision-making [20–22]. PFSs and q-ROFSs further expand uncertainty modeling capabilities, enabling more flexible decision-making strategies [10, 23]. The introduction of FFSs and their interval-valued variants has proven effective for handling real-world multi-attribute group decision-making problems, such as partner evaluation in recycling and sustainable development [24, 25]. Similarly, CFSs and their extensions—including CIFS, CPFSs, and Cq-ROFSs—provide a powerful framework for datasets characterized by incomplete, ambiguous, or multidimensional uncertainty, with applications in medical diagnostics, robotics, signal processing, environmental research, and supply chain management [7, 9, 26–28]. Moreover, BFSs and their extensions—including BIFSs, BPFSs, and BCFSs/BCIFSs—capture both positive and negative perspectives of information, enabling dynamic, context-aware decision-making in complex environments [19, 29–32]. These multidimensional fuzzy frameworks offer robust mathematical tools to model uncertainty and support decision-making across diverse real-world scenarios, from environmental management and healthcare to robotics and artificial intelligence.

### 1.1. Motivation

Decision-making in complex and uncertain environments often involves multidimensional uncertainty, conflicting objectives, and evaluations that are both positive and negative. Traditional MADM approaches, based on classical set theory, are limited in capturing the nuances of hesitation, dual preferences, and complex-valued uncertainties inherent in real-world problems. Although extensions like n,m-rung orthopair, complex n,m-rung orthopair, and bipolar n,m-rung orthopair fuzzy sets address parts of these challenges, they are insufficient for fully modeling dynamic, contradictory, or phase-dependent decision contexts. To bridge this gap, this study introduces BCn,m-ROFS, a unified framework capable of integrating positive and negative information, hesitation, and complex-valued uncertainty. The associated aggregation operators, BCn,m-ROFWA and BCn,m-ROFWG, allow decision-makers to effectively combine supporting and opposing evaluations across multiple attributes, such as resource efficiency, environmental impact, economic viability, and technological feasibility. Application to water crisis management demonstrates the framework's ability to produce robust, discriminative, and interpretable alternative rankings, supporting informed and sustainable decisions.

### 1.2. Key contributions

This work makes several notable contributions to multi-attribute decision-making under uncertainty:

(i) Proposes the BCn,m-ROFS framework, which unifies bipolarity, complex membership, and n,m-rung orthopair structures, enabling effective modeling of both quantitative and qualitative uncertainties.

(ii) Establishes a rigorous mathematical foundation for BCn,m-ROFS, including definitions of subset relationships, equality, complement, union, and intersection operations, illustrated through examples to ensure clarity and theoretical soundness.

(iii) Introduces two novel aggregation operators, BCn,m-ROFWA and BCn,m-ROFWG, designed to combine multi-attribute information while preserving the bipolar and complex characteristics of the data.

(iv) Validates the framework through a water crisis management application, demonstrating its ability to generate consistent, interpretable, and robust rankings under uncertain and conflicting conditions.

(v) Analyzes the effects of parameters $n$ and $m$ on aggregation outcomes, highlighting operator stability, monotonic behavior, and robustness, allowing decision-makers to fine-tune compensatory effects without impacting optimal rankings.

(vi) Compares BCn,m-ROFS with existing fuzzy MADM approaches, demonstrating superior discriminative power, ranking reliability, and the capability to handle intricate interdependencies among alternatives.

(vii) Provides visualizations of aggregation behavior, offering intuitive insight into how BCn,m-ROFS integrates and processes bipolar and complex information.

(viii) Expands the theoretical landscape of fuzzy MADM, paving the way for future research in large-scale decision problems, dynamic environments, and integration with intelligent decision-support systems.

In summary, BCn,m-ROFS and its aggregation operators offer a versatile, theoretically grounded, and practical framework for enhanced decision-making in complex, uncertainty-prone, and sustainability-focused contexts.

### 1.3. Paper organization

The structure of this paper is as follows:

- **Section 2** reviews prior studies on MADM, with emphasis on bipolar, complex, and n,m-rung orthopair fuzzy sets.
- **Section 3** presents the BCn,m-ROFS framework, detailing its definitions, algebraic operations, subset relations, and complement characteristics, accompanied by illustrative examples.
- **Section 4** introduces BCn,m-ROFS aggregation operators, including BCn,m-ROFWA and BCn,m-ROFWG, along with their formal definitions and key properties.
- **Section 5** demonstrates the application of BCn,m-ROFS to water crisis management, highlighting its effectiveness in dealing with uncertain and bipolar information.
- **Section 6** provides a comparative analysis with existing fuzzy MADM methods, using numerical results and graphical illustrations to evaluate performance, ranking stability, and interpretability.
- **Section 7** presents a sensitivity study of BCn,m-ROFWA and BCn,m-ROFWG, analyzing stability under different $n$ and $m$ values and discussing practical considerations.
- **Section 8** concludes the paper by summarizing main findings, contributions, and proposing future research directions.

This organization ensures a coherent flow from theoretical development to practical application, emphasizing the flexibility and advantages of BCn,m-ROFS in multi-attribute decision-making scenarios.

## 2. Preliminaries

This section introduces essential foundational concepts.

**Definition 1.** Let $\mathscr{S}$ be a universe of discourse. Suppose $\mathscr{Y}$ is defined as an object of the form:

$$\mathscr{Y} = \left\{ \left\langle \mathscr{T}, \mathscr{A}^+_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^+_{\mathscr{Y}}(\mathscr{T}), \mathscr{A}^-_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^-_{\mathscr{Y}}(\mathscr{T}) \right\rangle : \mathscr{T} \in \mathscr{S} \right\},$$

where $\mathscr{A}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} : \mathscr{S} \to [0,1]$, $\mathscr{A}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} : \mathscr{S} \to [-1,0]$, satisfying the condition $0 \leq \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n + \left(\mathscr{C}^+_{\mathscr{Y}}\right)^m \leq 1$, and $0 \leq \left|\mathscr{A}^-_{\mathscr{Y}}\right|^n + \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m \leq 1, \forall \mathscr{T} \in \mathscr{S}$ and $n, m \in N$. Then, $\mathscr{Y}$ is called a

(i) BIFS [33] if $n = m = 1$,

(ii) BPFS [34] if $n = m = 2$,

(iii) Bq-ROFS [35] if $n = m = q$,

(iv) Bn,m-ROFS [16] if $n, m \geq 1$.

Where $\mathscr{A}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}}$ represent the positive membership and non-membership degree, while $\mathscr{A}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}}$ define the negative membership and non-membership degree respectively.

**Definition 2.** Let $\mathscr{S}$ be a universe of discourse and $\mathscr{Y}$ be defined as an object of the form:

$$\mathscr{Y} = \left\{ \langle \mathscr{T}, \mathscr{A}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}_{\mathscr{Y}}(\mathscr{T}) \rangle : \mathscr{T} \in \mathscr{S} \right\},$$

where $\mathscr{A}_{\mathscr{Y}}(\mathscr{T}) = T_{\mathscr{Y}}(\mathscr{T}) \cdot e^{i.2\pi J_{T_{\mathscr{Y}}}(\mathscr{T})}$ and $\mathscr{C}_{\mathscr{Y}}(\mathscr{T}) = B_{\mathscr{Y}}(\mathscr{T}) \cdot e^{i.2\pi J_{B_{\mathscr{Y}}}(\mathscr{T})}$.
$\mathscr{Y}$ is called a

(i) CIFS [8] if $0 \leq T_{\mathscr{Y}}(\mathscr{T}) + B_{\mathscr{Y}}(\mathscr{T}) \leq 1$ and $0 \leq J_{T_{\mathscr{Y}}}(\mathscr{T}) + J_{B_{\mathscr{Y}}}(\mathscr{T}) \leq 1$,

(ii) CPFS [36] if $0 \leq T^2_{\mathscr{Y}}(\mathscr{T}) + B^2_{\mathscr{Y}}(\mathscr{T}) \leq 1$ and $0 \leq J^2_{T_{\mathscr{Y}}}(\mathscr{T}) + J^2_{B_{\mathscr{Y}}}(\mathscr{T}) \leq 1$,

(iii) Cq-ROFS [10] if $0 \leq T^q_{\mathscr{Y}}(\mathscr{T}) + B^q_{\mathscr{Y}}(\mathscr{T}) \leq 1$ and $0 \leq J^q_{T_{\mathscr{Y}}}(\mathscr{T}) + J^q_{B_{\mathscr{Y}}}(\mathscr{T}) \leq 1$,

(iv) Cn,m-ROFS [11] if $0 \leq T^n_{\mathscr{Y}}(\mathscr{T}) + B^m_{\mathscr{Y}}(\mathscr{T}) \leq 1$ and $0 \leq J^n_{T_{\mathscr{Y}}}(\mathscr{T}) + J^m_{B_{\mathscr{Y}}}(\mathscr{T}) \leq 1$.

**Definition 3.** [18] Let $\mathscr{S}$ be a universe of discourse. Then,

$$\mathscr{Y} = \left\{ \langle \mathscr{T}, \mathscr{A}^+_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{B}^+_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^+_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{D}^+_{\mathscr{Y}}(\mathscr{T}), \right.$$

$$\left. \mathscr{A}^-_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{B}^-_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^-_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{D}^-_{\mathscr{Y}}(\mathscr{T}) \rangle : \mathscr{T} \in \mathscr{S} \right\}$$

is called a BCIFS if

$$0 \leq \mathscr{A}^+_{\mathscr{Y}}(\mathscr{T}) + \mathscr{C}^+_{\mathscr{Y}}(\mathscr{T}) \leq 1, \; 0 \leq \mathscr{B}^+_{\mathscr{Y}}(\mathscr{T}) + \mathscr{D}^+_{\mathscr{Y}}(\mathscr{T}) \leq 1,$$

$$0 \leq \left|\mathscr{A}^{-}_{\mathscr{Y}}(\mathscr{T})\right| + \left|\mathscr{C}^{-}_{\mathscr{Y}}(\mathscr{T})\right| \leq 1 \text{ and } 0 \leq |\mathscr{B}^{-}_{\mathscr{Y}}(\mathscr{T})| + |\mathscr{D}^{-}_{\mathscr{Y}}(\mathscr{T})| \leq 1.$$

where

$$\mathscr{A}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{B}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{D}^{+}_{\mathscr{Y}}(\mathscr{T}) \in [0,1],$$

$$\mathscr{A}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{B}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{D}^{-}_{\mathscr{Y}}(\mathscr{T}) \in [-1,0],$$

## 3. Bipolar complex n,m-rung orthopair fuzzy sets

This section provides a comprehensive explanation of the fundamental principles and essential operations associated with bipolar complex n,m-ROFS.

**Definition 4.** A BCn,m-ROFS $\mathscr{Y}$ over a universe of discourse $\mathscr{S}$ is defined as an object of the form:

$$\mathscr{Y} = \big\{\langle \mathscr{T}, \mathscr{A}^{+}_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{B}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{+}_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{D}^{+}_{\mathscr{Y}}(\mathscr{T}),$$
$$\mathscr{A}^{-}_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{B}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{-}_{\mathscr{Y}}(\mathscr{T}) + i\mathscr{D}^{-}_{\mathscr{Y}}(\mathscr{T})\rangle : \mathscr{T} \in \mathscr{S}\big\}$$

where

$$\mathscr{A}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{B}^{+}_{\mathscr{Y}}(\mathscr{T}), \mathscr{D}^{+}_{\mathscr{Y}}(\mathscr{T}) \in [0,1],$$

$$\mathscr{A}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{C}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{B}^{-}_{\mathscr{Y}}(\mathscr{T}), \mathscr{D}^{-}_{\mathscr{Y}}(\mathscr{T}) \in [-1,0],$$

$$0 \leq (\mathscr{A}^{+}_{\mathscr{Y}}(\mathscr{T}))^n + (\mathscr{C}^{+}_{\mathscr{Y}}(\mathscr{T}))^m \leq 1,\ 0 \leq (\mathscr{B}^{+}_{\mathscr{Y}}(\mathscr{T}))^n + (\mathscr{D}^{+}_{\mathscr{Y}}(\mathscr{T}))^m \leq 1,$$

$$0 \leq \left|\mathscr{A}^{-}_{\mathscr{Y}}(\mathscr{T})\right|^n + \left|\mathscr{C}^{-}_{\mathscr{Y}}(\mathscr{T})\right|^m \leq 1 \text{ and } 0 \leq |\mathscr{B}^{-}_{\mathscr{Y}}(\mathscr{T})|^n + |\mathscr{D}^{-}_{\mathscr{Y}}(\mathscr{T})|^m \leq 1.$$

The calculation for $\mathscr{T}$, $\mathscr{Y} = \langle \mathscr{A}^{+}_{\mathscr{Y}} + i\mathscr{B}^{+}_{\mathscr{Y}}, \mathscr{C}^{+}_{\mathscr{Y}} + i\mathscr{D}^{+}_{\mathscr{Y}}, \mathscr{A}^{-}_{\mathscr{Y}} + i\mathscr{B}^{-}_{\mathscr{Y}}, \mathscr{C}^{-}_{\mathscr{Y}} + i\mathscr{D}^{-}_{\mathscr{Y}}\rangle$ denotes the BCn,m-ROF number (BCn,m-ROFN).

**Example 1.** Consider a decision-making problem where a municipality evaluates two smart water management alternatives, $\mathscr{A}_1$ and $\mathscr{A}_2$, based on two attributes: water efficiency ($\mathscr{A}\mathscr{T}_1$) and environmental sustainability ($\mathscr{A}\mathscr{T}_2$). The evaluations are expressed as BC2,3-ROFNs as follows:

$$\mathscr{Y}_{11} = \langle 0.75 + 0.55i, 0.25 + 0.15i, -0.20 - 0.10i, -0.05 - 0.02i\rangle,$$
$$\mathscr{Y}_{12} = \langle 0.65 + 0.45i, 0.30 + 0.10i, -0.15 - 0.05i, -0.10 - 0.03i\rangle,$$
$$\mathscr{Y}_{21} = \langle 0.60 + 0.50i, 0.35 + 0.20i, -0.10 - 0.05i, -0.15 - 0.08i\rangle,$$
$$\mathscr{Y}_{22} = \langle 0.85 + 0.65i, 0.15 + 0.10i, -0.30 - 0.25i, -0.05 - 0.02i\rangle.$$

Here, each BC2,3-ROFN $\mathscr{Y}_{ij}$ is represented as

$$\mathscr{Y}_{ij} = \langle \mathscr{A}^{+}_{\mathscr{Y}} + i\mathscr{B}^{+}_{\mathscr{Y}}, \mathscr{C}^{+}_{\mathscr{Y}} + i\mathscr{D}^{+}_{\mathscr{Y}}, \mathscr{A}^{-}_{\mathscr{Y}} + i\mathscr{B}^{-}_{\mathscr{Y}}, \mathscr{C}^{-}_{\mathscr{Y}} + i\mathscr{D}^{-}_{\mathscr{Y}}\rangle$$

where:

- $\mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}$: positive membership degree, indicating the extent of supportive evidence toward the alternative;
- $\mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}$: positive non-membership degree, reflecting the level of non-support or uncertainty in positive assessment;
- $\mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}$: negative membership degree, signifying the amount of opposing evidence against the alternative;
- $\mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}}$: negative non-membership degree, describing the extent of non-opposition or hesitation in negative assessment.

**Definition 5.** For any two BCn,m-ROFSs
$\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1}(\mathscr{T}) + i\mathscr{B}^+_{\mathscr{Y}_1}(\mathscr{T}), \mathscr{C}^+_{\mathscr{Y}_1}(\mathscr{T}) + i\mathscr{D}^+_{\mathscr{Y}_1}(\mathscr{T}), \mathscr{A}^-_{\mathscr{Y}_1}(\mathscr{T}) + i\mathscr{B}^-_{\mathscr{Y}_1}(\mathscr{T}), \mathscr{C}^-_{\mathscr{Y}_1}(\mathscr{T}) + i\mathscr{D}^-_{\mathscr{Y}_1}(\mathscr{T}) \right\rangle$
and
$\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2}(\mathscr{T}) + i\mathscr{B}^+_{\mathscr{Y}_2}(\mathscr{T}), \mathscr{C}^+_{\mathscr{Y}_2}(\mathscr{T}) + i\mathscr{D}^+_{\mathscr{Y}_2}(\mathscr{T}), \mathscr{A}^-_{\mathscr{Y}_2}(\mathscr{T}) + i\mathscr{B}^-_{\mathscr{Y}_2}(\mathscr{T}), \mathscr{C}^-_{\mathscr{Y}_2}(\mathscr{T}) + i\mathscr{D}^-_{\mathscr{Y}_2}(\mathscr{T}) \right\rangle$,
we have:

(i) $\mathscr{Y}_1 \subseteq \mathscr{Y}_2$ if and only if $\mathscr{A}^+_{\mathscr{Y}_1}(\mathscr{T}) \le \mathscr{A}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{A}^-_{\mathscr{Y}_1}(\mathscr{T}) \ge \mathscr{A}^-_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{C}^+_{\mathscr{Y}_1}(\mathscr{T}) \ge \mathscr{C}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{C}^-_{\mathscr{Y}_1}(\mathscr{T}) \le \mathscr{C}^-_{\mathscr{Y}}(\mathscr{T})$ for real terms and $\mathscr{B}^+_{\mathscr{Y}_1}(\mathscr{T}) \le \mathscr{B}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{B}^-_{\mathscr{Y}_1}(\mathscr{T}) \ge \mathscr{B}^-_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{D}^+_{\mathscr{Y}_1}(\mathscr{T}) \ge \mathscr{D}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{D}^-_{\mathscr{Y}_1}(\mathscr{T}) \le \mathscr{D}^-_{\mathscr{Y}_2}(\mathscr{T})$ for imaginary terms.

(ii) $\mathscr{Y}_1 = \mathscr{Y}_2$ if and only if $\mathscr{A}^+_{\mathscr{Y}_1}(\mathscr{T}) = \mathscr{A}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{A}^-_{\mathscr{Y}_1}(\mathscr{T}) = \mathscr{A}^-_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{C}^+_{\mathscr{Y}_1}(\mathscr{T}) = \mathscr{C}^+_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{C}^-_{\mathscr{Y}_1}(\mathscr{T}) = \mathscr{C}^-_{\mathscr{Y}_2}(\mathscr{T})$, $\mathscr{B}^+_{\mathscr{Y}_1} = \mathscr{B}^+_{\mathscr{Y}_2}$, $\mathscr{B}^-_{\mathscr{Y}_1} = \mathscr{B}^-_{\mathscr{Y}_2}$, $\mathscr{D}^+_{\mathscr{Y}_1} = \mathscr{D}^+_{\mathscr{Y}_2}$, and $\mathscr{D}^-_{\mathscr{Y}_1} = \mathscr{D}^-_{\mathscr{Y}_2}$.

**Example 2.** Let $\mathscr{S} = \{\mathscr{T}\}$, then

$$\mathscr{Y}_1 = \{\langle \mathscr{T}, 0.8 + i(0.3), 0.5 + i(0.4), -0.2 + i(-0.7), -0.7 + i(-0.3)\rangle : \mathscr{T} \in \mathscr{S}\}$$
and
$$\mathscr{Y}_2 = \{\langle \mathscr{T}, 0.9 + i(0.4), 0.3 + i(0.3), -0.3 + i(-0.9), -0.6 + i(-0.2)\rangle : \mathscr{T} \in \mathscr{S}\}$$

are two BC3,2-ROF sets, and it is clear that $\mathscr{Y}_1 \subseteq \mathscr{Y}_2$

**Definition 6.** For any two BCn,m-ROFSs $\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \right\rangle$ and $\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \right\rangle$, we have:

(i) $$\mathscr{Y}_1 \cup \mathscr{Y}_2 = \begin{array}{l} \langle \max\{\mathscr{A}^+_{\mathscr{Y}_1}, \mathscr{A}^+_{\mathscr{Y}_2}\} + i\max\{\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{B}^+_{\mathscr{Y}_2}\}, \\ \min\{\mathscr{C}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_2}\} + i\min\{\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{D}^+_{\mathscr{Y}_2}\}, \\ \min\{\mathscr{A}^-_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_2}\} + i\min\{\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{B}^-_{\mathscr{Y}_2}\}, \\ \max\{\mathscr{C}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_2}\} + i\max\{\mathscr{D}^-_{\mathscr{Y}_1}, \mathscr{D}^-_{\mathscr{Y}_2}\}\rangle. \end{array}$$

(ii) $$\mathscr{Y}_1 \cap \mathscr{Y}_2 = \begin{array}{l} \langle \min\{\mathscr{A}^+_{\mathscr{Y}_1}, \mathscr{A}^+_{\mathscr{Y}_2}\} + i\min\{\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{B}^+_{\mathscr{Y}_2}\}, \\ \max\{\mathscr{C}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_2}\} + i\max\{\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{D}^+_{\mathscr{Y}_2}\}, \\ \max\{\mathscr{A}^-_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_2}\} + i\max\{\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{B}^-_{\mathscr{Y}_2}\}, \\ \min\{\mathscr{C}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_2}\} + i\min\{\mathscr{D}^-_{\mathscr{Y}_1}, \mathscr{D}^-_{\mathscr{Y}_2}\}\rangle. \end{array}$$

(iii) $\mathscr{Y}_1{}^c = \begin{array}{l} \langle(\mathscr{C}^+_{\mathscr{Y}_1})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}_1})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}_1})^{\frac{n}{m}} + i(\mathscr{B}^+_{\mathscr{Y}_1})^{\frac{n}{m}}, \\ -|\mathscr{C}^-_{\mathscr{Y}_1}|^{\frac{m}{n}} + i(-|\mathscr{D}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}_1}|^{\frac{n}{m}} + i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\frac{n}{m}})\rangle. \end{array}$

**Example 3.** Let $\mathscr{S} = \{\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \mathscr{T}_4\}$, then

$$\mathscr{Y}_1 = \left\{ \begin{array}{l} \langle \mathscr{T}_1, 0.9 + i(0.32), 0.4 + i(0.2), -0.3 + i(-0.3), -0.8 + i(-0.18)\rangle\,, \\ \langle \mathscr{T}_2, 0.5 + i(0.2), 0.7 + i(0.3), -0.1 + i(-0.1), -0.3 + i(-0.4)\rangle\,, \\ \langle \mathscr{T}_3, 0.4 + i(0.4), 0.9 + i(0.32), -0.7 + i(-0.2), -0.6 + i(-0.29)\rangle\,, \\ \langle \mathscr{T}_4, 0.8 + i(0.5), 0.9 + i(0.3), -0.9 + i(-0.2), -0.8 + i(-0.4)\rangle \end{array} \right\}$$

and

$$\mathscr{Y}_2 = \left\{ \begin{array}{l} \langle \mathscr{T}_1, 0.4 + i(0.2), 0.9 + i(0.3), -0.8 + i(-0.4), -0.3 + i(-0.6)\rangle\,, \\ \langle \mathscr{T}_2, 0.7 + i(0.3), 0.5 + i(0.2), -0.3 + i(-0.6), -0.1 + i(-0.4)\rangle\,, \\ \langle \mathscr{T}_3, 0.7 + i(0.2), 0.6 + i(0.3), -0.9 + i(-0.4), -0.4 + i(-0.6)\rangle\,, \\ \langle \mathscr{T}_4, 0.9 + i(0.4), 0.8 + i(0.2), -0.8 + i(-0.3), -0.9 + i(-0.5)\rangle \end{array} \right\}$$ are two BC5,6-ROFSs. Thus,

(i) $$\mathscr{Y}_1 \cup \mathscr{Y}_2 = \left\{ \begin{array}{l} \langle \mathscr{T}_1, 0.9 + i(0.32), 0.4 + i(0.2), -0.8 + i(-0.4), -0.3 + i(-0.6)\rangle \\ \langle \mathscr{T}_2, 0.7 + i(0.3), 0.5 + i(0.2), -0.3 + i(-0.6), -0.1 + i(-0.4)\rangle \\ \langle \mathscr{T}_3, 0.7 + i(0.4), 0.6 + i(0.3), -0.9 + i(-0.4), -0.4 + i(-0.29)\rangle \\ \langle \mathscr{T}_4, 0.9 + i(0.5), 0.8 + i(0.2), -0.9 + i(-0.3), -0.8 + i(-0.4)\rangle \end{array} \right\}.$$

(ii) $$\mathscr{Y}_1 \cap \mathscr{Y}_2 = \left\{ \begin{array}{l} \langle \mathscr{T}_1, 0.4 + i(0.2), 0.9 + i(0.3), -0.3 + i(-0.3), -0.8 + i(-0.6)\rangle \\ \langle \mathscr{T}_2, 0.5 + i(0.2), 0.7 + i(0.3), -0.1 + i(-0.1), -0.3 + i(-0.4)\rangle \\ \langle \mathscr{T}_3, 0.4 + i(0.2), 0.9 + i(0.32), -0.7 + i(-0.2), -0.6 + i(-0.6)\rangle \\ \langle \mathscr{T}_4, 0.8 + i(0.4), 0.9 + i(0.3), -0.8 + i(-0.3), -0.9 + i(-0.6)\rangle \end{array} \right\}.$$

(iii) $$\mathscr{Y}_1^c = \left\{ \begin{array}{l} \langle \mathscr{T}_1, 0.3 + i(0.14), 0.9 + i(0.4), -0.8 + i(-0.13), -0.4 + i(-0.4)\rangle \\ \langle \mathscr{T}_2, 0.7 + i(0.23), 0.6 + i(0.3), -0.24 + i(-0.3), -0.15 + i(-0.1)\rangle \\ \langle \mathscr{T}_3, 0.9 + i(0.3), 0.5 + i(0.5), -0.54 + i(-0.2), -0.74 + i(-0.3)\rangle \\ \langle \mathscr{T}_4, 0.9 + i(0.23), 0.83 + i(0.6), -0.8 + i(-0.3), -0.9 + i(-0.3)\rangle \end{array} \right\}.$$

**Theorem 1.** Let $\mathscr{Y} = \langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}}\rangle$,
$\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \right\rangle$ and
$\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \right\rangle$ be BCn,m-ROFSs, then

(i) $\mathscr{Y}^c$ is also a BCn,m-ROFS and $(\mathscr{Y}^c)^c = \mathscr{Y}$.

(ii) $\mathscr{Y}_1 \cup \mathscr{Y}_2$ is also a BCn,m-ROFS.

(iii) $\mathscr{Y}_1 \cap \mathscr{Y}_2$ is also a BCn,m-ROFS.

