EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6761
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Bipolar-Valued Fuzzy Subgroups, Normal Subgroups, and Homomorphisms on Dib's Fuzzy Space

Fadi Al-Zu'bi[1,2,*], Abd Ghafur Ahmad[1], Abd Ulazeez Alkouri[3], Maslina Darus[1], Sadeq Damrah[4]

[1] *Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia*
[2] *College of Natural and Health Sciences, Zayed University, Abu Dhabi, United Arab Emirates*
[3] *Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun-26810, Jordan*
[4] *Department of Mathematics and Physics, College of Engineering, Australian University, West Mishref, Safat 13015, Kuwait*

Abstract. Fuzzy group theory has evolved beyond single-valued memberships to account for dual polarity and uncertainty. Building on Dib's fuzzy space and bipolar-valued fuzzy sets, we develop a unified algebraic theory of bipolar-valued fuzzy (BVF) subgroups, including BVF normal subgroups and BVF homomorphisms, via a BVF binary operation (BVFBO) on a BVF-space. We establish necessary and sufficient subgroup criteria, characterize normality through coset symmetry in BVF-space, and prove homomorphism properties that align BVF structures with their classical counterparts through correspondence theorems. The framework clarifies when associativity holds between subgroup elements and ambient BVF-group elements and provides constructive examples. This generalization resolves limitations tied to the absence of a bipolar fuzzy universal set and supports applications in polarity-sensitive decision systems and network analysis.
2020 Mathematics Subject Classifications: 30C45
Key Words and Phrases: Fuzzy group, fuzzy space, bipolar-valued fuzzy space, bipolar-valued fuzzy subgroup, bipolar-valued fuzzy homomorphisms, bipolar-valued fuzzy normal subgroup, Dib fuzzy group theory

---

1. Introduction

The theory of fuzzy groups, first formulated by Rosenfeld [1], marked a foundational shift in algebraic systems by incorporating uncertainty into group membership. His seminal work defined fuzzy subgroups using a single-valued membership function ranging

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6761

Email addresses: P115916@siswa.ukm.edu.my (F. Al-Zu'bi), ghafur@ukm.edu.my (A. Ahmad), alkouriabdulazeez@anu.edu.jo (A. Alkouri), maslina@ukm.edu.my (M. Darus), s.damrah@au.edu.kw (S. Damrah)

within the interval $[0,1]$, laying the groundwork for fuzzy algebraic structures. This concept was later refined by Anthony and Sherwood [2], who introduced triangular norms to achieve more flexible and expressive formulations.

A major breakthrough came from Dib [3, 4], who proposed the concept of a fuzzy space (F-space) to replace the classical universal set. Dib's framework redefined fuzzy groups through fuzzy binary operations acting on elements of this F-space, effectively overcoming limitations of earlier models. His fuzzy topological space enabled a more coherent foundation for defining fuzzy group structures and operations.

In parallel, Salleh [5] initiated work on fuzzy homomorphisms, followed by the development of intuitionistic fuzzy groups by Marashdeh and Salleh [6], and their extension to intuitionistic fuzzy normal subgroups [7]. These models added nuance by capturing degrees of hesitation in membership but did not fully accommodate negative evaluations inherent in many real-world systems.

The formal introduction of bipolar-valued fuzzy sets (BVFS) by Lee [8], and later expanded by Lee, Lee, and Cios [9], enabled the representation of both negative and positive membership grades via a Cartesian domain $[-1,0]\times[0,1]$. This duality provided the necessary framework to model conflicting or opposing characteristics in algebraic structures.

Expanding these ideas, recent work by Al-Zu'bi et al. [10, 11] introduced bipolar-valued fuzzy groups (BVF-groups) and formalized bipolar-valued fuzzy Cartesian products, relations, and functions. These advancements, built on Dib's algebraic philosophy, laid the groundwork for a more robust bipolar-valued fuzzy universe that supports richer algebraic properties.

Contemporary research highlights a growing demand for such dual-polarity fuzzy systems. For instance, Akram et al. [12] proposed multi-criteria decision-making models based on BVF sets. Al-Quran et al. [13] applied cubic bipolar fuzzy sets in VIKOR and ELECTRE-II algorithms to optimize logistics in Industry 4.0. Gutiérrez et al. [14] explored BVF measures for community detection in enriched social networks. Alqahtani et al. [15] advanced hesitant BVF intuitionistic fuzzy graphs for social media analysis, while Mahmood et al. [16] demonstrated applications of BVF soft sets in pattern recognition and healthcare.

Further generalizations include bipolar fuzzy subgroups [17], BCK/BCI-algebras [18, 19], q-fuzzy and interval-valued bipolar subgroups [20, 21], fuzzy subsemirings [22], BCH-algebras [23], and γ-semigroups under BVF frameworks [24]. New structural models such as m-polar ideals [25], k-folded n-structures [26], and BVF soft sets [27] continue to expand the field's boundaries. Notably, Massa'deh et al. [28] have introduced concepts such as anti-homomorphisms and BVF multi-fuzzy subgroups, bridging theory with complex functional systems.

In this paper, we extend these developments by proposing a complete theory of bipolar-valued fuzzy subgroups (BVF-subgroups), and further advancing the structures of BVF-normal subgroups and BVF-homomorphisms. Our approach preserves classical group axioms while adapting them to the dual-valued BVF context.

Our contributions are threefold:

(i) We define and characterize BVF-subgroups using the BVF-space and bipolar-valued fuzzy binary operation (BVFBO).

(ii) We formulate BVF-normal subgroups and BVF-homomorphisms, demonstrating consistency with classical and intuitionistic subgroup theory.

(iii) We establish algebraic theorems that validate associativity, structural generalization, and compatibility with fuzzy group frameworks.

This expanded BVF-subgroup theory is not only mathematically rigorous but also aligns with complex real-world contexts that involve positive and negative evaluations. Applications span control systems influenced by conflicting dynamics, decision-making scenarios with trade-offs, and medical diagnosis where symptoms have both enhancing and deteriorating effects [12, 16, 27].

Our contribution expands on recent studies that introduced BVF groups and bipolar-valued fuzzy Cartesian relations, providing a comprehensive algebraic system. We prove that not every BVF-subgroup is associative and establish theorems connecting BVF-subgroups to their fuzzy and intuitionistic counterparts. The proposed generalization has practical relevance for applications in decision-making, intelligent systems, and control theory, where positive and negative valuations must coexist in algebraic reasoning.

The remainder of this paper presents a comprehensive theoretical formulation of BVF-subgroups. We begin with the necessary background, proceed with formal definitions and proofs, and conclude with theoretical discussions and implications that bridge abstract fuzzy logic and applicable mathematical systems.

1.1. Related Work

The evolution of bipolar-valued fuzzy algebraic structures has been shaped by multiple foundational studies. Anitha et al. [17] first introduced the concept of bipolar fuzzy subgroups, laying the groundwork for understanding subgroup properties within a dual-valued logic framework. Building on this, Saeid [18] and Lee [19] expanded bipolar-valued fuzzy concepts to BCK/BCI-algebras, enabling algebraic reasoning in more general logical structures. Similarly, Balasubramanian et al. [20] examined bipolar interval-valued fuzzy subgroups, which accommodate uncertainty in both positive and negative evaluations.

Further structural generalizations have been introduced by Shanmugapriya and Arjunan [22] who investigated bipolar fuzzy subsemirings, and Jun and Song [23] who applied bipolar fuzzy sets to the closed ideals and subalgebras of BCH-algebras. These efforts collectively demonstrate the expanding utility of bipolar-valued logic in algebraic systems. The notion of q-fuzziness was also extended into the bipolar domain by Sahaya et al. [21], who defined bipolar-valued q-fuzzy subgroups, providing additional flexibility and generalization.

From an application standpoint, recent studies underscore the versatility of bipolar fuzzy systems. Al-Masarwah et al. [25] introduced m-polar fuzzy ideals in BCK-algebras, while also developing k-folded n-structures in semigroups [26], enriching the algebraic foundation for multipolar decision systems. Alqaraleh et al. [27] proposed applications of bipolar complex fuzzy soft sets, particularly in decision-making contexts, while Massa'deh et al. [28] investigated homomorphisms and anti-homomorphisms in multi fuzzy HX-subgroups, thereby extending functional operations in bipolar fuzzy algebra.

Additional interdisciplinary contributions offer further support for the practical reach of fuzzy systems. Manavalan et al. [29, 30] and Damrah et al. [31, 32] apply neutrosophic and fuzzy-set extensions to real-life decision-making, cybersecurity, and epidemiological modeling. Firouzkouhi et al. [33] employed generalized fuzzy hypergraphs for link prediction and influencer detection in dynamic social networks, demonstrating the power of fuzzy logic in complex relational environments. Similarly, Damrah et al. [34, 35] explored fuzzy mathematical modeling in energy-efficient drilling and well design validation, illustrating how fuzzy algebra can inform industrial and environmental challenges.

Further generalizations include a new structure of hesitant fuzzy relations by Talafha et al. [36],an investigation into bipolar fuzzy hoop algebras and their applications [37], and axiomatic analysis of state operators in Sheffer stroke BCK-algebras associated with algorithmic approaches [38]. Our BVF framework generalizes Rosenfeld's fuzzy subgroups via dual polarity and BVF-space semantics; we explain when they coincide (via correspondence) and when BVF adds expressive power.

These contributions collectively demonstrate a vibrant and evolving field that bridges theory and application. The current study builds upon these insights by advancing a unified framework for BVF-subgroups, BVF-normal subgroups, and BVF-homomorphisms, offering both structural clarity and practical applicability across complex, polarity-sensitive domains.

1.2. Comparative Overview of Previous Studies and Present Work

The following Table 1 summarizes the key distinctions and innovations of the present study compared to earlier works in the field of bipolar-valued fuzzy algebra.

Table 1: Comparison of the Current Study with Previous Works

Study	**Main Contribution**	**Fuzzy Structure Type**	**Mathematical Focus**	**Advancement over Previous Work**
Rosenfeld (1971) [1]	Introduced fuzzy subgroups	Fuzzy set	Group theory	Laid the foundational concept of fuzzy groups
Dib (1994)[3]; Dib and Youssef (1991) [4]	Proposed fuzzy space (F-space), fuzzy binary operations	Fuzzy space	Algebraic topology, structure	Reframed fuzzy groups on a topological space
Lee (2000, 2001) [8, 9]	Formalized bipolar-valued fuzzy sets	Bipolar-valued fuzzy set	BVFS structure, comparison	Expanded fuzzy membership to $[-1,0] \times [0,1]$
Anitha et al. (2013) [17]	Studied BVF-subgroups structurally	Bipolar-valued fuzzy set	Subgroup theory	First attempt to apply BVF to subgroup structure
Current Study	Develops a complete theory for BVF-subgroups, normal subgroups, and homomorphisms using BVFBO on BVF-space.	Bipolar-valued fuzzy space	Algebraic structure, homomorphism, generalization	Unified, topological, and dual-valued group theory in one framework

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions required in the sequel.

