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Employing Cubic $\mathcal{B}$-Spline Functions to Solve Linear Systems of Volterra Integro-Fractional Differential Equations with Variable Coefficients

Diar Kh. Abdullah[1,*], Shazad Sh. Ahmed[1], Karwan H. F. Jwamer[1]

[1] *Mathematics Department, College of Science, University of Sulaimani, Sulaimanyah, Iraq*

Abstract. In this study, we present a collocation method based on cubic $\mathcal{B}$-spline functions for solving systems of Volterra integro-differential equations involving both classical and fractional derivatives in the Caputo sense (LSVIDEs-CF). The approach begins by dividing the problem domain into a finite number of subintervals, followed by the construction of cubic $\mathcal{B}$-spline basis functions within each segment. Control points are introduced as the unknowns in the approximate numerical solution, which is expressed as a cubic combination of these basis functions. The given system of VIFDEs-CF is then reduced to a system of algebraic equations, which are efficiently solved using the Jacobian matrix method. In practice, the integrals are approximated using the Clenshaw–Curtis quadrature rule. The implementation of this method is supported by Python software, ensuring effective computational processing. Numerical examples are provided to demonstrate the efficiency of the proposed method. An itemized version of the algorithm is also presented to facilitate its implementation.

2020 Mathematics Subject Classifications: 65R20, 65M70, 34A08, 45J05

Key Words and Phrases: System of Volterra integro-fractional differential equations, cubic $\mathcal{B}$-spline function, collocation method, Jacobian matrix algorithm, Clenshaw-Curtis quadrature rule

1. Introduction

Integro-differential equations (IDEs) are widely used in mathematical modeling because they combine integration and differentiation of an unknown function. Scholars such as Volterra, Fredholm, Malthus, Verhulst, Abel, and Lotka have employed IDEs to study problems in fields including mathematical biology, physics, and economics [1]. Over the past several decades, extensive research has been conducted on IDEs, resulting in numerous publications and theoretical advancements. One notable application of IDEs is in

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Email addresses: Diar.Khalid85@gmail.com (D. Kh. Abdullah),
Shazad.ahmed@univsul.edu.iq (Sh. Sh. Ahmed), karwan.jwamer@univsul.edu.iq (K. H. F. Jwamer)

modeling the dynamic spread of viruses using statistical methods [2, 3]. Fractional integro-differential equations, in particular, have gained attention due to their ability to represent memory effects and hereditary behavior in complicated systems, and hence, the search for correct approximate solutions has developed into one of the central fields of study in numerical analysis, which led to significant results in the numerical solution of linear types of these equations [4]. Various kinds of fractional integral equations have been resolved successfully by building various approaches in recent times [5]. For such complex systems, reliable numerical methods are necessary to avoid misleading predictions. The flexibility of the cubic $\mathcal{B}$-spline collocation method is that it can be effectively applied to both linear and nonlinear FIDEs, arising in viscoelasticity and population control, demonstrating its usefulness in applications [6]-[7]. Current research has been towards the development of numerical methods that can provide precise approximations of the solutions of FIDEs. Among them, the cubic $\mathcal{B}$-spline collocation technique was recently popularized, as cubic $\mathcal{B}$-splines inherently possess the flexibility and smoothness that would be suitable for approximating fractional derivative functions [8]. Some researchers have presented extremely powerful methods to derive efficient and viable approximate solutions of these equations. Yi et al. presented a graphical illustration of the CAS wavelet method for solving FIDEs [9]. Quantic B-spline techniques were proposed by Raslan et al. and Fazal et al. see [10, 11]. Diar et al. illustrated quadratic and linear spline functions, see [7, 12]. While Mohammad et al. demonstrated the spline collocation method [13]. Abbas et al. introduced an efficient spline technique for solving time-fractional problems [14]. Ghosh et al. presented an iterative difference scheme [15]. Masti et al. worked on the collocation-Galerkin method [16]. Khan et al. applied the LADM procedure [17]. Khader et al. used the Chebyshev pseudo-spectral method [18]. Huang et al. presented the Taylor expansion method [19]. Yn H et al. demonstrated the Taylor series method [20]. Singh et al. applied the adaptive mesh method [21], Li et al. explored deep learning techniques [22], and Shazad worked on the system of linear Volterra integro-fractional differential equations [23]. In this paper, our focus is on approximating a (SVIDE's-CF) using a new extended cubic $\mathcal{B}$-spline based collocation method. The target (SVIDE's-CF) with the general forms defined on any closed-bounded interval $[a, b]$ is described by the equations:

$$\mathcal{P}_i(x)\mathcal{Y}_i'(x) + a_{i0}(x)\,{}^{\mathcal{C}}_{a}\mathcal{D}^{\sigma_{i0}}_{x}\mathcal{Y}_i(x) + a_{i1}(x)\mathcal{Y}_i(x) = \mathcal{F}_i(x) + \sum_{j=0}^{m}\omega_{ij}\int_a^x \mathcal{K}_{ij}(x,s)\,{}^{\mathcal{C}}_{a}\mathcal{D}^{\beta_{ij}}_{s}\mathcal{Y}_j(s)\,ds. \tag{1}$$

Under the following conditions:

$$\left[\mathcal{D}^{k_i}_{x}\mathcal{Y}_i(x)\right]_{x=a} = \vartheta_{ik_i}, \forall k_i = 0, 1, \ldots, \mu_i - 1, \text{ and } i = 0, 1, \ldots, m. \tag{2}$$

The variable coefficients $\mathcal{P}_i(x)(\not\equiv 0)$, $a_{i0}(x)$ and $a_{i1}(x) \in C([a, b], \mathbb{R})$ and $\mathcal{K}_{ij} \in C(\Theta, \mathbb{R})$, $\Theta = \{(x, s) : a \leq s < x \leq b\}$, for each $i, j = 0, 1, \ldots, m$, with fractional orders: $\sigma_{i1} > \sigma_{i0} > 0$ and $\beta_{im} > \beta_{i(m-1)} > \cdots > \beta_{i1} > \beta_{i0} = 0$, and for all $i, j = 0, 1, \ldots, m$. Furthermore, the $\mu_i = \max\left\{1, m^{\beta}_{im}\right\}$ for all $i = 0, 1, \ldots, m$, where $m^{\beta}_{ij} - 1 < \beta_{ij} \leq m^{\beta}_{ij}, m^{\beta}_{ij} = \lceil \beta_{ij} \rceil, \omega_{ij} \in \mathbb{R}$ also for all $i, j = 0, 1, \ldots, m$. In this work, we examined and assessed a system of

Volterra integro-differential equations for classical and fractional orders (SVIDE's-CF), using quadrature techniques in conjunction with cubic $\mathcal{B}$-spline functions. The information was summarized by an algorithm, and we later produced Python software to implement the method and a few test instances that were resolved with the help of these algorithms. This paper is structured in the following manner: Section 2 presents background information on fractional calculus and highlights its key features. Section 3 discusses the materials used and introduces cubic $\mathcal{B}$-spline functions along with new lemmas. Section 4 explores the derivation of the proposed method. Section 5 provides the algorithms. Section 6 compares the numerical results with those from existing methods. Finally, Section 7 offers a concise summary of the study's findings.

2. Basic definitions of fractional derivatives

This section discusses fractional calculus and its properties and defines the numerical integration as the Clenshaw-Curtis quadrature rule:

Definition 1. *([3], [24]). The fractional operators ${}_aJ_x^\alpha$ and ${}_a^R\mathcal{D}_x^\alpha$ of order α for a function $\mathcal{Y}(x)$ are called the Riemann-Liouville fractional integral and differential operators, respectively, for all $n-1<\alpha\leq n$ ($n\in\mathbb{Z}^+$, $\alpha\in\mathbb{R}^+$).*

$$ {}_aJ_x^\alpha\mathcal{Y}(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\mathcal{Y}(s)}{(x-s)^{1-\alpha}}\,ds;\quad {}_aJ_x^0\mathcal{Y}(x)=\mathcal{Y}(x),\quad x>a \tag{3}$$

$$ {}_a^R\mathcal{D}_x^\alpha\mathcal{Y}(x)=\frac{d^n}{dx^n}\left({}_aJ_x^{n-\alpha}\mathcal{Y}(x)\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x\frac{\mathcal{Y}(s)}{(x-s)^{\alpha+1-n}}\,ds,\quad x>a \tag{4}$$

Definition 2. *([3], [24]). The fractional operator ${}_a^C\mathcal{D}_x^\alpha$ of order α for a function $\mathcal{Y}(x)$ is called the Caputo fractional differential operator, for all $n-1<\alpha\leq n$ ($n\in\mathbb{Z}^+$, $\alpha\in\mathbb{R}^+$), and is defined as:*

$$ {}_a^C\mathcal{D}_x^\alpha\mathcal{Y}(x)={}_aJ_x^{n-\alpha}\left(\frac{d^n}{dx^n}\mathcal{Y}(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{\mathcal{Y}^{(n)}(s)}{(x-s)^{\alpha+1-n}}\,ds,\quad x>a \tag{5}$$

The following are some interesting properties of the operator ${}_a^C\mathcal{D}_x^\alpha$:

- *For $n=\alpha$, we have*
$$ {}_a^C\mathcal{D}_x^\alpha\mathcal{Y}(x)=\frac{d^n}{dx^n}\mathcal{Y}(x).$$
- *The operator ${}_a^CD_x^\alpha$ is linear, i.e., for any $\varphi_1,\varphi_2\in\mathbb{R}$,*
$$ {}_a^C\mathcal{D}_x^\alpha\left[\varphi_1\mathcal{Y}_1(x)\pm\varphi_2\mathcal{Y}_2(x)\right]=\varphi_1{}_a^C\mathcal{D}_t^\alpha\mathcal{Y}_1(x)\pm\varphi_2{}_a^C\mathcal{D}_x^\alpha\mathcal{Y}_2(x).$$
- *For any constant $\varphi\in\mathbb{R}$,*
$$ {}_a^C\mathcal{D}_x^\alpha\varphi=0.$$

- *If* $\mathcal{Y}(x) = (x-a)^p$*, then:*

$$ {}_a^C\mathcal{D}_x^{\alpha}(x-a)^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha}, & \textit{if } p \in \mathbb{N},\, p \geq \lceil\alpha\rceil \textit{ or } p \notin \mathbb{N},\, p > \lceil\alpha\rceil - 1 \\ 0, & \textit{if } p \in \{0,1,2,\ldots,\lceil\alpha\rceil - 1\} \end{cases} $$

Definition 3. *([12], [25], [26]). Clenshaw and Curtis established a method for approximating a definite integral by expressing the integrand as a finite Chebyshev series. This approach is especially effective when applied to integral equations.*

$$ \int_{-1}^{1} \mathcal{Y}(x)\,dx \approx \sum_{r=0,\,even}^{\mathcal{N}}{}'' \left(\frac{2}{\mathcal{N}} \sum_{k=0}^{\mathcal{N}} \cos\left(\frac{rk\pi}{\mathcal{N}}\right) \mathcal{Y}\left(\cos\left(\frac{k\pi}{\mathcal{N}}\right)\right)\right), \quad k = 0,1,\ldots,\mathcal{N}. \tag{6} $$

Remark 2.1 *([26]). The following observations can be made:*

- *The notation* $\sum''$ *indicates that the first and last terms in the summation are to be halved.*
- *The transformation*

$$ x = \frac{b-a}{2}t + \frac{b+a}{2} $$

maps $x \in [-1,1]$ *to* $t \in [a,b]$*.*

3. Materials and cubic $\mathcal{B}$-spline function

In this section, we will define the cubic $\mathcal{B}$-spline function, the $\mathcal{B}$-spline curve and knot vectors, starting with the definition of a $\mathcal{B}$-spline of degree k.