*Proof.*

(i) Since

$$0 \le (\mathscr{A}^+_{\mathscr{Y}})^n + (\mathscr{C}^+_{\mathscr{Y}})^m \le 1,$$

$$0 \le \left|\mathscr{A}_{\mathscr{Y}}^{-}\right|^{n} + \left|\mathscr{C}_{\mathscr{Y}}^{-}\right|^{m} \le 1,$$

$$0 \le (\mathscr{B}_{\mathscr{Y}}^{+})^{n} + (\mathscr{D}_{\mathscr{Y}}^{+})^{m} \le 1,$$

and

$$0 \le |\mathscr{B}_{\mathscr{Y}}^{-}|^{n} + |\mathscr{D}_{\mathscr{Y}}^{-}|^{m} \le 1$$

then

$$0 \le ((\mathscr{C}_{\mathscr{Y}}^{+})^{\frac{m}{n}})^{n} + ((\mathscr{A}_{\mathscr{Y}}^{+})^{\frac{n}{m}})^{m} = (\mathscr{A}_{\mathscr{Y}}^{+})^{n} + (\mathscr{C}_{\mathscr{Y}}^{+})^{m} \le 1,$$

$$0 \le \left|- \left|\mathscr{C}_{\mathscr{Y}}^{-}\right|^{\frac{m}{n}}\right|^{n} + \left|- \left|\mathscr{A}_{\mathscr{Y}}^{-}\right|^{\frac{n}{m}}\right|^{m} = \left|\mathscr{A}_{\mathscr{Y}}^{-}\right|^{n} + \left|\mathscr{C}_{\mathscr{Y}}^{-}\right|^{m} \le 1,$$

$$0 \le ((\mathscr{D}_{\mathscr{Y}}^{+})^{\frac{m}{n}})^{n} + ((\mathscr{B}_{\mathscr{Y}}^{+})^{\frac{n}{m}})^{m} = (\mathscr{B}_{\mathscr{Y}}^{+})^{n} + (\mathscr{D}_{\mathscr{Y}}^{+})^{m} \le 1,$$

and

$$0 \le \left|-|\mathscr{D}_{\mathscr{Y}}^{-}|^{\frac{m}{n}}\right|^{n} + \left|-|\mathscr{B}_{\mathscr{Y}}^{-}|^{\frac{n}{m}}\right|^{m} = |\mathscr{B}_{\mathscr{Y}}^{-}|^{n} + |\mathscr{D}_{\mathscr{Y}}^{-}|^{m} \le 1.$$

It follows that $\mathscr{Y}^{c}$ is BCn,m-ROF set, and it is also obvious that $(\mathscr{Y}^{c})^{c} = \langle(\mathscr{C}_{\mathscr{Y}}^{+})^{\frac{m}{n}} + i(\mathscr{D}_{\mathscr{Y}}^{+})^{\frac{m}{n}}, (\mathscr{A}_{\mathscr{Y}}^{+})^{\frac{n}{m}} + i(\mathscr{B}_{\mathscr{Y}}^{+})^{\frac{n}{m}}, -|\mathscr{C}_{\mathscr{Y}}^{-}|^{\frac{m}{n}} + i(-|\mathscr{D}_{\mathscr{Y}}^{-}|^{\frac{m}{n}}), -|\mathscr{A}_{\mathscr{Y}}^{-}|^{\frac{n}{m}} + i(-|\mathscr{B}_{\mathscr{Y}}^{-}|^{\frac{n}{m}})\rangle^{c}$
$= \langle((\mathscr{A}_{\mathscr{Y}}^{+})^{\frac{n}{m}})^{\frac{m}{n}} + i((\mathscr{B}_{\mathscr{Y}}^{+})^{\frac{n}{m}})^{\frac{m}{n}}, ((\mathscr{C}_{\mathscr{Y}}^{+})^{\frac{m}{n}})^{\frac{n}{m}} + i((\mathscr{D}_{\mathscr{Y}}^{+})^{\frac{m}{n}})^{\frac{n}{m}}, -|-|\mathscr{A}_{\mathscr{Y}}^{-}|^{\frac{n}{m}}|^{\frac{m}{n}} + i(-|-|\mathscr{B}_{\mathscr{Y}}^{-}|^{\frac{n}{m}}|^{\frac{m}{n}}), -|-|\mathscr{C}_{\mathscr{Y}}^{-}|^{\frac{m}{n}}|^{\frac{n}{m}} + i(-|-|\mathscr{D}_{\mathscr{Y}}^{-}|^{\frac{m}{n}}|^{\frac{n}{m}})\rangle$

$$= \langle \mathscr{A}_{\mathscr{Y}}^{+} + i\mathscr{B}_{\mathscr{Y}}^{+}, \mathscr{C}_{\mathscr{Y}}^{+} + i\mathscr{D}_{\mathscr{Y}}^{+}, \mathscr{A}_{\mathscr{Y}}^{-} + i\mathscr{B}_{\mathscr{Y}}^{-}, \mathscr{C}_{\mathscr{Y}}^{-} + i\mathscr{D}_{\mathscr{Y}}^{-}\rangle,$$

where

$$\mathscr{A}_{\mathscr{Y}}^{-} = -\left|\mathscr{A}_{\mathscr{Y}}^{-}\right|, \quad \mathscr{C}_{\mathscr{Y}}^{-} = -\left|\mathscr{C}_{\mathscr{Y}}^{-}\right|,$$

$$\mathscr{B}_{\mathscr{Y}}^{-} = -\left|\mathscr{B}_{\mathscr{Y}}^{-}\right|, \text{ and } \quad \mathscr{D}_{\mathscr{Y}}^{-} = -\left|\mathscr{D}_{\mathscr{Y}}^{-}\right|.$$

(ii) Since we have

$$0 \le \mathscr{A}_{\mathscr{Y}_1}^{+} \le 1, \quad 0 \le \mathscr{A}_{\mathscr{Y}_2}^{+} \le 1, \quad 0 \le \mathscr{C}_{\mathscr{Y}_1}^{+} \le 1, \quad 0 \le \mathscr{C}_{\mathscr{Y}_2}^{+} \le 1,$$

$$0 \le | \mathscr{A}_{\mathscr{Y}_1}^{-} | \le 1, \quad 0 \le | \mathscr{A}_{\mathscr{Y}_2}^{-} | \le 1, \quad 0 \le | \mathscr{C}_{\mathscr{Y}_1}^{-} | \le 1, \quad 0 \le | \mathscr{C}_{\mathscr{Y}_2}^{-} | \le 1,$$

$$0 \le \mathscr{B}_{\mathscr{Y}_1}^{+} \le 1, \quad 0 \le \mathscr{B}_{\mathscr{Y}_2}^{+} \le 1, \quad 0 \le \mathscr{D}_{\mathscr{Y}_1}^{+} \le 1, \quad 0 \le \mathscr{D}_{\mathscr{Y}_2}^{+} \le 1,$$

$$0 \leq |\mathscr{D}^-_{\mathscr{Y}_1}| \leq 1, \quad 0 \leq |\mathscr{D}^-_{\mathscr{Y}_2}| \leq 1, \quad 0 \leq |\mathscr{B}^-_{\mathscr{Y}_1}| \leq 1, \text{ and } \quad 0 \leq |\mathscr{B}^-_{\mathscr{Y}_2}| \leq 1.$$

then it is clear that
$0 \leq \max\left\{\mathscr{A}^+_{\mathscr{Y}_1}, \mathscr{A}^+_{\mathscr{Y}_2}\right\}^n + \min\left\{\mathscr{C}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_2}\right\}^m \leq 1,$
$0 \leq \min\left\{|\mathscr{A}^-_{\mathscr{Y}_1}|, |\mathscr{A}^-_{\mathscr{Y}_2}|\right\}^n + \max\left\{|\mathscr{C}^-_{\mathscr{Y}_1}|, |\mathscr{C}^-_{\mathscr{Y}_2}|\right\}^m \leq 1,$
$0 \leq \max\left\{\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{B}^+_{\mathscr{Y}_2}\right\}^n + \min\left\{\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{D}^+_{\mathscr{Y}_2}\right\}^m \leq 1,$
and $0 \leq \max\left\{|\mathscr{D}^-_{\mathscr{Y}_1}|, |\mathscr{D}^-_{\mathscr{Y}_2}|\right\}^n + \min\left\{|\mathscr{B}^-_{\mathscr{Y}_1}|, |\mathscr{B}^-_{\mathscr{Y}_2}|\right\}^m \leq 1.$
Hence $\mathscr{Y}_1 \cup \mathscr{Y}_2$ is BCn,m-ROFS.

(iii) Obvious.

**Theorem 2.** Let $\mathscr{Y} = \langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}} \rangle$,
$\mathscr{Y}_1 = \langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \rangle$ and
$\mathscr{Y}_2 = \langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \rangle$ be BCn,m-ROFSs, then

(i) $\mathscr{Y}_1 \cap \mathscr{Y}_2 = \mathscr{Y}_2 \cap \mathscr{Y}_1$.

(ii) $\mathscr{Y}_1 \cup \mathscr{Y}_2 = \mathscr{Y}_2 \cup \mathscr{Y}_1$.

(iii) $(\mathscr{Y}_1 \cup \mathscr{Y}_2) \cap \mathscr{Y}_2 = \mathscr{Y}_2$.

(iv) $(\mathscr{Y}_1 \cap \mathscr{Y}_2) \cup \mathscr{Y}_2 = \mathscr{Y}_2$.

(v) $\mathscr{Y} \cap (\mathscr{Y}_1 \cap \mathscr{Y}_2) = (\mathscr{Y} \cap \mathscr{Y}_1) \cap \mathscr{Y}_2$.

(vi) $\mathscr{Y} \cup (\mathscr{Y}_1 \cup \mathscr{Y}_2) = (\mathscr{Y} \cup \mathscr{Y}_1) \cup \mathscr{Y}_2$.

(vii) $(\mathscr{Y} \cap \mathscr{Y}_1) \cup \mathscr{Y}_2 = (\mathscr{Y} \cup \mathscr{Y}_2) \cap (\mathscr{Y}_1 \cup \mathscr{Y}_2)$.

(viii) $(\mathscr{Y} \cup \mathscr{Y}_1) \cap \mathscr{Y}_2 = (\mathscr{Y} \cap \mathscr{Y}_2) \cup (\mathscr{Y}_1 \cap \mathscr{Y}_2)$.

*Proof.* Obvious.

**Theorem 3.** Let $\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \right\rangle$ and
$\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \right\rangle$ be BCn,m-ROFSs, then

(i) $(\mathscr{Y}_1 \cap \mathscr{Y}_2)^c = {\mathscr{Y}_1}^c \cup {\mathscr{Y}_2}^c$.

(ii) $(\mathscr{Y}_1 \cup \mathscr{Y}_2)^c = {\mathscr{Y}_1}^c \cap {\mathscr{Y}_2}^c$.

*Proof.*

(i) $(\mathscr{Y}_1 \cap \mathscr{Y}_2)^c = \langle \{\min\left\{\mathscr{A}^+_{\mathscr{Y}_1}, \mathscr{A}^+_{\mathscr{Y}_2}\right\} + i\min\left\{\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{B}^+_{\mathscr{Y}_2}\right\},$
$\max\left\{\mathscr{C}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_2}\right\} + i\max\left\{\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{D}^+_{\mathscr{Y}_2}\right\},$
$\max\left\{\mathscr{A}^-_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_2}\right\} + i\max\left\{\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{B}^-_{\mathscr{Y}_2}\right\},$
$\min\left\{\mathscr{C}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_2}\right\} + i\min\left\{\mathscr{D}^-_{\mathscr{Y}_1}, \mathscr{D}^-_{\mathscr{Y}_2}\right\}\}\rangle^c$
$= \langle \max\{(\mathscr{C}^+_{\mathscr{Y}_1})^{\frac{m}{n}}, (\mathscr{C}^+_{\mathscr{Y}_2})^{\frac{m}{n}}\} + i\max\{(\mathscr{D}^+_{\mathscr{Y}_1})^{\frac{m}{n}}, (\mathscr{D}^+_{\mathscr{Y}_2})^{\frac{m}{n}}\},$
$\min\{(\mathscr{A}^+_{\mathscr{Y}_1})^{\frac{n}{m}}, (\mathscr{A}^+_{\mathscr{Y}_2})^{\frac{n}{m}}\} + i\min\{(\mathscr{B}^+_{\mathscr{Y}_1})^{\frac{n}{m}}, (\mathscr{B}^+_{\mathscr{Y}_2})^{\frac{n}{m}}\},$
$\min\{-|\mathscr{C}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}, -|\mathscr{C}^-_{\mathscr{Y}_2}|^{\frac{m}{n}}\} + i\min\{-|\mathscr{D}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}, -|\mathscr{D}^-_{\mathscr{Y}_2}|^{\frac{m}{n}}\},$
$\max\{-|\mathscr{A}^-_{\mathscr{Y}_1}|^{\frac{n}{m}}, -|\mathscr{A}^-_{\mathscr{Y}_2}|^{\frac{n}{m}}\} + i\max\{-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\frac{n}{m}}, -|\mathscr{B}^-_{\mathscr{Y}_2}|^{\frac{n}{m}}\}\rangle$

$$= \langle (\mathscr{C}^+_{\mathscr{Y}_1})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}_1})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}_1})^{\frac{n}{m}} + i(\mathscr{B}^+_{\mathscr{Y}_1})^{\frac{n}{m}}, -|\mathscr{C}^-_{\mathscr{Y}_1}|^{\frac{m}{n}} + i(-|\mathscr{D}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}_1}|^{\frac{n}{m}} + i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\frac{n}{m}})\rangle$$

$$\cup$$

$$\langle (\mathscr{C}^+_{\mathscr{Y}_2})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}_2})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}_2})^{\frac{n}{m}} + i(\mathscr{B}^+_{\mathscr{Y}_2})^{\frac{n}{m}}, -|\mathscr{C}^-_{\mathscr{Y}_2}|^{\frac{m}{n}} + i(-|\mathscr{D}^-_{\mathscr{Y}_2}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}_2}|^{\frac{n}{m}} + i(-|\mathscr{B}^-_{\mathscr{Y}_2}|^{\frac{n}{m}})\rangle$$

$= {\mathscr{Y}_1}^c \cup {\mathscr{Y}_2}^c.$

(ii) Can be proven similar to (1).

**Definition 7.** Let $\mathscr{Y} = \left\langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}}\right\rangle$,
$\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1}\right\rangle$ and
$\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2}\right\rangle$ be BCn,m-ROFSs, and $\zeta$ be a positive real number ($\zeta > 0$), then

(i) $\mathscr{Y}_1 \oplus \mathscr{Y}_2 =$
$\langle (((\mathscr{A}^+_{\mathscr{Y}_1})^n + (\mathscr{A}^+_{\mathscr{Y}_2})^n - (\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}} + i((\mathscr{B}^+_{\mathscr{Y}_1})^n + (\mathscr{B}^+_{\mathscr{Y}_2})^n - (\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2}) + i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}),$
$-(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2}) + i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})), -(|\mathscr{C}^-_{\mathscr{Y}_1}|^m + |\mathscr{C}^-_{\mathscr{Y}_2}|^m - |\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}} + i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m + |\mathscr{D}^-_{\mathscr{Y}_2}|^m - |\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})\rangle.$

(ii) $\mathscr{Y}_1 \otimes \mathscr{Y}_2 =$
$\langle (\mathscr{A}^+_{\mathscr{Y}_1}\mathscr{A}^+_{\mathscr{Y}_2}) + i(\mathscr{B}^+_{\mathscr{Y}_1}\mathscr{B}^+_{\mathscr{Y}_2}), ((\mathscr{C}^+_{\mathscr{Y}_1})^m + (\mathscr{C}^+_{\mathscr{Y}_2})^m - (\mathscr{C}^+_{\mathscr{Y}_1})^m(\mathscr{C}^+_{\mathscr{Y}_2})^m)^{\frac{1}{m}} + i((\mathscr{D}^+_{\mathscr{Y}_1})^m + (\mathscr{D}^+_{\mathscr{Y}_2})^m - (\mathscr{D}^+_{\mathscr{Y}_1})^m(\mathscr{D}^+_{\mathscr{Y}_2})^m)^{\frac{1}{m}},$
$-(|\mathscr{A}^-_{\mathscr{Y}_1}|^n + |\mathscr{A}^-_{\mathscr{Y}_2}|^n - |\mathscr{A}^-_{\mathscr{Y}_1}|^n|\mathscr{A}^-_{\mathscr{Y}_2}|^n)^{\frac{1}{n}} + i(-(|\mathscr{B}^-_{\mathscr{Y}_1}|^n + |\mathscr{B}^-_{\mathscr{Y}_2}|^n - |\mathscr{B}^-_{\mathscr{Y}_1}|^n|\mathscr{B}^-_{\mathscr{Y}_2}|^n)^{\frac{1}{n}}), -(\mathscr{C}^-_{\mathscr{Y}_1}\mathscr{C}^-_{\mathscr{Y}_2}) + i(-(\mathscr{D}^-_{\mathscr{Y}_1}\mathscr{D}^-_{\mathscr{Y}_2}))\rangle.$

(iii) $\zeta\mathscr{Y} =$
$\langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}} + i(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^\zeta + i(\mathscr{D}^+_{\mathscr{Y}})^\zeta,$
$-|\mathscr{A}^-_{\mathscr{Y}}|^\zeta + i(-|\mathscr{B}^-_{\mathscr{Y}}|^\zeta), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^\zeta)^{\frac{1}{m}} + i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^\zeta)^{\frac{1}{m}})\rangle.$

(iv) $\mathscr{Y}^\zeta =$
$\langle(\mathscr{A}^+_{\mathscr{Y}})^\zeta + i(\mathscr{B}^+_{\mathscr{Y}})^\zeta, (1-(1-(\mathscr{C}^+_{\mathscr{Y}})^m)^\zeta)^{\frac{1}{m}} + i(1-(1-(\mathscr{D}^+_{\mathscr{Y}})^m)^\zeta)^{\frac{1}{m}},$
$-(1-(1-|\mathscr{A}^-_{\mathscr{Y}}|^n)^\zeta)^{\frac{1}{n}} + i(-(1-(1-|\mathscr{B}^-_{\mathscr{Y}}|^n)^\zeta)^{\frac{1}{n}}), -|\mathscr{C}^-_{\mathscr{Y}}|^\zeta + i(-|\mathscr{D}^-_{\mathscr{Y}}|^\zeta)\rangle.$

**Example 4.** Consider the two BC3,4-ROFSs
$\mathscr{Y}_1 = \{\langle\mathscr{T}, 0.8+i(0.3), 0.7+i(0.9), -0.7+i(-0.9), -0.5+i(-0.3)\rangle : \mathscr{T}\in\mathscr{S}\}$ and
$\mathscr{Y}_2 = \{\langle\mathscr{T}, 0.4+i(0.15), 0.8+i(0.32), -0.4+i(-0.7), -0.5+i(-0.2)\rangle : \mathscr{T}\in\mathscr{S}\}$ for $\mathscr{S} = \{\mathscr{T}\}$ and $\zeta = 7$, we have

(i) $\mathscr{Y}_1\oplus\mathscr{Y}_2 = \langle(((\mathscr{A}^+_{\mathscr{Y}_1})^n + (\mathscr{A}^+_{\mathscr{Y}_2})^n - (\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}} + i((\mathscr{B}^+_{\mathscr{Y}_1})^n + (\mathscr{B}^+_{\mathscr{Y}_2})^n - (\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2}) + i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}), -(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2}) + i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})),$
$-(|\mathscr{C}^-_{\mathscr{Y}_1}|^m + |\mathscr{C}^-_{\mathscr{Y}_2}|^m - |\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}} + i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m + |\mathscr{D}^-_{\mathscr{Y}_2}|^m - |\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})\rangle$
$= \langle(((0.8)^3 + (0.4)^3 - (0.8)^3(0.4)^3)^{\frac{1}{3}} + i((0.3)^3 + (0.15)^3 - (0.3)^3(0.15)^3)^{\frac{1}{3}},$
$((0.7)(0.8)) + i((0.9)(0.32)), -((-0.7)(-0.4)) + i(-((-0.9)(-0.7))),$
$-(|-0.5|^4 + |-0.5|^4 - |-0.5|^4|-0.5|^4)^{\frac{1}{4}} + i(-(|-0.3|^4 + |-0.2|^4 - |-0.3|^4|-0.2|^4)^{\frac{1}{4}})\rangle$
$\approx \langle(0.816) + i(0.31), (0.49) + i(0.81), -(0.28) + i(-0.63), (-0.59) + i(-0.314)\rangle.$

(ii) $\mathscr{Y}_1\otimes\mathscr{Y}_2 = \langle(\mathscr{A}^+_{\mathscr{Y}_1}\mathscr{A}^+_{\mathscr{Y}_2}) + i(\mathscr{B}^+_{\mathscr{Y}_1}\mathscr{B}^+_{\mathscr{Y}_2}),$
$((\mathscr{C}^+_{\mathscr{Y}_1})^m + (\mathscr{C}^+_{\mathscr{Y}_2})^m - (\mathscr{C}^+_{\mathscr{Y}_1})^m(\mathscr{C}^+_{\mathscr{Y}_2})^m)^{\frac{1}{m}} + i((\mathscr{D}^+_{\mathscr{Y}_1})^m + (\mathscr{D}^+_{\mathscr{Y}_2})^m - (\mathscr{D}^+_{\mathscr{Y}_1})^m(\mathscr{D}^+_{\mathscr{Y}_2})^m)^{\frac{1}{m}},$
$-(|\mathscr{A}^-_{\mathscr{Y}_1}|^n + |\mathscr{A}^-_{\mathscr{Y}_2}|^n - |\mathscr{A}^-_{\mathscr{Y}_1}|^n|\mathscr{A}^-_{\mathscr{Y}_2}|^n)^{\frac{1}{n}} + i(-(|\mathscr{B}^-_{\mathscr{Y}_1}|^n + |\mathscr{B}^-_{\mathscr{Y}_2}|^n - |\mathscr{B}^-_{\mathscr{Y}_1}|^n|\mathscr{B}^-_{\mathscr{Y}_2}|^n)^{\frac{1}{n}}),$
$-(\mathscr{C}^-_{\mathscr{Y}_1}\mathscr{C}^-_{\mathscr{Y}_2}) + i(-(\mathscr{D}^-_{\mathscr{Y}_1}\mathscr{D}^-_{\mathscr{Y}_2}))\rangle$
$= \langle((0.8)(0.4)) + i((0.3)(0.15)),$
$((0.7)^4 + (0.8)^4 - (0.7)^4(0.8)^4)^{\frac{1}{4}} + i((0.9)^4 + (0.32)^4 - (0.9)^4(0.32)^4)^{\frac{1}{4}},$
$-(|-0.7|^3 + |-0.4|^3 - |-0.7|^3|-0.4|^3)^{\frac{1}{3}} + i(-(|-0.9|^3 + |-0.7|^3 - |-0.9|^3|-0.7|^3)^{\frac{1}{3}}),$
$-((-0.5)(-0.5) + i(-((-0.3)(-0.2))))\rangle$
$\approx \langle(0.64) + i(0.09), (0.86) + i(0.9), -(0.73) + i(-0.94), (-0.25) + i(-0.06)\rangle.$

(iii) $7\mathscr{Y}_1 = \langle(1-(1-(0.8)^3)^7)^{\frac{1}{3}} + i(1-(1-(0.3)^3)^7)^{\frac{1}{3}}, (0.7)^7 + i(0.9)^7,$
$-|-0.7|^7 + i(-|-0.9|^7), -(1-(1-|-0.5|^4)^7)^{\frac{1}{4}} + i(-(1-(1-|-0.3|^4)^7)^{\frac{1}{4}})\rangle$
$\approx \langle(0.99) + i(0.6), (0.08) + i(0.5), -(0.08) + i(-0.5), (-0.78) + i(-0.5)\rangle.$

(iv) $\mathscr{Y}^7 = \langle(0.8)^7 + i(0.3)^7, (1-(1-(0.7)^4)^7)^{\frac{1}{4}} + i(1-(1-(0.9)^4)^7)^{\frac{1}{4}},$
$-(1-(1-|-0.7|^3)^7)^{\frac{1}{3}} + i(-(1-(1-|-0.9|^3)^7)^{\frac{1}{3}}), -|-0.5|^7 + i(-|-0.3|^7)\rangle$
$\approx \langle(0.21) + i(0.0002), (0.96) + i(0.99), -(0.98) + i(-0.99), (-0.008) + i(-0.0002)\rangle.$

**Theorem 4.** If $\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \right\rangle$ and $\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \right\rangle$ are BCn,m-ROFSs, then $\mathscr{Y}_1\oplus\mathscr{Y}_2$ and $\mathscr{Y}_1\otimes\mathscr{Y}_2$ are also BCn,m-ROFSs.