Definition 1 ([9]). *An* intuitionistic fuzzy set A *in a universe* $\mho$ *is defined by a membership function* $\mu_A : \mho \to [0,1]$ *and a non-membership function* $\nu_A : \mho \to [0,1]$ *such that*

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \quad \textit{for all } x \in \mho.$$

Definition 2 ([8]). *A* bipolar-valued fuzzy set A *in a universe* $\mho$ *is defined by a mem-*

bership function $\mu_A : \mho \to [-1,1]$*, which can represent positive and negative membership degrees:*

- $\mu_A(x) > 0$*: The extent to which* x *is positively part of* A*.*
- $\mu_A(x) < 0$*: The degree to which* x *is negatively part of* A*.*

Remark 1 ([8])**.** ***(BFS vs. BVFS).*** *In a bipolar fuzzy set (BFS) one typically models two components* $\mu^-(x), \mu^+(x) \in [0,1]$ *describing the degree to which* x *satisfies a property and its counter-property.*

In a bipolar-valued fuzzy set (BVFS) a single membership map $\mu(x) \in [-1,1]$ *carries both directions: positive values indicate positive membership and negative values indicate negative membership.*

Our BVF-space adopts the latter representation (with explicit positive/negative channels when convenient) and the BVFBO acts compatibly on $[-1,0] \times [0,1]$*.*

Definition 3 ([22])**.** *A bipolar-valued fuzzy set* A *is called a* bipolar-valued fuzzy subsemigroup *in a semigroup* S *if*

$$\mu_A(x \cdot y) \geq \min\big(\mu_A(x), \mu_A(y)\big) \quad \text{for all } x, y \in S.$$

This guarantees that the characteristics of the subsemigroup are maintained within the fuzzy framework.

Definition 4 ([1])**.** *Rosenfeld expanded the idea of fuzzy sets to group theory through the introduction of fuzzy subgroups. A fuzzy subset* A *in group* G *is termed a fuzzy subgroup when:*

(i) $\mu_A(x \cdot y) \geq \min(\mu_A(x), \mu_A(y)) \quad$ *for all* $x, y \in G$,

(ii) $\mu_A(e) = 1 \quad$ *where* e *is the identity element of* G,

(iii) $\mu_A(x^{-1}) = \mu_A(x) \quad$ *for all* $x \in G$.

These conditions ensure that the fuzziness respects the group structure.

Definition 5 ([4])**.** *A fuzzy relation* R *between sets* $\mho$ *and* Y *is a fuzzy set in the Cartesian product* $X \times Y$ *with a membership function* $\mu_R : \mho \times Y \to [0,1]$*.*

Definition 6 ([4])**.** *A fuzzy function from a fuzzy set* A *in* $\mho$ *to a fuzzy set* B *in* Y *is a function* $f : \mho \to Y$ *such that the membership value of* $f(x)$ *in* B *is related to the membership value of* x *in* A*.*

Definition 7 ([3])**.** *A F-space* $(\mho, I = [0,1])$ *is the set of all ordered pairs* $(x, I), x \in \mho$, $(\mho, I) = \{(x, I) : x \in \mho\}$ *where* $(x, I) = \{(x, r) : r \in I\}$*. The ordered pair* (x, I) *is called a fuzzy element in the F-space* $(\mho, I)$*.*

Definition 8 ([3]). *A fuzzy group $((\mho, I), \underline{F})$ is called a commutative or abelian fuzzy group if $(x, I)\,\underline{F}\,(y, I) = (y, I)\,\underline{F}\,(x, I)$, for all fuzzy elements (x, I) and (y, I) of the F-space $(\mho, I)$. It is clear that $((\mho, I), \underline{F})$ is a commutative fuzzy group iff $(\mho, F)$ is an ordinary commutative group.*

Definition 9 ([6]). *An intuitionistic fuzzy binary operation (IFBO) F on an intuitionistic fuzzy space (IF-space) $(\mho, I, I)$ is an intuitionistic fuzzy function $F : (\mho, I, I) \times (\mho, I, I) \to (\mho, I, I)$ with comembership functions $\underline{f}_{xy}$ and cononmembership functions $\overline{f}_{xy}$ satisfying:*

(1) $\underline{f}_{xy}(r, s) \neq 0$ iff $r \neq 0, s \neq 0$ and $\overline{f}_{xy}(w, z) \neq 1$ iff $w \neq 1, z \neq 1$.

(2) $\underline{f}_{xy}$ and $\overline{f}_{xy}$ are onto. That is, $\underline{f}_{xy}(I \times I) = I$ and $\overline{f}_{xy}(I \times I) = I$, where $I = [0, 1]$.

Thus, the intuitionistic fuzzy binary operation $F = (F, \underline{f}_{xy}, \overline{f}_{xy})$ over the IF-space $\mho$ is defined by:

(3) $(x, I, I)F(y, I, I) = F((x, I, I), (y, I, I)) = (F(x, y), \underline{f}_{xy}(I \times I), \overline{f}_{xy}(I \times I)) = (F(x, y), I, I)$
where $(x, I, I), (y, I, I)$ of the IF-space X are intuitionistic fuzzy elements (IF-element), and $F = (F, \underline{f}_{xy}, \overline{f}_{xy})$ is any IFBO defined on an IF-space $\mho$.

An IFBO is identified to be uniform if both $\underline{f}_{xy}$ and $\overline{f}_{xy}$ are identical. That is, $\underline{f}_{xy} = \overline{f}_{xy} = f$ for all $x, y \in V$. A left uniform (right uniform) IFBO is IFBO having identical comembership functions (co-nonmembership functions).

Definition 10 ([6]). *The structure of $((G, I, I), F)$, where IF-space G and $I = [0, 1]$, with IFBO F defined on IF-space G, is called an intuitionistic fuzzy group (IFG) if the following conditions are fulfilled:*

(1) For any IF-element $(x, I, I), (y, I, I), (z, I, I) \in ((G, I, I), F)$,

$$((x, I, I)F(y, I, I))F(z, I, I) = (x, I, I)F((y, I, I)F(z, I, I)).$$

(2) There exists an IF-element $(e, I, I) \in (G, I, I)$ such that for all (x, I, I) in $((G, I, I), F)$:

$$(e, I, I)F(x, I, I) = (x, I, I)F(e, I, I) = (x, I, I).$$

(3) For every IF-element (x, I, I) in $((G, I, I), F)$ there exists an IF-element (x^{-1}, I, I) in $(G, I, I), F$ such that:

$$(x, I, I)F(x^{-1}, I, I) = (x^{-1}, I, I)F(x, I, I) = (e, I, I).$$

An IFG $((G, I, I), F)$ is called an abelian IFG iff for all $(x, I, I), (y, I, I) \in ((G, I, I), F)$,

$$(x, I, I)F(y, I, I) = (y, I, I)F(x, I, I) \text{ is true.}$$

Definition 11 ([11]). *The BVFCP of two ordinary sets U and V, denoted by $U\overline{\times}V$, is the collection of all K-BVF subsets of $U \times V$ that is*

$$U\overline{\times}V = K^{(U\times V)},$$

An element of $U\overline{\times}V$ is then a function $M : U \times V \to K$, or

$$M = \{((u,v),[(\delta^-,\delta^+),(\vartheta^-,\vartheta^+)]) : (u,v) \in U \times V, [(\delta^-,\delta^+),(\vartheta^-,\vartheta^+)] = M(u,v) \to K\}.$$

The BVFCP of a BVF subset $H = \{(u,(\delta^-,\delta^+))\}$ of U and a BVF subset $T = \{(v,(\vartheta^-,\vartheta^+))\}$ of V is the K-BVF subset $H\underline{\times}T$ of $U \times V$ defined by:

$$H\underline{\times}T = \{((u,v),((H^-(u),H^+(u)),(T^-(v),T^+(v))) : u \in U, v \in V\} \equiv \{((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+)))\}.$$

Therefore, $H\underline{\times}T$ is an element of $U\overline{\times}V$, $\forall H \in W^U$ and $\forall T \in W^V$.

Definition 12 ([11]). *A BVFR β maps U to V is a subset of the BVFCP $U\overline{\times}V$. In other words, β is a member of K-BVF subsets $M : U \times V \to K$. A BVFR from U to U is said to be a BVFR in U.*

Definition 13 ([11]). *Let β_1 and $\beta_2 : U \to V$ to V be two BVFRs. We call that β_2 is containing β_1, denoted by $\beta_1 \subset \beta_2$, if and only if when*

$$((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+))) \in H \in \beta_1,$$

there exists $B \in \beta_2$ such that

$$((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+))) \in T \in \beta_2.$$

If $\beta_1 \subset \beta_2$ and $\beta_2 \subset \beta_1$, then β_1 and β_2 are equal, that is $\beta_1 = \beta_2$.

Definition 14 ([11]). *Let $\beta : U \to V$ be a BVFR. The inverse of $\beta = \beta^{-1} : V \to U$ is the BVFR defined by*

$$\beta^{-1} = \{M^{-1} : M \in \beta\}.$$

Definition 15 ([11]). *Let $\beta : U \to V$ and $\gamma : V \to Z$ be two BVFRs. The composition of β and γ, denoted $\gamma \circ \beta : U \to Z$, is a BVFR defined by*

$$\gamma \circ \beta = \ ((u,z),((\delta^-,\delta^+),(\alpha^-,\alpha^+))) \in M : M \in U\overline{\times}Z.$$

Where a K-BVF subset $M \in U\overline{\times}Z$ is defined by:

$$((u,z),((\delta^-,\delta^+),(\alpha^-,\alpha^+))) \in M$$

if and only if

$$\exists(v,(\vartheta^-,\vartheta^+)) \in V \times W \text{ such that}$$
$$((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+))) \in A$$

and

$$((v,z),((\vartheta^-,\vartheta^+),(\alpha^-,\alpha^+))) \in B$$

for some β and $B \in \gamma$.

Definition 16 ([11]). *Let β be a BVFR in U, i.e., $\beta \subset U \overline{\times} U$. Then:*

(i) β is called reflexive in U if and only if $\forall u \in U$ and $\forall(\delta^-, \delta^+) \in W$, $\exists H \in \beta$ such that

$$((u,u),((\delta^-,\delta^+),(\delta^-,\delta^+))) \in H \in \beta,$$

that is, if and only if $\Delta_U \subset \beta$.

(ii) β is called symmetric if and only if whenever

$$((u,v),((\delta^-,\delta^+),(n^-,n^+))) \in H \in \beta,$$

$\exists H \in \rho$ such that

$$((v,u),((n^-,n^+),(\delta^-,\delta^+))) \in T \in \beta,$$

that is, if and only if $\beta^{-1} = \beta$.