Definition 4. *([27], [28]). Let* $\mathcal{T}_{\mathcal{N}} = \{x_0, x_1, \ldots, x_N\}$ *be a uniform or non-uniform partition of the interval* $[a,b]$*. The k-degree B-spline basis function* $\mathcal{B}_r^k(x)$*, for* $r \geq 0$*, is defined recursively as:*

$$ \mathcal{B}_r^k(x) = \frac{x - x_r}{x_{r+k} - x_r}\mathcal{B}_r^{k-1}(x) + \frac{x_{r+k+1} - x}{x_{r+k+1} - x_{r+1}}\mathcal{B}_{r+1}^{k-1}(x) \tag{7} $$

Remark 1 *([29]). We note the following facts about the* $\mathcal{B}$*-spline basis functions:*

- ***Local support:*** $\mathcal{B}_r^k(x) = 0$ *for all* $x \notin [x_r, x_{r+k+1})$*.*
- ***Nonnegativity:*** $\mathcal{B}_r^k(x) > 0$ *for all* $x \in [x_r, x_{r+k+1})$*.*

Remark 2 *([27], [30]). In the context of* $\mathcal{B}$*-spline curves, the knot vector* $\mathcal{T} = \{x_0, x_1, x_2, \ldots, x_m\}$ *plays a crucial role in determining the shape of the curve.*

Definition 5. *([27], [30]). An open uniform knot vector is a knot vector that contains k equal knot values at each end. The parametric knot values x_r for an open curve are given by:*

$$x_r = \begin{cases} 0, & r < k+1 \\ r-k, & k+1 \le r \le n \\ n-k+1, & r > n \end{cases} \tag{8}$$

where $0 \le r \le n+k+1$, and the parameter x lies in the range $0 \le x \le n-k+1$.

Definition 6. *([27]). A B-spline curve is a piecewise-defined polynomial curve. Mathematically, a $\mathcal{B}$-spline curve $\mathcal{B}^{(k,n)}(x)$ of degree k, with control points $\{\vartheta_0, \vartheta_1, \dots, \vartheta_n\}$ and a non-decreasing knot vector $\mathcal{T} = \{x_0, x_1, x_2, \dots, x_m\}$, is defined as:*

$$\mathcal{B}^{\{k,n\}}(x) = \sum_{z=0}^{n} \vartheta_z \, \mathcal{B}_z^k(x), \quad x_0 \le x \le x_m \tag{9}$$

Here, ϑ_z are the control points that influence the shape of the curve, and $\mathcal{B}_z^k(x)$ are the B-spline basis functions of degree k, which are defined recursively and determine the influence of each control point on the curve for any parameter value x.

Lemma 1. *General expression of a cubic $\mathcal{B}$-spline curve. Let $\vartheta_0, \vartheta_1, \vartheta_2$, and ϑ_3 be four consecutive control points of a third-degree $\mathcal{B}$-spline curve defined on the interval $[a, b]$. Then, the curve is given by:*

$$\begin{aligned} \mathcal{B}^{\{k,n\}}(x^\star) = \vartheta_0 \left(\frac{b-x^\star}{b-a}\right)^3 + 3\vartheta_1 \left(\frac{x^\star-a}{b-a}\right)\left(\frac{b-x^\star}{b-a}\right)^2 \\ + 3\vartheta_2 \left(\frac{x^\star-a}{b-a}\right)^2 \left(\frac{b-x^\star}{b-a}\right) + \vartheta_3 \left(\frac{x^\star-a}{b-a}\right)^3, \quad x^\star \in [a, b] \end{aligned} \tag{10}$$

Proof. *First, we convert the given knots $x_r \in \mathcal{T}$, defined on the original interval $0 \le x \le n-k+1$ as in (8), to a new knot vector $x_r^\star \in \mathcal{T}^\star$ defined on a new interval $x^\star \in [a, b]$. This is achieved by applying the following linear transformation to each element of the original knot vector $\mathcal{T} = \{x_0, x_1, x_2, \dots, x_m\}$, yielding a new knot vector $\mathcal{T}^\star = \{x_0^\star, x_1^\star, \dots, x_m^\star\}$:*

$$x_r^\star = a + \left(\frac{b-a}{n-k+1}\right) x_r \tag{11}$$

By applying the transformation defined in (11) to each case in (8), we obtain:

$$x_r^\star = \begin{cases} a, & r < k+1 \\ a + \left(\dfrac{b-a}{n-k+1}\right)(r-k), & k+1 \le r \le n \\ b, & r > n \end{cases} \tag{12}$$

Where:

- *For* $r < k+1$, $x_r = 0$ *implies* $x_r^\star = a$
- *For* $k+1 \leq r \leq n$, $x_r = r-k$ *implies* $x_r^\star = a + \left(\dfrac{b-a}{n-k+1}\right)(r-k)$
- *For* $r > n$, $x_r = n-k+1$ *implies* $x_r^\star = b$

Substituting $k = 3$ *(cubic* $\mathcal{B}$*-spline) into the recurrence relation from* (7)*, we obtain the cubic* $\mathcal{B}$*-spline basis functions used in the expression above.*

$$\mathcal{B}_r^3(x^\star) = \frac{x^\star - x_r^\star}{x_{r+3}^\star - x_r^\star}\mathcal{B}_r^2(x^\star) + \frac{x_{r+4}^\star - x^\star}{x_{r+4}^\star - x_{r+1}^\star}\mathcal{B}_{r+1}^2(x^\star) \tag{13}$$

We substitute the expressions for $\mathcal{B}_r^2(x^\star)$ *and* $\mathcal{B}_{r+1}^2(x^\star)$*, which are computed using the first and zero degree* $\mathcal{B}$*-spline basis functions. After simplification and substitution, we obtain:*

$$\mathcal{B}_r^3(x^\star) = \begin{cases} \dfrac{(x^\star - x_r^\star)^3}{(x_{r+3}^\star - x_r^\star)(x_{r+2}^\star - x_r^\star)(x_{r+1}^\star - x_r^\star)}, & \text{if } x_r^\star \leq x^\star < x_{r+1}^\star \\[2ex] \dfrac{(x^\star - x_r^\star)^2(x_{r+2}^\star - x^\star)}{(x_{r+3}^\star - x_r^\star)(x_{r+2}^\star - x_r^\star)(x_{r+2}^\star - x_{r+1}^\star)} \\ +\dfrac{(x^\star - x_r^\star)(x_{r+3}^\star - x^\star)(x^\star - x_{r+1}^\star)}{(x_{r+3}^\star - x_r^\star)(x_{r+3}^\star - x_{r+1}^\star)(x_{r+2}^\star - x_{r+1}^\star)} & \text{if } x_{r+1}^\star \leq x^\star < x_{r+2}^\star \\ +\dfrac{(x_{r+4}^\star - x^\star)(x^\star - x_{r+1}^\star)^2}{(x_{r+4}^\star - x_{r+1}^\star)(x_{r+3}^\star - x_{r+1}^\star)(x_{r+2}^\star - x_{r+1}^\star)} \\ \dfrac{(x_{r+3}^\star - x^\star)^2(x^\star - x_r^\star)}{(x_{r+3}^\star - x_r^\star)(x_{r+3}^\star - x_{r+1}^\star)(x_{r+3}^\star - x_{r+2}^\star)} \\ +\dfrac{(x_{r+4}^\star - x^\star)(x^\star - x_{r+1}^\star)(x_{r+3}^\star - x^\star)}{(x_{r+4}^\star - x_{r+1}^\star)(x_{r+3}^\star - x_{r+1}^\star)(x_{r+3}^\star - x_{r+2}^\star)} & \text{if } x_{r+2}^\star \leq x^\star < x_{r+3}^\star \\ +\dfrac{(x_{r+4}^\star - x^\star)^2(x^\star - x_{r+2}^\star)}{(x_{r+4}^\star - x_{r+1}^\star)(x_{r+4}^\star - x_{r+2}^\star)(x_{r+3}^\star - x_{r+2}^\star)} \\ \dfrac{(x_{r+4}^\star - x^\star)^3}{(x_{r+4}^\star - x_{r+1}^\star)(x_{r+4}^\star - x_{r+2}^\star)(x_{r+4}^\star - x_{r+3}^\star)}, & \text{if } x_{r+3}^\star \leq x^\star < x_{r+4}^\star \\[2ex] 0, & \text{otherwise} \end{cases} \tag{14}$$

Now, if we set $k = 3$ *and* $n = 3$*, then from* (12)*, we obtain the open uniform knot vector:* $\mathcal{T}^\star = \{a, a, a, a, b, b, b, b\}$*. The parameter* $x^\star$ *lies in the range:* $x^\star \in [a, b]$*.*

Since the knot vector is uniform and open, $x_r^\star \in \mathcal{T}^\star$ *for* $r = 0, \ldots, n$*. Substituting into* (9) *and using the basis functions from* (13)*, we get:*

$$\mathcal{B}^{\{k,n\}}(x^\star) = \sum_{z=0}^{3} \vartheta_z \, \mathcal{B}_z^3(x^\star)$$

$$= \vartheta_0 \mathcal{B}_0^3(x^\star) + \vartheta_1 \mathcal{B}_1^3(x^\star) + \vartheta_2 \mathcal{B}_2^3(x^\star) + \vartheta_3 \mathcal{B}_3^3(x^\star) \tag{15}$$