*Proof.* For the two BCn,m-ROFSs $\mathscr{Y}_1$and $\mathscr{Y}_2$ the following relations are evident:

$$0 \leq \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \leq 1, \quad 0 \leq \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \leq 1, \quad 0 \leq \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n + \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \leq 1,$$

$$0 \leq \left|\mathscr{A}^-_{\mathscr{Y}_1}\right|^n \leq 1, \quad 0 \leq \left|\mathscr{C}^-_{\mathscr{Y}_1}\right|^m \leq 1, \quad 0 \leq \left|\mathscr{A}^-_{\mathscr{Y}_1}\right|^n + \left|\mathscr{C}^-_{\mathscr{Y}_1}\right|^m \leq 1,$$

$$0 \leq \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \leq 1, \quad 0 \leq \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m \leq 1, \quad 0 \leq \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n + \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m \leq 1,$$

$$0 \leq \left|\mathscr{A}^-_{\mathscr{Y}_2}\right|^n \leq 1, \quad 0 \leq \left|\mathscr{C}^-_{\mathscr{Y}_2}\right|^m \leq 1, \quad and \quad 0 \leq \left|\mathscr{A}^-_{\mathscr{Y}_2}\right|^n + \left|\mathscr{C}^-_{\mathscr{Y}_2}\right|^m \leq 1.$$

Furthermore, since

$$0 \leq \mathscr{B}^+_{\mathscr{Y}_1} \leq 1, \quad 0 \leq \mathscr{B}^+_{\mathscr{Y}_2} \leq 1, \quad 0 \leq \mathscr{D}^+_{\mathscr{Y}_1} \leq 1, \quad 0 \leq \mathscr{D}^+_{\mathscr{Y}_2} \leq 1,$$

$$0 \leq (\mathscr{B}^+_{\mathscr{Y}_1})^n + (\mathscr{D}^+_{\mathscr{Y}_1})^m \leq 1, \quad 0 \leq (\mathscr{B}^+_{\mathscr{Y}_2})^n + (\mathscr{D}^+_{\mathscr{Y}_2})^m \leq 1,$$

$$0 \leq |\mathscr{D}^-_{\mathscr{Y}_1}| \leq 1, \quad 0 \leq |\mathscr{D}^-_{\mathscr{Y}_2}| \leq 1, \quad 0 \leq |\mathscr{B}^-_{\mathscr{Y}_1}| \leq 1, \quad 0 \leq |\mathscr{B}^-_{\mathscr{Y}_2}| \leq 1,$$

$$0 \leq |\mathscr{B}^-_{\mathscr{Y}_1}|^n + |\mathscr{D}^-_{\mathscr{Y}_1}|^m \leq 1, \quad and \quad 0 \leq |\mathscr{B}^-_{\mathscr{Y}_2}|^n + |\mathscr{D}^-_{\mathscr{Y}_2}|^m \leq 1.$$

Then, we have

$$\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \geq \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n, \quad \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \geq \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n, \quad 1 \geq \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \quad \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \geq 0,$$

$$\left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \geq \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m, \quad \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m \geq \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m, \quad and \quad 1 \geq \quad \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m \geq 0$$

which indicates that $\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \geq 0$ implies

$$\left(\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n\right)^{\frac{1}{n}} \geq 0, \, and$$

$$\left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m + \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m - \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m \quad \geq 0 \quad \text{implies}$$
$$\left(\left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m + \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m - \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m\right)^{\frac{1}{m}} \geq 0.$$

Since $\left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \leq 1$ and $0 \leq 1-\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n$, then $\left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n\right) \leq \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n\right)$ and we get

$$\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n \leq 1$$

and hence

$$\left(\left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^+_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^+_{\mathscr{Y}_2}\right)^n\right)^{\frac{1}{n}} \leq 1.$$

Similarly, we can get

$$\left(\left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m + \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m - \left(\mathscr{C}^+_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^+_{\mathscr{Y}_2}\right)^m\right)^{\frac{1}{m}} \leq 1.$$

It is obvious that

$$0 \leq \left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)^m \leq 1 - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n \text{ and } 0 \leq \left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)^m \leq 1 - \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n,$$

then we can get

$$\begin{aligned}&\left(\left(\left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n\right)^{\frac{1}{n}}\right)^n + \left(\left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)\left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)\right)^m \\ &\leq \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n + \left(1 - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n\right)\left(1 - \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n\right) = 1.\end{aligned}$$

Therefore,

$$\begin{aligned}&0 \leq \left(\left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n\right)^{\frac{1}{n}} \leq 1, \\ &0 \leq \left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)\left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right) \leq 1\end{aligned}$$

and

$$0 \leq \left(\left(\left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n + \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n - \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)^n \left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)^n\right)^{\frac{1}{n}}\right)^n + \left(\left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)\left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)\right)^m \leq 1.$$

Similarly, we have

(i)

$$0 \leq \left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)\left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right) \leq 1, \quad 0 \leq \left(\left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)^m + \left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)^m - \left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)^m\right)^{\frac{1}{m}} \leq 1$$

and

$$0 \leq \left(\left(\mathscr{A}^{+}_{\mathscr{Y}_1}\right)\left(\mathscr{A}^{+}_{\mathscr{Y}_2}\right)\right)^n + \left(\left(\left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)^m + \left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)^m - \left(\mathscr{C}^{+}_{\mathscr{Y}_1}\right)^m \left(\mathscr{C}^{+}_{\mathscr{Y}_2}\right)^m\right)^{\frac{1}{m}}\right)^m \leq 1,$$

(ii)

$$-1 \leq -\left(\mathscr{A}^{-}_{\mathscr{Y}_1}\right)\left(\mathscr{A}^{-}_{\mathscr{Y}_2}\right) \leq 0, \quad -1 \leq -\left(\left|\mathscr{C}^{-}_{\mathscr{Y}_1}\right|^m + \left|\mathscr{C}^{-}_{\mathscr{Y}_2}\right|^m - \left|\mathscr{C}^{-}_{\mathscr{Y}_1}\right|^m \left|\mathscr{C}^{-}_{\mathscr{Y}_2}\right|^m\right)^{\frac{1}{m}} \leq 0$$

and

$$0 \leq \left|-\left(\mathscr{A}^{-}_{\mathscr{Y}_1}\right)\left(\mathscr{A}^{-}_{\mathscr{Y}_2}\right)\right|^n + \left|-\left(\left|\mathscr{C}^{-}_{\mathscr{Y}_1}\right|^m + \left|\mathscr{C}^{-}_{\mathscr{Y}_2}\right|^m - \left|\mathscr{C}^{-}_{\mathscr{Y}_1}\right|^m \left|\mathscr{C}^{-}_{\mathscr{Y}_2}\right|^m\right)^{\frac{1}{m}}\right|^m \leq 1.$$

(iii)

$$-1 \leq -\left(\left|\mathscr{A}^{-}_{\mathscr{Y}_1}\right|^n + \left|\mathscr{A}^{-}_{\mathscr{Y}_2}\right|^n - \left|\mathscr{A}^{-}_{\mathscr{Y}_1}\right|^n \left|\mathscr{A}^{-}_{\mathscr{Y}_2}\right|^n\right)^{\frac{1}{n}} \leq 0, \quad -1 \leq -\left(\mathscr{C}^{-}_{\mathscr{Y}_1}\right)\left(\mathscr{C}^{-}_{\mathscr{Y}_2}\right) \leq 0$$

and

$$0 \leq \left| -\left( \left| \mathscr{A}^{-}_{\mathscr{Y}_1} \right|^{n} + \left| \mathscr{A}^{-}_{\mathscr{Y}_2} \right|^{n} - \left| \mathscr{A}^{-}_{\mathscr{Y}_1} \right|^{n} \left| \mathscr{A}^{-}_{\mathscr{Y}_2} \right|^{n} \right)^{\frac{1}{n}} \right|^{n} + \left| -\left( \mathscr{C}^{-}_{\mathscr{Y}_1} \right) \left( \mathscr{C}^{-}_{\mathscr{Y}_2} \right) \right|^{m} \leq 1.$$

In a similar manner we have
(i)

$$0 \leq \left( \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right)^{n} + \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right)^{n} - \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right)^{n} \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right)^{n} \right)^{\frac{1}{n}} \leq 1, \quad 0 \leq \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right) \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right) \leq 1$$

and

$$0 \leq \left( \left( \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right)^{n} + \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right)^{n} - \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right)^{n} \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right)^{n} \right)^{\frac{1}{n}} \right)^{n} + \left( \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right) \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right) \right)^{m} \leq 1.$$

(ii)

$$0 \leq \left| -\left( |\mathscr{D}^{-}_{\mathscr{Y}_1}|^{m} + |\mathscr{D}^{-}_{\mathscr{Y}_2}|^{m} - |\mathscr{D}^{-}_{\mathscr{Y}_1}|^{m} |\mathscr{D}^{-}_{\mathscr{Y}_2}|^{m} \right) \right|^{\frac{1}{m}} \leq 1, \quad 0 \leq |\mathscr{B}^{-}_{\mathscr{Y}_1}||\mathscr{B}^{-}_{\mathscr{Y}_2}| \leq 1$$

and

$$0 \leq \left| -\left( |\mathscr{D}^{-}_{\mathscr{Y}_1}|^{m} + |\mathscr{D}^{-}_{\mathscr{Y}_2}|^{m} - |\mathscr{D}^{-}_{\mathscr{Y}_1}|^{m} |\mathscr{D}^{-}_{\mathscr{Y}_2}|^{m} \right)^{\frac{1}{m}} \right|^{m} + \left( |\mathscr{B}^{-}_{\mathscr{Y}_1}||\mathscr{B}^{-}_{\mathscr{Y}_2}| \right)^{n} \leq 1.$$

(iii)

$$0 \leq \left| -\left( |\mathscr{B}^{-}_{\mathscr{Y}_1}|^{n} + |\mathscr{B}^{-}_{\mathscr{Y}_2}|^{n} - |\mathscr{B}^{-}_{\mathscr{Y}_1}|^{n} |\mathscr{B}^{-}_{\mathscr{Y}_2}|^{n} \right) \right|^{\frac{1}{n}} \leq 1, \quad 0 \leq |\mathscr{D}^{-}_{\mathscr{Y}_1}||\mathscr{D}^{-}_{\mathscr{Y}_2}| \leq 1$$

and

$$0 \leq \left| -\left( |\mathscr{B}^{-}_{\mathscr{Y}_1}|^{n} + |\mathscr{B}^{-}_{\mathscr{Y}_2}|^{n} - |\mathscr{B}^{-}_{\mathscr{Y}_1}|^{n} |\mathscr{B}^{-}_{\mathscr{Y}_2}|^{n} \right)^{\frac{1}{n}} \right|^{n} + \left( |\mathscr{D}^{-}_{\mathscr{Y}_1}||\mathscr{D}^{-}_{\mathscr{Y}_2}| \right)^{m} \leq 1.$$

(iv)

$$0 \leq \left( \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right)^{m} + \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right)^{m} - \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right)^{m} \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right)^{m} \right)^{\frac{1}{m}} \leq 1, \quad 0 \leq \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right) \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right) \leq 1$$

and

$$0 \leq \left( \left( \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right)^{m} + \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right)^{m} - \left( \mathscr{D}^{+}_{\mathscr{Y}_1} \right)^{m} \left( \mathscr{D}^{+}_{\mathscr{Y}_2} \right)^{m} \right)^{\frac{1}{m}} \right)^{m} + \left( \left( \mathscr{B}^{+}_{\mathscr{Y}_1} \right) \left( \mathscr{B}^{+}_{\mathscr{Y}_2} \right) \right)^{n} \leq 1.$$

Therefore, $\mathscr{Y}_1 \oplus \mathscr{Y}_2$ and $\mathscr{Y}_1 \otimes \mathscr{Y}_2$ are BCn,m-ROFSs.

**Theorem 5.** If $\mathscr{Y} = \left\langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}} \right\rangle$ is a BCn,m-ROFS, then $\zeta\mathscr{Y}$ and $\mathscr{Y}^\zeta$ are also BCn,m-ROFSs.

*Proof.* Since

$$0 \leq \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n \leq 1, \quad 0 \leq \left(\mathscr{C}^+_{\mathscr{Y}}\right)^m \leq 1, \quad 0 \leq \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n + \left(\mathscr{C}^+_{\mathscr{Y}}\right)^m \leq 1,$$

$$0 \leq \left|\mathscr{A}^-_{\mathscr{Y}}\right|^n \leq 1, 0 \leq \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m \leq 1 \text{ and } 0 \leq \left|\mathscr{A}^-_{\mathscr{Y}}\right|^n + \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m \leq 1,$$

then

$$0 \leq \left(\mathscr{C}^+_{\mathscr{Y}}\right)^m \leq 1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n \text{ and } 0 \leq \left|\mathscr{A}^-_{\mathscr{Y}}\right|^n \leq 1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m$$

$$\Rightarrow 0 \leq \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta \text{ and } 0 \leq \left(1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta$$

$$\Rightarrow 1 - \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta \leq 1 \text{ and } 1 - \left(1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta \leq 1$$

$$\Rightarrow 0 \leq \left(1 - \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta\right)^{\frac{1}{n}} \leq (1)^{\frac{1}{n}} = 1 \text{ and } 0 \leq \left(1 - \left(1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta\right)^{\frac{1}{m}} \leq (1)^{\frac{1}{m}} = 1.$$

It is obvious that

$$0 \leq \left(\mathscr{C}^+_{\mathscr{Y}}\right)^\zeta \leq 1 \quad \textit{and} \quad -1 \leq -\left|\mathscr{A}^-_{\mathscr{Y}}\right|^\zeta \leq 0,$$

then we can get

$$0 \leq \left(\left(1 - \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta\right)^{\frac{1}{n}}\right)^n + \quad \left(\left(\mathscr{C}^+_{\mathscr{Y}}\right)^\zeta\right)^m \leq 1 - \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta + \left(1 - \left(\mathscr{A}^+_{\mathscr{Y}}\right)^n\right)^\zeta = 1,$$

and

$$0 \leq \left|-\left|\mathscr{A}^-_{\mathscr{Y}}\right|^\zeta\right|^n + \left|-\left(1 - \left(1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta\right)^{\frac{1}{m}}\right|^m \leq \left(\left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta + \left(1 - \left(1 - \left|\mathscr{C}^-_{\mathscr{Y}}\right|^m\right)^\zeta\right) = 1$$

by the same way we can have

(i)

$$0 \leq \left(1 - \left(1 - \left(\mathscr{B}^+_{\mathscr{Y}}\right)^n\right)^\zeta\right)^{\frac{1}{n}} \leq 1, \quad 0 \leq (\mathscr{D}^+_{\mathscr{Y}})^\zeta \leq 1,$$

and

$$0 \leq \left(\left(1 - \left(1 - \left(\mathscr{B}^+_{\mathscr{Y}}\right)^n\right)^\zeta\right)^{\frac{1}{n}}\right)^n + \quad \left(\left(\mathscr{D}^+_{\mathscr{Y}}\right)^\zeta\right)^m \leq 1$$

(ii)

$$0 \leq |\mathscr{B}^-_{\mathscr{Y}}|^\zeta \leq 1, \quad -1 \leq -\left(1 - \left(1 - |\mathscr{D}^-_{\mathscr{Y}}|^m\right)^\zeta\right)^{\frac{1}{m}} \leq 0,$$

and

$$0 \leq \left(|\mathscr{B}^-_{\mathscr{Y}}|^\zeta\right)^n + \left|-\left(1 - \left(1 - |\mathscr{D}^-_{\mathscr{Y}}|^m\right)^\zeta\right)^{\frac{1}{m}}\right|^m \leq 1.$$

Similarly, we can also get

$$0 \leq \left((\mathscr{A}_{\mathscr{Y}}^{+})^{\zeta}\right)^{n} + \left(\left(1-\left(1-(\mathscr{C}_{\mathscr{Y}}^{+})^{m}\right)^{\zeta}\right)^{\frac{1}{m}}\right)^{m} \leq 1,$$

and

$$0 \leq \left|-\left(1-\left(1-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}\right)^{\zeta}\right)^{\frac{1}{n}}\right|^{n} + \left|-|\mathscr{C}_{\mathscr{Y}}^{-}|^{\zeta}\right|^{m} \leq 1.$$

Similarly we can get

(i)

$$0 \leq (\mathscr{B}_{\mathscr{Y}}^{+})^{\zeta} \leq 1, \quad 0 \leq \left(1-\left(1-(\mathscr{D}_{\mathscr{Y}}^{+})^{m}\right)^{\zeta}\right)^{\frac{1}{m}} \leq 1,$$

and

$$0 \leq \left((\mathscr{B}_{\mathscr{Y}}^{+})^{\zeta}\right)^{n} + \left(\left(1-\left(1-(\mathscr{D}_{\mathscr{Y}}^{+})^{m}\right)^{\zeta}\right)^{\frac{1}{m}}\right)^{m} \leq 1.$$

(ii)

$$-1 \leq -\left(1-\left(1-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}\right)^{\zeta}\right)^{\frac{1}{n}} \leq 0, \quad 0 \leq |\mathscr{D}_{\mathscr{Y}}^{-}|^{\zeta} \leq 1,$$

and

$$0 \leq \left|-\left(1-\left(1-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}\right)^{\zeta}\right)^{\frac{1}{n}}\right|^{n} + \left(|\mathscr{D}_{\mathscr{Y}}^{-}|^{\zeta}\right)^{m} \leq 1.$$

Therefore, $\zeta\mathscr{Y}$ and $\mathscr{Y}^{\zeta}$ are BCn,m-ROFSs.

**Theorem 6.** Let $\mathscr{Y} = \langle \mathscr{A}_{\mathscr{Y}}^{+} + i\mathscr{B}_{\mathscr{Y}}^{+}, \mathscr{C}_{\mathscr{Y}}^{+} + i\mathscr{D}_{\mathscr{Y}}^{+}, \mathscr{A}_{\mathscr{Y}}^{-} + i\mathscr{B}_{\mathscr{Y}}^{-}, \mathscr{C}_{\mathscr{Y}}^{-} + i\mathscr{D}_{\mathscr{Y}}^{-} \rangle$, $\mathscr{Y}_1 = \left\langle \mathscr{A}_{\mathscr{Y}_1}^{+} + i\mathscr{B}_{\mathscr{Y}_1}^{+}, \mathscr{C}_{\mathscr{Y}_1}^{+} + i\mathscr{D}_{\mathscr{Y}_1}^{+}, \mathscr{A}_{\mathscr{Y}_1}^{-} + i\mathscr{B}_{\mathscr{Y}_1}^{-}, \mathscr{C}_{\mathscr{Y}_1}^{-} + i\mathscr{D}_{\mathscr{Y}_1}^{-} \right\rangle$ and $\mathscr{Y}_2 = \left\langle \mathscr{A}_{\mathscr{Y}_2}^{+} + i\mathscr{B}_{\mathscr{Y}_2}^{+}, \mathscr{C}_{\mathscr{Y}_2}^{+} + i\mathscr{D}_{\mathscr{Y}_2}^{+}, \mathscr{A}_{\mathscr{Y}_2}^{-} + i\mathscr{B}_{\mathscr{Y}_2}^{-}, \mathscr{C}_{\mathscr{Y}_2}^{-} + i\mathscr{D}_{\mathscr{Y}_2}^{-} \right\rangle$ be BCn,m-ROFSs, then the following properties hold:

(i) $\mathscr{Y}_1 \oplus \mathscr{Y}_2 = \mathscr{Y}_2 \oplus \mathscr{Y}_1$.

(ii) $\mathscr{Y}_1 \otimes \mathscr{Y}_2 = \mathscr{Y}_2 \otimes \mathscr{Y}_1$.

(iii) $\mathscr{Y} \oplus (\mathscr{Y}_1 \oplus \mathscr{Y}_2) = (\mathscr{Y} \oplus \mathscr{Y}_1) \oplus \mathscr{Y}_2$.

(iv) $\mathscr{Y} \otimes (\mathscr{Y}_1 \otimes \mathscr{Y}_2) = (\mathscr{Y} \otimes \mathscr{Y}_1) \otimes \mathscr{Y}_2$.

(v) $\mathscr{Y} \oplus (\mathscr{Y}_1 \cup \mathscr{Y}_2) = (\mathscr{Y} \oplus \mathscr{Y}_1) \cup (\mathscr{Y} \oplus \mathscr{Y}_2)$.

(vi) $\mathscr{Y} \oplus (\mathscr{Y}_1 \cap \mathscr{Y}_2) = (\mathscr{Y} \oplus \mathscr{Y}_1) \cap (\mathscr{Y} \oplus \mathscr{Y}_2)$.

(vii) $\mathscr{Y} \otimes (\mathscr{Y}_1 \cup \mathscr{Y}_2) = (\mathscr{Y} \otimes \mathscr{Y}_1) \cup (\mathscr{Y} \otimes \mathscr{Y}_2)$.

(viii) $\mathscr{Y} \otimes (\mathscr{Y}_1 \cap \mathscr{Y}_2) = (\mathscr{Y} \otimes \mathscr{Y}_1) \cap (\mathscr{Y} \otimes \mathscr{Y}_2)$.

*Proof.* Parts (1), (3), (5) and (8) will be exhibited here. Likewise, the remaining parts can be presented.

(1) $\mathscr{Y}_1\oplus\mathscr{Y}_2=\langle(((\mathscr{A}^+_{\mathscr{Y}_1})^n+(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}}+i((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})+i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}),-(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2})+i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})),$
$-(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})\rangle$
$=\langle(((\mathscr{A}^+_{\mathscr{Y}_2})^n+(\mathscr{A}^+_{\mathscr{Y}_1})^n-(\mathscr{A}^+_{\mathscr{Y}_2})^n(\mathscr{A}^+_{\mathscr{Y}_1})^n)^{\frac{1}{n}}+i((\mathscr{B}^+_{\mathscr{Y}_2})^n+(\mathscr{B}^+_{\mathscr{Y}_1})^n-(\mathscr{B}^+_{\mathscr{Y}_2})^n(\mathscr{B}^+_{\mathscr{Y}_1})^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}_2}\mathscr{C}^+_{\mathscr{Y}_1})+i(\mathscr{D}^+_{\mathscr{Y}_2}\mathscr{D}^+_{\mathscr{Y}_1}),-(\mathscr{A}^-_{\mathscr{Y}_2}\mathscr{A}^-_{\mathscr{Y}_1})+i(-(\mathscr{B}^-_{\mathscr{Y}_2}\mathscr{B}^-_{\mathscr{Y}_1})),$
$-(|\mathscr{C}^-_{\mathscr{Y}_2}|^m+|\mathscr{C}^-_{\mathscr{Y}_1}|^m-|\mathscr{C}^-_{\mathscr{Y}_2}|^m|\mathscr{C}^-_{\mathscr{Y}_1}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}_2}|^m+|\mathscr{D}^-_{\mathscr{Y}_1}|^m-|\mathscr{D}^-_{\mathscr{Y}_2}|^m|\mathscr{D}^-_{\mathscr{Y}_1}|^m)^{\frac{1}{m}})\rangle$
$=\mathscr{Y}_2\oplus\mathscr{Y}_1.$

(3) $\mathscr{Y}\oplus(\mathscr{Y}_1\oplus\mathscr{Y}_2)=\big\{\langle\mathscr{A}^+_{\mathscr{Y}}+i\mathscr{B}^+_{\mathscr{Y}},\mathscr{A}^-_{\mathscr{Y}}+i\mathscr{B}^-_{\mathscr{Y}},\mathscr{C}^+_{\mathscr{Y}}+i\mathscr{D}^+_{\mathscr{Y}},\mathscr{C}^-_{\mathscr{Y}}+i\mathscr{D}^-_{\mathscr{Y}}\rangle\big\}\oplus\langle(((\mathscr{A}^+_{\mathscr{Y}_1})^n+$
$(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}}+i((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})+i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}),-(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2})+i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})),$
$-(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})$
$=\langle(((\mathscr{A}^+_{\mathscr{Y}})^n+((\mathscr{A}^+_{\mathscr{Y}_1})^n+(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)-(\mathscr{A}^+_{\mathscr{Y}})^n((\mathscr{A}^+_{\mathscr{Y}_1})^n+(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n))^{\frac{1}{n}}$
$+i((\mathscr{B}^+_{\mathscr{Y}})^n+((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)-(\mathscr{B}^+_{\mathscr{Y}})^n((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n))^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}}\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})+i(\mathscr{D}^+_{\mathscr{Y}}\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}),-(|\mathscr{A}^-_{\mathscr{Y}}||\mathscr{A}^-_{\mathscr{Y}_1}||\mathscr{A}^-_{\mathscr{Y}_2}|)+i(-(|\mathscr{B}^-_{\mathscr{Y}}||\mathscr{B}^-_{\mathscr{Y}_1}||\mathscr{B}^-_{\mathscr{Y}_2}|)),$
$-(|\mathscr{C}^-_{\mathscr{Y}}|+(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)-|\mathscr{C}^-_{\mathscr{Y}}|(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m))^{\frac{1}{m}}$
$+i(-(|\mathscr{D}^-_{\mathscr{Y}}|+(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)-|\mathscr{D}^-_{\mathscr{Y}}|(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m))^{\frac{1}{m}})\rangle$
$=\langle(((\mathscr{A}^+_{\mathscr{Y}})^n+(\mathscr{A}^+_{\mathscr{Y}_1})^n-(\mathscr{A}^+_{\mathscr{Y}})^n(\mathscr{A}^+_{\mathscr{Y}_1})^n)^{\frac{1}{n}}+i((\mathscr{B}^+_{\mathscr{Y}})^n+(\mathscr{B}^+_{\mathscr{Y}_1})^n-(\mathscr{B}^+_{\mathscr{Y}})^n(\mathscr{B}^+_{\mathscr{Y}_1})^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}}\mathscr{C}^+_{\mathscr{Y}_1})+i(\mathscr{D}^+_{\mathscr{Y}}\mathscr{D}^+_{\mathscr{Y}_1}),-(\mathscr{A}^-_{\mathscr{Y}}\mathscr{A}^-_{\mathscr{Y}_1})+i(-(\mathscr{B}^-_{\mathscr{Y}}\mathscr{B}^-_{\mathscr{Y}_1})),$
$-(|\mathscr{C}^-_{\mathscr{Y}}|^m+|\mathscr{C}^-_{\mathscr{Y}_1}|^m-|\mathscr{C}^-_{\mathscr{Y}}|^m|\mathscr{C}^-_{\mathscr{Y}_1}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}}|^m+|\mathscr{D}^-_{\mathscr{Y}_1}|^m-|\mathscr{D}^-_{\mathscr{Y}}|^m|\mathscr{D}^-_{\mathscr{Y}_1}|^m)^{\frac{1}{m}})\rangle$

$$\oplus$$

$$\langle\mathscr{A}^+_{\mathscr{Y}_2}+i\mathscr{B}^+_{\mathscr{Y}_2},\mathscr{A}^-_{\mathscr{Y}_2}+i\mathscr{B}^-_{\mathscr{Y}_2},\mathscr{C}^+_{\mathscr{Y}_2}+i\mathscr{D}^+_{\mathscr{Y}_2},\mathscr{C}^-_{\mathscr{Y}_2}+i\mathscr{D}^-_{\mathscr{Y}_2}\rangle$$

$=(\mathscr{Y}\oplus\mathscr{Y}_1)\oplus\mathscr{Y}_2.$

(5) $\mathscr{Y}\oplus(\mathscr{Y}_1\cup\mathscr{Y}_2)=\langle\mathscr{A}^+_{\mathscr{Y}}+i\mathscr{B}^+_{\mathscr{Y}},\mathscr{A}^-_{\mathscr{Y}}+i\mathscr{B}^-_{\mathscr{Y}},\mathscr{C}^+_{\mathscr{Y}}+i\mathscr{D}^+_{\mathscr{Y}},\mathscr{C}^-_{\mathscr{Y}}+i\mathscr{D}^-_{\mathscr{Y}}\rangle$

$$\oplus$$

$\langle\max\{\mathscr{A}^+_{\mathscr{Y}_1},\mathscr{A}^+_{\mathscr{Y}_2}\}+i\max\{\mathscr{B}^+_{\mathscr{Y}_1},\mathscr{B}^+_{\mathscr{Y}_2}\},\min\{\mathscr{C}^+_{\mathscr{Y}_1},\mathscr{C}^+_{\mathscr{Y}_2}\}+i\min\{\mathscr{D}^+_{\mathscr{Y}_1},\mathscr{D}^+_{\mathscr{Y}_2}\},$
$\min\{\mathscr{A}^-_{\mathscr{Y}_1},\mathscr{A}^-_{\mathscr{Y}_2}\}+i\min\{\mathscr{B}^-_{\mathscr{Y}_1},\mathscr{B}^-_{\mathscr{Y}_2}\},\max\{\mathscr{C}^-_{\mathscr{Y}_1},\mathscr{C}^-_{\mathscr{Y}_2}\}+i\max\{\mathscr{D}^-_{\mathscr{Y}_1},\mathscr{D}^-_{\mathscr{Y}_2}\}\rangle$
$=\langle(((\mathscr{A}^+_{\mathscr{Y}})^n+\max\{\mathscr{A}^+_{\mathscr{Y}_1},\mathscr{A}^+_{\mathscr{Y}_2}\}^n-(\mathscr{A}^+_{\mathscr{Y}})^n\max\{\mathscr{A}^+_{\mathscr{Y}_1},\mathscr{A}^+_{\mathscr{Y}_2}\}^n)^{\frac{1}{n}}$
$+i((\mathscr{B}^+_{\mathscr{Y}})^n+\max\{\mathscr{B}^+_{\mathscr{Y}_1},\mathscr{B}^+_{\mathscr{Y}_2}\}^n-(\mathscr{B}^+_{\mathscr{Y}})^n\max\{\mathscr{B}^+_{\mathscr{Y}_1},\mathscr{B}^+_{\mathscr{Y}_2}\}^n)^{\frac{1}{n}},$
$(\mathscr{C}^+_{\mathscr{Y}}\min\{\mathscr{C}^+_{\mathscr{Y}_1},\mathscr{C}^+_{\mathscr{Y}_2}\})+i(\mathscr{D}_{\mathscr{Y}}\min\{\mathscr{D}^+_{\mathscr{Y}_1},\mathscr{D}^+_{\mathscr{Y}_2}\}),-(\mathscr{A}^-_{\mathscr{Y}}\min\{\mathscr{A}^-_{\mathscr{Y}_1},\mathscr{A}^-_{\mathscr{Y}_2}\})+i(-(\mathscr{B}^-_{\mathscr{Y}}\min\{\mathscr{B}^-_{\mathscr{Y}_1},\mathscr{B}^-_{\mathscr{Y}_2}\})),$
$-(|\mathscr{C}^-_{\mathscr{Y}}|^m+|\max\{\mathscr{C}^-_{\mathscr{Y}_1},\mathscr{C}^-_{\mathscr{Y}_2}\}|^m-|\mathscr{C}^-_{\mathscr{Y}}|^m|\max\{\mathscr{C}^-_{\mathscr{Y}_1},\mathscr{C}^-_{\mathscr{Y}_2}\}|^m)^{\frac{1}{m}}$
$+i(-(|\mathscr{D}^-_{\mathscr{Y}}|^m+|\max\{\mathscr{D}^-_{\mathscr{Y}_1},\mathscr{D}^-_{\mathscr{Y}_2}\}|^m-|\mathscr{D}^-_{\mathscr{Y}}|^m|\max\{\mathscr{D}^-_{\mathscr{Y}_1},\mathscr{D}^-_{\mathscr{Y}_2}\}|^m)^{\frac{1}{m}})\rangle$