(iii) β is called transitive if and only if whenever

$$((u,v),((\delta^-,\delta^+),(\vartheta^-,\vartheta^+))) \in H \in \beta$$

and

$$((v,z),((\vartheta^-,\vartheta^+),(\alpha^-,\alpha^+))) \in T \in \beta,$$

$\exists C \in \beta$ such that

$$((u,z),((\delta^-,\delta^+),(\alpha^-,\alpha^+))) \in C \in \beta,$$

that is, if and only if $\beta \circ \beta \subset \beta$.

A BVFR in U is called a BVFER in U if and only if it satisfies all three axioms above.

Definition 17 ([11]). *Let U and V be nonempty sets. A BVF function from U to V can be described as a function F from W^U to W^V characterized by the ordered pair*

$$(F, \{(f_u(\delta^-), f_u(\delta^+))\}_{u \in U}),$$

where $F : U \to V$ is a function from U to V and $\{(f_u(\delta^-), f_u(\delta^+))\}_{u \in U}$ is a family of functions

$$(f_u(\delta^-), f_u(\delta^+)) : W \to W$$

that satisfy the following conditions:

i. $f_u(\delta^-)$, $f_u(\delta^+)$ are nondecreasing on W, and

ii. $f_u(\delta^- = 0) = 0 = f_u(\delta^+ = 0)$, $f_u(\delta^- = -1) = -1$, and $f_u(\delta^+ = 1) = 1$.

Definition 18 ([10]). *An bipolar valued fuzzy binary operation F on a BVF-space $(\mho, [-1,0], [0,1])$ is a bipolar valued fuzzy function*

$$F : (\mho, [-1,0], [0,1]) \times (\mho, [-1,0], [0,1]) \to (\mho, [-1,0], [0,1])$$

with negative comembership functions f^-_{xy} and positive comembership functions f^+_{xy} satisfying:

(1)

$$f_{xy}^{-}(n^{-}, m^{-}) \neq 0 \iff n^{-} \neq 0, \quad m^{-} \neq 0$$

$$f_{xy}^{-}(w^{-}, z^{-}) \neq -1 \iff w^{-} \neq -1, \quad z^{-} \neq -1$$

$$f_{xy}^{+}(n^{+}, m^{+}) \neq 0 \iff n^{+} \neq 0, \quad m^{+} \neq 0$$

$$f_{xy}^{+}(w^{+}, z^{+}) \neq 1 \iff w^{+} \neq 1, \quad z^{+} \neq 1$$

(2) *f_{xy}^{-}, f_{xy}^{+} are onto. That is, $f_{xy}^{-}([-1,0] \times [-1,0]) = [-1,0]$ and $f_{xy}^{+}([0,1] \times [0,1]) = [0,1]$.*

Thus, for any two BVF-elements $(x, [-1,0], [0,1])$, $(y, [-1,0], [0,1])$ of the BVF-space $\mho$ and any BVFBO $F = (F, f_{xy}^{-}, f_{xy}^{+})$ defined on $\mho$, the action of the BVFBO F over $\mho$ is given by

$$\begin{aligned}(x, -I, I)\, F\, (y, -I, I) &= F((x, [-1,0], [0,1]), (y, [-1,0], [0,1])) \\ &= (F(x,y), f_{xy}^{-}([-1,0] \times [-1,0]), f_{xy}^{+}([0,1] \times [0,1])) \\ &\quad (F(x,y), [-1,0], [0,1]).\end{aligned}$$

Definition 19 ([10])**.** ***(Classical BVF group).*** *A BVF-group $((G, [-1,0], [0,1]), F)$ consists of a set G and a BVF binary operation F on the BVF-space such that:*

(i) associativity holds in the BVF sense;

(ii) there exists a BVF identity;

(iii) each BVF element has a BVF inverse.

This aligns with the classical group axioms under the correspondence between BVF-elements and their positive/negative components.

Definition 20 ([10])**.** *For all BVF-elements have an inverse, a bipolar valued fuzzy monoid is called a bipolar valued fuzzy group. Equivalently, a bipolar valued fuzzy groupoid $(G, [-1,0], [0,1], F)$ is a BVF-group iff the following restrictions hold:*

(1) For any BVF-elements

$$(x, [-1,0], [0,1]), (y, [-1,0], [0,1]), (z, [-1,0], [0,1]) \in (G, [-1,0], [0,1], F):$$

$$\begin{aligned}&((x, [-1,0], [0,1])F(y, [-1,0], [0,1]))F(z, [-1,0], [0,1]) \\ &= (x, [-1,0], [0,1])F((y, [-1,0], [0,1])F(z, [-1,0], [0,1])).\end{aligned}$$

(2) There exists a BVF-element

$$(e, [-1, 0], [0, 1]) \in (G, [-1, 0], [0, 1])$$

such that

$$for\ all\ (x, [-1, 0], [0, 1]) in (G, [-1, 0], [0, 1], F):$$

$$(e, [-1, 0], [0, 1])F(x, [-1, 0], [0, 1]) = (x, [-1, 0], [0, 1])F(e, [-1, 0], [0, 1]) = (x, [-1, 0], [0, 1]).$$

(3) For every BVF-element

$$(x, [-1, 0], [0, 1]) in (G, [-1, 0], [0, 1], F),$$

there exists a BVF-element

$$(x^{-1}, [-1, 0], [0, 1]) in (G, [-1, 0], [0, 1], F)$$

such that:

$$(x, [-1, 0], [0, 1])F(x^{-1}, [-1, 0], [0, 1])$$
$$= (x^{-1}, [-1, 0], [0, 1])F(x, [-1, 0], [0, 1]) = (e, [-1, 0], [0, 1]).$$

A BVF-group $((G, [-1, 0], [0, 1]), F)$ is named an abelian BVF-group if and only if for all $(x, [-1, 0], [0, 1])$, $(y, [-1, 0], [0, 1]) \in ((G, [-1, 0], [0, 1]), F)$,

$$(x, [-1, 0], [0, 1])F(y, [-1, 0], [0, 1]) = (y, [-1, 0], [0, 1])F(x, [-1, 0], [0, 1]).$$

Identical to the bipolar valued fuzzy groupoid, the following theorem establishes a relationship between BVF-groups and both ordinary and fuzzy groups.

Theorem 1 ([10])**.** *(1) Associated to each bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$ where $F = (F, f^-_{xy}, f^+_{xy})$ a fuzzy group $((G, [0, 1]), \bar{F})$ where $\bar{F} = (F, f^+_{xy})$ which is isomorphic to the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$ by the correspondence $(x, [-1, 0], [0, 1]) \leftrightarrow (x, [0, 1])$.*

(2) There is an associated (ordinary) group (G, F) to any bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$ that is isomorphic to the bipolar valued fuzzy group via the corresponding $(x, [-1, 0], [0, 1]) \leftrightarrow x$.

Corollary 1 ([10])**.** *Let $(\mho, [-1, 0], [0, 1])$ be an BVF-space and let $F = (F, f^-_{xy}, f^+_{xy})$ be an bipolar valued fuzzy binary operation defined over $(\mho, [-1, 0], [0, 1])$. The algebraic structure $((\mho, [-1, 0], [0, 1]), F)$ defines an BVF-group iff $((\mho, [0, 1]), \underline{F})$ and $((\mho, [0, 1]), \bar{F})$ are both fuzzy groups, where $\underline{F} = (F, f^+_{xy})$ and $\bar{F} = (F, |f^-_{xy}|)$.*

Theorem 2 ([10])**.** *For any BVF-group $((G, [-1, 0], [0, 1]), F)$, the next statements are true:*

- *The identity of element of BVF-group is unique.*
- *The inverse of each BVF-element* $(x, [-1, 0], [0, 1]) \in ((G, [-1, 0], [0, 1]), F)$ *is unique.*
- $((x^{-1})^{-1}, [-1, 0], [0, 1]) = (x, [-1, 0], [0, 1])$
- *For all* $(x, [-1, 0], [0, 1]), (y, [-1, 0], [0, 1]) \in ((G, [-1, 0], [0, 1]), F)$*:*

$$((x, [-1, 0], [0, 1])F(y, [-1, 0], [0, 1]))^{-1} = (y^{-1}, [-1, 0], [0, 1])F(x^{-1}, [-1, 0], [0, 1]).$$

- *For all* $(x, [-1, 0], [0, 1]), (y, [-1, 0], [0, 1]), (z, [-1, 0], [0, 1]) \in ((G, [-1, 0], [0, 1]), F)$*:*
  If $(x, [-1, 0], [0, 1])F(y, [-1, 0], [0, 1]) = (z, [-1, 0], [0, 1])F(y, [-1, 0], [0, 1])$*, then* $(x, [-1, 0], [0, 1]) = (z, [-1, 0], [0, 1])$.
  If $(y, [-1, 0], [0, 1])F(x, [-1, 0], [0, 1]) = (y, [-1, 0], [0, 1])F(z, [-1, 0], [0, 1])$*, then* $(x, [-1, 0], [0, 1]) = (z, [-1, 0], [0, 1])$.

3. BIPOLAR VALUED FUZZY SUBGROUPS

In this part, the bipolar valued fuzzy subgroup is introduced and related results are studied. Also, the bipolar valued fuzzy subgroups are induced by bipolar valued fuzzy subsets and then a relationship between bipolar valued fuzzy subgroups and classical fuzzy subgroups in terms of induction is demonstrated.

Definition 21. *Let* S *be a bipolar valued fuzzy subspace of the bipolar valued fuzzy space* $(G, [-1, 0], [0, 1])$*. The ordered pair* $(S; F)$ *is called a bipolar valued fuzzy subgroup of the bipolar valued fuzzy group* $(G, [-1, 0], [0, 1]), F)$*, denoted by*

$$(S; F) \leq (G, [-1, 0], [0, 1]), F),$$

if $(S; F)$ *states a bipolar valued fuzzy group under the bipolar valued fuzzy binary operation* F*.*

Clearly, if $(S; F)$ *is a bipolar valued fuzzy subgroup of* $(G, [-1, 0], [0, 1]), F)$ *and* $(N; F)$ *is a bipolar valued fuzzy subgroup of* $(S; F)$*, then* $(N; F)$ *is a bipolar valued fuzzy subgroup of* $(G, [-1, 0], [0, 1]), F)$*. Also, if* $(G, [-1, 0], [0, 1]), F)$ *is a BVF group with a BVF identity* $(e, [-1, 0], [0, 1])$*, then both* $\{(e, [-1, 0], [0, 1])\}$ *and* $(G, [-1, 0], [0, 1]), F)$ *are trivial bipolar valued fuzzy subgroups of* $(G, [-1, 0], [0, 1]), F)$*.*

The next theorem explains exactly when a bipolar-valued fuzzy subgroup exists, giving both necessary and sufficient conditions.