That is:

$$\mathcal{B}^{\{k,n\}}(x^\star) =$$

$$\vartheta_0 \cdot \begin{cases} \dfrac{(x^\star - x_0^\star)^3}{(x_3^\star - x_0^\star)(x_2^\star - x_0^\star)(x_1^\star - x_0^\star)}, & \textit{if } x_0^\star \le x^\star < x_1^\star \\ \dfrac{(x^\star - x_0^\star)^2(x_2^\star - x^\star)}{(x_3^\star - x_0^\star)(x_2^\star - x_0^\star)(x_2^\star - x_1^\star)} + \dfrac{(x^\star - x_0^\star)(x_3^\star - x^\star)(x^\star - x_1^\star)}{(x_3^\star - x_0^\star)(x_3^\star - x_1^\star)(x_2^\star - x_1^\star)} + \dfrac{(x_4^\star - x^\star)(x^\star - x_1^\star)^2}{(x_4^\star - x_1^\star)(x_3^\star - x_1^\star)(x_2^\star - x_1^\star)} & \textit{if } x_1^\star \le x^\star < x_2^\star \\ \dfrac{(x_3^\star - x^\star)^2(x^\star - x_0^\star)}{(x_3^\star - x_0^\star)(x_3^\star - x_1^\star)(x_3^\star - x_2^\star)} + \dfrac{(x_4^\star - x^\star)(x^\star - x_1^\star)(x_3^\star - x^\star)}{(x_3^\star - x_1^\star)^2(x_3^\star - x_2^\star)} + \dfrac{(x_4^\star - x^\star)^2(x^\star - x_2^\star)}{(x_4^\star - x_1^\star)(x_4^\star - x_2^\star)(x_3^\star - x_2^\star)} & \textit{if } x_2^\star \le x^\star < x_3^\star \\ \dfrac{(x_4^\star - x^\star)^3}{(x_4^\star - x_1^\star)(x_4^\star - x_2^\star)(x_4^\star - x_3^\star)}, & \textit{if } x_3^\star \le x^\star < x_4^\star \\ 0, & \textit{otherwise} \end{cases}$$

$$+\vartheta_1 \cdot \begin{cases} \dfrac{(x^\star - x_1^\star)^3}{(x_4^\star - x_1^\star)(x_3^\star - x_1^\star)(x_2^\star - x_1^\star)}, & \textit{if } x_1^\star \le x^\star < x_2^\star \\ \dfrac{(x^\star - x_1^\star)^2(x_3^\star - x^\star)}{(x_4^\star - x_1^\star)(x_3^\star - x_1^\star)(x_3^\star - x_2^\star)} + \dfrac{(x^\star - x_1^\star)(x_4^\star - x^\star)(x^\star - x_2^\star)}{(x_4^\star - x_1^\star)(x_4^\star - x_2^\star)(x_3^\star - x_2^\star)} + \dfrac{(x_5^\star - x^\star)(x^\star - x_2^\star)^2}{(x_5^\star - x_2^\star)(x_4^\star - x_2^\star)(x_3^\star - x_2^\star)} & \textit{if } x_2^\star \le x^\star < x_3^\star \\ \dfrac{(x_4^\star - x^\star)^2(x^\star - x_1^\star)}{(x_4^\star - x_1^\star)(x_4^\star - x_2^\star)(x_4^\star - x_3^\star)} + \dfrac{(x_5^\star - x^\star)(x^\star - x_2^\star)(x_4^\star - x^\star)}{(x_5^\star - x_2^\star)(x_4^\star - x_2^\star)(x_4^\star - x_3^\star)} + \dfrac{(x_5^\star - x^\star)^2(x^\star - x_3^\star)}{(x_5^\star - x_2^\star)(x_5^\star - x_3^\star)(x_4^\star - x_3^\star)} & \textit{if } x_3^\star \le x^\star < x_4^\star \\ \dfrac{(x_5^\star - x^\star)^3}{(x_5^\star - x_2^\star)(x_5^\star - x_3^\star)(x_5^\star - x_4^\star)}, & \textit{if } x_4^\star \le x^\star < x_5^\star \\ 0, & \textit{otherwise} \end{cases}$$

$$+\vartheta_2 \cdot \begin{cases} \dfrac{(x^\star - x_2^\star)^3}{(x_5^\star - x_2^\star)(x_4^\star - x_2^\star)(x_3^\star - x_2^\star)}, & \textit{if } x_2^\star \le x^\star < x_3^\star \\ \dfrac{(x^\star - x_2^\star)^2(x_4^\star - x^\star)}{(x_5^\star - x_2^\star)(x_4^\star - x_2^\star)(x_4^\star - x_3^\star)} + \dfrac{(x^\star - x_2^\star)(x_5^\star - x^\star)(x^\star - x_3^\star)}{(x_5^\star - x_2^\star)(x_5^\star - x_3^\star)(x_4^\star - x_3^\star)} + \dfrac{(x_6^\star - x^\star)(x^\star - x_3^\star)^2}{(x_6^\star - x_3^\star)(x_5^\star - x_3^\star)(x_4^\star - x_3^\star)} & \textit{if } x_3^\star \le x^\star < x_4^\star \\ \dfrac{(x_5^\star - x^\star)^2(x^\star - x_2^\star)}{(x_5^\star - x_2^\star)(x_5^\star - x_3^\star)(x_5^\star - x_4^\star)} + \dfrac{(x_6^\star - x^\star)(x^\star - x_3^\star)(x_5^\star - x^\star)}{(x_6^\star - x_3^\star)(x_5^\star - x_3^\star)(x_5^\star - x_4^\star)} + \dfrac{(x_6^\star - x^\star)^2(x^\star - x_4^\star)}{(x_6^\star - x_3^\star)(x_6^\star - x_4^\star)(x_5^\star - x_4^\star)} & \textit{if } x_4^\star \le x^\star < x_5^\star \\ \dfrac{(x_6^\star - x^\star)^3}{(x_6^\star - x_3^\star)(x_6^\star - x_4^\star)(x_6^\star - x_5^\star)}, & \textit{if } x_5^\star \le x^\star < x_6^\star \\ 0, & \textit{otherwise} \end{cases}$$

$$+\vartheta_3 \cdot \begin{cases} \dfrac{(x^\star - x_3^\star)^3}{(x_6^\star - x_3^\star)(x_5^\star - x_3^\star)(x_4^\star - x_3^\star)}, & \textit{if } x_3^\star \le x^\star < x_4^\star \\ \dfrac{(x^\star - x_3^\star)^2(x_5^\star - x^\star)}{(x_6^\star - x_3^\star)(x_5^\star - x_3^\star)(x_5^\star - x_4^\star)} + \dfrac{(x^\star - x_3^\star)(x_6^\star - x^\star)(x^\star - x_4^\star)}{(x_6^\star - x_3^\star)(x_6^\star - x_4^\star)(x_5^\star - x_4^\star)} + \dfrac{(x_7^\star - x^\star)(x^\star - x_4^\star)^2}{(x_7^\star - x_4^\star)(x_6^\star - x_4^\star)(x_5^\star - x_4^\star)} & \textit{if } x_4^\star \le x^\star < x_5^\star \\ \dfrac{(x_6^\star - x^\star)^2(x^\star - x_3^\star)}{(x_6^\star - x_3^\star)(x_6^\star - x_4^\star)(x_6^\star - x_5^\star)} + \dfrac{(x_7^\star - x^\star)(x^\star - x_4^\star)(x_6^\star - x^\star)}{(x_7^\star - x_4^\star)(x_6^\star - x_4^\star)(x_6^\star - x_5^\star)} + \dfrac{(x_7^\star - x^\star)^2(x^\star - x_5^\star)}{(x_7^\star - x_4^\star)(x_7^\star - x_5^\star)(x_6^\star - x_5^\star)} & \textit{if } x_5^\star \le x^\star < x_6^\star \\ \dfrac{(x_7^\star - x^\star)^3}{(x_7^\star - x_4^\star)(x_7^\star - x_5^\star)(x_7^\star - x_6^\star)}, & \textit{if } x_6^\star \le x^\star < x_7^\star \\ 0, & \textit{otherwise} \end{cases}$$

Through the utilization of $\mathcal{T}^\star$ and carrying out subsequent computations, we obtain (10)*:*

$$\mathcal{B}^{\{C,3\}}(x^\star) = \begin{cases} \dfrac{\vartheta_0(b - x^\star)^3}{(b-a)^3} + \dfrac{3\vartheta_1(b - x^\star)^2(x^\star - a)}{(b-a)^3} + \dfrac{3\vartheta_2(x^\star - a)^2(b - x^\star)}{(b-a)^3} + \dfrac{\vartheta_3(x^\star - a)^3}{(b-a)^3} & \textit{if } a \le x^\star \le b \\ 0, & \textit{otherwise} \end{cases} \tag{16}$$

The proof is done.

Lemma 2. *Let $\mathcal{B}^{\{C,3\}}(x^\star)$ represent a cubic $\mathcal{B}$-spline curve defined over the closed bounded interval $[a,b]$ with control points $\vartheta_0, \vartheta_1, \vartheta_2$, and ϑ_3 being four consecutive control points. The α-Caputo fractional derivative of $\mathcal{B}^{\{C,3\}}(x^\star)$, for $\alpha \in (0,1]$, is expressed as:*

$$
\begin{aligned}
{}_{a}^{\mathcal{C}}\mathcal{D}_{x^\star}^{\alpha}\mathcal{B}^{\{C,3\}}(x^\star) = \frac{3(x^\star-a)^{1-\alpha}}{(\mathcal{N}h)^3\,\Gamma(4-\alpha)}\Big[\, & \vartheta_0\Big(2(\mathcal{N}h)(3-\alpha)(x^\star-a) - (\mathcal{N}h)^2(2-\alpha)(3-\alpha) \\
& \quad - 2(x^\star-a)^2\Big) \\
& + \vartheta_1\Big((\mathcal{N}h)^2(2-\alpha)(3-\alpha) - 4(\mathcal{N}h)(3-\alpha)(x^\star-a) \\
& \quad + 6(x^\star-a)^2\Big) \\
& + \vartheta_2\Big((\mathcal{N}h)(3-\alpha)(x^\star-a) - 6(x^\star-a)^2\Big) \\
& \quad + 2\vartheta_3(x^\star-a)^2\,\Big]
\end{aligned}
\tag{17}
$$

Proof: *Use a general expression of a cubic $\mathcal{B}$-spline curve defined on the interval $x^\star \in [a,b]$. From Equation* (16)*, it is given by:*

$$
\begin{aligned}
\mathcal{B}^{\{C,3\}}(x^\star) = {} & \vartheta_0\left(\frac{b-x^\star}{b-a}\right)^3 + 3\vartheta_1\left(\frac{x^\star-a}{b-a}\right)\left(\frac{b-x^\star}{b-a}\right)^2 \\
& + 3\vartheta_2\left(\frac{x^\star-a}{b-a}\right)^2\left(\frac{b-x^\star}{b-a}\right) + \vartheta_3\left(\frac{x^\star-a}{b-a}\right)^3
\end{aligned}
\tag{18}
$$