$= \langle((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(1-\mathscr{A}_{\mathscr{Y}}^{+})^{n}\max\{(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n},(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n}\})^{\frac{1}{n}}+i((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(1-\mathscr{B}_{\mathscr{Y}}^{+})^{n}\max\{(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n},(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n}\})^{\frac{1}{n}},$
$\min\{\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_2}^{+}\}+i(\min\{\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_2}^{+}\}),$
$\min\{-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_1}^{-}),-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_2}^{-})\}+i(\min\{-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_1}^{-}),-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_2}^{-})\}),$
$-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+(1-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m})\max\{|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m},|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m}\})^{\frac{1}{m}}+i(-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+(1-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m})\max\{|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m},|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m}\})^{\frac{1}{m}})\rangle$
and
$(\mathscr{Y}\oplus\mathscr{Y}_1)\cup(\mathscr{Y}\oplus\mathscr{Y}_2)=\langle((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n}-(\mathscr{A}_{\mathscr{Y}}^{+})^{n}(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n})^{\frac{1}{n}}+i((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n}-$
$(\mathscr{B}_{\mathscr{Y}}^{+})^{n}(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n})^{\frac{1}{n}},(\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_1}^{+})+i(\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_1}^{+}),-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_1}^{-})+i(-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_1}^{-})),$
$-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m}-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m})^{\frac{1}{m}}+i(-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m}-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m})^{\frac{1}{m}})\rangle$

$$\cup$$

$\langle((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n}-(\mathscr{A}_{\mathscr{Y}}^{+})^{n}(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}}+i((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(\mathscr{B}_{\mathscr{Y}}^{+})^{n}-(\mathscr{B}_{\mathscr{Y}}^{+})^{n}(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}},$
$(\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_2}^{+})+i(\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_2}^{+}),-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_2}^{-})+i(-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_2}^{-})),$
$-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m}-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m})^{\frac{1}{m}}+i(-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m}-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m})^{\frac{1}{m}})\rangle$
$=\langle\max\{((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n}-(\mathscr{A}_{\mathscr{Y}}^{+})^{n}(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n})^{\frac{1}{n}},((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n}-(\mathscr{A}_{\mathscr{Y}}^{+})^{n}(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}}\}$
$+i(\max\{((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n}-(\mathscr{B}_{\mathscr{Y}}^{+})^{n}(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n})^{\frac{1}{n}},((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(\mathscr{B}_{\mathscr{Y}}^{+})^{n}-(\mathscr{B}_{\mathscr{Y}}^{+})^{n}(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}}\}),$
$\min\{\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_2}^{+}\}+i(\min\{\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_2}^{+}\}),$
$\min\{-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_1}^{-}),-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_2}^{-})\}+i(\min\{-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_1}^{-}),-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_2}^{-})\}),$
$\max\{-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m}-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m})^{\frac{1}{m}},-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m}-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m})^{\frac{1}{m}}\}$
$+i(\max\{-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m}-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m})^{\frac{1}{m}},-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m}-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m})^{\frac{1}{m}}\})\rangle$
$= \langle((\mathscr{A}_{\mathscr{Y}}^{+})^{n}+(1-\mathscr{A}_{\mathscr{Y}}^{+})^{n}\max\{(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n},(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n}\})^{\frac{1}{n}}+i((\mathscr{B}_{\mathscr{Y}}^{+})^{n}+(1-\mathscr{B}_{\mathscr{Y}}^{+})^{n}\max\{(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n},(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n}\})^{\frac{1}{n}},$
$\min\{\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}}^{+}\mathscr{C}_{\mathscr{Y}_2}^{+}\}+i(\min\{\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{D}_{\mathscr{Y}}^{+}\mathscr{D}_{\mathscr{Y}_2}^{+}\}),$
$\min\{-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_1}^{-}),-(\mathscr{A}_{\mathscr{Y}}^{-}\mathscr{A}_{\mathscr{Y}_2}^{-})\}+i(\min\{-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_1}^{-}),-(\mathscr{B}_{\mathscr{Y}}^{-}\mathscr{B}_{\mathscr{Y}_2}^{-})\}),$
$-(|\mathscr{C}_{\mathscr{Y}}^{-}|^{m}+(1-|\mathscr{C}_{\mathscr{Y}}^{-}|^{m})\max\{|\mathscr{C}_{\mathscr{Y}_1}^{-}|^{m},|\mathscr{C}_{\mathscr{Y}_2}^{-}|^{m}\})^{\frac{1}{m}}+i(-(|\mathscr{D}_{\mathscr{Y}}^{-}|^{m}+(1-|\mathscr{D}_{\mathscr{Y}}^{-}|^{m})\max\{|\mathscr{D}_{\mathscr{Y}_1}^{-}|^{m},|\mathscr{D}_{\mathscr{Y}_2}^{-}|^{m}\})^{\frac{1}{m}})\rangle.$

(8) $\mathscr{Y}\otimes(\mathscr{Y}_1\cap\mathscr{Y}_2)=\langle\mathscr{A}_{\mathscr{Y}}^{+}+i\mathscr{B}_{\mathscr{Y}}^{+},\mathscr{A}_{\mathscr{Y}}^{-}+i\mathscr{B}_{\mathscr{Y}}^{-},\mathscr{C}_{\mathscr{Y}}^{+}+i\mathscr{D}_{\mathscr{Y}}^{+},\mathscr{C}_{\mathscr{Y}}^{-}+i\mathscr{D}_{\mathscr{Y}}^{-}\rangle$

$$\otimes$$

$\langle\min\{\mathscr{A}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}_2}^{+}\}+i\min\{\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{B}_{\mathscr{Y}_2}^{+}\},\max\{\mathscr{C}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}_2}^{+}\}+i\max\{\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{D}_{\mathscr{Y}_2}^{+}\},$
$\max\{\mathscr{A}_{\mathscr{Y}_1}^{-},\mathscr{A}_{\mathscr{Y}_2}^{-}\}+i\max\{\mathscr{B}_{\mathscr{Y}_1}^{-},\mathscr{B}_{\mathscr{Y}_2}^{-}\},\min\{\mathscr{C}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}_2}^{-}\}+i\min\{\mathscr{D}_{\mathscr{Y}_1}^{-},\mathscr{D}_{\mathscr{Y}_2}^{-}\}\rangle$
$=\langle(\mathscr{A}_{\mathscr{Y}}^{+}\min\{\mathscr{A}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}_2}^{+}\})+i(\mathscr{B}_{\mathscr{Y}}^{+}\min\{\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{B}_{\mathscr{Y}_2}^{+}\}),$
$((\mathscr{C}_{\mathscr{Y}}^{+})^{m}+\max\{(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m}\}-(\mathscr{C}_{\mathscr{Y}}^{+})^{m}(\max\{(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}}$
$+i((\mathscr{D}_{\mathscr{Y}}^{+})^{m}+\max\{(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m}\}-(\mathscr{D}_{\mathscr{Y}}^{+})^{m}(\max\{(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}},$
$-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}+\max\{|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n}\}-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}\max\{|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}$
$+i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}+\max\{|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n}\}-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}\max\{|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}),$
$-(\mathscr{C}_{\mathscr{Y}}^{-}\min\{\mathscr{C}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}_2}^{-}\})+i(-(\mathscr{D}_{\mathscr{Y}}^{-}\min\{\mathscr{D}_{\mathscr{Y}_1}^{-},\mathscr{D}_{\mathscr{Y}_2}^{-}\}))\rangle$
$=\langle(\min\{\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_2}^{+}\})+i(\min\{\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_2}^{+}\}),$
$((\mathscr{C}_{\mathscr{Y}}^{+})^{m}+(1-(\mathscr{C}_{\mathscr{Y}}^{+})^{m})(\max\{(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}}+i((\mathscr{D}_{\mathscr{Y}}^{+})^{m}+(1-(\mathscr{D}_{\mathscr{Y}}^{+})^{m})(\max\{(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}},$

$-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}+(1-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n})\max\{|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}+i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}+(1-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n})\max\{|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}),$
$-(\min\{\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_2}^{-}\})+i(-(\min\{\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_1}^{-},\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_2}^{-}\}))\rangle.$

On the other hand

$(\mathscr{Y}\otimes\mathscr{Y}_1)\cap(\mathscr{Y}\otimes\mathscr{Y}_2)=$
$\langle(\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_1}^{+})+i(\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_1}^{+}),((\mathscr{C}_{\mathscr{Y}}^{+})^{m}+(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m}-(\mathscr{C}_{\mathscr{Y}}^{+})^{m}(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m})^{\frac{1}{m}}+i((\mathscr{D}_{\mathscr{Y}}^{+})^{m}+(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m}-$
$(\mathscr{D}_{\mathscr{Y}}^{+})^{m}(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m})^{\frac{1}{m}},-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}+|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n}-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n})^{\frac{1}{n}}+i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}+|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n}-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n})^{\frac{1}{n}}),-(\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_1}^{-})+$
$i(-(\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_1}^{-}))\rangle$

$$\cap$$

$\langle(\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_2}^{+})+i(\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_2}^{+}),((\mathscr{C}_{\mathscr{Y}}^{+})^{m}+(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m}-(\mathscr{C}_{\mathscr{Y}}^{+})^{m}(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m})^{\frac{1}{m}}+i((\mathscr{D}_{\mathscr{Y}}^{+})^{m}+(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m}-$
$(\mathscr{D}_{\mathscr{Y}}^{+})^{m}(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m})^{\frac{1}{m}},-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}+|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n}-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n})^{\frac{1}{n}}+i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}+|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n}-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n})^{\frac{1}{n}}),-(\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_2}^{-})+$
$i(-(\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_2}^{-}))\rangle$
$=\langle(\min\{\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}}^{+}\mathscr{A}_{\mathscr{Y}_2}^{+}\})+i(\min\{\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{B}_{\mathscr{Y}}^{+}\mathscr{B}_{\mathscr{Y}_2}^{+}\}),((\mathscr{C}_{\mathscr{Y}}^{+})^{m}+(1-(\mathscr{C}_{\mathscr{Y}}^{+})^{m})(\max\{(\mathscr{C}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{C}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}}$
$+i((\mathscr{D}_{\mathscr{Y}}^{+})^{m}+(1-(\mathscr{D}_{\mathscr{Y}}^{+})^{m})(\max\{(\mathscr{D}_{\mathscr{Y}_1}^{+})^{m},(\mathscr{D}_{\mathscr{Y}_2}^{+})^{m}\}))^{\frac{1}{m}},-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{n}+(1-|\mathscr{A}_{\mathscr{Y}}^{-}|^{n})\max\{|\mathscr{A}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{A}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}+$
$i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{n}+(1-|\mathscr{B}_{\mathscr{Y}}^{-}|^{n})\max\{|\mathscr{B}_{\mathscr{Y}_1}^{-}|^{n},|\mathscr{B}_{\mathscr{Y}_2}^{-}|^{n}\})^{\frac{1}{n}}),-(\min\{\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}}^{-}\mathscr{C}_{\mathscr{Y}_2}^{-}\})+i(-(\min\{\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_1}^{-},\mathscr{D}_{\mathscr{Y}}^{-}\mathscr{D}_{\mathscr{Y}_2}^{-}\}))\rangle.$

**Theorem 7.** Let $\mathscr{Y}_1=\left\langle\mathscr{A}_{\mathscr{Y}_1}^{+}+i\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}_1}^{+}+i\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}_1}^{-}+i\mathscr{B}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}_1}^{-}+i\mathscr{D}_{\mathscr{Y}_1}^{-}\right\rangle$ and $\mathscr{Y}_2=\left\langle\mathscr{A}_{\mathscr{Y}_2}^{+}+i\mathscr{B}_{\mathscr{Y}_2}^{+},\mathscr{C}_{\mathscr{Y}_2}^{+}+i\mathscr{D}_{\mathscr{Y}_2}^{+},\mathscr{A}_{\mathscr{Y}_2}^{-}+i\mathscr{B}_{\mathscr{Y}_2}^{-},\mathscr{C}_{\mathscr{Y}_2}^{-}+i\mathscr{D}_{\mathscr{Y}_2}^{-}\right\rangle$ be BCn,m-ROFSs, then

(i) $(\mathscr{Y}_1\cup\mathscr{Y}_2)\oplus(\mathscr{Y}_1\cap\mathscr{Y}_2)=\mathscr{Y}_1\oplus\mathscr{Y}_2$.

(ii) $(\mathscr{Y}_1\cup\mathscr{Y}_2)\otimes(\mathscr{Y}_1\cap\mathscr{Y}_2)=\mathscr{Y}_1\otimes\mathscr{Y}_2$.

*Proof.* Obvious.

**Theorem 8.** Let $\mathscr{Y}=\langle\mathscr{A}_{\mathscr{Y}}^{+}+i\mathscr{B}_{\mathscr{Y}}^{+},\mathscr{C}_{\mathscr{Y}}^{+}+i\mathscr{D}_{\mathscr{Y}}^{+},\mathscr{A}_{\mathscr{Y}}^{-}+i\mathscr{B}_{\mathscr{Y}}^{-},\mathscr{C}_{\mathscr{Y}}^{-}+i\mathscr{D}_{\mathscr{Y}}^{-}\rangle$, $\mathscr{Y}_1=\left\langle\mathscr{A}_{\mathscr{Y}_1}^{+}+i\mathscr{B}_{\mathscr{Y}_1}^{+},\mathscr{C}_{\mathscr{Y}_1}^{+}+i\mathscr{D}_{\mathscr{Y}_1}^{+},\mathscr{A}_{\mathscr{Y}_1}^{-}+i\mathscr{B}_{\mathscr{Y}_1}^{-},\mathscr{C}_{\mathscr{Y}_1}^{-}+i\mathscr{D}_{\mathscr{Y}_1}^{-}\right\rangle$ and $\mathscr{Y}_2=\left\langle\mathscr{A}_{\mathscr{Y}_2}^{+}+i\mathscr{B}_{\mathscr{Y}_2}^{+},\mathscr{C}_{\mathscr{Y}_2}^{+}+i\mathscr{D}_{\mathscr{Y}_2}^{+},\mathscr{A}_{\mathscr{Y}_2}^{-}+i\mathscr{B}_{\mathscr{Y}_2}^{-},\mathscr{C}_{\mathscr{Y}_2}^{-}+i\mathscr{D}_{\mathscr{Y}_2}^{-}\right\rangle$ be BCn,m-ROFSs, then the following properties hold:

(1) $(\mathscr{Y}_1\oplus\mathscr{Y}_2)^{c}=\mathscr{Y}_1^{c}\otimes\mathscr{Y}_2^{c}$.

(2) $(\mathscr{Y}_1\otimes\mathscr{Y}_2)^{c}=\mathscr{Y}_1^{c}\oplus\mathscr{Y}_2^{c}$.

(3) $(\mathscr{Y}^{c})^{\zeta}=(\zeta\mathscr{Y})^{c}$.

(4) $\zeta(\mathscr{Y})^{c}=(\mathscr{Y}^{\zeta})^{c}$.

*Proof.* Parts (1) and (3) will be exhibited here. Likewise, the remaining parts can be presented.

(1) $(\mathscr{Y}_1\oplus\mathscr{Y}_2)^{c}=\langle(((\mathscr{A}_{\mathscr{Y}_1}^{+})^{n}+(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n}-(\mathscr{A}_{\mathscr{Y}_1}^{+})^{n}(\mathscr{A}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}}+i((\mathscr{B}_{\mathscr{Y}_1}^{+})^{n}+(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n}-(\mathscr{B}_{\mathscr{Y}_1}^{+})^{n}(\mathscr{B}_{\mathscr{Y}_2}^{+})^{n})^{\frac{1}{n}},(\mathscr{C}_{\mathscr{Y}_1}^{+}\mathscr{C}_{\mathscr{Y}_2}^{+})+$

$i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}), -(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2})+i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})), -(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})\rangle^c$
$= \langle(\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2})^{\frac{m}{n}}, (((\mathscr{A}^+_{\mathscr{Y}_1})^n + (\mathscr{A}^+_{\mathscr{Y}_2})^n - (\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}})^{\frac{n}{m}} + i(((\mathscr{B}^+_{\mathscr{Y}_1})^n + (\mathscr{B}^+_{\mathscr{Y}_2})^n - (\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}})^{\frac{n}{m}}, -((|\mathscr{C}^-_{\mathscr{Y}_1}|^m + |\mathscr{C}^-_{\mathscr{Y}_2}|^m - |\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})^{\frac{m}{n}} + i(-((|\mathscr{D}^-_{\mathscr{Y}_1}|^m + |\mathscr{D}^-_{\mathscr{Y}_2}|^m - |\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}})^{\frac{m}{n}}), -|-(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2})|^{\frac{n}{m}} + i(-|-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2}))|^{\frac{n}{m}})\rangle$

$$= \langle(\mathscr{C}^+_{\mathscr{Y}_1})^{\frac{m}{n}}+i(\mathscr{D}^+_{\mathscr{Y}_1})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}_1})^{\frac{n}{m}}+i(\mathscr{B}^+_{\mathscr{Y}_1})^{\frac{n}{m}}, -|\mathscr{C}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}+i(-|\mathscr{D}^-_{\mathscr{Y}_1}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}_1}|^{\frac{n}{m}}+i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\frac{n}{m}})\rangle$$

$$\otimes$$

$\langle(\mathscr{C}^+_{\mathscr{Y}_2})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}_2})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}_2})^{\frac{n}{m}} + i(\mathscr{B}^+_{\mathscr{Y}_2})^{\frac{n}{m}}, -|\mathscr{C}^-_{\mathscr{Y}_2}|^{\frac{m}{n}} + i(-|\mathscr{D}^-_{\mathscr{Y}_2}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}_2}|^{\frac{n}{m}} + i(-|\mathscr{B}^-_{\mathscr{Y}_2}|^{\frac{n}{m}})\rangle$
$= \mathscr{Y}_1^c \otimes \mathscr{Y}_2^c$.

(3) $(\mathscr{Y}^c)^\zeta = \langle(\mathscr{C}^+_{\mathscr{Y}})^{\frac{m}{n}} + i(\mathscr{D}^+_{\mathscr{Y}})^{\frac{m}{n}}, (\mathscr{A}^+_{\mathscr{Y}})^{\frac{n}{m}} + i(\mathscr{B}^+_{\mathscr{Y}})^{\frac{n}{m}}, -|\mathscr{C}^-_{\mathscr{Y}}|^{\frac{m}{n}} + i(-|\mathscr{D}^-_{\mathscr{Y}}|^{\frac{m}{n}}), -|\mathscr{A}^-_{\mathscr{Y}}|^{\frac{n}{m}} + i(-|\mathscr{B}^-_{\mathscr{Y}}|^{\frac{n}{m}})\rangle^\zeta$
$= \langle(((\mathscr{C}^+_{\mathscr{Y}})^{\frac{m}{n}})^\zeta + i((\mathscr{D}^+_{\mathscr{Y}})^{\frac{m}{n}})^\zeta, (1-(1-((\mathscr{A}^+_{\mathscr{Y}})^{\frac{n}{m}})^m)^\zeta)^{\frac{1}{m}} + i(1-(1-((\mathscr{B}^+_{\mathscr{Y}})^{\frac{n}{m}})^m)^\zeta)^{\frac{1}{m}},$
$-(1-(1-|-|\mathscr{C}^-_{\mathscr{Y}}|^{\frac{m}{n}}|^n)^\zeta)^{\frac{1}{n}}+i(-(1-(1-|-|\mathscr{D}^-_{\mathscr{Y}}|^{\frac{m}{n}}|^n)^\zeta)^{\frac{1}{n}}), -|-|\mathscr{A}^-_{\mathscr{Y}}|^{\frac{n}{m}}|^\zeta+i(-|-|\mathscr{B}^-_{\mathscr{Y}}|^{\frac{n}{m}}|^\zeta)\rangle$
$= \langle(((\mathscr{C}^+_{\mathscr{Y}})^\zeta)^{\frac{m}{n}} + i((\mathscr{D}^+_{\mathscr{Y}})^\zeta)^{\frac{m}{n}}, ((1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}})^{\frac{n}{m}} + i((1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}})^{\frac{n}{m}},$
$-|-(1-(1-|-|\mathscr{C}^-_{\mathscr{Y}}|^m|^\zeta)^{\frac{1}{m}}|^{\frac{m}{n}} + i(-|-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^\zeta)^{\frac{1}{m}}|^{\frac{m}{n}}), -|-|\mathscr{A}^-_{\mathscr{Y}}|^\zeta|^{\frac{n}{m}} + i(-|-|\mathscr{B}^-_{\mathscr{Y}}|^\zeta|^{\frac{n}{m}})\rangle$
$= \langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}} + i(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^\zeta)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^\zeta + i(\mathscr{D}^+_{\mathscr{Y}})^\zeta,$
$-|\mathscr{A}^-_{\mathscr{Y}}|^\zeta + i(-|\mathscr{B}^-_{\mathscr{Y}}|^\zeta), (1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m|)^\zeta)^{\frac{1}{m}} + i((1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m|)^\zeta)^{\frac{1}{m}})\rangle^c = (\zeta\mathscr{Y})^c$.

**Theorem 9.** Let $\mathscr{Y} = \left\langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}}\right\rangle$,
$\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1}\right\rangle$ and
$\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2}\right\rangle$ be BCn,m-ROFSs. For $\zeta, \zeta_1, \zeta_2 > 0$ the following are valid:

(1) $\zeta(\mathscr{Y}_1 \oplus \mathscr{Y}_2) = \zeta\mathscr{Y}_1 \oplus \zeta\mathscr{Y}_2$.

(2) $(\zeta_1 + \zeta_2)\mathscr{Y} = \zeta_1\mathscr{Y} \oplus \zeta_2\mathscr{Y}$.

(3) $(\mathscr{Y}_1 \otimes \mathscr{Y}_2)^\zeta = \mathscr{Y}_1^\zeta \otimes \mathscr{Y}_2^\zeta$.

(4) $\mathscr{Y}^{(\zeta_1+\zeta_2)} = \mathscr{Y}^{\zeta_1} \otimes \mathscr{Y}^{\zeta_2}$.

*Proof.* Parts (1) and (2) will be exhibited here. Likewise, the remaining parts can be presented.
(1) $\zeta(\mathscr{Y}_1 \oplus \mathscr{Y}_2) = \zeta(((\mathscr{A}^+_{\mathscr{Y}_1})^n+(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}}+i((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})+$

$i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2}), -(\mathscr{A}^-_{\mathscr{Y}_1}\mathscr{A}^-_{\mathscr{Y}_2})+i(-(\mathscr{B}^-_{\mathscr{Y}_1}\mathscr{B}^-_{\mathscr{Y}_2})), -(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}}+i(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\frac{1}{m}}))$
$= \langle(1-(1-((\mathscr{A}^+_{\mathscr{Y}_1})^n+(\mathscr{A}^+_{\mathscr{Y}_2})^n-(\mathscr{A}^+_{\mathscr{Y}_1})^n(\mathscr{A}^+_{\mathscr{Y}_2})^n))^\zeta)^{\frac{1}{n}}+i(1-(1-((\mathscr{B}^+_{\mathscr{Y}_1})^n+(\mathscr{B}^+_{\mathscr{Y}_2})^n-(\mathscr{B}^+_{\mathscr{Y}_1})^n(\mathscr{B}^+_{\mathscr{Y}_2})^n))^\zeta)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})^\zeta+i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2})^\zeta, -(|\mathscr{A}^-_{\mathscr{Y}_1}||\mathscr{A}^-_{\mathscr{Y}_2}|)^\zeta+i(-(|\mathscr{B}^-_{\mathscr{Y}_1}||\mathscr{B}^-_{\mathscr{Y}_2}|)^\zeta), -(1-(1-(-(|\mathscr{C}^-_{\mathscr{Y}_1}|^m+|\mathscr{C}^-_{\mathscr{Y}_2}|^m-|\mathscr{C}^-_{\mathscr{Y}_1}|^m|\mathscr{C}^-_{\mathscr{Y}_2}|^m)))^\zeta)^{\frac{1}{m}}+i(-(1-(1-(-(|\mathscr{D}^-_{\mathscr{Y}_1}|^m+|\mathscr{D}^-_{\mathscr{Y}_2}|^m-|\mathscr{D}^-_{\mathscr{Y}_1}|^m|\mathscr{D}^-_{\mathscr{Y}_2}|^m)))^\zeta)^{\frac{1}{m}})\rangle.$

On the other hand,

$\zeta\mathscr{Y}_1\oplus\zeta\mathscr{Y}_2 = \langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}_1})^n)^\zeta)^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^+_{\mathscr{Y}_1})^n)^\zeta)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1})^\zeta+i(\mathscr{D}^+_{\mathscr{Y}_1})^\zeta, -|\mathscr{A}^-_{\mathscr{Y}_1}|^\zeta+i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^\zeta), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}_1}|^m)^\zeta)^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}_1}|^m)^\zeta)^{\frac{1}{m}})\rangle$

$$\oplus$$

$\langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}_2})^n)^\zeta)^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^+_{\mathscr{Y}_2})^n)^\zeta)^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_2})^\zeta+i(\mathscr{D}^+_{\mathscr{Y}_2})^\zeta, -|\mathscr{A}^-_{\mathscr{Y}_2}|^\zeta+i(-|\mathscr{B}^-_{\mathscr{Y}_2}|^\zeta), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^\zeta)^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^\zeta)^{\frac{1}{m}})\rangle$
$= \langle(((1-(1-(\mathscr{A}^+_{\mathscr{Y}_1})^n)^\zeta)+(1-(1-(\mathscr{A}^+_{\mathscr{Y}_2})^n)^\zeta)-(1-(1-(\mathscr{A}^+_{\mathscr{Y}_1})^n)^\zeta)(1-(1-(\mathscr{A}^+_{\mathscr{Y}_2})^n)^\zeta))^{\frac{1}{n}}+i((1-(1-(\mathscr{B}^+_{\mathscr{Y}_1})^n)^\zeta)+(1-(1-(\mathscr{B}^+_{\mathscr{Y}_2})^n)^\zeta)-(1-(1-(\mathscr{B}^+_{\mathscr{Y}_1})^n)^\zeta)(1-(1-(\mathscr{B}^+_{\mathscr{Y}_2})^n)^\zeta))^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1}\mathscr{C}^+_{\mathscr{Y}_2})^\zeta+i(\mathscr{D}^+_{\mathscr{Y}_1}\mathscr{D}^+_{\mathscr{Y}_2})^\zeta, -(|\mathscr{A}^-_{\mathscr{Y}_1}||\mathscr{A}^-_{\mathscr{Y}_2}|)^\zeta+i(-(|\mathscr{B}^-_{\mathscr{Y}_1}||\mathscr{B}^-_{\mathscr{Y}_2}|)^\zeta), -((1-(1-|\mathscr{C}^-_{\mathscr{Y}_1}|^m)^\zeta)+(1-(1-|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^\zeta)-(1-(1-|\mathscr{C}^-_{\mathscr{Y}_1}|^m)^\zeta)(1-(1-|\mathscr{C}^-_{\mathscr{Y}_2}|^m)^\zeta))^{\frac{1}{m}}$
$+i(-((1-(1-|\mathscr{D}^-_{\mathscr{Y}_1}|^m)^\zeta)+(1-(1-|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^\zeta)-(1-(1-|\mathscr{D}^-_{\mathscr{Y}_1}|^m)^\zeta)(1-(1-|\mathscr{D}^-_{\mathscr{Y}_2}|^m)^\zeta))^{\frac{1}{m}})\rangle$
$= \zeta(\mathscr{Y}_1\oplus\mathscr{Y}_2).$