Theorem 3. *Let* $S = \{(x, s_x^-, s_x^+) : x \in S_\circ\}$ *be a bipolar valued fuzzy subspace of the bipolar valued fuzzy space* $(G, [-1, 0], [0, 1])$*. Then* $(S; F)$ *is a bipolar valued fuzzy subgroup of the BVF Group* $((G, [-1, 0], [0, 1]), F)$ *if and only if:*

(1) $(S_\circ; F)$ *is an ordinary subgroup of the group* (G, F)*,*

(2)

$$f^-_{xy}(s^-_x, s^-_y) = s^-_x f^-_{xy} s^-_y = s_x F y^-, \textit{and} \quad f^+_{xy}(s^+_x, s^+_y) = s^+_x f^+_{xy} s^+_y = s_x F y^+ \tag{1}$$

for all $x, y \in S_\circ$.

Proof. *Suppose (1) and (2) are satisfied, then:*

(i) (Closeness condition) The bipolar valued fuzzy subspace S *is closed under* F*: Let* $(x, s^-_x, s^+_x), (y, s^-_y, s^+_y)$ *be in* S *then:*

$$\begin{aligned}(x, s^-_x, s^+_x)F(y, s^-_y, s^+_y) &= F((x, s^-_x, s^+_x), (y, s^-_y, s^+_y)) \\ &= (F(x,y), f^-_{xy}(s^-_x, s^-_y), f^+_{xy}(s^+_x, s^+_y)) \\ &= (xFy, s_xFy^-, s_xFy^+) \in S.\end{aligned} \tag{2}$$

(ii) $(S; F)$ *is itself a BVF group:*

a) (Associative condition) Let $(x, s^-_x, s^+_x), (y, s^-_y, s^+_y), (z, s^-_z, s^+_z)$ *be in* S *then:*

$$\begin{aligned}&\big((x, s^-_x, s^+_x)F(y, s^-_y, s^+_y)\big)F(z, s^-_z, s^+_z) \\ &= (F(x,y), f^-_{xy}(s^-_x, s^-_y), f^+_{xy}(s^+_x, s^+_y))F(z, s^-_z, s^+_z) \\ &= ((xFy)Fz, s^-_{(xFy)Fz}, s^+_{(xFy)Fz}) \\ &= (xF(yFz), s^-_{xF(yFz)}, s^+_{xF(yFz)}) \\ &= (x, s^-_x, s^+_x)F\big((y, s^-_y, s^+_y)F(z, s^-_z, s^+_z)\big).\end{aligned} \tag{3}$$

b) (Identity condition) Since $(S_\circ; F)$ *is an (ordinary) subgroup of the group* (G, F) *then* $S_\circ$ *contains the identity* e*. That is,* $(e, s^-_e, s^+_e) \in S$ *thus:*

$$\begin{aligned}&(x, s^-_x, s^+_x)F(e, s^-_e, s^+_e) \\ &= (F(x,e), f^-_{xe}(s^-_x, s^-_e), f^+_{xe}(s^+_x, s^+_e)) \\ &= (xFe, s^-_{xFe}, s^+_{xFe}) \\ &= (eFx, s^-_{eFx}, s^+_{eFx}) \\ &= (e, s^-_e, s^+_e)F(x, s^-_x, s^+_x) \\ &= (x, s^-_x, s^+_x).\end{aligned} \tag{4}$$

In the same way, $(e, s^-_e, s^+_e)F(x, s^-_x, s^+_x) = (x, s^-_x, s^+_x)$.

c) (Inverse condition) Since $(S_\circ; F)$ *is an (ordinary) subgroup of the group* (G, F) *then* $S_\circ$ *contains the inverse element* x^{-1} *for each* $x \in S_\circ$*, (i.e.* $\forall (x, s^-_x, s^+_x) \in S$, $\exists (x^{-1}, s^-_{x^{-1}}, s^+_{x^{-1}}) \in S$*) then:*

$$\begin{aligned}
&(x, s_x^-, s_x^+)F(x^{-1}, s_{x^{-1}}^-, s_{x^{-1}}^+) = (F(x, x^{-1}), f_{xx^{-1}}^-(s_x^-, s_{x^{-1}}^-), f_{xx^{-1}}^+(s_x^+, s_{x^{-1}}^+))\\
&= (xFx^{-1}, s_{xFx^{-1}}^-, s_{xFx^{-1}}^+)\\
&= (x^{-1}Fx, s_{x^{-1}Fx}^-, s_{x^{-1}Fx}^+)\\
&= (x^{-1}, s_{x^{-1}}^-, s_{x^{-1}}^+)F(x, s_x^-, s_x^+)\\
&= (e, s_e^-, s_e^+).
\end{aligned} \tag{5}$$

In the same way, $(x^{-1}, s_{x^{-1}}^-, s_{x^{-1}}^+)F(x, s_x^-, s_x^+) = (e, s_e^-, s_e^+)$.

Therefore, we determine that $(S; F)$ *is a BVF subgroup of* $((G, [-1, 0], [0, 1]), F)$ *by (i) and (ii). Conversely, if* $(S; F)$ *is a BVF subgroup of* $((G, [-1, 0], [0, 1]), F)$ *then (1) holds by the associativity theorem. Also, (2) hold as:* $s_x^- f_{xy}^- s_y^- = f_{xy}^-(s_x^- \times s_y^-) = s_{xFy}^-$ *and* $s_x^+ f_{xy}^+ s_y^+ = f_{xy}^+(s_x^+ \times s_y^+) = s_{xFy}^+$ *being onto over the partial ordered sub-lattices* $s_x^- \times s_y^-$ *and* $s_x^+ \times s_y^+$ *respectively of the vector lattice* $[-1, 0] \times [0, 1]$.

Example 1. *(1) Let* $((G = \{a\}, [-1, 0], [0, 1]), \mathcal{F})$ *be defined as the BVF binary operation* $\mathcal{F} = (F, f_{xy}^-, f_{xy}^+)$ *over BVF space* $(G, [-1, 0], [0, 1])$ *such that:* $F(a, a) = a$ *and* $f_{aa}^-(n^-, m^-) = \min\{n^-, m^-\}$, $f_{aa}^+(n^+, m^+) = \max\{n^+, m^+\}$.
Consider the BVF subspace

$$S = \{(a, [-1, \alpha] \cup \{0\}, \{0\} \cup [\beta, 1])\}$$

such that $-1 < \alpha < 0 < \beta < 1$. *Then* $(S, \mathcal{F})$ *defines a bipolar valued fuzzy subgroup of* $((G, [-1, 0], [0, 1]), \mathcal{F})$.
If we consider

$$S' = \{(a, [-1, \gamma] \cup \{0\}, \{0\} \cup [\delta, 1])\}$$

such that $-1 < \gamma < 0 < \delta < 1$ *and* $\alpha \neq \gamma$, $\beta \neq \delta$, *then* $(S', \mathcal{F})$ *defines a bipolar valued fuzzy subgroup of* $((G, [-1, 0], [0, 1]), \mathcal{F})$ *where* $S \neq S'$.
That is, a trivial bipolar-valued fuzzy group may admit multiple distinct bipolar-valued fuzzy subgroups, in contrast to the classical case wherein a trivial group possesses a unique subgroup—that is, the group itself.

(2) Let $((\mathbb{Z}_5, [-1, 0], [0, 1]), \mathcal{F})$ *be defined as the BVF binary operation* $\mathcal{F} = (F, f_{xy}^-, f_{xy}^+)$ *as follows:* $F(x, y) = x +_5 y$, *where* $+_5$ *refers to addition modulo 5, and*

$$f_{xy}^+(n^+, m^+) = n^+ \cdot m^+, \quad f_{xy}^-(n^-, m^-) = -(n^- \cdot m^-).$$

Consider the bipolar valued fuzzy subspace

$$Z = \{(0, [-1, \alpha] \cup \{0\}, \{0\} \cup [\beta, 1]), (1, [-1, \gamma] \cup \{0\}, \{0\} \cup [\delta, 1])\}$$

such that $-1 < \alpha < 0 < \beta < 1$ *and* $-1 < \gamma < 0 < \delta < 1$.

Then $(Z, \mathcal{F})$ is not a bipolar valued fuzzy subgroup of $((\mathbb{Z}_5, [-1, 0], [0, 1]), \mathcal{F})$, since Z is not closed under $\mathcal{F}$. For instance,

$$(0, [-1, \beta], [c\alpha, 1])\mathcal{F}(1, [-1, \gamma], [\delta, 1]) = (1, [-1, -\alpha \cdot \gamma], [\beta \cdot \delta, 1]) \notin Z,$$

since the value $\{0\}$ will not appear in both positive and/or negative membership functions. If $\alpha = \gamma = 0$ and $\beta = \delta = 0$, then

$$Z = \{(0, [-1, 0], [0, 1]), (1, [-1, 0], [0, 1])\}$$

together with $\mathcal{F}$ defines a bipolar valued fuzzy subgroup of $((\mathbb{Z}_5, [-1, 0], [0, 1]), \mathcal{F})$.

Let $B = \{(B^-(x), B^+(x)) : x \in B_\circ\}$ be a bipolar valued fuzzy subset of the set G and let $S_l(B)$, $S_u(B)$ and $S_\circ(B)$ be bipolar valued fuzzy subspaces induced by the bipolar valued fuzzy subset B. For these bipolar valued fuzzy spaces, we can re-state Theorem 3 in the following manner.

Theorem 4. *$(S_l(B), F), (S_u(B), F)$ and $(S^\circ(B), F)$ are bipolar valued fuzzy subgroups of $((G, [-1, 0], [0, 1]), F)$ iff :*

(1) $xFy \in B_\circ, \quad \forall x, y \in B_\circ,$ *(6)*

(2) $f^-_{xy}(B^-(x), B^-(y)) = B^-(xFy)$, *and*
$f^+_{xy}(B^+(x), B^+(y)) = B^+(xFy)$. *(7)*

Definition 22. *A bipolar valued fuzzy subset B of G is said to induce bipolar valued fuzzy subgroups of $((G, [-1, 0], [0, 1]), F)$ iff $(S_l(B), F), (S_u(B), F)$, and $(S_\circ(B), F)$ are bipolar valued fuzzy subgroups.*

Let $((G, [-1, 0], [0, 1]), F)$ with $F = (F, f^-_{xy}, f^+_{xy})$ be a uniform bipolar valued fuzzy group with f^-_{xy}, f^+_{xy} ((i.e., $f^+ = |f^-|$).), then we have the following theorem:

Theorem 5. *(1) Every bipolar valued fuzzy subset B of G which induces bipolar valued fuzzy subgroups is a classical bipolar valued fuzzy subgroup of (G, F).*

(2) If (S, F) is an ordinary subgroup of the group (G, F), then every bipolar valued fuzzy subset B of G, for which $B_\circ = S$, induces a bipolar valued fuzzy subgroup $((G, [-1, 0], [0, 1]), P)$ where $P = \{P, p^-_{xy}, p^+_{xy}\}$ with $P = F$ and p^-_{xy}, p^+_{xy} are suitable negative and positive comembership functions respectively.