We simplify the given expression as follows:

$$
\begin{aligned}
\mathcal{B}^{\{C,3\}}(x^\star) = {} & \frac{\vartheta_0}{(b-a)^3}(b-x^\star)^3 + \frac{3\vartheta_1}{(b-a)^3}(x^\star-a)(b-x^\star)^2 \\
& + \frac{3\vartheta_2}{(b-a)^3}(x^\star-a)^2(b-x^\star) + \frac{\vartheta_3}{(b-a)^3}(x^\star-a)^3
\end{aligned}
\tag{19}
$$

From the properties of the Caputo fractional derivative in Definition 2, for $\alpha \in (0,1]$, we can express the cubic $\mathcal{B}$-spline curve for all $x^\star \in [a,b]$ as:

$$\begin{aligned}
{}_{a}^{C}\mathcal{D}_{x^\star}^{\alpha}\mathcal{B}^{\{C,3\}}(x^\star) = {} & \frac{\vartheta_0}{(b-a)^3}\left[\frac{-3(b-a)}{\Gamma(2-\alpha)}(x^\star-a)^{1-\alpha}+\frac{6(b-a)}{\Gamma(3-\alpha)}(x^\star-a)^{2-\alpha}\right. \\
& \left. -\frac{6}{\Gamma(4-\alpha)}(x^\star-a)^{3-\alpha}\right] \\
& +\frac{3\vartheta_1}{(b-a)^3}\left[\frac{(b-a)^2}{\Gamma(2-\alpha)}(x^\star-a)^{1-\alpha}-\frac{4(b-a)}{\Gamma(3-\alpha)}(x^\star-a)^{2-\alpha}\right. \\
& \left. +\frac{6}{\Gamma(4-\alpha)}(x^\star-a)^{3-\alpha}\right] \\
& +\frac{3\vartheta_2}{(b-a)^3}\left[\frac{2(b-a)}{\Gamma(3-\alpha)}(x^\star-a)^{2-\alpha}-\frac{6}{\Gamma(4-\alpha)}(x^\star-a)^{3-\alpha}\right] \\
& +\frac{6\vartheta_3}{(b-a)^3\Gamma(4-\alpha)}(x^\star-a)^{3-\alpha}
\end{aligned} \tag{20}$$

A cubic $\mathcal{B}$-spline curve defined over the closed, bounded interval $[a,b]$ with control points $\vartheta_0, \vartheta_1, \vartheta_2$, and ϑ_3 - where h is the step size and $\mathcal{N}$ is the number of iterations - therefore, (17) *yields the desired fractional derivative expression.*

4. Derivation of the method

The main aim of this section is to apply the cubic $\mathcal{B}$-spline collocation method in order to introduce a numerical approximation technique to solve the problem (1), under the initial condition given by (2). The method expresses the solution as a cubic combination of $\mathcal{B}$-spline basis functions.

Let $a = x_0^\star < x_1^\star < x_2^\star < \cdots < x_{\mathcal{N}-1}^\star < x_{\mathcal{N}}^\star = b$, be a uniform mesh partition of the domain $[a,b]$ defined by the knots $x_r^\star$, with

$$h = \frac{x_{r+1}^\star - x_r^\star}{\mathcal{N}} = \frac{b-a}{\mathcal{N}}, \quad r = 0,1,\ldots,\mathcal{N}-1.$$

For all $x^\star \in [a,b]$, let $\mathcal{Y}_i(x) \approx \mathcal{B}_i^{\{C,3\}}(x^\star)$ for each $i = 0,1,\ldots,m$, using the collocation technique with cubic $\mathcal{B}$-spline basis functions to find an approximate solution $\mathcal{B}_i^{\{C,3\}}(x^\star)$, given by:

$$\mathcal{B}_i^{\{C,3\}}(x^\star) = \sum_{z=0}^{n} \vartheta_{iz}\,\mathcal{B}_{iz}^{k}(x^\star), \quad i = 0,1,\ldots,m \tag{21}$$

In order to solve the approximate solution of (21), we determine the control points ϑ_{iz} for all $i = 0,1,\ldots,m$ and $z = 0,\ldots,n$.

- ϑ_{i0} is determined by the initial condition (2).
- ϑ_{i1} is determined from

$$\vartheta_{i1} = \frac{\mathcal{N}h}{3} \left. \frac{d\mathcal{B}_{i1}^{\{C,3\}}(x^\star)}{dx^\star} \right|_{x^\star=a} + \vartheta_{i0},$$

where

$$\begin{aligned} \left. \frac{d\,\mathcal{B}_{i1}^{\{C,3\}}(x^\star)}{dx^\star} \right|_{x^\star=a} &\approx \mathcal{Y}_i'(x^\star)\big|_{x^\star=a} \\ &= \frac{1}{\mathcal{P}_i(x^\star)} \Bigg[F_i(x^\star) + \sum_{j=0}^{m} \omega_{ij} \int_a^{x^\star} \mathcal{K}_{ij}(x^\star, s)\, {}_a^C D_s^{\beta_{ij}} \mathcal{Y}_j(s)\, ds \\ &\qquad - a_{i0}(x^\star)\, {}_a^C D_{x^\star}^{\sigma_{i0}} \mathcal{Y}_i(x^\star) - a_{i1}(x^\star)\, \mathcal{Y}_i(x^\star) \Bigg] \Bigg|_{x^\star=a} \end{aligned}$$

- ϑ_{i2} is determined from

$$\vartheta_{i2} = \frac{(\mathcal{N}h)^2}{6} \left. \frac{d^2\mathcal{B}_{i1}^{\{C,3\}}(x^\star)}{dx^{\star 2}} \right|_{x^\star=a} - \vartheta_{i0} + 2\vartheta_{i1},$$

where the second derivative is approximated using a forward difference formula:

$$\left. \frac{d^2\mathcal{B}_{i1}^{\{C,3\}}(x^\star)}{dx^{\star 2}} \right|_{x^\star=a} \approx \mathcal{Y}_i''(x_r^\star) \approx \frac{\mathcal{Y}_i^{(p)}(x_{r+2}^\star) - 2\mathcal{Y}_i^{(p)}(x_{r+1}^\star) + \mathcal{Y}_i(x_r^\star)}{h^2}$$

In this case, $\mathcal{Y}_i(x_r^\star)$ is known from the initial condition (2), and $\mathcal{Y}_i^{(p)}(x_{r+2}^\star)$ and $\mathcal{Y}_i^{(p)}(x_{r+1}^\star)$ are determined from the first-order $\mathcal{B}$-spline [31].

- ϑ_{i3} is determined from the system of (VIDE's-CF) in (1).

Let the unknown function $\mathcal{B}_i^{\{C,3\}}(x^\star)$ be approximated using cubic $\mathcal{B}$-spline interpolation of degree 3, as given in (21). Substituting this approximation into (1), we obtain:

$$\begin{aligned} \mathcal{P}_i(x^\star) \frac{d}{dx^\star} \left(\sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}_{iz}^k(x^\star) \right) &+ a_{i0}(x^\star)\, {}_a^{\mathcal{C}}\mathcal{D}_{x^\star}^{\sigma_{i0}} \left(\sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}_{iz}^k(x^\star) \right) \\ &+ a_{i1}(x^\star) \sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}_{iz}^k(x^\star) \\ &= \mathcal{F}_i(x^\star) + \sum_{j=0}^{m} \omega_{ij} \int_a^{x^\star} \mathcal{K}_{ij}(x^\star, s)\, {}_a^{\mathcal{C}}\mathcal{D}_s^{\beta_{ij}} \left(\sum_{z=0}^{n} \vartheta_{jz}\, \mathcal{B}_{jz}^k(s) \right) ds \end{aligned} \tag{22}$$

Consequently, applying the linearity property of the Caputo fractional derivative from Definition 2, along with the standard derivative, we derive the following expression:

$$\begin{aligned}\mathcal{P}_i(x^\star)\sum_{z=0}^{n}\vartheta_{iz}\frac{d}{dx^\star}\left[\mathcal{B}^k_{iz}(x^\star)\right] + a_{i0}(x^\star)\sum_{z=0}^{n}\vartheta_{iz}\,{}^{\mathcal{C}}_{a}\mathcal{D}^{\sigma_{i0}}_{x^\star}\left[\mathcal{B}^k_{iz}(x^\star)\right]&\\ + a_{i1}(x^\star)\sum_{z=0}^{n}\vartheta_{iz}\,\mathcal{B}^k_{iz}(x^\star)&\\ = \mathcal{F}_i(x^\star) + \sum_{j=0}^{m}\omega_{ij}\int_a^{x^\star}\mathcal{K}_{ij}(x^\star,s)\sum_{z=0}^{n}\vartheta_{jz}\,{}^{\mathcal{C}}_{a}\mathcal{D}^{\beta_{ij}}_{s}\left[\mathcal{B}^k_{jz}(s)\right]ds&\end{aligned} \tag{23}$$

By defining $x^\star = x^\star_{r+1}$ and analyzing the collocation points, we employ a cubic $\mathcal{B}$-spline function ($k = 3$, $n = 3$) within the interval $[x^\star_r, x^\star_{r+1}]$. We then apply the fractional derivative in the Caputo sense. From (16), the fractional orders $\sigma_{i0}, \beta_{ij} \in (0, 1]$ for all $i, j = 0, 1, \ldots, m$, yielding results for each $r = 0, 1, \ldots, \mathcal{N} - 1$.