(2) $(\zeta_1+\zeta_2)\mathscr{Y} = \langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_1+\zeta_2})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_1+\zeta_2})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^{\zeta_1+\zeta_2}+i(\mathscr{D}^+_{\mathscr{Y}})^{\zeta_1+\zeta_2}, -|\mathscr{A}^-_{\mathscr{Y}}|^{\zeta_1+\zeta_2}+i(-|\mathscr{B}^-_{\mathscr{Y}}|^{\zeta_1+\zeta_2}), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_1+\zeta_2})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_1+\zeta_2})^{\frac{1}{m}})\rangle$
$= \langle(((1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_1})+(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_2})-(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_1})(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_2}))^{\frac{1}{n}}$
$+i((1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_1})+(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_2})-(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_1})(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_2}))^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^{\zeta_1}(\mathscr{C}^+_{\mathscr{Y}})^{\zeta_2}+i(\mathscr{D}^+_{\mathscr{Y}})^{\zeta_1}(\mathscr{D}^+_{\mathscr{Y}})^{\zeta_2}, -((-|\mathscr{A}^-_{\mathscr{Y}}|^{\zeta_1})(-|\mathscr{A}^-_{\mathscr{Y}}|^{\zeta_2}))+i(-((-|\mathscr{B}^-_{\mathscr{Y}}|^{\zeta_1})(-|\mathscr{B}^-_{\mathscr{Y}}|^{\zeta_2}))), -(|-(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_1})|+|-(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_2})|-|-(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_1})||-(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_2})|)^{\frac{1}{m}}+i((-(|-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_1})|+|-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_2})|-|-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_1})||-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_2})|)^{\frac{1}{m}}))\rangle$
$= \langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_1})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_1})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^{\zeta_1}+i(\mathscr{D}^+_{\mathscr{Y}})^{\zeta_1}, -|\mathscr{A}^-_{\mathscr{Y}}|^{\zeta_1}+i(-|\mathscr{B}^-_{\mathscr{Y}}|^{\zeta_1}), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_1})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_1})^{\frac{1}{m}})\rangle$

$$\oplus$$

$\langle(1-(1-(\mathscr{A}^+_{\mathscr{Y}})^n)^{\zeta_2})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^+_{\mathscr{Y}})^n)^{\zeta_2})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}})^{\zeta_2}+i(\mathscr{D}^+_{\mathscr{Y}})^{\zeta_2}, -|\mathscr{A}^-_{\mathscr{Y}}|^{\zeta_2}+i(-|\mathscr{B}^-_{\mathscr{Y}}|^{\zeta_2}), -(1-(1-|\mathscr{C}^-_{\mathscr{Y}}|^m)^{\zeta_2})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^-_{\mathscr{Y}}|^m)^{\zeta_2})^{\frac{1}{m}})\rangle$
$= \zeta_1\mathscr{Y}\oplus\zeta_2\mathscr{Y}.$

## 4. Bipolar complex n,m-rung orthopair fuzzy aggregation operators

This section focuses on the application of BCn,m-ROF weighted averaging and geometric aggregation operators for processing and interpreting information within the BCn,m-

ROFS framework. It presents an in-depth examination of the mathematical foundations of these operators and emphasizes their importance in supporting effective decision-making.

**Definition 8.** Let $\mathscr{Y}_i = \langle \mathscr{A}^+_{\mathscr{Y}_i} + i\mathscr{B}^+_{\mathscr{Y}_i}, \mathscr{C}^+_{\mathscr{Y}_i} + i\mathscr{D}^+_{\mathscr{Y}_i}, \mathscr{A}^-_{\mathscr{Y}_i} + i\mathscr{B}^-_{\mathscr{Y}_i}, \mathscr{C}^-_{\mathscr{Y}_i} + i\mathscr{D}^-_{\mathscr{Y}_i}\rangle$, $(i = 1, 2, \ldots, k)$ be a collection of BCn,m-ROFNs and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T$ be weight vector of $\mathscr{Y}_i$ with $\epsilon_i > 0$, such that $\sum_{i=1}^k \epsilon_i = 1$. Then, the

(i) bipolar complex n,m-rung orthopair fuzzy weighted averaging (BCn,m-ROFWA) operator is a mapping BCn,m-ROFWA: $\mathscr{Y}^k \to \mathscr{Y}$ such that,
BCn,m-ROFWA$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \bigoplus_{i=1}^k \epsilon_i \mathscr{Y}_i = \epsilon_1\mathscr{Y}_1 \oplus \epsilon_2\mathscr{Y}_2 \oplus \cdots \oplus \epsilon_k\mathscr{Y}_k$.

(ii) bipolar complex n,m-rung orthopair fuzzy weighted geometric (BCn,m-ROFWG) operator is a mapping BCn,m-ROFWG: $\mathscr{Y}^k \to \mathscr{Y}$ such that,
BCn,m-ROFWG$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \bigotimes_{i=1}^k \mathscr{Y}_i^{\epsilon_i} = \mathscr{Y}_1^{\epsilon_1} \otimes \mathscr{Y}_2^{\epsilon_2} \otimes \cdots \otimes \mathscr{Y}_k^{\epsilon_k}$.

**Theorem 10.** Let $\mathscr{Y}_i = \langle \mathscr{A}^+_{\mathscr{Y}_i} + i\mathscr{B}^+_{\mathscr{Y}_i}, \mathscr{C}^+_{\mathscr{Y}_i} + i\mathscr{D}^+_{\mathscr{Y}_i}, \mathscr{A}^-_{\mathscr{Y}_i} + i\mathscr{B}^-_{\mathscr{Y}_i}, \mathscr{C}^-_{\mathscr{Y}_i} + i\mathscr{D}^-_{\mathscr{Y}_i}\rangle$, $(i = 1, 2, \ldots, k)$ be a collection of BCn,m-ROFNs and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T$ be weight vector of $\mathscr{Y}_i$ with $\epsilon_i > 0$, such that $\sum_{i=1}^k \epsilon_i = 1$. Then, the aggregation value of BCn,m-ROFNs $\mathscr{Y}_i$ by using the

(i) BCn,m-ROFWA operator is also an BCn,m-ROFN, and
BCn,m-ROFWA$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \langle (1 - \prod_{i=1}^k (1 - (\mathscr{A}^+_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}} + i(1 - \prod_{i=1}^k (1 - (\mathscr{B}^+_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}, \prod_{i=1}^k (\mathscr{C}^+_{\mathscr{Y}_i})^{\epsilon_i} + i\prod_{i=1}^k (\mathscr{D}^+_{\mathscr{Y}_i})^{\epsilon_i}, -(\prod_{i=1}^k |\mathscr{A}^-_{\mathscr{Y}_i}|^{\epsilon_i}) + i(-(\prod_{i=1}^k |\mathscr{B}^-_{\mathscr{Y}_i}|^{\epsilon_i})), -(1 - \prod_{i=1}^k (1 - |\mathscr{C}^-_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}} + i(-(1 - \prod_{i=1}^k (1 - |\mathscr{D}^-_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle$.

(ii) Bn,m-ROFWG operator is also an BCn,m-ROFN, and
BCn,m-ROFWG$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \langle \prod_{i=1}^k (\mathscr{A}^+_{\mathscr{Y}_i})^{\epsilon_i} + i(\prod_{i=1}^k \mathscr{B}^+_{\mathscr{Y}_i})^{\epsilon_i}$, $(1 - \prod_{i=1}^k (1 - (\mathscr{C}^+_{\mathscr{Y}_i})^m)^{\epsilon_i})^{\frac{1}{m}} + i(1 - \prod_{i=1}^k (1 - (\mathscr{D}^+_{\mathscr{Y}_i})^m)^{\epsilon_i})^{\frac{1}{m}}, -(1 - \prod_{i=1}^k (1 - |\mathscr{A}^-_{\mathscr{Y}_i}|^n)^{\epsilon_i})^{\frac{1}{n}} + i(-(1 - \prod_{i=1}^k (1 - |\mathscr{B}^-_{\mathscr{Y}_i}|^n)^{\epsilon_i})^{\frac{1}{n}}), -(\prod_{i=1}^k |\mathscr{C}^-_{\mathscr{Y}_i}|_i^{\epsilon_i}) + i(-(\prod_{i=1}^k |\mathscr{D}^-_{\mathscr{Y}_i}|_i^{\epsilon_i}))\rangle$.

*Proof.*

(i) We can prove the theorem by utilizing the technique of mathematical induction. Therefore, we follow as
(a) For $i = 2$, since $\epsilon_1\mathscr{Y}_1 = \langle (1 - (1 - (\mathscr{A}^+_{\mathscr{Y}_1})^n)^{\epsilon_1})^{\frac{1}{n}} + i(1 - (1 - (\mathscr{B}^+_{\mathscr{Y}_1})^n)^{\epsilon_1})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1})^{\epsilon_1} + i(\mathscr{D}^+_{\mathscr{Y}_1})^{\epsilon_1}, -|\mathscr{A}^-_{\mathscr{Y}_1}|^{\epsilon_1} + i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\epsilon_1}), -(1 - (1 - |\mathscr{C}^-_{\mathscr{Y}_1}|^m)^{\epsilon_1})^{\frac{1}{m}} + i(-(1 - (1 - |\mathscr{D}^-_{\mathscr{Y}_1}|^m)^{\epsilon_1})^{\frac{1}{m}})\rangle$ and

$$\epsilon_2\mathscr{Y}_2 = \langle (1 - (1 - (\mathscr{A}^+_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}} + i(1 - (1 - (\mathscr{B}^+_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_2})^{\epsilon_2} + i(\mathscr{D}^+_{\mathscr{Y}_2})^{\epsilon_2}, -|\mathscr{A}^-_{\mathscr{Y}_2}|^{\epsilon_2} + i(-|\mathscr{B}^-_{\mathscr{Y}_2}|^{\epsilon_2}), -(1 - (1 - |\mathscr{C}^-_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}} + i(-(1 - (1 - |\mathscr{D}^-_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}})\rangle$$

then
BCn,m-ROFWA$(\mathscr{Y}_1, \mathscr{Y}_2) = \epsilon_1\mathscr{Y}_1 \oplus \epsilon_2\mathscr{Y}_2 = \langle (1 - (1 - (\mathscr{A}^+_{\mathscr{Y}_1})^n)^{\epsilon_1})^{\frac{1}{n}} + i(1 - (1 - (\mathscr{B}^+_{\mathscr{Y}_1})^n)^{\epsilon_1})^{\frac{1}{n}}, (\mathscr{C}^+_{\mathscr{Y}_1})^{\epsilon_1} + i(\mathscr{D}^+_{\mathscr{Y}_1})^{\epsilon_1}, -|\mathscr{A}^-_{\mathscr{Y}_1}|^{\epsilon_1} + i(-|\mathscr{B}^-_{\mathscr{Y}_1}|^{\epsilon_1}), -(1 - (1 - |\mathscr{C}^-_{\mathscr{Y}_1}|^m)^{\epsilon_1})^{\frac{1}{m}} + i(-(1 - (1 - |\mathscr{D}^-_{\mathscr{Y}_1}|^m)^{\epsilon_1})^{\frac{1}{m}})\rangle$

$$\oplus$$

$\langle(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}},(\mathscr{C}^{+}_{\mathscr{Y}_2})^{\epsilon_2}+i(\mathscr{D}^{+}_{\mathscr{Y}_2})^{\epsilon_2},-|\mathscr{A}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}+i(-|\mathscr{B}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}),-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}})\rangle = \langle(((1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})+(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})-(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}))^{\frac{1}{n}}$
$+\,i((1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})+(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})-(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}))^{\frac{1}{n}},$
$(\mathscr{C}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{C}^{+}_{\mathscr{Y}_2})^{\epsilon_2}+i(\mathscr{D}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{D}^{+}_{\mathscr{Y}_2})^{\epsilon_2},-(|\mathscr{A}^{-}_{\mathscr{Y}_1}|^{\epsilon_1}|\mathscr{A}^{-}_{\mathscr{Y}_2}|^{\epsilon_2})+i(-(|\mathscr{B}^{-}_{\mathscr{Y}_1}|^{\epsilon_1}|\mathscr{B}^{-}_{\mathscr{Y}_2}|^{\epsilon_2})),$
$-((1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})+(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2}))^{\frac{1}{m}}$
$+\,i(-((1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})+(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2}))^{\frac{1}{m}})\rangle$
$=\langle(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1}+1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}-(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}))^{\frac{1}{n}}$
$+i(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1}+1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}-(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1})(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2}))^{\frac{1}{n}},$
$(\mathscr{C}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{C}^{+}_{\mathscr{Y}_2})^{\epsilon_2}+i(\mathscr{D}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{D}^{+}_{\mathscr{Y}_2})^{\epsilon_2},-((-|\mathscr{A}^{-}_{\mathscr{Y}_1}|^{\epsilon_1})(-|\mathscr{A}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}))+i(-((-|\mathscr{B}^{-}_{\mathscr{Y}_1}|^{\epsilon_1})(-|\mathscr{B}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}))),$
$-(|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})|+|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})|-(|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})|)(|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})|))^{\frac{1}{m}}+i(-(|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})|+|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})|-(|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1})|)(|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})|))^{\frac{1}{m}})\rangle$
$=\langle(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1}\cdot(1-(\mathscr{A}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_1})^n)^{\epsilon_1}\cdot(1-(\mathscr{B}^{+}_{\mathscr{Y}_2})^n)^{\epsilon_2})^{\frac{1}{n}},$
$(\mathscr{C}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{C}^{+}_{\mathscr{Y}_2})^{\epsilon_2}+i((\mathscr{D}^{+}_{\mathscr{Y}_1})^{\epsilon_1}(\mathscr{D}^{+}_{\mathscr{Y}_2})^{\epsilon_2}),-((|\mathscr{A}^{-}_{\mathscr{Y}_1}|^{\epsilon_1})(|\mathscr{A}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}))+i(-((|\mathscr{B}^{-}_{\mathscr{Y}_1}|^{\epsilon_1})(|\mathscr{B}^{-}_{\mathscr{Y}_2}|^{\epsilon_2}))),$
$-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1}(1-|\mathscr{C}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_1}|^m)^{\epsilon_1}(1-|\mathscr{D}^{-}_{\mathscr{Y}_2}|^m)^{\epsilon_2})^{\frac{1}{m}})\rangle$
$=\langle(1-\prod_{i=1}^{2}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{2}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}},\prod_{i=1}^{2}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i\prod_{i=1}^{2}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i},$
$-(\prod_{i=1}^{2}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})+i(-(\prod_{i=1}^{2}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})),-(1-\prod_{i=1}^{2}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{2}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle.$

(b) Suppose that outcome is true for $i=r$ that is,
BCn,m-ROFWA$(\mathscr{Y}_1,\mathscr{Y}_2,\ldots,\mathscr{Y}_r)=\epsilon_1\mathscr{Y}_1\oplus\epsilon_2\mathscr{Y}_2\oplus\cdots\oplus\epsilon_r\mathscr{Y}_r=\langle(1-\prod_{i=1}^{r}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{r}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}},\prod_{i=1}^{r}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i\prod_{i=1}^{r}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i},$
$-(\prod_{i=1}^{r}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})+i(-(\prod_{i=1}^{r}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})),-(1-\prod_{i=1}^{r}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{r}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle.$

(c) Now, we have to prove that outcome is true for $i=r+1$, by utilizing the (a) and (b) we have BCn,m-ROFWA$(\mathscr{Y}_1,\mathscr{Y}_2,\ldots,\mathscr{Y}_{r+1})=\epsilon_1\mathscr{Y}_1\oplus\epsilon_2\mathscr{Y}_2\oplus\cdots\oplus\epsilon_{r+1}\mathscr{Y}_{r+1}=$
$\langle(1-\prod_{i=1}^{r}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{r}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}},\prod_{i=1}^{r}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i\prod_{i=1}^{r}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i},$
$-(\prod_{i=1}^{r}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})+i(-(\prod_{i=1}^{r}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})),-(1-\prod_{i=1}^{r}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{r}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle$

$$\oplus$$

$\langle(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}})^{\frac{1}{n}}+i(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}})^{\frac{1}{n}},(\mathscr{C}^{+}_{\mathscr{Y}_{r+1}})^{\epsilon_{r+1}}+i(\mathscr{D}^{+}_{\mathscr{Y}_{r+1}})^{\epsilon_{r+1}},$
$-|\mathscr{A}^{-}_{\mathscr{Y}_{r+1}}|^{\epsilon_{r+1}}+i(-|\mathscr{B}^{-}_{\mathscr{Y}_{r+1}}|^{\epsilon_{r+1}}),-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})^{\frac{1}{m}}+i(-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})^{\frac{1}{m}})\rangle$
$=\langle((1-\prod_{i=1}^{r}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})+(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}})-(1-\prod_{i=1}^{r}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})(1-(1-(\mathscr{A}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}}))^{\frac{1}{n}}+i((1-\prod_{i=1}^{r}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})+(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}})-(1-\prod_{i=1}^{r}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})(1-(1-(\mathscr{B}^{+}_{\mathscr{Y}_{r+1}})^{n})^{\epsilon_{r+1}}))^{\frac{1}{n}},(\prod_{i=1}^{r}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}(\mathscr{C}^{+}_{\mathscr{Y}_{r+1}})^{\epsilon_{r+1}})+i(\prod_{i=1}^{r}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i}(\mathscr{D}^{+}_{\mathscr{Y}_{r+1}})^{\epsilon_{r+1}}),$
$-(-(\prod_{i=1}^{r}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})(-|\mathscr{A}^{-}_{\mathscr{Y}_{r+1}}|^{\epsilon_{r+1}}))+i(-(-(\prod_{i=1}^{r}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})(-|\mathscr{B}^{-}_{\mathscr{Y}_{r+1}}|^{\epsilon_{r+1}}))),-(|-(1-\prod_{i=1}^{r}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})|+(|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})|)-(|-(1-\prod_{i=1}^{r}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})|)(|-(1-(1-|\mathscr{C}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})|))^{\frac{1}{m}}+i(-(|-(1-\prod_{i=1}^{r}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})|+(|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})|)-(|-(1-\prod_{i=1}^{r}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})|)(|-(1-(1-|\mathscr{D}^{-}_{\mathscr{Y}_{r+1}}|^{m})^{\epsilon_{r+1}})|))^{\frac{1}{m}})\rangle$
$=\langle(1-\prod_{i=1}^{r+1}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{r+1}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})^{\frac{1}{n}},\prod_{i=1}^{r+1}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i\prod_{i=1}^{r+1}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i},$
$-(\prod_{i=1}^{r+1}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})+i(-(\prod_{i=1}^{r+1}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})),-(1-\prod_{i=1}^{r+1}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{r+1}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})^{\frac{1}{m}})\rangle.$

(ii) The proof is similar to (1).

**Example 5.** Let $\mathscr{S}=\{\mathscr{T}_1\}$. Let

$$\mathscr{Y}_1=\langle 0.8+i(0.3),0.5+i(0.9),-0.9+i(-0.9),-0.5+i(-0.3)\rangle$$
$$\mathscr{Y}_2=\langle 0.7+i(0.15),0.4+i(0.32),-0.3+i(-0.7),-0.2+i(-0.2)\rangle$$
and
$$\mathscr{Y}_3=\langle 0.6+i(0.2),0.8+i(0.3),-0.1+i(-0.5),-0.7+i(-0.2)\rangle$$

be the three BCn,m-ROFNs with weight vector $\epsilon=(0.3,0.5,0.2)^T$, respectively, then we have

(i) BCn,m-ROFWA$(\mathscr{Y}_1,\mathscr{Y}_2,\mathscr{Y}_3)=\langle(1-\prod_{i=1}^{3}(1-(\mathscr{A}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{3}(1-(\mathscr{B}^{+}_{\mathscr{Y}_i})^{n})^{\epsilon_i})^{\frac{1}{n}},\prod_{i=1}^{3}(\mathscr{C}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i\prod_{i=1}^{3}(\mathscr{D}^{+}_{\mathscr{Y}_i})^{\epsilon_i},-(\prod_{i=1}^{3}|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})+i(-(\prod_{i=1}^{3}|\mathscr{B}^{-}_{\mathscr{Y}_i}|^{\epsilon_i})),-(1-\prod_{i=1}^{3}(1-|\mathscr{C}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{3}(1-|\mathscr{D}^{-}_{\mathscr{Y}_i}|^{m})^{\epsilon_i})^{\frac{1}{m}})\rangle$

$\approx\langle(0.72)+i(0.22),(0.49)+i(0.43),-(0.334)+i(-0.71),-(0.54)+i(-0.25)\rangle$
$for\quad n=3\quad and\quad m=5$
$\approx\langle(0.73)+i(0.24),(0.49)+i(0.43),-(0.33)+i(-0.71),-(0.49)+i(-0.24)\rangle$
$for\quad n=5\quad and\quad m=3$
$\approx\langle(0.71)+i(0.21),(0.49)+i(0.43),-(0.33)+i(-0.71),-(0.59)+i(-0.26)\rangle$
$for\quad n=1\quad and\quad m=9$
$\approx\langle(0.73)+i(0.26),(0.49)+i(0.43),-(0.33)+i(-0.71),-(0.43)+i(-0.23)\rangle$
$for\quad n=9\quad and\quad m=1.$

(ii) BCn,m-ROFWG$(\mathscr{Y}_1,\mathscr{Y}_2,\ldots,\mathscr{Y}_k)=\langle\prod_{i=1}^{k}(\mathscr{A}^{+}_{\mathscr{Y}_i})^{\epsilon_i}+i(\prod_{i=1}^{k}\mathscr{B}^{+}_{\mathscr{Y}_i})^{\epsilon_i},(1-\prod_{i=1}^{k}(1-(\mathscr{C}^{+}_{\mathscr{Y}_i})^{m})^{\epsilon_i})^{\frac{1}{m}}+i(1-\prod_{i=1}^{k}(1-(\mathscr{D}^{+}_{\mathscr{Y}_i})^{m})^{\epsilon_i})^{\frac{1}{m}},-(1-\prod_{i=1}^{k}(1-|\mathscr{A}^{-}_{\mathscr{Y}_i}|^{n})^{\epsilon_i})^{\frac{1}{n}}+i(-(1-$

$\prod_{i=1}^{k}(1-|\mathscr{B}_{\mathscr{Y}_i}^{-}|^n)^{\epsilon_i})^{\frac{1}{n}}), -(\prod_{i=1}^{k}|\mathscr{C}_{\mathscr{Y}_i}^{-}|_i^{\epsilon_i}) + i(-(\prod_{i=1}^{k}|\mathscr{D}_{\mathscr{Y}_i}^{-}|_i^{\epsilon_i})))\rangle$
$\approx \langle(0.71) + i(0.2), (0.62) + i(0.75), -(0.69) + i(-0.78), -(0.34) + i(-0.23)\rangle$
$for \quad n = 3 \quad and \quad m = 5$
$\approx \langle(0.71) + i(0.2), (0.58) + i(0.7), -(0.75) + i(-0.79), -(0.34) + i(-0.23)\rangle$
$for \quad n = 5 \quad and \quad m = 3$
$\approx \langle(0.71) + i(0.2), (0.68) + i(0.8), -(0.59) + i(-0.76), -(0.34) + i(-0.23)\rangle$
$for \quad n = 1 \quad and \quad m = 9$
$\approx \langle(0.71) + i(0.2), (0.55) + i(0.62), -(0.8) + i(-0.81), -(0.34) + i(-0.23)\rangle$
$for \quad n = 9 \quad and \quad m = 1.$

**Theorem 11.** (Idempotency) Let $\mathscr{Y}_i = \langle \mathscr{A}_{\mathscr{Y}_i}^{+} + i\mathscr{B}_{\mathscr{Y}_i}^{+}, \mathscr{C}_{\mathscr{Y}_i}^{+} + i\mathscr{D}_{\mathscr{Y}_i}^{+}, \mathscr{A}_{\mathscr{Y}_i}^{-} + i\mathscr{B}_{\mathscr{Y}_i}^{-}, \mathscr{C}_{\mathscr{Y}_i}^{-} + i\mathscr{D}_{\mathscr{Y}_i}^{-}\rangle$, $(i = 1, 2, \ldots, k)$ be a collection of BCn,m-ROFNs, and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T$ be weight vector of $\mathscr{Y}_i$ with $\epsilon_i > 0$, such that $\sum_{i=1}^{k} \epsilon_i = 1$. If all $\mathscr{Y}_i = \left\langle \mathscr{A}_{\mathscr{Y}_i}^{+} + i\mathscr{B}_{\mathscr{Y}_i}^{+}, \mathscr{C}_{\mathscr{Y}_i}^{+} + i\mathscr{D}_{\mathscr{Y}_i}^{+}, \mathscr{A}_{\mathscr{Y}_i}^{-} + i\mathscr{B}_{\mathscr{Y}_i}^{-}, \mathscr{C}_{\mathscr{Y}_i}^{-} + i\mathscr{D}_{\mathscr{Y}_i}^{-}\right\rangle, (i = 1, 2, \ldots, k)$ are identical to $\mathscr{Y} = \left\langle \mathscr{A}_{\mathscr{Y}}^{+} + i\mathscr{B}_{\mathscr{Y}}^{+}, \mathscr{C}_{\mathscr{Y}}^{+} + i\mathscr{D}_{\mathscr{Y}}^{+}, \mathscr{A}_{\mathscr{Y}}^{-} + i\mathscr{B}_{\mathscr{Y}}^{-}, \mathscr{C}_{\mathscr{Y}}^{-} + i\mathscr{D}_{\mathscr{Y}}^{-}\right\rangle$ then,

(i) BCn,m-ROFWA$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \mathscr{Y}$.

(ii) BCn,m-ROFWG$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) = \mathscr{Y}$.