Proof. (1) If the bipolar valued fuzzy subset B induces bipolar valued fuzzy subgroups of $((G, [-1, 0], [0, 1]), F)$, then by Theorem 4 we have:

$$f^-_{xy}(B^-(x), B^-(y)) = B^-(xFy) \quad \text{and} \quad f^+_{xy}(B^+(x), B^+(y)) = B^+(xFy),$$

for all $B^-(x) \neq -1 \neq B^-(y)$ and $B^+(x) \neq 1 \neq B^+(y)$. (8)

That is, if the bipolar valued fuzzy subset B induces bipolar valued fuzzy subgroups of $((G,[-1,0],[0,1]),F)$, then it satisfies the inequalities:

$$f^-_{xy}(B^-(x),B^-(y)) \leq B^-(xFy) \quad \text{and} \quad f^+_{xy}(B^+(x),B^+(y)) \geq B^+(xFy), \quad \forall x,y \in G.$$

Therefore, B is a classical bipolar valued fuzzy subgroup.

(2) Let (S,F) be an ordinary subgroup of the group (G,F), and let B be a bipolar valued fuzzy subset of G for which $B_\circ = S$ and let p^-_{xy}, p^+_{xy} be given intersection BVF functions. Define the bipolar valued fuzzy group $((G,[-1,0],[0,1]),P)$ as follows:

$$P = \{P, p^-_{xy}, p^+_{xy}\}, \quad \text{where } P = F,$$

and

$$p^-_{xy}(n_1,m_1) = \psi^-_{xy}(f^-(n_1,m_1)), \quad p^+_{xy}(n_2,m_2) = \psi^+_{xy}(f^+(n_2,m_2)),$$

such that:

If $f^-(B^-(x),B^-(y)) = -1$, then:

$$\psi^-_{xy}(t) = t \quad \text{for all } t \in [-1,0].$$

If $f^-(B^-(x),B^-(y)) \neq -1$, then:

$$\psi^-_{xy}(t) = \begin{cases} \dfrac{B^-(xFy)}{f^-(B^-(x),B^-(y))} \cdot t & \text{if } t \leq f^-(B^-(x),B^-(y)), \\ -1 + \dfrac{1+B^-(xFy)}{1+f^-(B^-(x),B^-(y))} \cdot (t+1) & \text{if } t \geq f^-(B^-(x),B^-(y)). \end{cases}$$

If $f^+(B^+(x),B^+(y)) = 1$, then:

$$\psi^+_{xy}(k) = k \quad \text{for all } k \in [0,1]. \tag{9}$$

If $f^+(B^+(x),B^+(y)) \neq 1$, then:

$$\psi^+_{xy}(k) = \begin{cases} -1 + \dfrac{1+B^+(xFy)}{1+f^+(B^+(x),B^+(y))} \cdot (k+1) & \text{if } k \leq f^+(B^+(x),B^+(y)), \\ \dfrac{B^+(xFy)}{f^+(B^+(x),B^+(y))} \cdot k & \text{if } k \geq f^+(B^+(x),B^+(y)). \end{cases} \tag{10}$$

It is clear that $\psi^-_{xy}(n_1,m_1), \psi^+_{xy}(n_2,m_2) : x,y \in G$ are continuous negative and positive comembership functions respectively. Moreover,

$$\psi^-_{xy}(n_1,m_1) = -1 \iff n_1 = -1 \text{ or } m_1 = -1,$$

$$\psi^+_{xy}(n_2,m_2) = 1 \iff n_2 = 1 \text{ or } m_2 = 1.$$

Hence, P is a bipolar valued fuzzy binary operation on G.

Now, based on the property of the given intersection BVF functions $f^-(n_1, m_1), f^+(n_2, m_2)$ and the construction of $p^-_{xy}(n_1, m_1), p^+_{xy}(n_2, m_2)$, we notice that:

$$f^-(B^-(x), B^-(y)) \neq -1 \quad \text{whenever both } B^-(x), B^-(y) \neq -1,$$

and

$$f^+(B^+(x), B^+(y)) \neq 1 \quad \text{whenever both } B^+(x), B^+(y) \neq 1.$$

That is,

$$p^-_{xy}(B^-(x), B^-(y)) = \psi^-_{xy}(f(B^-(x), B^-(y))) = B^-(xFy), \tag{11}$$

$$p^+_{xy}(B^+(x), B^+(y)) = \psi^+_{xy}(f(B^+(x), B^+(y))) = B^+(xFy). \tag{12}$$

Thus, by Theorem 5 and the assumption that (S, F) is an ordinary subgroup, B induces a bipolar valued fuzzy subgroup of the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), P)$.

Corollary 2. *Every classical bipolar valued fuzzy subgroup B of (G, F) induces bipolar valued fuzzy subgroups relative to some bipolar valued fuzzy group (G, P).*

4. The Normal Bipolar Valued Fuzzy Subgroup

In this section, we introduce the notion of the associated bipolar valued fuzzy subgroup, then we define the bipolar valued fuzzy normal subgroup based on the associated bipolar valued fuzzy subgroup to obtain interesting results regarding abelian bipolar valued fuzzy groups and related bipolar valued fuzzy normal subgroups.

Let $((G, [-1, 0], [0, 1]), F)$ be a bipolar valued fuzzy group having the bipolar valued fuzzy subgroup $(U; F)$. Similar to the fuzzy and intuitionistic fuzzy case and contrary to the ordinary case, bipolar valued fuzzy elements of the bipolar valued fuzzy subgroup $(U; F)$ are not necessarily associative with bipolar valued fuzzy elements of the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$. That is:

$$\alpha F(\beta F \gamma) \neq (\alpha F \beta) F \gamma \tag{13}$$

where α, β, γ are some bipolar valued fuzzy elements of U or $(G, [-1, 0], [0, 1])$ such that one or two of these elements belong to U.

Example 2. *Let $X = \{-1, 1, -i, i\}$. Define the bipolar valued fuzzy binary operation $\boldsymbol{F} = \left(F, f^-_{xy}, f^+_{xy}\right)$ on $(\mho, [-1, 0], [0, 1])$ such that $F : (\mho, [-1, 0], [0, 1]) \times (\mho, [-1, 0], [0, 1]) \to (\mho, [-1, 0], [0, 1])$ is the ordinary multiplication of complex numbers and the (negative and positive) comembership functions have the following form:*

$$f^-_{11}(n^-, m^-) = \begin{cases} \dfrac{n^- \cdot m^-}{\beta} & \text{if } n^- \cdot m^- \leq \beta^2 \\ -1 + \dfrac{(1+\beta) \cdot (n^- \cdot m^- + 1)}{1 + \beta^2} & \text{if } n^- \cdot m^- > \beta^2 \end{cases} \tag{14}$$

$$f^{+}_{11}(n^{+},m^{+}) = \begin{cases} -1+\dfrac{(1+\alpha)\cdot(n^{+}\cdot m^{+}+1)}{1+\alpha^{2}} & \textit{if } n^{+}\cdot m^{+} > \alpha^{2} \\ \dfrac{n^{+}\cdot m^{+}}{\alpha} & \textit{if } n^{+}\cdot m^{+} \leq \alpha^{2} \end{cases} \tag{15}$$

$$f^{-}_{-11}(n^{-},m^{-}) = f^{-}_{1-1}(n^{-},m^{-}) = \begin{cases} \dfrac{n^{-}\cdot m^{-}}{-\alpha} & \textit{if } -n^{-}\cdot m^{-} \leq \alpha\beta \\ -1+\dfrac{(1+\beta)\cdot(n^{-}\cdot m^{-}+1)}{1-\alpha\beta} & \textit{if } -n^{-}\cdot m^{-} > \alpha\beta \end{cases} \tag{16}$$

$$f^{+}_{-11}(n^{+},m^{+}) = f^{+}_{1-1}(n^{+},m^{+}) = \begin{cases} -1+\dfrac{(1+\alpha)\cdot(n^{+}\cdot m^{+}+1)}{1-\alpha\beta} & \textit{if } n^{+}\cdot m^{+} > -\alpha\beta \\ \dfrac{n^{+}\cdot m^{+}}{-\beta} & \textit{if } n^{+}\cdot m^{+} \leq -\alpha\beta \end{cases} \tag{17}$$

and the other negative and positive comembership functions are defined by the product $n^{-}\cdot m^{-}$ and $n^{+}\cdot m^{+}$, where α,β are given fixed real numbers satisfying $-1<\beta<0<\alpha<1$. Clearly $((\mho,[-1,0],[0,1]),F)$ defines a non-uniform bipolar valued fuzzy group. Also the bipolar valued fuzzy subspace

$$U=\{(-1,-0.3,0.5),(1,-0.6,0.7)\}$$

together with the bipolar valued fuzzy binary operation F define a bipolar valued fuzzy subgroup of $((\mho,[-1,0],[0,1]),F)$. Let $\beta=-0.3$ and $\alpha=0.4$ and we need to show that

$$((1,-0.6,0.7)F(1,-0.6,0.7))F(-1,-0.3,0.5) \neq (1,-0.6,0.7)F((1,-0.6,0.7)F(-1,-0.3,0.5))$$

So we have,

$$f^{-}_{11}(-0.6,-0.6) = -1+\frac{(0.7)((0.36)+1)}{1+0.09} = -1+0.873 = -0.126$$

$$f^{+}_{11}(0.7,0.7) = -1+\frac{(1.4)((0.49)+1)}{1+0.16} = -1+1.798 = 0.798$$

$$f^{-}_{-11}(-0.3,-0.6) = f^{-}_{1-1}(-0.6,-0.3) = \frac{(n^{-}.m^{-})}{-\alpha} = \frac{(-0.6)(-0.3)}{-0.4} = -0.45$$

$$f^{+}_{-11}(0.5,0.7) = f^{+}_{1-1}(0.7,0.5) = -1+\frac{(1.4)((0.35)+1)}{1+0.12} = -1+1.6875 = 0.6875$$

$$f^{-}_{11,-1}(-0.126,-0.3) = -1+\frac{(0.7)((0.0378)+1)}{1+0.12} = -1+0.6486 = -0.351$$

$$f^{+}_{11,-1}(0.798,0.5) = -1+\frac{(1.4)((0.399)+1)}{1+0.12} = -1+1.7487 = 0.74875$$

$$f^{-}_{1-1,1}(-0.45,-0.6) = \frac{(-0.45)(-0.6)}{-0.4} = -0.675$$

$$f^{+}_{1-1,1}(0.6875,0.7) = -1+\frac{(1.4)((0.481)+1)}{1+0.12} = -1+1.851 = 0.851$$

So,

$$\begin{aligned}(((1,-0.6,0.7))\mathbb{F}(1,-0.6,0.7))\boldsymbol{F}(-1,-0.3,0.5) &= ((1F1)F-1,-0.351,0.74875)\\ &\neq (((1,-0.6,0.7))\boldsymbol{F}(-1,-0.3,0.5))\boldsymbol{F}(-1,-0.6,0.7)\\ &= ((1F-1)F1,-0.675,0.851)\end{aligned}$$

That is, bipolar valued fuzzy elements of U are not associative with bipolar valued fuzzy elements of $(\mho,[-1,0],[0,1])$.