$$\begin{aligned}\mathcal{P}_i(x^\star_{r+1})\Bigg\{ &\frac{-3\vartheta_{i0}(b-x^\star_{r+1})^2}{(\mathcal{N}h)^3} + 3\vartheta_{i1}\left[\frac{-2(x^\star_{r+1}-a)(b-x^\star_{r+1})}{(\mathcal{N}h)^3} + \frac{(b-x^\star_{r+1})^2}{(\mathcal{N}h)^3}\right]\\ &+3\vartheta_{i2}\left[\frac{-(x^\star_{r+1}-a)^2}{(\mathcal{N}h)^3} + \frac{2(b-x^\star_{r+1})(x^\star_{r+1}-a)}{(\mathcal{N}h)^3}\right] + 3\vartheta_{i3}\cdot\frac{(x^\star_{r+1}-a)^2}{(\mathcal{N}h)^3}\Bigg\}\\ &+a_{i0}(x^\star_{r+1})\cdot\frac{(x^\star_{r+1}-a)^{1-\sigma_{i0}}}{(\mathcal{N}h)^3\,\Gamma(4-\sigma_{i0})}\cdot\Bigg\{\vartheta_{i0}\Big[-3(\mathcal{N}h)^2(2-\sigma_{i0})(3-\sigma_{i0})\\ &+6(\mathcal{N}h)(3-\sigma_{i0}) - 6(x^\star_{r+1}-a)^2\Big] + \vartheta_{i1}\Big[(\mathcal{N}h)^2(2-\sigma_{i0})(3-\sigma_{i0})\\ &-4(\mathcal{N}h)(3-\sigma_{i0})(x^\star_{r+1}-a) + 6(x^\star_{r+1}-a)^2\Big]\\ &+3\vartheta_{i2}\Big[2(\mathcal{N}h)(3-\sigma_{i0})(x^\star_{r+1}-a) - 6(x^\star_{r+1}-a)^2\Big] + 6\vartheta_{i3}(x^\star_{r+1}-a)^2\Bigg\}\\ &+a_{i1}(x^\star_{r+1})\cdot\Bigg\{\vartheta_{i0}\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right)^3 + 3\vartheta_{i1}\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right)^2\\ &+3\vartheta_{i2}\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)^2\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right) + \vartheta_{i3}\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)^3\Bigg\}\\ = \;&\mathcal{F}_i(x^\star_{r+1}) + \sum_{j=1}^{m}\omega_{ij}\Bigg\{\sum_{q=0}^{r-1}\int_{x^\star_q}^{x^\star_{q+1}}\mathcal{K}_{ij}(x^\star_{r+1},s)\cdot\frac{(s-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^3\,\Gamma(4-\beta_{ij})}\cdot\Big[\\ &\vartheta_{j0}\left(-3(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) + 6(\mathcal{N}h)(3-\beta_{ij}) - 6(s-a)^2\right)\\ &+3\vartheta_{j1}\left((\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) - 4(\mathcal{N}h)(3-\beta_{ij})(s-a) + 6(s-a)^2\right)\end{aligned}$$

$$+3\vartheta_{j2}\left(2(\mathcal{N}h)(3-\beta_{ij})(s-a)-6(s-a)^2\right)+6\vartheta_{j3}(s-a)^2\Bigg]\,ds\Bigg\}$$

$$\int_{x_r^\star}^{x_{r+1}^\star}\mathcal{K}_{ij}(x_{r+1}^\star,s)\cdot\frac{(s-a)^{1-\beta_{ij}}}{(\mathcal{N}h)^3\,\Gamma(4-\beta_{ij})}\cdot\Bigg[$$
$$\vartheta_{j0}\left(-3(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij})+6(\mathcal{N}h)(3-\beta_{ij})-6(s-a)^2\right)$$
$$+3\vartheta_{j1}\left((\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij})-2(s-a)-4(\mathcal{N}h)(3-\beta_{ij})(s-a)\right)$$
$$+6(s-a)^2+3\vartheta_{j2}\left(2(\mathcal{N}h)(3-\beta_{ij})(s-a)-6(s-a)^2\right)+6\vartheta_{j3}(s-a)^2\Bigg]\,ds \tag{24}$$

The cubic $\mathcal{B}$-spline function over the interval $[x_r^\star, x_{r+1}^\star]$ is optimized to simplify its representation and promote efficient solution methods in practice, these integrals must be approximated using the Clenshaw-Curtis quadrature rule [12, 25, 26], hence, (25) is applicable for $r = 0, 1, \ldots, \mathcal{N}-1$ and $i, j = 0, 1, \ldots, m$.

$$\mathcal{V}^{\sigma_{i0}}_{i,x_{r+1}^\star}\cdot\vartheta_{i3}+\mathcal{W}^{\sigma_{i0}}_{i,x_{r+1}^\star}=\mathcal{F}_i(x_{r+1}^\star)+\sum_{j=1}^{m}\omega_{ij}\left\{\mathcal{E}^r_{ij}\right\}+\sum_{j=1}^{m}\omega_{ij}\left\{\check{\mathcal{E}}^r_{ij}\right\}\cdot\vartheta_{j3}+\mathcal{G}^r_{ij}+\check{\mathcal{G}}^r_i\cdot\vartheta_{03} \tag{25}$$

To obtain the unique solution of the system described in (26), we require three additional equations. These are obtained by computing the coefficients ϑ_{i0}, ϑ_{i1}, and ϑ_{i2}. Once these are determined, the remaining coefficient ϑ_{i3} can be calculated by solving the following linear system of algebraic equations of size $(m+1)\times(m+1)$:

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{C} \tag{26}$$

$$\mathbf{A}=\begin{bmatrix}\mathcal{V}^{\sigma_{00}}_{0,x_{r+1}^\star}-\check{\mathcal{G}}^r_0 & -\omega_{01}\check{\mathcal{E}}^r_{01} & -\omega_{02}\check{\mathcal{E}}^r_{02} & \cdots & -\omega_{0m}\check{\mathcal{E}}^r_{0m}\\ -\check{\mathcal{G}}^r_1 & \mathcal{V}^{\sigma_{10}}_{1,x_{r+1}^\star}-\omega_{11}\check{\mathcal{E}}^r_{11} & -\omega_{12}\check{\mathcal{E}}^r_{12} & \cdots & -\omega_{1m}\check{\mathcal{E}}^r_{1m}\\ -\check{\mathcal{G}}^r_2 & -\omega_{21}\check{\mathcal{E}}^r_{21} & \mathcal{V}^{\sigma_{20}}_{2,x_{r+1}^\star}-\omega_{22}\check{\mathcal{E}}^r_{22} & \cdots & -\omega_{2m}\check{\mathcal{E}}^r_{2m}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\check{\mathcal{G}}^r_m & -\omega_{m1}\check{\mathcal{E}}^r_{m1} & -\omega_{m2}\check{\mathcal{E}}^r_{m2} & \cdots & \mathcal{V}^{\sigma_{m0}}_{m,x_{r+1}^\star}-\omega_{mm}\check{\mathcal{E}}^r_{mm}\end{bmatrix} \tag{27}$$

$$\mathbf{B}=\begin{bmatrix}\vartheta_{03}\\ \vartheta_{13}\\ \vartheta_{23}\\ \vdots\\ \vartheta_{m3}\end{bmatrix} \tag{28}$$

$$\mathbf{C}=\begin{bmatrix}\mathcal{L}^0_r\\ \mathcal{L}^1_r\\ \mathcal{L}^2_r\\ \vdots\\ \mathcal{L}^m_r\end{bmatrix} \tag{29}$$

where,

- $$\mathcal{V}^{\sigma_{i0}}_{i,x^\star_{r+1}} = \frac{3\,\mathcal{P}_i(x^\star_{r+1})(x^\star_{r+1}-a)^2}{(\mathcal{N}h)^3} + \frac{6\,a_{i0}(x^\star_{r+1})(x^\star_{r+1}-a)^{3-\sigma_{i0}}}{(\mathcal{N}h)^3\Gamma(4-\sigma_{i0})} + a_{i1}(x^\star_{r+1})\cdot\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)^3.$$

- $$\begin{aligned}\mathcal{W}^{\sigma_{i0}}_{i,x^\star_{r+1}} \;=\;& \mathcal{P}_i(x^\star_{r+1})\Bigg\{\frac{-3\vartheta_{i0}(b-x^\star_{r+1})}{(\mathcal{N}h)^3} + 3\vartheta_{i1}\left[\frac{-2(x^\star_{r+1}-a)(b-x^\star_{r+1})}{(\mathcal{N}h)^3} + \frac{(b-x^\star_{r+1})^2}{(\mathcal{N}h)^3}\right]\\ & +3\vartheta_{i2}\left[\frac{-(x^\star_{r+1}-a)^2}{(\mathcal{N}h)^3} + \frac{2(b-x^\star_{r+1})(x^\star_{r+1}-a)}{(\mathcal{N}h)^3}\right]\Bigg\}\\ & +a_{i0}(x^\star_{r+1})\cdot\frac{(x^\star_{r+1}-a)^{1-\sigma_{i0}}}{(\mathcal{N}h)^3\Gamma(4-\sigma_{i0})}\cdot\Big[\vartheta_{i0}\left(-3(\mathcal{N}h)^2(2-\sigma_{i0})\right)\\ & (3-\sigma_{i0}) + 6(\mathcal{N}h)(3-\sigma_{i0}) - 6(x^\star_{r+1}-a)^2 + \vartheta_{i1}\left((\mathcal{N}h)^2\right)\\ & (2-\sigma_{i0})(3-\sigma_{i0}) - 4(\mathcal{N}h)(3-\sigma_{i0})(x^\star_{r+1}-a) + 6(x^\star_{r+1}-a)^2\\ & +3\vartheta_{i2}\left(2(\mathcal{N}h)(3-\sigma_{i0})(x^\star_{r+1}-a) - 6(x^\star_{r+1}-a)^2\right)\Big]\\ & +a_{i1}(x^\star_{r+1})\cdot\Bigg[\vartheta_{i0}\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right)^3 + 3\vartheta_{i1}\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right)^2\\ & +3\vartheta_{i2}\left(\frac{x^\star_{r+1}-a}{\mathcal{N}h}\right)^2\left(\frac{b-x^\star_{r+1}}{\mathcal{N}h}\right)\Bigg].\end{aligned}$$

- $$\mathcal{E}^r_{ij} = \sum_{q=0}^{r-1}\sum_{\xi=0}^{\mathcal{N}}\lambda_\xi\,\mathcal{K}_{ij}(x^\star_{r+1},s_\xi)\cdot\frac{(x^\star_{q+1}-x^\star_q)(s_\xi-a)^{1-\beta_{ij}}}{2(\mathcal{N}h)^3\Gamma(4-\beta_{ij})}\cdot$$

$$\left\{\begin{aligned}&\vartheta_{j0}\left[-3(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) + 6(\mathcal{N}h)(3-\beta_{ij}) - 6(s_\xi-a)^2\right]\\ &+3\vartheta_{j1}\left[(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) - 4(\mathcal{N}h)(3-\beta_{ij})(s_\xi-a) - 2(s_\xi-a) + 6(s_\xi-a)^2\right]\\ &+3\vartheta_{j2}\left[2(\mathcal{N}h)(3-\beta_{ij})(s_\xi-a) - 6(s_\xi-a)^2\right]\end{aligned}\right\}$$

$$+\sum_{\xi=0}^{\mathcal{N}}\lambda_\xi\,\mathcal{K}_{ij}(x^\star_{r+1},s_\xi)\cdot\frac{(x^\star_{r+1}-x^\star_r)(s_\xi-a)^{1-\beta_{ij}}}{2(\mathcal{N}h)^3\Gamma(4-\beta_{ij})}\cdot$$

$$\left\{\begin{aligned}&\vartheta_{j0}\left[-3(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) + 6(\mathcal{N}h)(3-\beta_{ij}) - 6(s_\xi-a)^2\right]\\ &+3\vartheta_{j1}\left[(\mathcal{N}h)^2(2-\beta_{ij})(3-\beta_{ij}) - 4(\mathcal{N}h)(3-\beta_{ij})(s_\xi-a) - 2(s_\xi-a) + 6(s_\xi-a)^2\right]\\ &+3\vartheta_{j2}\left[2(\mathcal{N}h)(3-\beta_{ij})(s_\xi-a) - 6(s_\xi-a)^2\right]\end{aligned}\right\}.$$