*Proof.*

(i) Since $\mathscr{Y}_i = \mathscr{Y} = \left\langle \mathscr{A}_{\mathscr{Y}}^{+} + i\mathscr{B}_{\mathscr{Y}}^{+}, \mathscr{C}_{\mathscr{Y}}^{+} + i\mathscr{D}_{\mathscr{Y}}^{+}, \mathscr{A}_{\mathscr{Y}}^{-} + i\mathscr{B}_{\mathscr{Y}}^{-}, \mathscr{C}_{\mathscr{Y}}^{-} + i\mathscr{D}_{\mathscr{Y}}^{-}\right\rangle (i = 1, 2, \ldots, k)$, then
BCn,m-ROFWA$(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k)$
$= \langle(1-\prod_{i=1}^{k}(1-(\mathscr{A}_{\mathscr{Y}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}} + i(1-\prod_{i=1}^{k}(1-(\mathscr{B}_{\mathscr{Y}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}},$
$\prod_{i=1}^{k}(\mathscr{C}_{\mathscr{Y}_i}^{+})^{\epsilon_i} + i\prod_{i=1}^{k}(\mathscr{D}_{\mathscr{Y}_i}^{+})^{\epsilon_i}, -(\prod_{i=1}^{k}|\mathscr{A}_{\mathscr{Y}_i}^{-}|^{\epsilon_i}) + i(-(\prod_{i=1}^{k}|\mathscr{B}_{\mathscr{Y}_i}^{-}|^{\epsilon_i})), -(1-\prod_{i=1}^{k}(1-$
$|\mathscr{C}_{\mathscr{Y}_i}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}} + i(-(1-\prod_{i=1}^{k}(1-|\mathscr{D}_{\mathscr{Y}_i}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle$
$= \langle(1-\prod_{i=1}^{k}(1-(\mathscr{A}_{\mathscr{Y}}^{+})^n)^{\epsilon_i})^{\frac{1}{n}} + i(1-\prod_{i=1}^{k}(1-(\mathscr{B}_{\mathscr{Y}}^{+})^n)^{\epsilon_i})^{\frac{1}{n}}, \prod_{i=1}^{k}(\mathscr{C}_{\mathscr{Y}}^{+})^{\epsilon_i} + i\prod_{i=1}^{k}(\mathscr{D}_{\mathscr{Y}}^{+})^{\epsilon_i},$
$-(\prod_{i=1}^{k}|\mathscr{A}_{\mathscr{Y}}^{-}|^{\epsilon_i}) + i(-(\prod_{i=1}^{k}|\mathscr{B}_{\mathscr{Y}}^{-}|^{\epsilon_i})), -(1-\prod_{i=1}^{k}(1-|\mathscr{C}_{\mathscr{Y}}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}} + i(-(1-\prod_{i=1}^{k}(1-$
$|\mathscr{D}_{\mathscr{Y}}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle$
$= \langle(1-(1-(\mathscr{A}_{\mathscr{Y}}^{+})^n)^{\sum_{i=1}^{k}\epsilon_i})^{\frac{1}{n}} + i(1-(1-(\mathscr{B}_{\mathscr{Y}}^{+})^n)^{\sum_{i=1}^{k}\epsilon_i})^{\frac{1}{n}}, (\mathscr{C}_{\mathscr{Y}}^{+})^{\sum_{i=1}^{k}\epsilon_i} + i(\mathscr{D}_{\mathscr{Y}}^{+})^{\sum_{i=1}^{k}\epsilon_i},$
$-(|\mathscr{A}_{\mathscr{Y}}^{-}|^{\sum_{i=1}^{k}\epsilon_i}) + i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|^{\sum_{i=1}^{k}\epsilon_i})), -(1-(1-|\mathscr{C}_{\mathscr{Y}}^{-}|^m)^{\sum_{i=1}^{k}\epsilon_i})^{\frac{1}{m}} + i(-(1-(1-$
$|\mathscr{D}_{\mathscr{Y}}^{-}|^m)^{\sum_{i=1}^{k}\epsilon_i})^{\frac{1}{m}})\rangle$
$= \langle(1-(1-(\mathscr{A}_{\mathscr{Y}}^{+})^n))^{\frac{1}{n}} + i(1-(1-(\mathscr{B}_{\mathscr{Y}}^{+})^n))^{\frac{1}{n}}, (\mathscr{C}_{\mathscr{Y}}^{+}) + i(\mathscr{D}_{\mathscr{Y}}^{+}), -(|\mathscr{A}_{\mathscr{Y}}^{-}|) + i(-(|\mathscr{B}_{\mathscr{Y}}^{-}|)), -(1-$
$(1-|\mathscr{C}_{\mathscr{Y}}^{-}|^m))^{\frac{1}{m}} + i(-(1-(1-|\mathscr{D}_{\mathscr{Y}}^{-}|^m))^{\frac{1}{m}})\rangle$

$= \langle \mathscr{A}_{\mathscr{Y}}^{+} + i\mathscr{B}_{\mathscr{Y}}^{+}, \mathscr{C}_{\mathscr{Y}}^{+} + i\mathscr{D}_{\mathscr{Y}}^{+}, \mathscr{A}_{\mathscr{Y}}^{-} + i\mathscr{B}_{\mathscr{Y}}^{-}, \mathscr{C}_{\mathscr{Y}}^{-} + i\mathscr{D}_{\mathscr{Y}}^{-}\rangle$,
where $-|\mathscr{A}_{\mathscr{Y}}^{-}| = \mathscr{A}_{\mathscr{Y}}^{-}$, $-|\mathscr{C}_{\mathscr{Y}}^{-}| = \mathscr{C}_{\mathscr{Y}}^{-}$, $-|\mathscr{B}_{\mathscr{Y}}^{-}| = \mathscr{B}_{\mathscr{Y}}^{-}$ and $-|\mathscr{D}_{\mathscr{Y}}^{-}| = \mathscr{D}_{\mathscr{Y}}^{-}$.

(2) The proof is similar to (1).

**Theorem 12.** (Boundedness) Let $\mathscr{Y}_i = \langle \mathscr{A}^+_{\mathscr{Y}_i} + i\mathscr{B}^+_{\mathscr{Y}_i}, \mathscr{C}^+_{\mathscr{Y}_i} + i\mathscr{D}^+_{\mathscr{Y}_i}, \mathscr{A}^-_{\mathscr{Y}_i} + i\mathscr{B}^-_{\mathscr{Y}_i}, \mathscr{C}^-_{\mathscr{Y}_i} + i\mathscr{D}^-_{\mathscr{Y}_i} \rangle$, $(i = 1, 2, \ldots, k)$ be a collection of BCn,m-ROFNs and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T$ be weight vector of $\mathscr{Y}_i$ with $\epsilon_i > 0$, such that $\sum_{i=1}^k \epsilon_i = 1$. If $\overline{\mathscr{Y}}$ and $\underline{\mathscr{Y}}$ are two BCn,m-ROFNs, such that
$\overline{\mathscr{Y}} = \langle \mathscr{A}^{+^+}_{\overline{\mathscr{Y}}} + i\mathscr{B}^{+^+}_{\overline{\mathscr{Y}}}, \mathscr{C}^{+^-}_{\overline{\mathscr{Y}}} + i\mathscr{D}^{+^-}_{\overline{\mathscr{Y}}}, \mathscr{A}^{-^-}_{\overline{\mathscr{Y}}} + i\mathscr{B}^{-^-}_{\overline{\mathscr{Y}}}, \mathscr{C}^{-^+}_{\overline{\mathscr{Y}}} + i\mathscr{D}^{-^+}_{\overline{\mathscr{Y}}} \rangle$
$= \langle max\mathscr{A}^+_{\mathscr{Y}_i} + imax\mathscr{B}^+_{\mathscr{Y}_i}, min\mathscr{C}^+_{\mathscr{Y}_i} + imin\mathscr{D}^+_{\mathscr{Y}_i}, min\mathscr{A}^-_{\mathscr{Y}_i} + imin\mathscr{B}^-_{\mathscr{Y}_i}, max\mathscr{C}^-_{\mathscr{Y}_i} + imax\mathscr{D}^-_{\mathscr{Y}_i} \rangle$,
where $1 \leq i \leq k$
and,
$\underline{\mathscr{Y}} = \langle \mathscr{A}^{+^-}_{\underline{\mathscr{Y}}} + i\mathscr{B}^{+^-}_{\underline{\mathscr{Y}}}, \mathscr{C}^{+^+}_{\underline{\mathscr{Y}}} + i\mathscr{D}^{+^+}_{\underline{\mathscr{Y}}}, \mathscr{A}^{-^+}_{\underline{\mathscr{Y}}} + i\mathscr{B}^{-^+}_{\underline{\mathscr{Y}}}, \mathscr{C}^{-^-}_{\underline{\mathscr{Y}}} + i\mathscr{D}^{-^-}_{\underline{\mathscr{Y}}} \rangle$
$= \langle min\mathscr{A}^+_{\underline{\mathscr{Y}}_i} + imin\mathscr{B}^+_{\underline{\mathscr{Y}}_i}, max\mathscr{C}^+_{\underline{\mathscr{Y}}_i} + imax\mathscr{D}^+_{\underline{\mathscr{Y}}_i}, max\mathscr{A}^-_{\underline{\mathscr{Y}}_i} + imax\mathscr{B}^-_{\underline{\mathscr{Y}}_i}, min\mathscr{C}^-_{\underline{\mathscr{Y}}_i} + imin\mathscr{D}^-_{\underline{\mathscr{Y}}_i} \rangle$,
where $1 \leq i \leq k$, then

(i) $\underline{\mathscr{Y}} \leq \text{BCn, m-ROFWA}(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) \leq \overline{\mathscr{Y}}$.

(ii) $\underline{\mathscr{Y}} \leq \text{BCn, m-ROFWG}(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) \leq \overline{\mathscr{Y}}$.

*Proof.*

(i) For any $\mathscr{Y}_i = \left\langle \mathscr{A}^+_{\mathscr{Y}_i} + i\mathscr{B}^+_{\mathscr{Y}_i}, \mathscr{C}^+_{\mathscr{Y}_i} + i\mathscr{D}^+_{\mathscr{Y}_i}, \mathscr{A}^-_{\mathscr{Y}_i} + i\mathscr{B}^-_{\mathscr{Y}_i}, \mathscr{C}^-_{\mathscr{Y}_i} + i\mathscr{D}^-_{\mathscr{Y}_i} \right\rangle, (i = 1, 2, \ldots, k)$ we can get
$\mathscr{A}^{+^-}_{\underline{\mathscr{Y}}} \leq \mathscr{A}^+_{\mathscr{Y}_i} \leq \mathscr{A}^{+^+}_{\overline{\mathscr{Y}}}, \mathscr{A}^{-^-}_{\underline{\mathscr{Y}}} \leq \mathscr{A}^-_{\mathscr{Y}_i} \leq \mathscr{A}^{-^+}_{\overline{\mathscr{Y}}}, \mathscr{C}^{+^-}_{\underline{\mathscr{Y}}} \leq \mathscr{C}^+_{\mathscr{Y}_i} \leq \mathscr{C}^{+^+}_{\overline{\mathscr{Y}}}$, and $\mathscr{C}^{-^-}_{\underline{\mathscr{Y}}} \leq \mathscr{C}^-_{\mathscr{Y}_i} \leq \mathscr{C}^{-^+}_{\overline{\mathscr{Y}}}$.
Similarly,
$\mathscr{B}^{+^-}_{\underline{\mathscr{Y}}} \leq \mathscr{B}^+_{\mathscr{Y}_i} \leq \mathscr{B}^{+^+}_{\overline{\mathscr{Y}}}, \mathscr{B}^{-^-}_{\underline{\mathscr{Y}}} \leq \mathscr{B}^-_{\mathscr{Y}_i} \leq \mathscr{B}^{-^+}_{\overline{\mathscr{Y}}}, \mathscr{D}^{+^-}_{\underline{\mathscr{Y}}} \leq \mathscr{D}^+_{\mathscr{Y}_i} \leq \mathscr{D}^{+^+}_{\overline{\mathscr{Y}}}$, and $\mathscr{D}^{-^-}_{\underline{\mathscr{Y}}} \leq \mathscr{D}^-_{\mathscr{Y}_i} \leq \mathscr{D}^{-^+}_{\overline{\mathscr{Y}}}$,
then we have
$\mathscr{A}^{+^-}_{\mathscr{Y}} = (1 - (1 - (\mathscr{A}^{+^-}_{\mathscr{Y}})^n)^{\sum_{i=1}^k \epsilon_i})^{\frac{1}{n}}$
$= (1 - \prod_{i=1}^k (1 - (\mathscr{A}^{+^-}_{\mathscr{Y}})^n)^{\epsilon_i})^{\frac{1}{n}}$
$\leq (1 - \prod_{i=1}^k (1 - (\mathscr{A}^+_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}$
$\leq (1 - \prod_{i=1}^k (1 - (\mathscr{A}^{+^+}_{\mathscr{Y}})^n)^{\epsilon_i})^{\frac{1}{n}}$
$= (1 - (1 - (\mathscr{A}^{+^+}_{\mathscr{Y}})^n)^{\sum_{i=1}^k \epsilon_i})^{\frac{1}{n}} = \mathscr{A}^{+^+}_{\mathscr{Y}}$,
$\mathscr{A}^{-^-}_{\mathscr{Y}} = -(|\mathscr{A}^{-^-}_{\mathscr{Y}}|^{\sum_{i=1}^k \epsilon_i})$
$= -(\prod_{i=1}^k (|\mathscr{A}^{-^-}_{\mathscr{Y}}|)^{\epsilon_i})$
$\leq -(\prod_{i=1}^k (|\mathscr{A}^-_{\mathscr{Y}}|)^{\epsilon_i})$
$\leq -(\prod_{i=1}^k (|\mathscr{A}^{-^+}_{\mathscr{Y}}|)^{\epsilon_i}) = -(|\mathscr{A}^{-^+}_{\mathscr{Y}}|^{\sum_{i=1}^k \epsilon_i}) = -|\mathscr{A}^{-^+}_{\mathscr{Y}}| = \mathscr{A}^{-^+}_{\mathscr{Y}}$,
$\mathscr{C}^{+^-}_{\mathscr{Y}} = (\mathscr{C}^{+^-}_{\mathscr{Y}})^{\sum_{i=1}^k \epsilon_i}$
$= \prod_{i=1}^k (\mathscr{C}^{+^-}_{\mathscr{Y}})^{\epsilon_i}$
$\leq \prod_{i=1}^k (\mathscr{C}^+_{\mathscr{Y}})^{\epsilon_i}$
$\leq \prod_{i=1}^k (\mathscr{C}^{+^+}_{\mathscr{Y}})^{\epsilon_i} = (\mathscr{C}^{+^+}_{\mathscr{Y}})^{\sum_{i=1}^k \epsilon_i} = \mathscr{C}^{+^+}_{\mathscr{Y}}$, and
$\mathscr{C}^{-^-}_{\mathscr{Y}} = -(1 - (1 - |\mathscr{C}^{-^-}_{\mathscr{Y}}|^m)^{\sum_{i=1}^k \epsilon_i})^{\frac{1}{m}}$

$= -(1 - \prod_{i=1}^{k}(1 - |\mathscr{C}_{\underline{\mathscr{Y}}}^{-^-}|^m)^{\epsilon_i})^{\frac{1}{m}}$
$\leq -(1 - \prod_{i=1}^{k}(1 - |\mathscr{C}_{\mathscr{Y}_i}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}}$
$\leq \quad -(1 - \prod_{i=1}^{k}(1 - |\mathscr{C}_{\overline{\mathscr{Y}}}^{-^+}|^m)^{\epsilon_i})^{\frac{1}{m}} = -(1-(1-(\mathscr{C}_{\overline{\mathscr{Y}}}^{-^+})^m)^{\sum_{i=1}^{k}\epsilon_i})^{\frac{1}{m}} = \mathscr{C}_{\overline{\mathscr{Y}}}^{-^+}.$
In the same way we can find that

$$\mathscr{B}_{\underline{\mathscr{Y}}}^{+^-} \leq \mathscr{B}_{\mathscr{Y}_i}^{+} \leq \mathscr{B}_{\overline{\mathscr{Y}}}^{+^+}, \mathscr{B}_{\underline{\mathscr{Y}}}^{-^-} \leq \mathscr{B}_{\mathscr{Y}_i}^{-} \leq \mathscr{B}_{\overline{\mathscr{Y}}}^{-^+}, \mathscr{D}_{\underline{\mathscr{Y}}}^{+^-} \leq \mathscr{D}_{\mathscr{Y}_i}^{+} \leq \mathscr{D}_{\overline{\mathscr{Y}}}^{+^+}, \mathscr{D}_{\underline{\mathscr{Y}}}^{-^-} \leq \mathscr{D}_{\mathscr{Y}_i}^{-} \leq \mathscr{D}_{\overline{\mathscr{Y}}}^{-^+}.$$

Therefore, $\underline{\mathscr{Y}} \leq \mathrm{BCn, m\text{-}ROFWA}(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) \leq \overline{\mathscr{Y}}$.

(ii) The proof is similar to (1).

**Theorem 13.** (Monotonicity) Let $\mathscr{Y}_i = \langle \mathscr{A}_{\mathscr{Y}_i}^{+} + i\mathscr{B}_{\mathscr{Y}_i}^{+}, \mathscr{C}_{\mathscr{Y}_i}^{+} + i\mathscr{D}_{\mathscr{Y}_i}^{+}, \mathscr{A}_{\mathscr{Y}_i}^{-} + i\mathscr{B}_{\mathscr{Y}_i}^{-}, \mathscr{C}_{\mathscr{Y}_i}^{-} + i\mathscr{D}_{\mathscr{Y}_i}^{-}\rangle$ and $\tilde{\mathscr{Y}}_i = \langle \mathscr{A}_{\tilde{\mathscr{Y}}_i}^{+} + i\mathscr{B}_{\tilde{\mathscr{Y}}_i}^{+}, \mathscr{C}_{\tilde{\mathscr{Y}}_i}^{+} + i\mathscr{D}_{\tilde{\mathscr{Y}}_i}^{+}, \mathscr{A}_{\tilde{\mathscr{Y}}_i}^{-} + i\mathscr{B}_{\tilde{\mathscr{Y}}_i}^{-}, \mathscr{C}_{\tilde{\mathscr{Y}}_i}^{-} + i\mathscr{D}_{\tilde{\mathscr{Y}}_i}^{-}\rangle$, $(i = 1, 2, \ldots, k)$ be any two collections of BCn,m-ROFNs. If $\mathscr{Y}_i \subseteq \tilde{\mathscr{Y}}_i$, then

(i) $\mathrm{BCn, m\text{-}ROFWA}(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) \leq \mathrm{BCn, m\text{-}ROFWA}(\tilde{\mathscr{Y}}_1, \tilde{\mathscr{Y}}_2, \ldots, \tilde{\mathscr{Y}}_k)$.

(ii) $\mathrm{BCn, m\text{-}ROFWG}(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k) \leq \mathrm{BCn, m\text{-}ROFWG}(\tilde{\mathscr{Y}}_1, \tilde{\mathscr{Y}}_2, \ldots, \tilde{\mathscr{Y}}_k)$.

*Proof.*

(i) Since for all $i$ we have

$$\mathscr{A}_{\mathscr{Y}_i}^{+} \leq \mathscr{A}_{\tilde{\mathscr{Y}}_i}^{+}, \mathscr{A}_{\mathscr{Y}_i}^{-} \geq \mathscr{A}_{\tilde{\mathscr{Y}}_i}^{-}, \mathscr{C}_{\mathscr{Y}_i}^{+} \geq \mathscr{C}_{\tilde{\mathscr{Y}}_i}^{+}, \mathscr{C}_{\mathscr{Y}_i}^{-} \leq \mathscr{C}_{\tilde{\mathscr{Y}}_i}^{-},$$

$$\mathscr{B}_{\mathscr{Y}_i}^{+} \leq \mathscr{B}_{\tilde{\mathscr{Y}}_i}^{+}, \mathscr{B}_{\mathscr{Y}_i}^{-} \geq \mathscr{B}_{\tilde{\mathscr{Y}}_i}^{-}, \mathscr{D}_{\mathscr{Y}_i}^{+} \geq \mathscr{D}_{\tilde{\mathscr{Y}}_i}^{+}, \text{ and } \mathscr{D}_{\mathscr{Y}_i}^{-} \leq \mathscr{D}_{\tilde{\mathscr{Y}}_i}^{-}$$

then

$$(1 - \prod_{i=1}^{k}(1 - (\mathscr{A}_{\mathscr{Y}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}} \leq (1 - \prod_{i=1}^{k}(1 - (\mathscr{A}_{\tilde{\mathscr{Y}}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}},$$

$$-(\prod_{i=1}^{k}|\mathscr{A}_{\mathscr{Y}_i}^{-}|^{\epsilon_i}) \leq -(\prod_{i=1}^{k}|\mathscr{A}_{\tilde{\mathscr{Y}}_i}^{-}|^{\epsilon_i}),$$

$$\prod_{i=1}^{k}(\mathscr{C}_{\mathscr{Y}_i}^{+})^{\epsilon_i} \geq \prod_{i=1}^{k}(\mathscr{C}_{\tilde{\mathscr{Y}}_i}^{+})^{\epsilon_i},$$

$$-(1 - \prod_{i=1}^{k}(1 - |\mathscr{C}_{\mathscr{Y}_i}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}} \geq -(1 - \prod_{i=1}^{k}(1 - |\mathscr{C}_{\tilde{\mathscr{Y}}_i}^{-}|^m)^{\epsilon_i})^{\frac{1}{m}},$$

$$(1 - \prod_{i=1}^{k}(1 - (\mathscr{B}_{\mathscr{Y}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}} \leq (1 - \prod_{i=1}^{k}(1 - (\mathscr{B}_{\tilde{\mathscr{Y}}_i}^{+})^n)^{\epsilon_i})^{\frac{1}{n}},$$

$$-(\prod_{i=1}^{k}|\mathscr{B}^-_{\mathscr{Y}_i}|^{\epsilon_i}) \leq -(\prod_{i=1}^{k}|\mathscr{B}^-_{\tilde{\mathscr{Y}}_i}|^{\epsilon_i}),$$

$$\prod_{i=1}^{k}(\mathscr{D}^+_{\mathscr{Y}_i})^{\epsilon_i} \geq \prod_{i=1}^{k}(\mathscr{D}^+_{\tilde{\mathscr{Y}}_i})^{\epsilon_i},$$

and

$$-(1-\prod_{i=1}^{k}(1-|\mathscr{D}^-_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}} \geq -(1-\prod_{i=1}^{k}(1-|\mathscr{D}^-_{\tilde{\mathscr{Y}}_i}|^m)^{\epsilon_i})^{\frac{1}{m}}.$$

Therefore,
BCn,m-ROFWA $(\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_k)$
$= \langle (1-\prod_{i=1}^{k}(1-(\mathscr{A}^+_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}} + i(1-\prod_{i=1}^{k}(1-(\mathscr{B}^+_{\mathscr{Y}_i})^n)^{\epsilon_i})^{\frac{1}{n}}, \prod_{i=1}^{k}(\mathscr{C}^+_{\mathscr{Y}_i})^{\epsilon_i} + i\prod_{i=1}^{k}(\mathscr{D}^+_{\mathscr{Y}_i})^{\epsilon_i}, -(\prod_{i=1}^{k}|\mathscr{A}^-_{\mathscr{Y}_i}|^{\epsilon_i}) + i(-(\prod_{i=1}^{k}|\mathscr{B}^-_{\mathscr{Y}_i}|^{\epsilon_i})), -(1-\prod_{i=1}^{k}(1-|\mathscr{C}^-_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}} + i(-(1-\prod_{i=1}^{k}(1-|\mathscr{D}^-_{\mathscr{Y}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle \leq \langle (1-\prod_{i=1}^{k}(1-(\mathscr{A}^+_{\tilde{\mathscr{Y}}_i})^n)^{\epsilon_i})^{\frac{1}{n}} + i(1-\prod_{i=1}^{k}(1-(\mathscr{B}^+_{\tilde{\mathscr{Y}}_i})^n)^{\epsilon_i})^{\frac{1}{n}}, \prod_{i=1}^{k}(\mathscr{C}^+_{\tilde{\mathscr{Y}}_i})^{\epsilon_i} + i\prod_{i=1}^{k}(\mathscr{D}^+_{\tilde{\mathscr{Y}}_i})^{\epsilon_i}, -(\prod_{i=1}^{k}|\mathscr{A}^-_{\tilde{\mathscr{Y}}_i}|^{\epsilon_i}) + i(-(\prod_{i=1}^{k}|\mathscr{B}^-_{\tilde{\mathscr{Y}}_i}|^{\epsilon_i})), -(1-\prod_{i=1}^{k}(1-|\mathscr{C}^-_{\tilde{\mathscr{Y}}_i}|^m)^{\epsilon_i})^{\frac{1}{m}} + i(-(1-\prod_{i=1}^{k}(1-|\mathscr{D}^-_{\tilde{\mathscr{Y}}_i}|^m)^{\epsilon_i})^{\frac{1}{m}})\rangle =$BCn,m-ROFWA$(\tilde{\mathscr{Y}}_1, \tilde{\mathscr{Y}}_2, \ldots, \tilde{\mathscr{Y}}_k)$.

(ii) The proof is similar to (1).

**Definition 9.** Let $\mathscr{Y} = \langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}} \rangle$ be any BCn,m-ROFN, then

(i) the score function of $\mathscr{Y}$ is defined as follow:
$S(\mathscr{Y}) = \frac{1}{4}\left([(\mathscr{A}^+_{\mathscr{Y}})^n - (\mathscr{C}^+_{\mathscr{Y}})^m] + [(\mathscr{B}^+_{\mathscr{Y}})^n - (\mathscr{D}^+_{\mathscr{Y}})^m] + [|\mathscr{A}^-_{\mathscr{Y}}|^n - |\mathscr{C}^-_{\mathscr{Y}}|^m] + [|\mathscr{B}^-_{\mathscr{Y}}|^n - |\mathscr{D}^-_{\mathscr{Y}}|^m]\right)$.

(ii) the accuracy function of $\mathscr{Y}$ is defined as follow:
$\mathrm{A}(\mathscr{Y}) = \frac{1}{4}\left([(\mathscr{A}^+_{\mathscr{Y}})^n + (\mathscr{C}^+_{\mathscr{Y}})^m] + [(\mathscr{B}^+_{\mathscr{Y}})^n + (\mathscr{D}^+_{\mathscr{Y}})^m] + [|\mathscr{A}^-_{\mathscr{Y}}|^n + |\mathscr{C}^-_{\mathscr{Y}}|^m] + [|\mathscr{B}^-_{\mathscr{Y}}|^n + |\mathscr{D}^-_{\mathscr{Y}}|^m]\right)$.

**Remark 1.** For any BCn,m-ROFN $\mathscr{Y} = \langle \mathscr{A}^+_{\mathscr{Y}} + i\mathscr{B}^+_{\mathscr{Y}}, \mathscr{C}^+_{\mathscr{Y}} + i\mathscr{D}^+_{\mathscr{Y}}, \mathscr{A}^-_{\mathscr{Y}} + i\mathscr{B}^-_{\mathscr{Y}}, \mathscr{C}^-_{\mathscr{Y}} + i\mathscr{D}^-_{\mathscr{Y}} \rangle$ we have

(i) $S(\mathscr{Y}) \in [-1, 1]$.

(ii) $\mathrm{A}(\mathscr{Y}) \in [0, 1]$.