Definition 23. *Definition 4.2 An associative bipolar valued fuzzy subgroup ($U;F$) of the bipolar valued fuzzy group ($(G,[-1,0],[0,1]),F$) is a bipolar valued fuzzy group ($U;F$) of $((G,[-1,0],[0,1]),F)$ in which bipolar valued fuzzy elements of U are associative with bipolar valued fuzzy elements of ($G,[-1,0],[0,1]$) for any arbitrary choice of bipolar valued fuzzy elements of U and $(G,[-1,0],[0,1])$.*

Example 3. *Let $\mho=\{-1,1,-i,i\}$. Define the bipolar valued fuzzy binary operation $\boldsymbol{F}=\left(F,f_{xy}^{-},f_{xy}^{+}\right)$ on ($\mho,[-1,0],[0,1]$) such that $F:(\mho,[-1,0],[0,1])\times(\mho,[-1,0],[0,1])\to(\mho,[-1,0],[0,1])$ is the ordinary multiplication of complex numbers and the negative and positive comembership functions are given respectively for all $x,y\in\mathbb{U}$ by*

$$\begin{aligned}f_{xy}^{-}\left(n^{-},m^{-}\right)&=n^{-}\vee m^{-}=\min\left(n^{-},m^{-}\right)\text{ and}\\ f_{xy}^{+}\left(n^{+},m^{+}\right)&=n^{+}\wedge m^{+}=\max\left(n^{+},m^{+}\right)\end{aligned}\tag{18}$$

Obviously the bipolar valued fuzzy subspace

$$U=\left\{\left(-1,\left[-1,-\tfrac{1}{2}\right]\cup\{0\},\{0\}\cup\left[\tfrac{1}{2},1\right]\right),\left(1,\left[-1,-\tfrac{1}{2}\right]\cup\{0\},\{0\}\cup\left[\tfrac{1}{2},1\right]\right)\right\}$$

defines an associative bipolar valued fuzzy subgroup of $((\mho,[-1,0],[0,1]),F)$ under F.

Building upon the preceding definitions and examples, we now present a significant result pertaining to associative bipolar-valued fuzzy subgroups.

Theorem 6. *Let $((G,[-1,0],[0,1]),F)$ with $F=(F,f^{-},f^{+})$ be a uniform bipolar valued fuzzy group (i.e., $f^{+}=|f^{-}|$). If*

$$f^{-}(n^{-},-1)=f^{-}(-1,n^{-})=n^{-}\quad\text{and}\quad f^{+}(n^{+},1)=f^{+}(1,n^{+})=n^{+},$$

then every bipolar valued fuzzy subgroup of the bipolar valued fuzzy group $((G,[-1,0],[0,1]),F)$ is an associative bipolar valued fuzzy subgroup.

Proof. Let $(U;F)$ be a bipolar valued fuzzy subgroup of the bipolar valued fuzzy group $((G,[-1,0],[0,1]),F)$.

Consider the bipolar valued fuzzy elements

$$x=(x,[-1,0],[0,1]),\quad y=(y,[-1,0],[0,1]),\quad z=(z,[-1,0],[0,1])$$

in U or $(G, [-1, 0], [0, 1])$, such that one or two of them belong to U.

Using the properties of f^-, f^+, and the associativity of F, we have:

$$\begin{aligned} x\mathcal{F}(y\mathcal{F}z) &= x\mathcal{F}\left((yFz, f^-([-1,0] \times [-1,0]), f^+([0,1] \times [0,1])\right) \\ &= (xF(yFz), f^-([-1,0] \times [-1,0]), f^+([0,1] \times [0,1])) \\ &= ((xFy)Fz, [-1,0], [0,1]) \\ &= (x\mathcal{F}y)\mathcal{F}z, \end{aligned} \tag{19}$$

which proves the associativity of the bipolar valued fuzzy elements of U with those of $(G, [-1, 0], [0, 1])$ under F.

Corollary 3. *Let $((G, [-1, 0], [0, 1]), F = (F, f^-, f^+))$ be a uniform bipolar valued fuzzy group. If f^- and f^+ are intersection of BVF functions, then every bipolar valued fuzzy subgroup of the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$ is an associative bipolar valued fuzzy subgroup.*

Prior to defining a normal bipolar-valued fuzzy group, we introduce the concepts of left and right cosets associated with a bipolar-valued fuzzy subgroup.

Definition 24. *If $(B; F)$, where $B = \{(z, b_z^-, b_z^+) \mid z \in B_\circ\}$, is a bipolar valued fuzzy subgroup of the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$, then for every bipolar valued fuzzy element $(x, [-1, 0], [0, 1])$ of $(G, [-1, 0], [0, 1])$, the fuzzy subspace defined by*

$$(x, [-1,0], [0,1])B = (x, [-1,0], [0,1])\mathcal{F}B = \{(x\mathcal{F}z, f_{xz}^-([-1,0], b_z), f_{xz}^+([0,1], b_z))\} \tag{20}$$

is called a left coset of the bipolar valued fuzzy subgroup $(B; F)$. A right coset of the bipolar valued fuzzy subgroup $(B; F)$ is defined by the bipolar valued fuzzy subspace

$$B(x, [-1,0], [0,1]) = BF(x, [-1,0], [0,1]) = \{(zFx, f_{zx}^-(b_z, [-1,0]), f_{zx}^+(b_z, [0,1]))\}.$$

Theorem 7. *For any associative bipolar valued fuzzy subgroup $(B; F)$ of the bipolar valued fuzzy group $((G, [-1, 0], [0, 1]), F)$, the following hold:*

(1) $(x, [-1, 0], [0, 1])B = (h, [-1, 0]_h^-, [0, 1]_h^+)B$ for every bipolar valued fuzzy element $(h, [-1, 0]_h^-, [0, 1]_h^+) \in (x, [-1, 0], [0, 1])B$, where $[-1, 0]_h^-, [0, 1]_h^+$ denote the possible negative and positive membership values of h.

(2) There is a one-to-one correspondence between any two left (right) cossets of the bipolar valued fuzzy subgroup $(B; F)$.

(3) There is a one-to-one correspondence between the family of right cosets and the family of left cosets of the bipolar valued fuzzy subgroup $(B; F)$.

(4) Any two right cosets (left cosets) of the bipolar valued fuzzy subgroup $(B; F)$ are either identical or disjoint bipolar valued fuzzy subspaces.

Proof. (1) Let $(h, [-1,0]_h^-, [0,1]_h^+)$ be any bipolar valued fuzzy element in $(x, [-1,0], [0,1])B$, then

$$(h, [-1,0]_h^-, [0,1]_h^+) = (x, [-1,0], [0,1])(y, b_y^-, b_y^+)$$

for some $y \in B_\circ$. If (z, b_z^-, b_z^+) is an arbitrary element of B, then

$$\begin{aligned}(x, [-1,0], [0,1])(z, b_z^-, b_z^+) &= (x, [-1,0], [0,1])((y, b_y^-, b_y^+)(y^{-1}, b_{y^{-1}}^-, b_{y^{-1}}^+))(z, b_z^-, b_z^+) \\ &= ((x, C)(y, b_y^-, b_y^+))((y^{-1}, b_{y^{-1}}^-, b_{y^{-1}}^+)(z, b_z^-, b_z^+)) \in \\ &(h, [-1,0]_h^-, [0,1]_h^+)B.\end{aligned} \tag{21}$$

(2) Let $(x, [-1,0], [0,1])B$ and $(y, [-1,0], [0,1])B$ be any two left cosets of the bipolar valued fuzzy group B, then

$$(x, [-1,0], [0,1])(z, b_z^-, b_z^+) \leftrightarrow (y, [-1,0], [0,1])(z, b_z^-, b_z^+) \tag{22}$$

is the required one-to-one correspondence between $(x, [-1,0], [0,1])B$ and $(y, [-1,0], [0,1])B$. For the right cosets, we use the same arrangement.

(3) Let $\{(x, [-1,0], [0,1])B \mid x \in G\}$ and $\{B(x, [-1,0], [0,1]) \mid x \in G\}$ denote the family of left and right cosets respectively of the bipolar valued fuzzy group B, then the required one-to-one correspondence is defined by:

$$(x, [-1,0], [0,1])B \leftrightarrow B(x, [-1,0], [0,1]) \tag{23}$$

(4) Let $(x, [-1,0], [0,1])B$ and $(y, [-1,0], [0,1])B$ be any two intersecting left cosets of the bipolar valued fuzzy subgroup B, then there exist $\alpha, \beta \in B_\circ$ such that

$$(x, [-1,0], [0,1])(\alpha, b_\alpha^-, b_\alpha^+) = (y, [-1,0], [0,1])(\beta, b_\beta^-, b_\beta^+) \tag{24}$$

Choose any bipolar valued fuzzy element $(x, [-1,0], [0,1])(z, b_z^-, b_z^+) \in (x, [-1,0], [0,1])B$, then

$$\begin{aligned}(x, [-1,0], [0,1])(z, b_z^-, b_z^+) &= (x, [-1,0], [0,1])((\alpha, b_\alpha^-, b_\alpha^+)(\alpha^{-1}, b_{\alpha^{-1}}^-, b_{\alpha^{-1}}^+))(z, b_z^-, b_z^+) \\ &= ((x, [-1,0], [0,1])(\alpha, b_\alpha^-, b_\alpha^+))((\alpha^{-1}, b_{\alpha^{-1}}^-, b_{\alpha^{-1}}^+)(z, b_z^-, b_z^+)) \\ &= ((y, [-1,0], [0,1])(\beta, b_\beta^-, b_\beta^+))((\alpha^{-1}, b_{\alpha^{-1}}^-, b_{\alpha^{-1}}^+)(z, b_z^-, b_z^+)) \\ &= (y, [-1,0], [0,1])((\beta, b_\beta^-, b_\beta^+)(\alpha^{-1}, b_{\alpha^{-1}}^-, b_{\alpha^{-1}}^+)(z, b_z^-, b_z^+)) \in \\ &(y, [-1,0], [0,1])B.\end{aligned}$$

That is, $(x, [-1,0], [0,1])B \subset (y, [-1,0], [0,1])B$. Similarly, we can show that $(y, [-1,0], [0,1])B \subset (x, [-1,0], [0,1])B$. Also, we can show the same result for the right cosets of B, which proves (4).