- $$\breve{\mathcal{E}}_{ij}^{r} = \sum_{q=0}^{r-1}\sum_{\xi=0}^{\mathcal{N}} \frac{3\lambda_\xi\,\mathcal{K}_{ij}(x_{r+1}^\star, s_\xi)\,(x_{q+1}^\star - x_q^\star)\,(s_\xi - a)^{3-\beta_{ij}}}{(\mathcal{N}h)^3\,\Gamma(4-\beta_{ij})} + \sum_{\xi=0}^{\mathcal{N}} \frac{3\lambda_\xi\,\mathcal{K}_{ij}(x_{r+1}^\star, s_\xi)\,(x_{r+1}^\star - x_r^\star)\,(s_\xi - a)^{3-\beta_{ij}}}{(\mathcal{N}h)^3\,\Gamma(4-\beta_{ij})}$$

- $$\mathcal{G}_i^r = \sum_{q=0}^{r-1}\sum_{\xi=0}^{\mathcal{N}} \lambda_\xi\,\mathcal{K}_{i0}(x_{r+1}^\star, s_\xi)\cdot\frac{(x_{q+1}^\star - x_q^\star)}{2}\cdot\left[\vartheta_{00}\left(\frac{b-s_\xi}{\mathcal{N}h}\right)^3 + 3\vartheta_{01}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)\left(\frac{b-s_\xi}{\mathcal{N}h}\right)^2 + 3\vartheta_{02}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)^2\left(\frac{b-s_\xi}{\mathcal{N}h}\right)\right] + \sum_{\xi=0}^{\mathcal{N}} \lambda_\xi\,\mathcal{K}_{i0}(x_{r+1}^\star, s_\xi)\cdot\frac{(x_{r+1}^\star - x_r^\star)}{2}\cdot\left[\vartheta_{00}\left(\frac{b-s_\xi}{\mathcal{N}h}\right)^3 + 3\vartheta_{01}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)\left(\frac{b-s_\xi}{\mathcal{N}h}\right)^2 + 3\vartheta_{02}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)^2\left(\frac{b-s_\xi}{\mathcal{N}h}\right)\right]$$

- $$\breve{\mathcal{G}}_i^r = \sum_{q=0}^{r-1}\sum_{\xi=0}^{\mathcal{N}} \lambda_\xi\,\mathcal{K}_{i0}(x_{r+1}^\star, s_\xi)\cdot\frac{(x_{q+1}^\star - x_q^\star)}{2}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)^3 + \sum_{\xi=0}^{\mathcal{N}} \lambda_\xi\,\mathcal{K}_{i0}(x_{r+1}^\star, s_\xi)\cdot\frac{(x_{r+1}^\star - x_r^\star)}{2}\left(\frac{s_\xi - a}{\mathcal{N}h}\right)^3$$

$$\mathcal{L}_r^i = \mathcal{F}_i(x_{r+1}^\star) + \sum_{j=1}^{m}\omega_{ij}\,\mathcal{E}_{ij}^r - \mathcal{W}_{i,x_{r+1}^\star}^{\sigma_{i0}} + \mathcal{G}_{ij}^r \tag{30}$$

Where,

$$t_\xi^\star = \sum_{\xi=0}^{\mathcal{N}''}\cos\left(\frac{\xi\pi}{\mathcal{N}}\right) \tag{31}$$

are the nodes evaluated to correspond to the extrema of the Chebyshev polynomials on the interval $[-1, 1]$.
The mapped node s_ξ is calculated by:

$$s_\xi = \frac{x_{q+1}^\star - x_q^\star}{2}\,t_\xi^\star + \frac{x_{q+1}^\star + x_q^\star}{2} \tag{32}$$

The weights λ_ξ are coefficients that multiply the function values at the nodes to approximate the integral, given by:

$$\lambda_\xi = \frac{2}{\mathcal{N}}\sum_{\substack{r=0\\ \text{even}}}^{\mathcal{N}''}\cos\left(\frac{r\xi\pi}{\mathcal{N}}\right) \tag{33}$$

5. Algorithms

In this section, we present a step-by-step algorithmic description of the proposed approach. To facilitate practical implementation, an itemized version of the algorithm is provided below.

Main Algorithm: LSVIFDE's (Using Cubic $\mathcal{B}$-spline)

Input Data:

- Domain bounds a, b, and total number of steps $\mathcal{N}$.
- Number of equations: $m+1$.
- Functions: $\mathcal{P}_i(x^\star)$, $a_{i0}(x^\star)$, $a_{i1}(x^\star)$, $\mathcal{F}_i(x^\star)$.
- Parameters: ω_{ij}, $\mathcal{K}_{ij}(x^\star, s)$, σ_{i0}, β_{ij}, for all $i, j = 0, 1, \ldots, m$.

Output:

Vector $\mathbf{B}$ from (28), containing control points $\vartheta_{i3},\ i = 0, 1, \ldots, m$.

Steps:

(i) Construct vectors $\mathbf{B}$ and $\mathbf{C}$ of size $m+1$, and matrix $\mathbf{A}$ of size $(m+1)\times(m+1)$.

(ii) Compute the step size:
$$h = \frac{b-a}{\mathcal{N}}, \quad \text{with } \mathcal{N} \in \mathbb{N}$$
and define the partition points:
$$x^\star_{r+1} = a + (r+1)h, \quad \text{for } r = 0, 1, \ldots, \mathcal{N}-1$$

(iii) Approximate:
$$\left.\frac{d\mathcal{B}_i^{\{C,3\}}(x^\star)}{dx^\star}\right|_{x^\star=a} \approx \mathcal{Y}_i'(a)$$
using the expression:
$$\frac{1}{\mathcal{P}_i(a)}\left[\mathcal{F}_i(a) + \sum_{j=0}^{m}\omega_{ij}\int_a^a \mathcal{K}_{ij}(a,s)\,{}^{\mathcal{C}}_{a}D_s^{\beta_{ij}}\mathcal{Y}_j(s)\,ds - a_{i0}(a)\,{}^{\mathcal{C}}_{a}D_a^{\sigma_{i0}}\mathcal{Y}_i(a) - a_{i1}(a)\mathcal{Y}_i(a)\right]$$

(iv) Compute:
$$\vartheta_{i0} = \text{from given initial condition (2)}$$

$$\vartheta_{i1} = \frac{\mathcal{N}h}{3} \cdot \left. \frac{d\mathcal{B}_i^{\{C,3\}}(x^\star)}{dx^\star} \right|_{x^\star=a} + \vartheta_{i0}$$

$$\vartheta_{i2} = \frac{(\mathcal{N}h)^2}{6} \cdot \left. \frac{d^2\mathcal{B}_i^{\{C,3\}}(x^\star)}{d(x^\star)^2} \right|_{x^\star=a} - \vartheta_{i0} + 2\vartheta_{i1}$$

(v) Use forward difference to approximate:

$$\left. \frac{d^2\mathcal{B}_i^{\{C,3\}}(x^\star)}{d(x^\star)^2} \right|_{x^\star=a} \approx \frac{\mathcal{Y}_i^{(p)}(x^\star_{r+2}) - 2\mathcal{Y}_i^{(p)}(x^\star_{r+1}) + \mathcal{Y}_i(x^\star_r)}{h^2}$$

(vi) Evaluate $\mathcal{Y}_i(x^\star_r)$ from (2), and $\mathcal{Y}_i^{(p)}(x^\star_{r+1})$, $\mathcal{Y}_i^{(p)}(x^\star_{r+2})$ using first-order $\mathcal{B}$-spline.

(vii) Compute entries of $\mathbf{C}$ using (29):

$$\mathbf{C} = \begin{bmatrix} \mathcal{L}_r^0 \\ \mathcal{L}_r^1 \\ \vdots \\ \mathcal{L}_r^m \end{bmatrix}$$

(viii) Construct the matrix $\mathbf{A}$ and modify its entries using the computed ϑ_{i0}, ϑ_{i1}, and ϑ_{i2}.

(ix) Solve the system $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$, using Clenshaw-Curtis quadrature for numerical integration.

Output:

$$\mathbf{B} = \begin{bmatrix} \vartheta_{03} & \vartheta_{13} & \vartheta_{23} & \dots & \vartheta_{m3} \end{bmatrix}^T$$

First Procedure: Normalize Finding Cubic $\mathcal{B}$-Spline for Each Interval (NFCBI)

We first perform all steps outlined in the **Main Algorithm (LSVIFDE's)**, and then proceed with the following additional steps:

(i) For $r = 0, 1, \dots, \mathcal{N} - 1$, set:

$$x^\star = x^\star_{r+1} = a + (r+1)h, \quad n = 3, \quad k = 3$$

(ii) For each $i = 0, 1, \dots, m$, evaluate the approximate cubic $\mathcal{B}$-spline solution as:

$$\mathcal{B}_i^{\{C,3\}}(x^\star_{r+1}) \approx \sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}_{iz}^k(x^\star_{r+1})$$

Output: The set of approximate solutions for each function:

$$\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1}),\quad \mathcal{B}_1^{\{C,3\}}(x^\star_{r+1}),\quad \ldots,\quad \mathcal{B}_m^{\{C,3\}}(x^\star_{r+1})$$

represent the cubic $\mathcal{B}$-spline approximation to each $\mathcal{Y}_i(x)$ over the interval $[x^\star_r, x^\star_{r+1}]$.

II. Second Procedure: First Control Point Cubic $\mathcal{B}$-Spline (FCPCBS)

We begin by performing all steps from the **Main Algorithm (LSVIFDE's)**, and then follow the additional steps below:

(i) Use:

$$\vartheta_{i3} = \vartheta_{i3}(x^\star_1),\quad i = 0, 1, \ldots, m$$

(ii) For $r = 0, 1, \ldots, \mathcal{N} - 1$, set:

$$x^\star = x^\star_{r+1} = a + (r+1)h,\quad n = 3,\quad k = 3$$

(iii) For each $i = 0, 1, \ldots, m$, evaluate:

$$\mathcal{B}_i^{\{C,3\}}(x^\star_{r+1}) \approx \sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}^k_{iz}(x^\star_{r+1})$$

Output: The approximate solutions for each function:

$$\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1}),\quad \mathcal{B}_1^{\{C,3\}}(x^\star_{r+1}),\quad \ldots,\quad \mathcal{B}_m^{\{C,3\}}(x^\star_{r+1})$$

III. Third Procedure: Average First and Final Control Point Cubic $\mathcal{B}$-Spline (FFCPCBS)

After completing the **Main Algorithm**, proceed with the following steps:

(i) Use:

$$\vartheta_{i3} = \frac{1}{2}\left(\vartheta_{i3}(x^\star_1) + \vartheta_{i3}(x^\star_{\mathcal{N}-1})\right),\quad i = 0, 1, \ldots, m$$

(ii) For $r = 0, 1, \ldots, \mathcal{N} - 1$, set:

$$x^\star = x^\star_{r+1} = a + (r+1)h,\quad n = 3,\quad k = 3$$

(iii) For each $i = 0, 1, \ldots, m$, compute:

$$\mathcal{B}_i^{\{C,3\}}(x^\star_{r+1}) \approx \sum_{z=0}^{n} \vartheta_{iz}\, \mathcal{B}^k_{iz}(x^\star_{r+1})$$

Output: The resulting approximations:

$$\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1}),\quad \mathcal{B}_1^{\{C,3\}}(x^\star_{r+1}),\quad \ldots,\quad \mathcal{B}_m^{\{C,3\}}(x^\star_{r+1})$$

6. Numerical Results

In this section, we present some numerical examples to demonstrate the accuracy and efficiency of the proposed method. We examine a set of test cases to evaluate the performance of the developed algorithm in solving a system of Volterra integro-differential equations with classical and fractional orders. The obtained results are compared with the exact solution; this method's implementation is supported by a Python program. All the calculations are executed in least square error.