**Remark 2.** For any two BCn,m-ROFNs $\mathscr{Y}_1 = \left\langle \mathscr{A}^+_{\mathscr{Y}_1} + i\mathscr{B}^+_{\mathscr{Y}_1}, \mathscr{C}^+_{\mathscr{Y}_1} + i\mathscr{D}^+_{\mathscr{Y}_1}, \mathscr{A}^-_{\mathscr{Y}_1} + i\mathscr{B}^-_{\mathscr{Y}_1}, \mathscr{C}^-_{\mathscr{Y}_1} + i\mathscr{D}^-_{\mathscr{Y}_1} \right\rangle$ and $\mathscr{Y}_2 = \left\langle \mathscr{A}^+_{\mathscr{Y}_2} + i\mathscr{B}^+_{\mathscr{Y}_2}, \mathscr{C}^+_{\mathscr{Y}_2} + i\mathscr{D}^+_{\mathscr{Y}_2}, \mathscr{A}^-_{\mathscr{Y}_2} + i\mathscr{B}^-_{\mathscr{Y}_2}, \mathscr{C}^-_{\mathscr{Y}_2} + i\mathscr{D}^-_{\mathscr{Y}_2} \right\rangle$ the comparison technique supposed as,

(i) if $S(\mathscr{Y}_1) < S(\mathscr{Y}_2)$, then $\mathscr{Y}_1 \prec \mathscr{Y}_2$,

(ii) if $S(\mathscr{Y}_1) > S(\mathscr{Y}_2)$, then $\mathscr{Y}_1 \succ \mathscr{Y}_2$,

(iii) if $S(\mathscr{Y}_1) = S(\mathscr{Y}_2)$, then

(a) if $\mathrm{A}(\mathscr{Y}_1) < \mathrm{A}(\mathscr{Y}_2)$, then $\mathscr{Y}_1 \prec \mathscr{Y}_2$,

(b) if $\mathrm{A}(\mathscr{Y}_1) > \mathrm{A}(\mathscr{Y}_2)$, then $\mathscr{Y}_1 \succ \mathscr{Y}_2$,

(c) if $\mathrm{A}(\mathscr{Y}_1) = \mathrm{A}(\mathscr{Y}_2)$, then $\mathscr{Y}_1 \approx \mathscr{Y}_2$.

**Example 6.** Consider $\mathscr{Y}_1 = \langle 1+i(1), 0+i(0), -1+i(-1), -0+i(-0)\rangle$ $\mathscr{Y}_2 = \langle 0+i(0), 1+i(1), 0+i(0), -1+i(-1)\rangle$ are two BC3,5-ROFNs for $\mathscr{S} = \{\mathscr{T}_1\}$, then

(i) $S(\mathscr{Y}_1) = 1$ and $S(\mathscr{Y}_2) = -1$, that is, $\mathscr{Y}_2 \prec \mathscr{Y}_1$.

(ii) $A(\mathscr{Y}_1) = 1$ and $A(\mathscr{Y}_2) = 1$, that is, $\mathscr{Y}_1 \approx \mathscr{Y}_2$.

**Example 7.** Consider $\mathscr{Y}_1 = \langle 1+i(1), 0+i(0), -1+i(-1), -0+i(-0)\rangle$ $\mathscr{Y}_2 = \langle 0+i(0), 0+i(0), 0+i(0), 0+i(0)\rangle$ are two BC3,5-ROFNs for $\mathscr{S} = \{\mathscr{T}_1\}$, then

(i) $S(\mathscr{Y}_1) = 1$ and $S(\mathscr{Y}_2) = 0$,

(ii) $A(\mathscr{Y}_1) = 1$ and $A(\mathscr{Y}_2) = 0$, that is, $\mathscr{Y}_2 \prec \mathscr{Y}_1$.

## 5. Decision making on bipolar complex n,m-rung orthopair fuzzy number

In this section, the developed algorithm has been used to apply the suggested operators to a multiple-attribute decision-making problem under the BCn,m-ROF environment.

For a MADM problem, assuming that $AL = \{al_1, al_2, \ldots, al_m\}$ is any finite collection of $m$ alternatives and $AT = \{at_1, at_2, \ldots, at_k\}$ is any finite collection of $k$ attributes. Let $D = [\mathscr{Y}_{ji}] = \left[\left\langle \mathscr{A}^{+}_{\mathscr{Y}_{ji}} \mathscr{B}^{+}_{\mathscr{Y}_{ji}}, \mathscr{C}^{+}_{\mathscr{Y}_{ji}} + i\mathscr{D}^{+}_{\mathscr{Y}_{ji}}, \mathscr{A}^{-}_{\mathscr{Y}_{ji}} + i\mathscr{B}^{-}_{\mathscr{Y}_{ji}}, \mathscr{C}^{-}_{\mathscr{Y}_{ji}} + i\mathscr{D}^{-}_{\mathscr{Y}_{ji}} \right\rangle\right]_{m\times k}$ be the decision maker's specified decision matrix (DM), where $\langle \mathscr{A}^{+}_{\mathscr{Y}_{ji}} + i\mathscr{B}^{+}_{\mathscr{Y}_{ji}}, \mathscr{C}^{+}_{\mathscr{Y}_{ji}} + i\mathscr{D}^{+}_{\mathscr{Y}_{ji}}, \mathscr{A}^{-}_{\mathscr{Y}_{ji}}$ $+ i\mathscr{B}^{-}_{\mathscr{Y}_{ji}}, \mathscr{C}^{-}_{\mathscr{Y}_{ji}} + i\mathscr{D}^{-}_{\mathscr{Y}_{ji}}\rangle$ reflect the evaluation data for each alternative and are the collection of BCn,m-ROFNs $al_j (j = 1, 2, \ldots, m)$ on attribute $at_i (i = 1, 2, \ldots, k)$ (Table 1). If $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T$ is the weight vector of attribute, with $\epsilon_i > 0$, such that $\sum_{i=1}^{k} \epsilon_i = 1$. Then, the primary method (Algorithm 1) for addressing MADM issues is as follows:

**Algorithm 1**

(1) Developing the BCn,m-ROF decision matrix $D = [\mathscr{Y}_{ji}]_{m\times k}$ for MADM problem, utilizing the values of BCn,m-ROFSs.

(2) Transforming the BCn,m-ROF Decision Matrix $D = [\mathscr{Y}_{ji}]_{m\times k}$ into a normalized BCn,m-ROF Decision Matrix.

(3) To determine alternative option values with associated weights, compute the suggested operator as follows:

$Al_j = BCn, m - ROFWA(\mathscr{Y}_{j1}, \mathscr{Y}_{j2}, \ldots, \mathscr{Y}_{jk}) =$

$\langle (1-\prod_{i=1}^{k}(1-(\mathscr{A}_{\mathscr{Y}_i}^{+})^{n})^{\epsilon_i})^{\frac{1}{n}}+i(1-\prod_{i=1}^{k}(1-(\mathscr{B}_{\mathscr{Y}_i}^{+})^{n})^{\epsilon_i})^{\frac{1}{n}}, \prod_{i=1}^{k}(\mathscr{C}_{\mathscr{Y}_i}^{+})^{\epsilon_i}+i\prod_{i=1}^{k}(\mathscr{D}_{\mathscr{Y}_i}^{+})^{\epsilon_i},$
$-(\prod_{i=1}^{k}|\mathscr{A}_{\mathscr{Y}_i}^{-}|^{\epsilon_i})+i(-(\prod_{i=1}^{k}|\mathscr{B}_{\mathscr{Y}_i}^{-}|^{\epsilon_i})), -(1-\prod_{i=1}^{k}(1-|\mathscr{C}_{\mathscr{Y}_i}^{-}|^{m})^{\epsilon_i})^{\frac{1}{m}}+i(-(1-\prod_{i=1}^{k}(1-$
$|\mathscr{D}_{\mathscr{Y}_i}^{-}|^{m})^{\epsilon_i})^{\frac{1}{m}})\rangle$
or
$Al_j = BCn,m-ROFWG(\mathscr{Y}_{j1},\mathscr{Y}_{j2},\ldots,\mathscr{Y}_{jk}) =$
$\langle \prod_{i=1}^{k}(\mathscr{A}_{\mathscr{Y}_i}^{+})^{\epsilon_i}+i(\prod_{i=1}^{k}\mathscr{B}_{\mathscr{Y}_i}^{+})^{\epsilon_i}, (1-\prod_{i=1}^{k}(1-(\mathscr{C}_{\mathscr{Y}_i}^{+})^{m})^{\epsilon_i})^{\frac{1}{m}}+i(1-\prod_{i=1}^{k}(1-(\mathscr{D}_{\mathscr{Y}_i}^{+})^{m})^{\epsilon_i})^{\frac{1}{m}}, -(1-$
$\prod_{i=1}^{k}(1-|\mathscr{A}_{\mathscr{Y}_i}^{-}|^{n})^{\epsilon_i})^{\frac{1}{n}}+i(-(1-\prod_{i=1}^{k}(1-|\mathscr{B}_{\mathscr{Y}_i}^{-}|^{n})^{\epsilon_i})^{\frac{1}{n}}), -(\prod_{i=1}^{k}|\mathscr{C}_{\mathscr{Y}_i}^{-}|_i^{\epsilon_i})+i(-(\prod_{i=1}^{k}|\mathscr{D}_{\mathscr{Y}_i}^{-}|_i^{\epsilon_i}))\rangle$
for each $j=1,2,\ldots,m$.
(4) Obtain the overall Bipolar Comlex n,m-ROFN of $Al_j(j=1,2,\ldots,m)$ score values calculated in Step 3.
(5) To identify the ideal ranking order of the possibilities, consider Remark 2 to decide which is the best alternative.

To facilitate a clear understanding of the BCn,m-ROFN-based decision-making process, Figure 1 illustrates a flowchart outlining the five procedural steps. The chart provides a visual overview beginning with the development of the BCn,m-ROF decision matrix (Step 1), followed by its normalization (Step 2), computation of alternative values using the BCn,m-ROFWA or BCn,m-ROFWG operators (Step 3), calculation of the overall score values for each alternative (Step 4), and finally, ranking the alternatives to identify the optimal option (Step 5). This visual representation helps readers clearly comprehend the systematic workflow and logical sequence of operations involved in the suggested MADM model within the BCn,m-ROF framework.

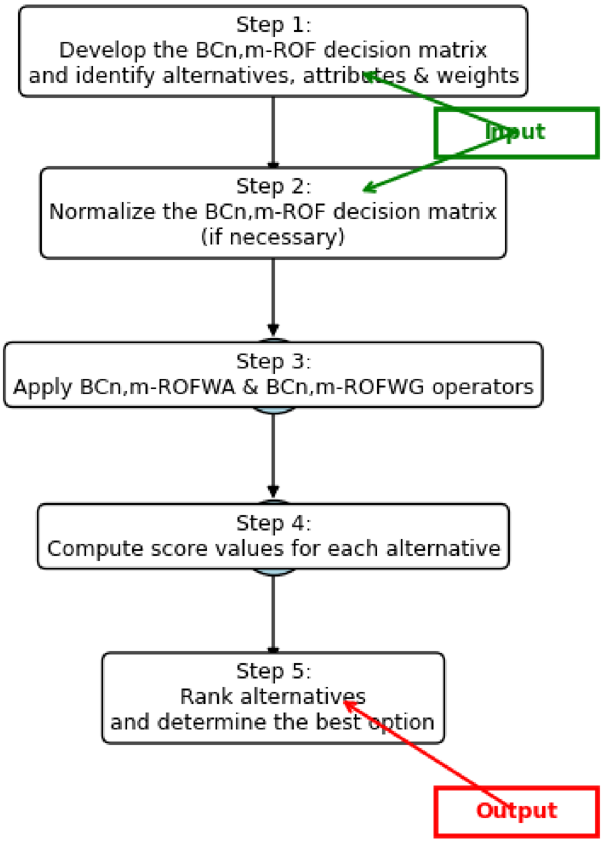


Figure 1: Illustrative flowchart of the suggested BCn,m-ROFN-based MADM procedure comprising five main steps.

### 5.1. Case study via bipolar complex n,m-rung orthopair fuzzy weighted averaging and weighted geometric operators

Iraq is experiencing a serious water crisis due to lower flows of the Tigris and Euphrates rivers (caused by upstream dam building in Turkey and Iran), climate change-induced droughts, ineffective irrigation techniques, and widespread pollution.The World Bank cautions that Iraq's availability of water per capita has decreased by 60% since 1970. According to the World Bank's "Country, Climate, and Development" study, Iraq faces losing a major amount of its freshwater resources by 2035, affecting its agriculture industry (which accounts for 20% of GDP) and displacing rural populations.

This section discusses a MADM issue to discover the ideal solutions balancing urgency, sustainability, and economic feasibility for Iraq's specific setting by examining six alterna-

tives for reducing water scarcity:
(1) Solar desalination ($al_1$),
(2) Wastewater recycling ($al_2$),
(3) Smart irrigation ($al_3$),
(4) Transboundary treaties ($al_4$),
(5) Public awareness ($al_5$),
(6) Rainwater harvesting ($al_6$).

The alternatives are evaluated upon the following five attributes:
$at_1 =$ Cost-effectiveness,
$at_2 =$Expeditious implementation
$at_3 =$Social recognition,
$at_4 =$ Geopolitical expediency,
$at_5 =$Ecological impact.

It is important to clarify that the numerical values assigned to the alternatives ($al_1$–$al_6$) and attributes ($at_1$–$at_5$) are illustrative and intended solely to demonstrate the application of the BCn,m-ROFWA and BCn,m-ROFWG operators. These values do not represent empirical measurements.

The corresponding weight vector of the attribute is $\epsilon = (0.14, 0.15, 0.20, 0.25, 0.26)^T$ respectively. The decision matrix $[\mathscr{Y}_{ji}]_{6\times5}$ is presented in Table 1, where $\mathscr{Y}_{ji}$ ($j = 1, 2, \ldots, 6$ and $i = 1, 2, \ldots, 5$) are in the form of BC3,5- ROFNs. Now we can calculate the following bipolar BC3,5- ROFN decision matrix as follow:

**Step 1.** & **Step 2.** Create the decision matrix utilizing the Bipolar complex 3,5-ROF data shown in Table 1. This step involves compiling the data provided in Table 2 into a decision matrix format that reflects the Bipolar complex 3,5-ROF structure.

**Step 3.** Implement the operators for evaluating the decision matrix:
$Al_j = \text{BC}3,5\text{-ROFWA}(\mathscr{Y}_{j1}, \mathscr{Y}_{j2}, \ldots, \mathscr{Y}_{j5})$
$Al_j = \text{BC}3,5\text{-ROFWG}(\mathscr{Y}_{j1}, \mathscr{Y}_{j2}, \ldots, \mathscr{Y}_{j5})$
for $j = 1, 2, 3, ..., 6$, utilizing weight vectors $\epsilon = (0.14, 0.15, 0.2, 0.25, 0.26)^T$ along with the parameters $n = 3$ and $m = 5$ as indicated in Table 2.

**Step 4.** Calculate the score values of $\text{BC}3,5\text{-ROFWA}(\mathscr{Y}_{j1}, \mathscr{Y}_{j2}, \ldots, \mathscr{Y}_{j5})$ and $\text{BC}3,5\text{-ROFWG}(\mathscr{Y}_{j1}, \mathscr{Y}_{j2}, \ldots, \mathscr{Y}_{j5})$ for $j = 1, 2, ..., 6$, as shown in Table 3.

**Step 5.** Determine the ranking of all the options based on the score values obtained in Step 4 by utilizing Definition 9, as shown in Table 4. The outcomes of the options' ranking determined by the BC3,5-ROFWA operator are presented as follows: $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$
while the outcomes of the options' ranking determined by the BC3,5-ROFWG operator are presented as follows: $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$
Consequently, $al_3$ is identified as the best option based on BC3,5-ROFWA operator and BC3,5-ROFWG operator.

The ranking outcomes from both methods clearly highlight $al_3$ as the most effective smart irrigation practice tailored to Iraq's conditions, among the six evaluated alternatives aimed at mitigating water scarcity. These results are visually depicted in Figure 2, based on the application of the BC3,5-ROFWA and BC3,5-ROFWG operators.

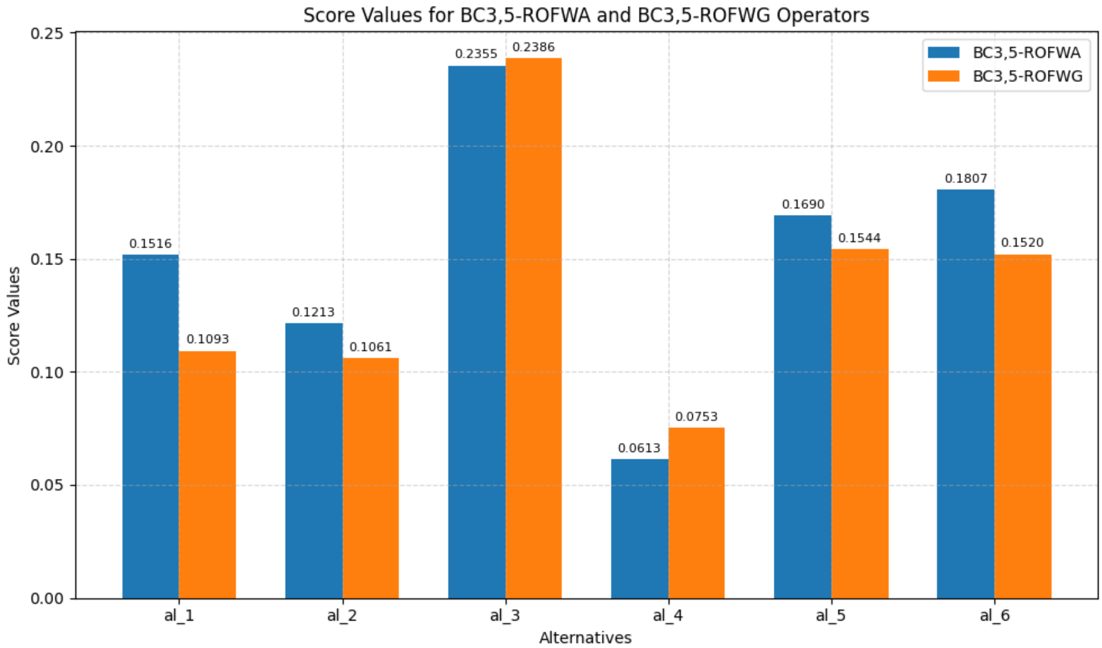


Figure 2: Score values of BC3,5-ROFWA and BC3,5-ROFWG operators.

Table 1: BC3,5-ROFS values

| | |
|---|---|
| | $at_1$ |
| $al_1$ | $\langle 0.8+i(0.7), 0.2+i(0.3), -0.6+i(-0.7), -0.7+i(-0.6)\rangle$ |
| $al_2$ | $\langle 0.4+i(0.3), 0.7+i(0.4), -0.5+i(-0.4), -0.6+i(-0.5)\rangle$ |
| $al_3$ | $\langle 0.6+i(0.6), 0.2+i(0.3), -0.6+i(-0.7), -0.4+i(-0.5)\rangle$ |
| $al_4$ | $\langle 0.5+i(0.5), 0.4+i(0.5), -0.5+i(-0.5), -0.4+i(-0.5)\rangle$ |
| $al_5$ | $\langle 0.5+i(0.5), 0.4+i(0.5), -0.5+i(-0.5), -0.4+i(-0.5)\rangle$ |
| $al_6$ | $\langle 0.9+i(0.5), 0.3+i(0.3), -0.5+i(-0.5), -0.3+i(-0.6)\rangle$ |
| | $at_2$ |
| $al_1$ | $\langle 0.9+i(0.6), 0.1+i(0.4), -0.7+i(-0.5), -0.5+i(-0.4)\rangle$ |
| $al_2$ | $\langle 0.3+i(0.5), 0.6+i(0.4), -0.4+i(-0.4), -0.7+i(-0.3)\rangle$ |
| $al_3$ | $\langle 0.7+i(0.3), 0.3+i(0.4), -0.7+i(-0.5), -0.4+i(-0.6)\rangle$ |
| $al_4$ | $\langle 0.4+i(0.3), 0.5+i(0.4), -0.6+i(-0.5), -0.4+i(-0.3)\rangle$ |
| $al_5$ | $\langle 0.4+i(0.3), 0.5+i(0.4), -0.6+i(-0.5), -0.4+i(-0.3)\rangle$ |
| $al_6$ | $\langle 0.7+i(0.6), 0.5+i(0.4), -0.6+i(-0.3), -0.4+i(-0.3)\rangle$ |
| | $at_3$ |
| $al_1$ | $\langle 0.2+i(0.3), 0.6+i(0.7), -0.1+i(-0.3), -0.5+i(-0.5)\rangle$ |
| $al_2$ | $\langle 0.6+i(0.4), 0.4+i(0.3), -0.7+i(-0.3), -0.6+i(-0.5)\rangle$ |
| $al_3$ | $\langle 0.4+i(0.3), 0.6+i(0.7), -0.3+i(-0.3), -0.4+i(-0.5)\rangle$ |
| $al_4$ | $\langle 0.2+i(0.1), 0.6+i(0.4), -0.3+i(-0.5), -0.6+i(-0.6)\rangle$ |
| $al_5$ | $\langle 0.5+i(0.3), 0.6+i(0.4), -0.8+i(-0.4), -0.3+i(-0.3)\rangle$ |
| $al_6$ | $\langle 0.6+i(0.4), 0.5+i(0.3), -0.7+i(-0.5), -0.3+i(-0.3)\rangle$ |
| | $at_4$ |
| $al_1$ | $\langle 0.7+i(0.5), 0.2+i(0.2), -0.7+i(-0.5), -0.3+i(-0.3)\rangle$ |
| $al_2$ | $\langle 0.7+i(0.6), 0.5+i(0.3), -0.8+i(-0.5), -0.4+i(-0.3)\rangle$ |
| $al_3$ | $\langle 0.8+i(0.5), 0.1+i(0.2), -0.9+i(-0.6), -0.3+i(-0.3)\rangle$ |
| $al_4$ | $\langle 0.3+i(0.1), 0.7+i(0.6), -0.8+i(-0.5), -0.3+i(-0.3)\rangle$ |
| $al_5$ | $\langle 0.8+i(0.6), 0.3+i(0.4), -0.8+i(-0.5), -0.4+i(-0.6)\rangle$ |
| $al_6$ | $\langle 0.7+i(0.4), 0.4+i(0.4), -0.9+i(-0.5), -0.3+i(-0.5)\rangle$ |
| | $at_5$ |
| $al_1$ | $\langle 0.1+i(0.1), 0.6+i(0.4), -0.8+i(-0.5), -0.4+i(-0.3)\rangle$ |
| $al_2$ | $\langle 0.1+i(0.1), 0.6+i(0.4), -0.8+i(-0.5), -0.4+i(-0.3)\rangle$ |
| $al_3$ | $\langle 0.9+i(0.6), 0.2+i(0.4), -0.8+i(-0.5), -0.2+i(-0.3)\rangle$ |
| $al_4$ | $\langle 0.1+i(0.2), 0.7+i(0.4), -0.7+i(-0.6), -0.5+i(-0.3)\rangle$ |
| $al_5$ | $\langle 0.4+i(0.5), 0.6+i(0.4), -0.7+i(-0.4), -0.4+i(-0.6)\rangle$ |
| $al_6$ | $\langle 0.2+i(0.3), 0.7+i(0.5), -0.6+i(-0.3), -0.3+i(-0.4)\rangle$ |

Table 2: Aggregated BC3,5-ROF information matrix

| | BC3,5-ROFWA |
|---|---|
| $al_1$ | $\langle 0.6923 + i(0.5010), 0.2988 + i(0.3613), -0.3483 + i(-0.4732), -0.5236 + i(-0.4585)\rangle$ |
| $al_2$ | $\langle 0.5361 + i(0.4539), 0.5401 + i(0.3514), -0.6573 + i(-0.4232), -0.5670 + i(-0.4151)\rangle$ |
| $al_3$ | $\langle 0.7758 + i(0.5078), 0.2226 + i(0.3613), -0.6375 + i(-0.4953), -0.3560 + i(-0.4738)\rangle$ |
| $al_4$ | $\langle 0.3321 + i(0.2925), 0.5967 + i(0.4567), -0.5695 + i(-0.5243), -0.4879 + i(-0.4661)\rangle$ |
| $al_5$ | $\langle 0.6061 + i(0.4884), 0.4638 + i(0.4127), -0.6929 + i(-0.4512), -0.3870 + i(-0.5390)\rangle$ |
| $al_6$ | $\langle 0.6916 + i(0.4447), 0.4805 + i(0.3844), -0.6675 + i(-0.4055), -0.3246 + i(-0.4684)\rangle$ |
| | BC3,5ROFGA |
| $al_1$ | $\langle 0.3476 + i(0.3200), 0.5163 + i(0.5200), -0.6855 + i(-0.5209), -0.4353 + i(-0.3823)\rangle$ |
| $al_2$ | $\langle 0.3332 + i(0.3066), 0.5817 + i(0.3678), -0.7245 + i(-0.4458), -0.4993 + i(-0.3569)\rangle$ |
| $al_3$ | $\langle 0.6761 + i(0.4498), 0.4403 + i(0.5269), -0.7717 + i(-0.5481), -0.31109 + i(-0.3960)\rangle$ |
| $al_4$ | $\langle 0.2332 + i(0.1769), 0.6428 + i(0.4975), -0.6655 + i(-0.5309), -0.4278 + i(-0.3702)\rangle$ |
| $al_5$ | $\langle 0.5132 + i(0.4376), 0.5332 + i(0.4209), -0.7279 + i(-0.4602), -0.3776 + i(-0.4589)\rangle$ |
| $al_6$ | $\langle 0.5076 + i(0.4070), 0.5697 + i(0.4203), -0.7433 + i(-0.4415), -0.3132 + i(-0.4048)\rangle$ |

Table 3: Score values

| | $S(\text{BC3, 5-ROFWA})$ | $S(\text{BC3, 5-ROFWG})$ |
|---|---|---|
| $al_1$ | 0.1516 | 0.1093 |
| $al_2$ | 0.1213 | 0.1061 |
| $al_3$ | 0.2355 | 0.2386 |
| $al_4$ | 0.0613 | 0.0753 |
| $al_5$ | 0.1690 | 0.1544 |
| $al_6$ | 0.1807 | 0.1520 |

Table 4: Rankings for proposed application

| **Models** | **Ranking order** | **Best solution** |
|---|---|---|
| BC3,5-ROFWA | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC3,5-ROFWG | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |

### 5.2. Evaluation of parameter influence

This subsection analyzes how the parameters $n$ and $m$ affect the performance and output behavior of the suggested BCn,m-ROF aggregation operators. These two parameters are fundamental in determining the operators' degree of flexibility and compensatory capability during aggregation. Generally, smaller values of $n$ and $m$ correspond to a more conservative and non-compensatory aggregation behavior, whereas larger values permit higher levels of compensation between the membership and nonmembership components. To investigate these effects, a series of experiments were carried out by varying $n$ and $m$ in the BCn,m-ROFWA and BCn,m-ROFWG operators. The corresponding results are summarized in Tables 5 and 6. In this analysis, we employed several different combinations of $n$ and $m$ values to examine how the score values and the corresponding rankings of alternatives evolve with varying parameter settings. This broad range provides a comprehensive sensitivity assessment that spans from highly restrictive to strongly compensatory aggregation behavior.