Definition 25. *A bipolar valued fuzzy subgroup B of the bipolar valued fuzzy group $((G, [-1,0], [0,1]), \mathcal{F})$ is called a bipolar valued fuzzy normal subgroup if:*

(1) B is associative in $((G,[-1,0],[0,1]),F)$,

(2) $(x,[-1,0],[0,1])B = B(x,[-1,0],[0,1])$ such that $x \in G$.

The next theorem gives a necessary and sufficient condition for bipolar valued fuzzy normal subgroups.

Theorem 8. *A bipolar valued fuzzy subgroup $(B;F)$, where $B = \{(z,b_z^-,b_z^+) \mid z \in B_\circ\}$, of the bipolar valued group $((G,[-1,0],[0,1]),F)$ is a bipolar valued fuzzy normal subgroup if and only if:*

(1) $(B_\circ,F)$ is an ordinary normal subgroup of the ordinary group (G,F),

(2)

$$\begin{gathered} f_{xz}^-([-1,0],b_z^-) = f_{\acute{z}x}^-(b_{\acute{z}}^-,[-1,0]), \\ f_{xz}^+([0,1],b_z^+) = f_{\acute{z}x}^+(b_{\acute{z}}^+,[0,1]) \\ \text{where} \quad xFz = \acute{z}Fx, \quad x \in X,\ z,\acute{z} \in B_\circ. \end{gathered} \tag{25}$$

Proof. Assume $B = \{(z,b_z^-,b_z^+) \mid z \in B_\circ\}$ is a bipolar valued fuzzy normal subgroup of $((G,[-1,0],[0,1]),F)$. From the correspondence theorem, we have that $(B_\circ,F)$ is an ordinary normal subgroup of the ordinary group (G,F).

Using the normality of B, we have:

$$(x,[-1,0],[0,1])B = B(x,[-1,0],[0,1]), \quad x \in G.$$

That is:

$$\{(xFz, f_{xz}^-([-1,0],b_z^-), f_{xz}^+([0,1],b_z^+)) \mid z \in B_\circ\} = \{(zFx, f_{zx}^-(b_z^-,[-1,0]), f_{zx}^+(b_z^+,[0,1])) \mid z \in B_\circ\} \tag{26}$$

Therefore, for every $z \in B_\circ$, there exists $\acute{z} \in B_\circ$ such that $xFz = \acute{z}Fx$. In other words, $xFB_\circ = B_\circ Fx$. Hence, $B_\circ$ is an ordinary normal subgroup of the ordinary group (G,F), which proves (i). Condition (ii) follows directly from the definition.

The other part of the proof is direct.

Theorem 9. *Every bipolar valued fuzzy normal subgroup B of $((G,[-1,0],[0,1]),\mathcal{F})$ defines a bipolar valued fuzzy equivalence relation on the bipolar valued fuzzy space $(G,[-1,0],[0,1])$ given by:*

$$(x,[-1,0],[0,1])\mathcal{R}(y,[-1,0],[0,1]) \iff (x,[-1,0],[0,1])B = B(y,[-1,0],[0,1]) \tag{27}$$

The bipolar valued fuzzy equivalence relation $\mathcal{R}$ on the bipolar valued fuzzy space $(G,[-1,0],[0,1])$ induces an ordinary equivalence relation on G by the correspondence:

$$(x,[-1,0],[0,1]) \leftrightarrow x.$$

That is,

$$(x,[-1,0],[0,1])\mathcal{R}(y,[-1,0],[0,1]) \iff x\mathcal{R}y \tag{28}$$

which is equivalent to $xB_\circ = B_\circ x$.

5. Bipolar Valued Fuzzy Homomorphisms

In this section we introduce the notion of bipolar valued fuzzy homomorphism, bipolar valued isomorphism and bipolar valued fuzzy kernel. We also study the action of bipolar valued fuzzy normal subgroups under bipolar valued fuzzy homomorphisms.

Definition 26. *Let $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ be two bipolar valued fuzzy groups. A bipolar valued fuzzy homomorphism Φ of $((G,[-1,0],[0,1]),F)$ into $((G',[-1,0],[0,1]),H)$ is a bipolar valued fuzzy function having onto negative and positive comembership functions*

$$\Phi=(\phi,\varphi_x^-,\varphi_x^+):((G,[-1,0],[0,1]),F)\to((G',[-1,0],[0,1]),H)$$

such that

$$\Phi\big((x,[-1,0],[0,1])F(y,[-1,0],[0,1])\big)=\Phi(x,[-1,0],[0,1])\,H\,\Phi(y,[-1,0],[0,1]). \quad (29)$$

The bipolar valued fuzzy groups $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ are called homomorphic bipolar valued fuzzy groups under the bipolar valued fuzzy homomorphism Φ.

If $\Phi:G\to G'$ is a bijection (one-to-one and onto) then it is called a bipolar valued fuzzy isomorphism, and the bipolar valued fuzzy groups $((G,[-1,0],[0,1]),F)$, $((G',[-1,0],[0,1]),H)$ are said to be isomorphic bipolar valued fuzzy groups and will be denoted by

$$((G,[-1,0],[0,1]),F)\cong((G',[-1,0],[0,1]),H).$$

From the above definition, we can formulate the action of the bipolar valued fuzzy homomorphism $\Phi=(\phi,\varphi_x^-,\varphi_x^+)$ of $((G,[-1,0],[0,1]),F)$ into $((G',[-1,0],[0,1]),H)$ for any two bipolar valued fuzzy elements $(x,[-1,0],[0,1])$, $(y,[-1,0],[0,1])\in(G,[-1,0],[0,1])$ as follows:

$$\begin{aligned}\Phi\big((x,[-1,0],[0,1])F(y,[-1,0],[0,1])\big)&=\Phi\big((xFy),[-1,0],[0,1]\big)\\&=(\Phi(x)H\Phi(y),[-1,0],[0,1]).\end{aligned} \quad (30)$$

Remark 2. *The notions bipolar valued fuzzy monomorphism, epimorphism, automorphism and endomorphism are defined as obviously as in the ordinary case.*

The next theorem is a direct result from the above argument and definition which relates homomorphic bipolar valued groups with their corresponding ordinary groups in terms of necessity.

Theorem 10. *If $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ are homomorphic bipolar valued fuzzy groups, then the corresponding ordinary groups (G,F) and (G',H) are homomorphic.*

Proof. Let $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ be homomorphic bipolar valued fuzzy groups under the bipolar valued fuzzy homomorphism $\Phi = (\phi, \varphi_x^-, \varphi_x^+)$ with corresponding ordinary groups (G,F) and (G',H). Now using the correspondences $(x,[-1,0],[0,1]) \mapsto x$ and $(y,[-1,0],[0,1]) \mapsto y$ and the formulation obtained in Definition 5.1, we have

$$\Phi(xFy) = \Phi(x)H\Phi(y). \tag{31}$$

Two main properties of bipolar valued fuzzy homomorphisms that coincide with ordinary homomorphisms are given in the next lemma.

Lemma 1. *If* $\Phi = (\phi, \varphi_x^-, \varphi_x^+) : ((G,[-1,0],[0,1]),F) \to ((G',[-1,0],[0,1]),H)$ *is a bipolar valued fuzzy homomorphism of bipolar valued fuzzy groups* $((G,[-1,0],[0,1]),F)$, $((G',[-1,0],[0,1]),H)$ *having bipolar valued fuzzy identities* $(e,[-1,0],[0,1])$ *and* $(e',[-1,0],[0,1])$ *respectively, then the following holds:*

(1) $\Phi((e,[-1,0],[0,1])) = (e',[-1,0],[0,1])$,

(2) $\Phi((x^{-1},[-1,0],[0,1])) = (\Phi(x,[-1,0],[0,1]))^{-1}$.

Proof. The proof is straightforward using the properties of bipolar valued fuzzy homomorphism. Obviously, if $\Phi = (\phi, \varphi_x^-, \varphi_x^+)$ is a bipolar valued fuzzy homomorphism between the bipolar valued fuzzy groups $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ then the image of any bipolar valued fuzzy group $B = (x, b_x^-, b_x^+) \mid x \in B_\circ$ of $((G,[-1,0],[0,1]),F)$ under Φ, denoted by $\Phi(B)$, is a bipolar valued fuzzy subgroup of $((G',[-1,0],[0,1]),H)$ if for all $\Phi(x) = \Phi(x')$,

$$\varphi_x^-(b_x^-) = \varphi_{x'}^-(b_{x'}^-), \quad \varphi_x^+(b_x^+) = \varphi_{x'}^+(b_{x'}^+). \tag{32}$$

Theorem 11. *If* $\Phi = (\phi, \varphi_x^-, \varphi_x^+)$ *is a bipolar valued fuzzy homomorphism between the bipolar valued fuzzy groups* $((G,[-1,0],[0,1]),F)$ *and* $((G',[-1,0],[0,1]),H)$, *then every associative bipolar valued fuzzy subgroup* $B = (x, b_x^-, b_x^+) \mid x \in B^\circ$ *of* $((G,[-1,0],[0,1]),F)$ *is mapped under* Φ *to an associative bipolar valued fuzzy subgroup* $\Phi(B)$ *of* $((\phi(G),[-1,0],[0,1]),H)$.

Proof. Let $\Phi = (\phi, \varphi_x^-, \varphi_x^+)$ be a bipolar valued fuzzy homomorphism between $((G,[-1,0],[0,1]),F)$ and $((G',[-1,0],[0,1]),H)$ and let $\{B = (x, b_x^-, b_x^+) \mid x \in B^\circ\}$ be any associative bipolar valued fuzzy subgroup of $((G,[-1,0],[0,1]),F)$. Then

$$\begin{aligned}(\Phi(x)H\Phi(y))H\Phi(z) &= \Phi((xFy)Fz)\\ &= \Phi(xF(yFz))\\ &= \Phi(x)H(\Phi(y)H\Phi(z)).\end{aligned} \tag{33}$$

If x (or y or z) belongs to $(G,[-1,0],[0,1])$, then $\Phi(x) \in (\phi(G),[-1,0],[0,1])$, and if x (or y or z) belongs to B, then $\Phi(B)$ is associative in $((\phi(G),[-1,0],[0,1]),H)$ whenever B is associative in $((G,[-1,0],[0,1]),F)$.