Example 1. *Consider the following classical and fractional order system of Volterra integro-differential equations (CFVIDEs):*

$$\begin{aligned}
e^x\,\mathcal{Y}_0'(x) + x^2\,{}_a^{\mathcal{C}}\mathcal{D}_x^{\sigma_{00}}\mathcal{Y}_0(x) + \cos(x)\,\mathcal{Y}_0(x) &= \mathcal{F}_0(x) + \omega_{00}\int_0^x e^x s\,\mathcal{Y}_0(s)\,ds \\
&\quad + \omega_{01}\int_0^x (sx^2-1)\,{}_a^{\mathcal{C}}\mathcal{D}_s^{\beta_{01}}\mathcal{Y}_1(s)\,ds \\
&\quad + \omega_{02}\int_0^x s^2\sin(x)\,{}_a^{\mathcal{C}}\mathcal{D}_s^{\beta_{02}}\mathcal{Y}_2(s)\,ds,
\end{aligned}$$

$$\begin{aligned}
\cos(x)\,\mathcal{Y}_1'(x) + x^3\,{}_0^{\mathcal{C}}\mathcal{D}_x^{0.6}\mathcal{Y}_1(x) + e^x\,\mathcal{Y}_1(x) &= \mathcal{F}_1(x) + \omega_{10}\int_0^x (x^2 s+1)\,\mathcal{Y}_0(s)\,ds \\
&\quad + \omega_{11}\int_0^x s\sin(x)\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{11}}\mathcal{Y}_1(s)\,ds \\
&\quad + \omega_{12}\int_0^x x^2(s-1)\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{12}}\mathcal{Y}_2(s)\,ds,
\end{aligned}$$

$$\begin{aligned}
2\,\mathcal{Y}_2'(x) + e^x\,{}_0^{\mathcal{C}}\mathcal{D}_x^{0.5}\mathcal{Y}_2(x) + \sin(x)\,\mathcal{Y}_2(x) &= \mathcal{F}_2(x) + \omega_{20}\int_0^x (x^3 s^2-1)\,\mathcal{Y}_0(s)\,ds \\
&\quad + \omega_{21}\int_0^x se^x\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{21}}\mathcal{Y}_1(s)\,ds \\
&\quad + \omega_{22}\int_0^x (s^2+1)\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{22}}\mathcal{Y}_2(s)\,ds.
\end{aligned}$$

The given functions $\mathcal{F}_0(x)$, $\mathcal{F}_1(x)$, and $\mathcal{F}_2(x)$ are defined as follows:

$$\begin{aligned}
\mathcal{F}_0(x) &= e^x + \frac{x^{2.3}}{\Gamma(1.3)} + (x+1)\cos(x) \\
&\quad - \omega_{00}\left(\frac{e^x x^3}{3} + \frac{e^x x^2}{2}\right) + \frac{2\omega_{01}}{\Gamma(1.2)}\left(\frac{x^{4.2}}{2.2} - \frac{x^{1.2}}{1.2}\right) - \frac{3\omega_{02}}{\Gamma(1.5)}\cdot\frac{x^{3.5}}{3.5}\sin(x), \\
\mathcal{F}_1(x) &= -2\cos(x) - \frac{2x^{3.4}}{\Gamma(1.4)} + (1-2x)e^x \\
&\quad - \omega_{10}\left(\frac{x^5}{3} + \frac{x^4}{2} + \frac{x^2}{2} + x\right) + \frac{2\omega_{11}}{\Gamma(1.3)}\cdot\frac{x^{2.3}}{2.3}\sin(x) - \frac{3\omega_{12}}{\Gamma(1.6)}\left(\frac{x^{4.6}}{2.6} - \frac{x^{3.6}}{1.6}\right),
\end{aligned}$$

$$\mathcal{F}_2(x) = 6 + \frac{3e^x x^{0.5}}{\Gamma(1.5)} + (3x-1)\sin(x)$$
$$- \omega_{20}\left(\frac{x^7}{4} + \frac{x^6}{3} - \frac{x^2}{2} - x\right) + \frac{2\omega_{21}}{\Gamma(1.1)} \cdot \frac{e^x x^{2.1}}{2.1} - \frac{3\omega_{22}}{\Gamma(1.7)}\left(\frac{x^{3.7}}{3.7} + \frac{x^{1.7}}{1.7}\right).$$

Initial Conditions:

$$\mathcal{Y}_0(0) = 1, \quad \mathcal{Y}_1(0) = 1, \quad \mathcal{Y}_2(0) = -1$$

Exact Solutions:

$$\mathcal{Y}_0(x) = x+1, \quad \mathcal{Y}_1(x) = 1-2x, \quad \mathcal{Y}_2(x) = 3x-1$$

Parameters:

$$\begin{aligned}
&\sigma_{00} = 0.7, \quad \sigma_{10} = 0.6, \quad \sigma_{20} = 0.5,\\
&\beta_{01} = 0.8, \quad \beta_{02} = 0.5, \quad \beta_{11} = 0.7,\\
&\beta_{12} = 0.4, \quad \beta_{21} = 0.9, \quad \beta_{22} = 0.3,\\
&\omega_{00} = \frac{\sin(0.3)}{\Gamma(13)}, \quad \omega_{01} = \frac{\sinh(0.7)}{\Gamma^4(16)}, \quad \omega_{02} = \frac{\cosh(30)}{\Gamma^5(14)},\\
&\omega_{10} = \frac{\cos(89)}{\Gamma(12)}, \quad \omega_{11} = \frac{\ln(5)}{\Gamma(12)}, \quad \omega_{12} = \frac{\sinh(0.3)}{\Gamma(11)},\\
&\omega_{20} = \frac{\cos(89)}{\Gamma^2(9)}, \quad \omega_{21} = \frac{\sin(179)}{\Gamma(11)}, \quad \omega_{22} = \frac{\sin(30)}{\Gamma(12)},\\
&\mathcal{N} = 10, \quad h = 0.1, \quad x_r^\star = a + rh, \quad r = 0,1,\ldots,\mathcal{N}-1.
\end{aligned}$$

We aim to approximate the solutions $\mathcal{B}_i^{\{C,3\}}(x_{r+1}^\star)$, $i = 0,1,2$ *as defined in* (21). *We execute the programs NFCBI, FCPCBS, and FFCPCBS to compute the unknown control points* $\vartheta_{i0}, \vartheta_{i1}, \vartheta_{i2}, \vartheta_{i3}$ *for* $i = 0,1,2$. *These control points are used to construct the approximate solutions for the given system.*

The first table demonstrates values of all control points for $\mathcal{B}_0^{\{C,3\}}(x_{r+1}^\star)$, $\mathcal{B}_1^{\{C,3\}}(x_{r+1}^\star)$, *and* $\mathcal{B}_2^{\{C,3\}}(x_{r+1}^\star)$ *according to the three algorithms NFCBI, FCPCBS, and FFCPCBS, respectively.*

Example 2. *Consider the following classical and fractional order system of Volterra integro-differential equations (CFVIDEs):*

$$e^x\,\mathcal{Y}_0'(x) + x^4\,{}_0^{\mathcal{C}}\mathcal{D}_x^{\sigma_{00}}\mathcal{Y}_0(x) + \cos(x)\,\mathcal{Y}_0(x) = \mathcal{F}_0(x) + \omega_{00}\int_0^x (\cos(x) - s^2)\,\mathcal{Y}_0(s)\,ds$$
$$+ \omega_{01}\int_0^x (3x - s^2)\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{01}}\mathcal{Y}_1(s)\,ds,$$

$$\cos(x)\,\mathcal{Y}_1'(x) + x\,{}_0^{\mathcal{C}}\mathcal{D}_x^{\sigma_{10}}\mathcal{Y}_1(x) + e^x\,\mathcal{Y}_1(x) = \mathcal{F}_1(x) + \omega_{10}\int_0^x (x^3 - s^4 + 4)\,\mathcal{Y}_0(s)\,ds$$

$$+\,\omega_{11}\int_0^x (\sin(x)+1)\,{}_0^{\mathcal{C}}\mathcal{D}_s^{\beta_{11}}\mathcal{Y}_1(s)\,ds.$$

The given functions $\mathcal{F}_0(x)$, and $\mathcal{F}_1(x)$ are defined as follows:

$$\begin{aligned}
\mathcal{F}_0(x) &= 2xe^x + \frac{2}{\Gamma(2.3)}x^{5.3} + (x^2-1)\cos(x)\\
&\quad - \omega_{00}\left(\frac{x^3}{3}\cos(x) - x\cos(x) - \frac{x^5}{5} - \frac{x^3}{3}\right) + \frac{2\omega_{01}}{\Gamma(2.4)}\left(\frac{3x^{3.4}}{2.4} - \frac{x^{4.4}}{4.4}\right),\\
\mathcal{F}_1(x) &= -2x\cos(x) - \frac{2}{\Gamma(2.6)}x^{2.6} + (-x^2+1)e^x\\
&\quad - \omega_{10}\left(\frac{x^6}{3} - \frac{x^7}{7} - \frac{4x^3}{3} - x^4 + \frac{x^5}{5} - 4x\right) - \frac{2\omega_{11}}{\Gamma(2.2)}\left(\frac{x^{2.2}}{2.4}\sin(x) + \frac{x^{2.2}}{2.2}\right).
\end{aligned}$$

Initial Conditions:

$$\mathcal{Y}_0(0) = -1, \quad \mathcal{Y}_1(0) = 1$$

Exact Solutions:

$$\mathcal{Y}_0(x) = x^2 - 1, \quad \mathcal{Y}_1(x) = -x^2 + 1$$

Parameters:

$$\begin{aligned}
&\sigma_{00} = 0.7, \quad \sigma_{10} = 0.4, \quad \beta_{01} = 0.6, \quad \beta_{11} = 0.8,\\
&\omega_{00} = \frac{\sin(0.3)}{\Gamma(13)}, \quad \omega_{01} = \frac{\sinh(0.7)}{\Gamma^4(16)}, \quad \omega_{10} = \frac{\cos(89)}{\Gamma(15)}, \quad \omega_{11} = \frac{\ln(2)}{\Gamma(14)},\\
&\mathcal{N} = 10, \quad h = 0.1, \quad x_r^\star = a + rh, \quad r = 0, 1, \ldots, \mathcal{N}-1.
\end{aligned}$$

We aim to approximate the solutions $\mathcal{B}_i^{\{C,3\}}(x_{r+1}^\star)$, $i = 0, 1$ as defined in (21). *We execute the programs NFCBI, FCPCBS, and FFCPCBS to compute the unknown control points $\vartheta_{i0}, \vartheta_{i1}, \vartheta_{i2}, \vartheta_{i3}$ for $i = 0, 1$. These control points are used to construct the approximate solutions for the given system.*

The sixth table demonstrates values of all control points for $\mathcal{B}_0^{\{C,3\}}(x_{r+1}^\star)$ and $\mathcal{B}_1^{\{C,3\}}(x_{r+1}^\star)$ according to the three algorithms NFCBI, FCPCBS, and FFCPCBS, respectively.