As observed in Table 5, when $n$ and $m$ increase, the score values produced by the BCn,m-ROFWA operator gradually decline. This decrease indicates that higher $n, m$ values lessen the relative influence of large membership degrees within the aggregation process. A consistent trend is evident in Table 6 for the BCn,m-ROFWG operator, where the scores also diminish steadily as $n$ and $m$ increase, eventually stabilizing into smoother and less dispersed values. Such monotonic behavior confirms the operators' consistent response to parameter variations—although the magnitude of scores changes, the ranking order of alternatives remains intact. A noteworthy observation is that the optimal alternative ($al_3$) remains the best choice across all tested $n$ and $m$ combinations. This invariance signifies the isotonic stability of the proposed aggregation operators, implying that the relative variations in numerical scores do not affect the final decision outcome. Consequently, decision-makers can flexibly adjust $n$ and $m$ according to their desired decision attitude—whether cautious or optimistic—without compromising the robustness of the final ranking.

The graphical illustrations in Figures 3 and 4 visually depict these trends, showing that while the numerical distances among alternatives gradually shrink with increasing $n$ and $m$, the ranking order remains unchanged. These visual results reinforce the numerical findings and provide intuitive evidence of the operators' adaptability and consistency under different parameter configurations. In summary, the sensitivity analysis validates the reliability and resilience of both BCn,m-ROFWA and BCn,m-ROFWG operators. The stable ranking of alternatives under a wide spectrum of parameter values demonstrates their capability to produce consistent and interpretable outcomes, making them suitable for multi-attribute decision-making scenarios characterized by uncertainty and varying decision preferences.

Table 5: Sensitivity analysis results for the BCn,m-ROFWA operator

| $n, m$ | Score values of | | | | | | Option |
|---|---|---|---|---|---|---|---|
| | $al_1$ | $al_2$ | $al_3$ | $al_4$ | $al_5$ | $al_6$ | |
| 2,4 | 0.2457 | 0.2078 | 0.3514 | 0.1245 | 0.2704 | 0.2804 | $al_3$ |
| 4,2 | -0.0473 | -0.1220 | -0.0477 | -0.1861 | -0.0767 | -0.0382 | $al_3$ |
| 6,7 | 0.0502 | 0.0255 | 0.0887 | 0.0035 | 0.0484 | 0.0608 | $al_3$ |
| 7,6 | 0.0332 | 0.0055 | 0.0659 | -0.0121 | 0.0266 | 0.0403 | $al_3$ |
| 9,12 | 0.0267 | 0.0087 | 0.0451 | 0.0017 | 0.0189 | 0.0279 | $al_3$ |
| 12,9 | 0.0139 | -0.0008 | 0.0261 | -0.0034 | 0.0059 | 0.0139 | $al_3$ |
| 15,20 | 0.0102 | 0.0008 | 0.0172 | 0.0001 | 0.0033 | 0.0090 | $al_3$ |
| 20,15 | 0.0051 | -0.0001 | 0.0090 | -0.0002 | 0.0008 | 0.0046 | $al_3$ |
| 21,22 | 0.0047 | 0.0001 | 0.0080 | -0.0000 | 0.0007 | 0.0041 | $al_3$ |
| 22,21 | 0.0041 | 0.0000 | 0.0071 | -0.0000 | 0.0005 | 0.0037 | $al_3$ |

Table 6: Sensitivity analysis results for the BCn,m-ROFWG operator

| $n, m$ | Score values of | | | | | | Option |
|---|---|---|---|---|---|---|---|
| | $al_1$ | $al_2$ | $al_3$ | $al_4$ | $al_5$ | $al_6$ | |
| 2,4 | 0.1877 | 0.1769 | 0.3499 | 0.1293 | 0.2542 | 0.2468 | $al_3$ |
| 4,2 | -0.1005 | -0.1192 | -0.0363 | -0.1619 | -0.0860 | -0.0627 | $al_3$ |
| 6,7 | 0.0307 | 0.0359 | 0.0922 | 0.0182 | 0.0442 | 0.0529 | $al_3$ |
| 7,6 | 0.0137 | 0.0184 | 0.0659 | -0.0007 | 0.0249 | 0.0352 | $al_3$ |
| 9,12 | 0.0143 | 0.0192 | 0.0476 | 0.0108 | 0.0194 | 0.0316 | $al_3$ |
| 12,9 | 0.0031 | 0.0065 | 0.0250 | -0.0011 | 0.0073 | 0.0178 | $al_3$ |
| 15,20 | 0.0029 | 0.0047 | 0.0172 | 0.0025 | 0.0043 | 0.0142 | $al_3$ |
| 20,15 | 0.0006 | 0.0013 | 0.0086 | 0.0001 | 0.0013 | 0.0077 | $al_3$ |
| 21,22 | 0.0007 | 0.0012 | 0.0078 | 0.0006 | 0.0011 | 0.0071 | $al_3$ |
| 22,21 | 0.0005 | 0.0009 | 0.0069 | 0.0004 | 0.0009 | 0.0064 | $al_3$ |

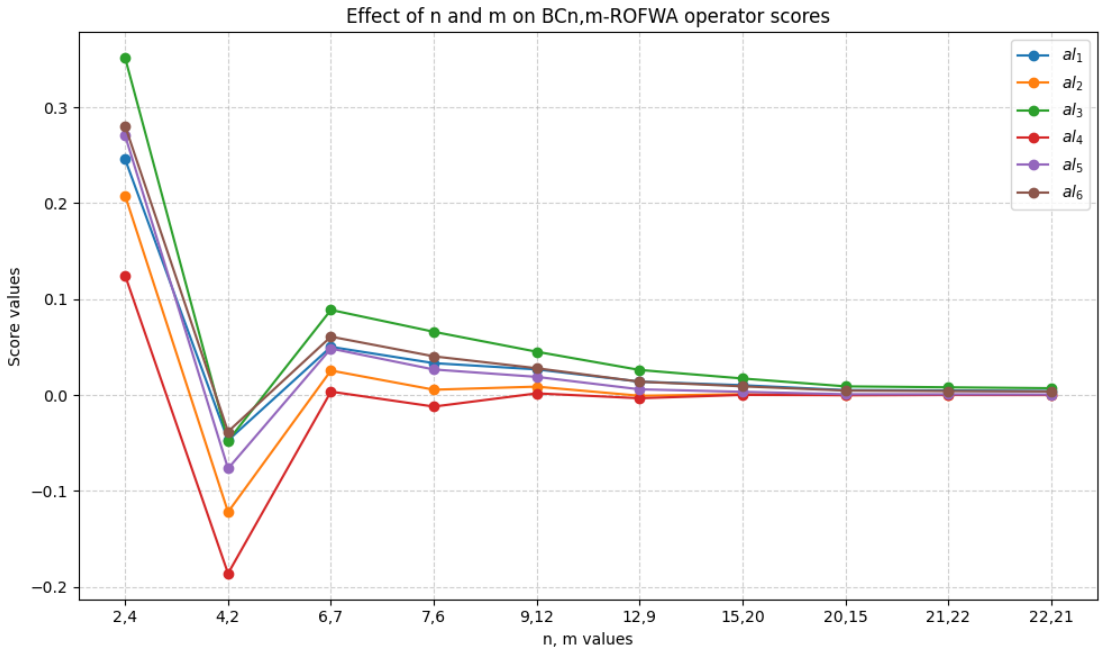


Figure 3: Stability of rankings under varying parameters $n$ and $m$ for the BCn,m-ROFWA

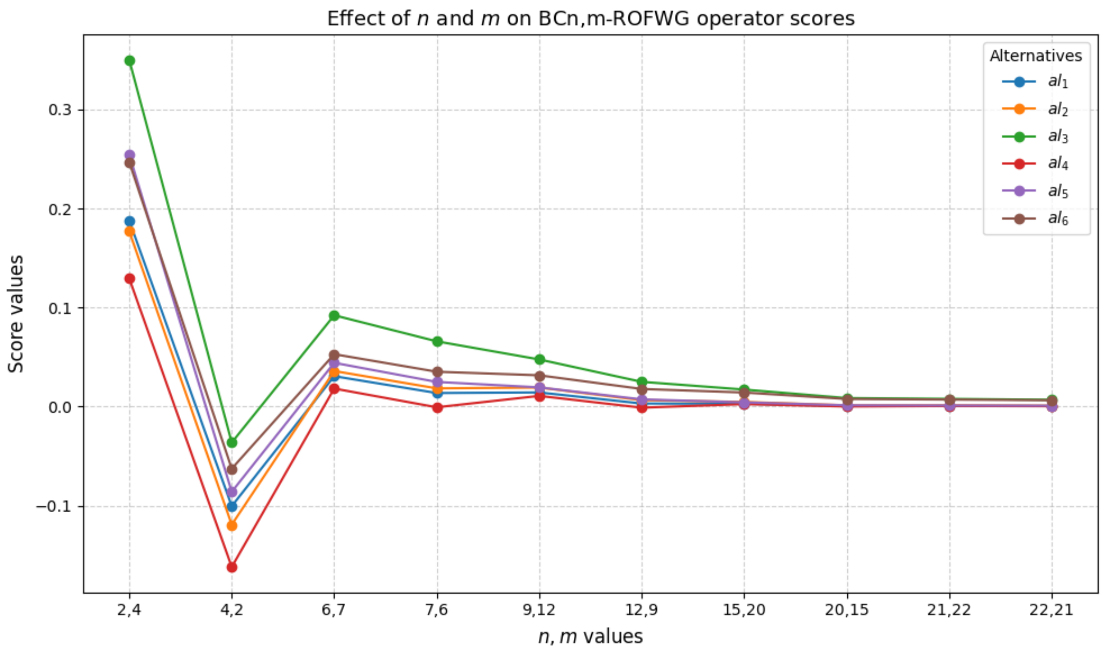


Figure 4: Stability of rankings under varying parameters $n$ and $m$ for the BCn,m-ROFWG

## 6. Comparison

To demonstrate the effectiveness of the suggested models, a comprehensive comparison is performed against various existing aggregation approaches. This evaluation is essential for validating both the accuracy and practical utility of the suggested models using internal and application-specific data.

The case study in Section 5.1 is initially analyzed using standard aggregation operators, including bipolar Pythagorean fuzzy weighted average (BPFWA), bipolar Pythagorean fuzzy weighted geometric (BPFWG), bipolar fuzzy weighted average (BFWA), bipolar fuzzy weighted geometric (BFWG), Pythagorean fuzzy weighted average (PFWA), Pythagorean fuzzy weighted geometric (PFWG), Fermatean fuzzy weighted average (FFWA), Fermatean fuzzy weighted geometric (FFWG), q-rung orthopair fuzzy weighted average (q-ROFWA), and q-rung orthopair fuzzy weighted geometric (q-ROFWG).

Subsequently, the current score function is applied alongside more advanced operators, including bipolar complex fuzzy credibility weighted average (BCFCWA), bipolar complex fuzzy credibility weighted geometric (BCFCWG), bipolar complex fuzzy weighted averaging (BCFWAA), bipolar complex fuzzy weighted geometric (BCFWGA), bipolar n,m-rung orthopair fuzzy weighted average (Bn,m-ROFWA), and bipolar n,m-rung orthopair fuzzy weighted geometric (Bn,m-ROFWG), as summarized in Table 7. In addition, the newly introduced operators—BC2,3-ROFWA, BC2,3-ROFWG, BC3,2-ROFWA, BC3,2-ROFWG, BC2,5-ROFWA, BC2,5-ROFWG, BC5,2-ROFWA, and BC5,2-ROFWG—are also employed to provide a more extensive assessment of model performance.

It is worth noting that certain operators, such as IFWA, IFWG, BCIFWA, and BCIFWG, are incompatible with the application described in Section 5.1. The incompatibility arises because the combined membership and nonmembership degrees in some alternatives exceed the permissible limit of 1. This limitation highlights that these operators are not always suitable for complex-valued bipolar fuzzy environments, restricting their practical applicability.

Across all applicable methods, alternative $al_3$ consistently emerges as the most effective solution for water conservation in Iraq, while $al_4$ remains the least favorable. The rankings of $al_1, al_2, al_5$, and $al_6$ vary slightly depending on the scoring function and operator used, emphasizing the benefits of the proposed framework in providing consistent and reliable decision support. For clearer interpretation, Figure 5 visually illustrates the performance differences among the aggregation operators discussed above.

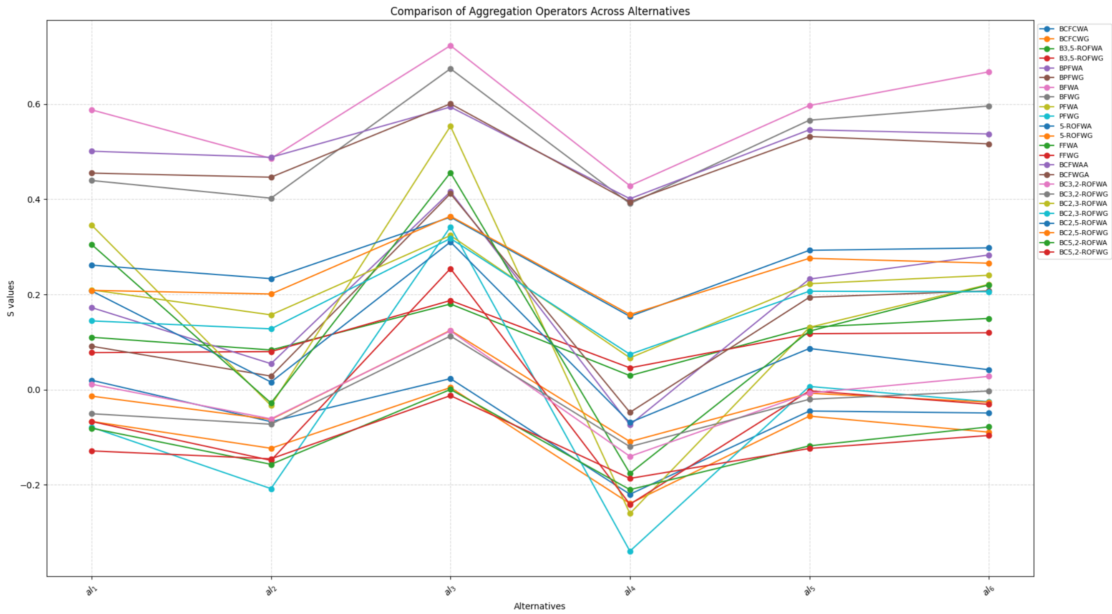


Figure 5: Results of the comparative analysis.

Table 7: Comparison for suggested application

| | $S(al_1)$, $S(al_2)$, $S(al_3)$, $S(al_4)$, $S(al_5)$, $S(al_6)$ | Ranking | best solution |
|---|---|---|---|
| BCFCWA [31] | 0.0198, -0.0672, 0.0232, -0.2197, -0.0449, -0.0490 | $al_3 \succ al_1 \succ al_5 \succ al_6 \succ al_2 \succ al_4$ | $al_3$ |
| BCFCWG [31] | -0.0667, -0.1231, 0.0045, -0.2392, -0.0556, -0.0892 | $al_3 \succ al_5 \succ al_1 \succ al_6 \succ al_2 \succ al_4$ | $al_3$ |
| B3,5-ROFWA [16] | 0.1101, 0.0834, 0.1799, 0.0295, 0.1313, 0.1496 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| B3,5-ROFWG [16] | 0.0779, 0.0799, 0.1873, 0.0458, 0.1175, 0.1196 | $al_3 \succ al_6 \succ al_5 \succ al_2 \succ al_1 \succ al_4$ | $al_3$ |
| BPFWA [34] | 0.1725, 0.0546, 0.4158, -0.0739, 0.2323, 0.2829 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BPFWG [34] | 0.0913, 0.0285, 0.4118, -0.0477, 0.1941, 0.2076 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BFWA [37] | 0.5878, 0.4858, 0.7228, 0.4284, 0.5969, 0.6676 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BFWG [37] | 0.4393, 0.4022, 0.6742, 0.3908, 0.566, 0.5958 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| IFWA [38] | —, —, —, —, —, — | Faild | |
| IFWG [39] | —, —, —, —, —, — | Faild | |
| PFWA [40] | 0.3459, -0.0334, 0.5534, -0.2602, 0.1309, 0.2212 | $al_3 \succ al_1 \succ al_6 \succ al_5 \succ al_2 \succ al_4$ | $al_3$ |
| PFWG [40] | -0.0788, -0.2079, 0.3421, -0.3392, 0.0068, -0.0253 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| 5-ROFWA [41] | 0.2075, 0.0160, 0.3103, -0.0690, 0.0865, 0.0419 | $al_3 \succ al_1 \succ al_6 \succ al_5 \succ al_2 \succ al_4$ | $al_3$ |
| 5-ROFWG [41] | -0.0136, -0.0625, 0.1247, -0.1091, -0.0075, -0.0263 | $al_3 \succ al_5 \succ al_1 \succ al_6 \succ al_2 \succ al_4$ | $al_3$ |
| FFWA [42] | 0.3051, -0.0279, 0.4559, -0.1758, 0.1229, 0.2199 | $al_3 \succ al_1 \succ al_6 \succ al_5 \succ al_2 \succ al_4$ | $al_3$ |
| FFWG [42] | -0.0668, -0.1485, 0.2544, -0.2407, -0.0025, -0.0308 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BCIFWA [19] | —, —, —, —, —, — | Faild | |
| BCIFWG [19] | —, —, —, —, —, — | Faild | |
| BCFWAA [43] | 0.5009, 0.4881, 0.5937, 0.4010, 0.5459, 0.5371 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BCFWGA [43] | 0.4549, 0.4463, 0.6005, 0.3945, 0.5317, 0.5163 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC3,2-ROFWA | 0.0115, -0.0613, 0.1235, -0.1401, -0.0070, 0.0279 | $al_3 \succ al_6 \succ al_1 \succ al_5 \succ al_2 \succ al_4$ | $al_3$ |
| BC3,2-ROFWG | -0.0505, -0.0725, 0.1127, -0.1197, -0.0202, -0.0029 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC2,3-ROFWA | 0.2096, 0.1571, 0.3239, 0.0668, 0.2226, 0.2403 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC2,3-ROFWG | 0.1445, 0.1277, 0.3176, 0.0743, 0.2068, 0.2058 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC2,5-ROFWA | 0.2616, 0.2332, 0.3624, 0.1539, 0.2927, 0.2979 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC2,5-ROFWG | 0.2086, 0.2009, 0.3644, 0.1578, 0.2759, 0.2657 | $al_3 \succ al_5 \succ al_6 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |
| BC5,2-ROFWA | -0.0809, -0.1568, 0.0004, -0.2100, -0.1180, -0.0782 | $al_3 \succ al_6 \succ al_1 \succ al_5 \succ al_2 \succ al_4$ | $al_3$ |
| BC5,2-ROFWG | -0.1286, -0.1452, -0.0125, -0.1864, -0.1235, -0.0962 | $al_3 \succ al_6 \succ al_5 \succ al_1 \succ al_2 \succ al_4$ | $al_3$ |

## 7. Sensitivity analysis and limitations of aggregation operators

This section provides a comprehensive examination of the sensitivity of the proposed BCn,m-ROFWA and BCn,m-ROFWG aggregation operators. We assess their robustness under varying parameter settings and discuss inherent limitations, highlighting aspects that warrant careful consideration in practical MADM applications.

### 7.1. Sensitivity evaluation of aggregation parameters

To evaluate the responsiveness of the BCn,m-ROFWA and BCn,m-ROFWG operators, the parameters $n$ and $m$ were systematically varied across a broad spectrum. These parameters directly control the compensatory behavior of the aggregation process: smaller values of $n$ and $m$ yield more restrictive, non-compensatory aggregation, while larger values allow greater compensation between membership and nonmembership degrees.

Table 8 shows the resulting scores for a representative set of alternatives across different $n$ and $m$ settings. The analysis reveals a general decreasing trend in the score values as $n$ and $m$ increase. Despite the reduction in absolute scores, the ranking of alternatives remains stable, with $al_3$ consistently emerging as the top-ranked alternative. This indicates the isotonic stability of the proposed operators, allowing decision-makers to adjust $n$ and $m$ to control aggregation intensity without affecting the final ranking.

Furthermore, the decreasing score trend highlights the sensitivity of the operators to extreme parameter values. Very large $n$ and $m$ produce scores approaching zero for all alternatives, potentially reducing discriminative power. Moderate values, however, maintain sufficient variation among alternatives, ensuring meaningful differentiation and interpretable outcomes. Figure 6 visually illustrates how variations in $n$ and $m$ influence aggregated results, confirming the trends observed in Table 8.

### 7.2. Limitations of the proposed aggregation operators

While the BCn,m-ROFWA and BCn,m-ROFWG operators offer a flexible and robust framework for MADM, the sensitivity analysis highlights several limitations:

(i) Convergence at extreme $n$ and $m$ values: Very large values of $n$ and $m$ lead to near-zero scores for all alternatives, reducing the ability to effectively distinguish between options. Decision-makers should avoid extreme settings to preserve meaningful differentiation.

(ii) Optimal performance at moderate $n$ and $m$: The operators provide the highest discriminative power when $n$ and $m$ are chosen within a moderate range. Such values allow the model to reflect nuanced differences among alternatives, producing accurate and reliable rankings.

(iii) Computational and interpretational challenges: As the number of alternatives or the magnitude of $n$ and $m$ increases, the computational cost rises due to more complex aggregation operations. Interpreting results for large sets of alternatives may require additional strategies, such as weighting schemes or dimensionality reduction, to maintain clarity and usability.

In conclusion, the sensitivity assessment confirms that the BCn,m-ROFWA and BCn,m-ROFWG operators are stable and robust across a wide range of practical $n$ and $m$ values, with rankings remaining consistent despite parameter variation. Future work may focus on enhancing discriminative capability under extreme parameter settings and extending the framework to efficiently handle large-scale, high-dimensional decision-making problems.

Table 8: Effect of $n$ and $m$ on aggregation results

| $n, m$ | Score values of | | | | | | Operator |
|---|---|---|---|---|---|---|---|
| | $al_1$ | $al_2$ | $al_3$ | $al_4$ | $al_5$ | $al_6$ | |
| 13,16 | 0.0135 | 0.0017 | 0.0229 | 0.0001 | 0.0057 | 0.0124 | BCn,m-ROFWA |
| | 0.0048 | 0.0074 | 0.0232 | 0.0038 | 0.0070 | 0.0181 | BCn,m-ROFWG |
| 30,45 | 0.0017 | 0.0000 | 0.0029 | 0.0000 | 0.0001 | 0.0015 | BCn,m-ROFWA |
| | 0.0001 | 0.0002 | 0.0028 | 0.0001 | 0.0001 | 0.0027 | BCn,m-ROFWG |
| 50,55 | 0.0002 | 0.0000 | 0.0003 | -0.0000 | 0.0000 | 0.0001 | BCn,m-ROFWA |
| | 0.0000 | 0.0000 | 0.0003 | 0.0000 | 0.0000 | 0.0003 | BCn,m-ROFWG |
| 60,70 | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | BCn,m-ROFWA |
| | 0.0000 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0001 | BCn,m-ROFWG |
| 70,60 | 0.0000 | -0.0000 | 0.0000 | -0.0000 | 0.0000 | 0.0000 | BCn,m-ROFWA |
| | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | BCn,m-ROFWG |
| 90,100 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | BCn,m-ROFWA |
| | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | BCn,m-ROFWG |

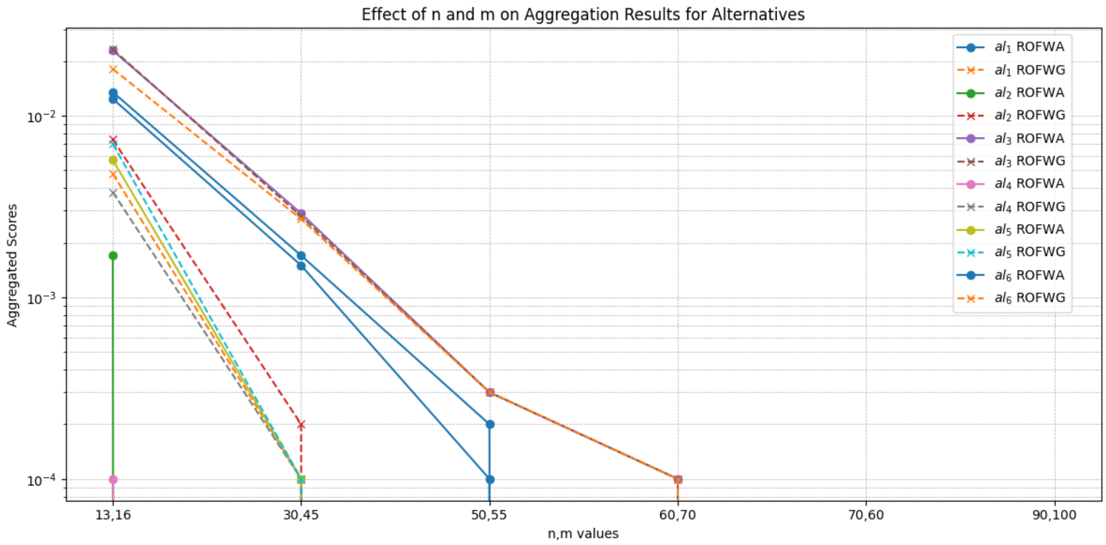


Figure 6: Aggregated score trends of alternatives under varying $n$ and $m$ values based on Table 8

## 8. Conclusions

This study presented a comprehensive MADM framework based on the BCn,m-ROFS, designed to handle complex, uncertain, and dual (positive-negative) evaluations in decision-making. The work established the rigorous mathematical foundations of BCn,m-ROFS, demonstrating its theoretical consistency and supporting the development of two novel aggregation operators: the BCn,m-ROF weighted averaging (BCn,m-ROFWA) and weighted geometric (BCn,m-ROFWG) operators. These operators are specifically formulated to capture both bipolar evaluations and complex-valued uncertainties, while offering flexible compensatory behavior through the $n$ and $m$ parameters. The practical applicability of the proposed framework was demonstrated through a MADM case study in smart water resource management, where several alternative strategies were evaluated across multiple environmental, economic, and social dimensions. Results consistently identified the most balanced and effective alternative, highlighting the operators' capability to integrate positive and negative perspectives across multiple criteria. Comparative analyses with existing aggregation approaches confirmed that BCn,m-ROFWA and BCn,m-ROFWG improve ranking stability, discriminative power, and interpretability, thereby validating the robustness and adaptability of the BCn,m-ROFS framework.

Key contributions of this work include its unified and generalized modeling capability, allowing the proposed operators to reduce to existing fuzzy set-based aggregation methods—such as IFS, PFS, FFS, BPFS, and BCFS—under specific parameter configurations. By incorporating bipolarity, complex-valued membership degrees, and the flexible $n$-$m$ structure, the framework provides a more realistic and comprehensive representation of conflicting, oscillatory, and dual-imaginary information compared to conventional approaches. Sensitivity analyses further confirmed that the rankings of alternatives remain robust under varying $n$ and $m$ values, while extreme parameter settings highlight practical considerations regarding operator discriminative capacity and computational efficiency. Despite these strengths, several limitations remain. The framework was tested on a single application domain, and the sensitivity analysis relied on expert-assigned weights and static attribute evaluations. In large-scale problems with numerous alternatives or criteria, computational requirements may increase, necessitating efficient implementation strategies.

Future research may explore the development of additional aggregation operators, extensions to dynamic or time-varying decision scenarios, applications in domains such as risk assessment or healthcare, and the integration of BCn,m-ROFS with machine learning techniques to further enhance decision-making performance in complex, high-stakes environments.

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