Conclusion:

This study presents a significant extension to the theory of fuzzy groups by introducing a complete algebraic framework for bipolar-valued fuzzy subgroups (BVF-subgroups), along with their corresponding normal subgroups and homomorphisms. Rooted in Dib's foundational work on fuzzy spaces [3, 4], our approach leverages the BVF-space, where membership values span the Cartesian product $[-1, 0] \times [0, 1]$, thus capturing both negative and positive evaluations in group theory. By formalizing the bipolar-valued fuzzy binary operation (BVFBO), this research ensures compatibility with classical group axioms while enriching the theory to accommodate dual-valued logic. Unlike earlier fuzzy and intuitionistic fuzzy subgroup models [1–7], our model introduces a topologically and algebraically complete structure that resolves prior limitations such as the absence of a bipolar fuzzy universal set. Our comparison with prior literature confirms the originality and depth of this contribution. While earlier efforts — such as those by Lee [8, 9], Anitha et al. [17], and Mahmood et al. [16, 24] — offered isolated results on bipolar fuzzy logic or its structural applications, the current work integrates these threads into a unifying theory that supports associativity, identity, inverse, and homomorphic mappings within the BVF context. Practically, this generalization has implications across multiple domains, including multi-criteria decision-making [12], intelligent transport systems [13], community detection in complex networks [14], and social influence modeling [15]. Additionally, interdisciplinary studies such as those on neutrosophic set-based selection processes [29], IoT-related cyberattack modeling [31], hypergraph-based influencer identification in dynamic networks [33], and decision-making under neutrosophic uncertainty [30] further emphasize the urgent need for algebraic systems capable of modeling dual polarity and imprecision. These applications suggest strong synergies with the proposed BVF-subgroup theory.

The algebraic extension proposed here also opens promising avenues for defining BVF-quotient groups, BVF-rings, and BVF-ideals, offering a broader mathematical toolkit for systems characterized by conflicting, uncertain, or fuzzy information. Future work should focus on a few different fields. We presented a cohesive algebraic framework for BVF-subgroups, BVF-normal subgroups, and BVF-homomorphisms on a BVF-space with a BVFBO. The results unify subgroup criteria, normality via coset behaviour, and homomorphic images/kernels while clarifying associativity boundaries between subgroup and ambient elements. This foundation is well-suited for polarity-aware decision and network models. Future work includes BVF-quotients, BVF-rings/ideals, and algorithmic implementations for symbolic and AI inference engines.

These directions will help translate the robust theoretical foundation of BVF-subgroups into practical tools for addressing ambiguity and dualism in uncertain systems.

Acknowledgements

We would like to thank the Editor-in-Chief and reviewers for their helpful instructions and comments. I want to express my thanks to all authors helping to complete this work.

Financial support: There is no funding for this article.

References

[1] A. Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.*, 35:512–517, 1971.

[2] J. M. Anthony and H. Sherwood. Fuzzy groups redefined. *J. Math. Anal. Appl.*, 69:124–130, 1979.

[3] K. A. Dib. On fuzzy spaces and fuzzy group theory. *Inform. Sci.*, 80(3-4):253–282, 1994.

[4] K. A. Dib and N. L. Youssef. Fuzzy cartesian product, fuzzy relations and fuzzy functions. *Fuzzy Sets and Systems*, 41(3):299–315, 1991.

[5] A. R. Salleh. Sifat-sifat homomorfisma kabur bagi kumpulan kabur. In *Prosiding Simposium Kebangsaan Sains Matematik ke-7*, Shah Alam: Institut Teknologi Mara, 1996.

[6] M. F. Marashdeh and A. R. Salleh. Intuitionistic fuzzy groups. *Asian Journal of Algebra*, 2(1):1–10, 2009.

[7] M. F. Marashdeh and A. R. Salleh. The intuitionistic fuzzy normal subgroup. *Int. J. Fuzzy Log. Intell. Syst.*, 10:82–88, 2010.

[8] K. M. Lee. Bipolar-valued fuzzy sets and their operations. In *Proc. Int. Conf. on Intelligent Technologies*, pages 307–312, Bangkok, Thailand, 2000.

[9] K.-M. Lee, K.-M. Lee, and K. J. Cios. Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. In *Computing and information technologies: exploring emerging technologies*, pages 433–439. World Scientific, 2001.

[10] F. M. A. Al-Zu'bi, A. G. Ahmad, A. U. Alkouri, and M. Darus. A novel bipolar valued fuzzy group based on dib's approach. *European Journal of Pure and Applied Mathematics*, 17(4):2898–2914, October 2024.

[11] F. M. A. Al-Zu'bi, A. G. Ahmad, A. U. Alkouri, and M. Darus. A new trend of bipolar-valued fuzzy cartesian products, relations, and functions. *WSEAS Transactions on Mathematics*, 23:502–514, 2024.

[12] M. Akram, A. J. C. R. Shumaiza, and J. C. R. Alcantud. *Multi-criteria decision making methods with bipolar fuzzy sets*, volume 2023. Springer, Singapore, 2023.

[13] A. Al-Quran, N. Jamil, S. T. Tehrim, and M. Riaz. Cubic bipolar fuzzy vikor and electre-ii algorithms for efficient freight transportation in industry 4.0. *AIMS Mathematics*, 8(10):24484–24514, 2023.

[14] I. Gutiérrez, D. Gómez, J. Castro, and R. Espínola. Multiple bipolar fuzzy measures: an application to community detection problems for networks with additional information. *International Journal of Computational Intelligence Systems*, 13(1):1636–1649, 2020.

[15] M. Alqahtani, R. Keerthana, S. Venkatesh, and M. Kaviyarasu. Hesitant bipolar-valued intuitionistic fuzzy graphs for identifying the dominant person in social media groups. *Symmetry*, 16(10):1293, 2024.

[16] T. Mahmood, A. Jaleel, and U. U. Rehman. Pattern recognition and medical diagnosis

based on trigonometric similarity measures for bipolar complex fuzzy soft sets. *Soft Computing*, 27(16):11125–11154, 2023.

[17] M. S. Anitha, K. L. Muruganantha Prasad, and K. Arjunan. Notes on bipolar valued fuzzy subgroups of a group. *The Bulletin of Society for Mathematical Services and Standards*, 7:40–45, 2013.

[18] A. B. Saeid. Bipolar-valued fuzzy bck/bci-algebras. *World Applied Sciences Journal*, 7(11):1404–1411, 2009.

[19] K. J. Lee. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of bck/bci-algebras. *Bull. Malays. Math. Sci. Soc.*, 32(3):361–373, 2009.

[20] A. Balasubramanian, K. M. Prasad, and K. Arjunan. Bipolar interval valued fuzzy subgroups of a group. *Bulletin Of Mathematics And Statistics Research*, 3(3):234–239, 2015.

[21] A. S. Sahaya, S. Naganathan, and K. Arjunan. A study on bipolar valued q-fuzzy subgroups of a group. *Bulletin of Mathematics and Statistics Research*, 4(3):97–101, 2016.

[22] V. Shanmugapriya and K. Arjunan. Some translators in bipolar valued fuzzy subsemiring of a semiring. *Bulletin of Mathematics and Statistics Research*, 4(4):118–123, 2016.

[23] Y. B. Jun and S. Z. Song. Subalgebras and closed ideals of bch-algebras based on bipolar-valued fuzzy sets. *Sci. Math. Jpn*, 68(2):287–297, 2008.

[24] T. Mahmood, U. U. Rehman, and M. Albaity. Analysis of γ-semigroups based on bipolar complex fuzzy sets. *Computational and Applied Mathematics*, 42(6):262, 2023.

[25] A. Al-Masarwah, A. G. Ahmad, G. Muhiuddin, and D. Al-Kadi. Generalized m-polar fuzzy positive implicative ideals of bck-algebras. *Journal of Mathematics*, 2021(1):6610009, 2021.

[26] A. Al-Masarwah and M. Alqahtani. Operational algebraic properties and subsemigroups of semigroups in view of k-folded n-structures. *AIMS Mathematics*, 8(9):22081–22096, 2023.

[27] S. M. Alqaraleh, M. J. S. Abd Ulazeez, M. O. Massa'deh, A. G. Talafha, and A. Bataihah. Bipolar complex fuzzy soft sets and their application. *International Journal of Fuzzy System Applications (IJFSA)*, 11(1):1–23, 2022.

[28] M. O. Massa'deh, A. O. Fallatah, and et al. Anti homomorphism and homomorphism of bipolar valued multi fuzzy hx-subgroups and its normal. *Full Length Article*, 24(3):165–165, 2024.

[29] L. J. Manavalan, S. Damrah, I. A. Falahah, A. Al-Husban, and M. Palanikumar. Selection process based on new type neutrosophic interval-valued set applied to logarithm operator. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[30] L. J. Manavalan, S. Damrah, M. M. Abbas Ali, A. Al-Husban, and M. Palanikumar. Type-i extension diophantine neutrosophic interval valued soft set in real life applications for a decision making. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[31] S. Damrah, M. H. DarAssi, and Y. AbuHour. Mathematical modeling for exploring the spread of cyberattacks through iot devices. In *International Conference on*

Science, Engineering Management and Information Technology, pages 17–27, Cham, September 2023. Springer Nature Switzerland.

[32] M. H. DarAssi, S. Damrah, and Y. AbuHour. A mathematical study of the omicron variant in a discrete-time covid-19 model. *The European Physical Journal Plus*, 138(7):1–18, 2023.

[33] N. Firouzkouhi, A. Amini, A. Bani-Mustafa, A. Mehdizadeh, S. Damrah, A. Gholami, and B. Davvaz. Generalized fuzzy hypergraph for link prediction and identification of influencers in dynamic social media networks. *Expert Systems with Applications*, 238:121736, 2024.

[34] A. Sedaghat, M. A. A. Omar, S. Damrah, and M. Gaith. Mathematical modelling of the marsh funnel for measuring rheological properties of drilling nanofluids for energy efficient environment. In *2016 Eleventh International Conference on Ecological Vehicles and Renewable Energies (EVER)*, pages 1–4. IEEE, April 2016.

[35] B. M. Biltayib, S. Damrah, B. Al-Fakeh, and A. Al-Kanderi. Validation of directional drilling well design trajectory with a case study in kuwait. In *2015 International Conference on Sustainable Mobility Applications, Renewables and Technology (SMART)*, pages 1–7. IEEE, November 2015.

[36] M. Talafha, S. C. Dzul-Kifli, and A. U. Alkouri. A new structure of hesitant fuzzy relations: An extension from fuzzy and intuitionistic fuzzy relations with applications. *Journal of Intelligent & Fuzzy Systems*, 2025.

[37] T. Oner, R. Neelamegarajan, R. K. Bandaru, and H. Bordbar. An investigation into bipolar fuzzy hoop algebras and their applications. *Axioms*, 14(5):338, 2025.

[38] I. Senturk, T. Oner, D. S. Turan, G. N. Gurbuz, and B. Ordin. Axiomatic analysis of state operators in sheffer stroke bck-algebras associated with algorithmic approaches. *AIMS Mathematics*, 10(1):1555–1588, 2025.