Example 3. *Consider the following classical and fractional order system of Volterra integro-differential equations (CFVIDEs):*

$$\begin{aligned}
\mathcal{Y}_0'(x) + x\,{}_0^{\mathcal{C}}\mathcal{D}_x^{0.4}\mathcal{Y}_0(x) + \sin(x)\,\mathcal{Y}_0(x) &= \mathcal{F}_0(x) + \omega_{00}\int_0^x xs\,\mathcal{Y}_0(s)\,ds\\
&\quad + \omega_{01}\int_0^x e^x\,{}_0^{\mathcal{C}}\mathcal{D}_s^{0.9}\mathcal{Y}_1(s)\,ds,
\end{aligned}$$

$$\mathcal{Y}_1'(x) + x^2\,{}_0^{\mathcal{C}}\mathcal{D}_x^{0.5}\mathcal{Y}_1(x) + e^x\,\mathcal{Y}_1(x) = \mathcal{F}_1(x) + \omega_{10}\int_0^x (1+3x)\,\mathcal{Y}_0(s)\,ds$$

$$+ \omega_{11} \int_0^x \cos(x) \, {}_0^{\mathcal{C}}\mathcal{D}_s^{0.7} \mathcal{Y}_1(s) \, ds.$$

The given functions $\mathcal{F}_0(x)$ and $\mathcal{F}_1(x)$ are defined as follows:

$$\mathcal{F}_0(x) = 3x^2 + \frac{6}{\Gamma(3.6)} x^{3.6} + (x^3 + 1)\sin(x) - \omega_{00} \left(\frac{x^6}{5} + \frac{x^3}{2} \right) - \frac{2\omega_{01} e^x}{1.1\Gamma(1.1)} x^{1.1}$$

$$\mathcal{F}_1(x) = 2 + \frac{2}{\Gamma(1.5)} x^{2.5} + e^x(1 + 2x) - \omega_{10} \left(\frac{x^4}{4} + x + \frac{3x^5}{4} + 3x^2 \right) - \frac{2\omega_{11}\cos(x)}{1.3\Gamma(1.3)} x^{1.3}$$

Initial Conditions:

$$\mathcal{Y}_0(0) = 1, \quad \mathcal{Y}_1(0) = 1$$

Exact Solutions:

$$\mathcal{Y}_0(x) = x^3 + 1, \quad \mathcal{Y}_1(x) = 1 + 2x$$

Parameters:

$$\begin{aligned}
&\sigma_{00} = 0.4, \quad \sigma_{10} = 0.5, \quad \beta_{01} = 0.9, \quad \beta_{11} = 0.7, \\
&\omega_{00} = \frac{\sin(0.3)}{\Gamma^2(10)}, \quad \omega_{01} = \frac{-\ln(0.4 \times 0.7)}{\Gamma(20)}, \\
&\omega_{10} = \frac{\cos(89)}{\Gamma^2(10)}, \quad \omega_{11} = \frac{\log(0.99999)}{\Gamma(1.5)}, \\
&\mathcal{N} = 10, \quad h = 0.1, \quad x_r^\star = a + rh, \quad r = 0, 1, \ldots, \mathcal{N} - 1
\end{aligned}$$

We aim to approximate the solutions $\mathcal{B}_i^{\{C,3\}}(x_{r+1}^\star)$, for $i = 0, 1$, as defined in equation (21). *We execute the programs NFCBI, FCPCBS, and FFCPCBS to compute the unknown control points $\vartheta_{i0}, \vartheta_{i1}, \vartheta_{i2}, \vartheta_{i3}$ for $i = 0, 1$. These control points are used to construct the approximate solutions for the given system.*

The 10th table demonstrates values of all control points for $\mathcal{B}_0^{\{C,3\}}(x_{r+1}^\star)$ and $\mathcal{B}_1^{\{C,3\}}(x_{r+1}^\star)$, according to the three algorithms: NFCBI, FCPCBS, and FFCPCBS, respectively.

Table 1: The values of control points of $\mathcal{B}_0^{\{C,3\}}(x_{r+1}^\star)$, $\mathcal{B}_1^{\{C,3\}}(x_{r+1}^\star)$, and $\mathcal{B}_2^{\{C,3\}}(x_{r+1}^\star)$ for three methods NFCBI, FCPCBS and FFCPCBS, respectively, for example 1.

Method	Interval	CP of the $\mathcal{B}_0^{\{C,3\}}(x_{r+1}^\star)$	CP of the $\mathcal{B}_1^{\{C,3\}}(x_{r+1}^\star)$	CP of the $\mathcal{B}_2^{\{C,3\}}(x_{r+1}^\star)$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.0, 0.1]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9998843842$	$\vartheta_{13} = -1.000000104$	$\vartheta_{23} = 2.0000000055$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.1, 0.2]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$

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Method	Interval	CP of the $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of the $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$	CP of the $\mathcal{B}_2^{\{C,3\}}(x^\star_{r+1})$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999513774$	$\vartheta_{13} = -1.000000044$	$\vartheta_{23} = 2.0000000024$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.2, 0.3]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999738009$	$\vartheta_{13} = -1.000000024$	$\vartheta_{23} = 2.0000000014$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.3, 0.4]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999850724$	$\vartheta_{13} = -1.000000014$	$\vartheta_{23} = 2.0000000010$
NFCBI		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.4, 0.5]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999918765$	$\vartheta_{13} = -1.000000007$	$\vartheta_{23} = 2.0000000010$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.5, 0.6]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999964421$	$\vartheta_{13} = -1.000000003$	$\vartheta_{23} = 2.0000000011$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.6, 0.7]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999997248$	$\vartheta_{13} = -1.000000000$	$\vartheta_{23} = 2.0000000013$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.7, 0.8]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 2.0000022027$	$\vartheta_{13} = -0.999999997$	$\vartheta_{23} = 2.0000000017$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.8, 0.9]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 2.0000041419$	$\vartheta_{13} = -0.999999995$	$\vartheta_{23} = 2.0000000023$
		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	]0.9, 1.0]	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$

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Method	Interval	CP of the $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of the $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$	CP of the $\mathcal{B}_2^{\{C,3\}}(x^\star_{r+1})$
		$\vartheta_{03} = 2.000005702$	$\vartheta_{13} = -0.999999994$	$\vartheta_{23} = 2.0000000029$
FCPCBS		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.0000000000$
	$]0.0,\ 1.0]$	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.66667334902$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9998843842$	$\vartheta_{13} = -1.000000104$	$\vartheta_{23} = 2.0000000055$
FFCPCBS		$\vartheta_{00} = 1.0000000000$	$\vartheta_{10} = 1.0000000000$	$\vartheta_{20} = -1.000000000$
	$]0.0,\ 1.0]$	$\vartheta_{01} = 1.3333333333$	$\vartheta_{11} = 0.3333333333$	$\vartheta_{21} = 0.0000000000$
		$\vartheta_{02} = 1.6666733490$	$\vartheta_{12} = -0.333333327$	$\vartheta_{22} = 0.9999999996$
		$\vartheta_{03} = 1.9999450432$	$\vartheta_{13} = -1.000000049$	$\vartheta_{23} = 2.0000000042$

From (21), we obtain the approximate solution for the classical and fractional order linear system of Volterra integro-differential equations (SVIDEs-CF) with variable coefficients, for Example 1 as shown below:

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} -0.00013566283\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } 0 < x^\star \le \frac{1}{10} \\ -0.0000486226\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{1}{10} < x^\star \le \frac{1}{5} \\ -0.00002619903\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{1}{5} < x^\star \le \frac{3}{10} \\ -0.00001492755\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{3}{10} < x^\star \le \frac{2}{5} \\ -0.00000812343\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{2}{5} < x^\star \le \frac{1}{2} \\ -0.00000355785\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{1}{2} < x^\star \le \frac{3}{5} \\ -0.0000002752\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{3}{5} < x^\star \le \frac{7}{10} \\ \ \ 0.00000220271\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{7}{10} < x^\star \le \frac{4}{5} \\ \ \ 0.00000414190\, x^{\star 3} + 0.00002004706\, x^{\star 2} + x^\star + 1 & \text{if } \frac{4}{5} < x^\star \le \frac{9}{10} \\ \ \ 0.00000570220\, x^{\star 3} + 0.0000000181\, x^{\star 2} + x^\star + 1 & \text{if } \frac{9}{10} < x^\star \le 1 \end{cases}$$

$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} -0.0000001223\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } 0 < x^\star \le \frac{1}{10} \\ -0.0000000442\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{1}{10} < x^\star \le \frac{1}{5} \\ -0.0000000242\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{1}{5} < x^\star \le \frac{3}{10} \\ -0.0000000140\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{3}{10} < x^\star \le \frac{2}{5} \\ -0.0000000077\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{2}{5} < x^\star \le \frac{1}{2} \\ -0.0000000033\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{1}{2} < x^\star \le \frac{3}{5} \\ -0.0000000001\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{3}{5} < x^\star \le \frac{7}{10} \\ \ \ 0.0000000024\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{7}{10} < x^\star \le \frac{4}{5} \\ \ \ 0.0000000042\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{4}{5} < x^\star \le \frac{9}{10} \\ \ \ 0.0000000054\, x^{\star 3} + 0.00002004706\, x^{\star 2} - 2x^\star + 1 & \text{if } \frac{9}{10} < x^\star \le 1 \end{cases}$$

$$\mathcal{B}_2^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} 0.00000000652\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } 0 < x^\star \le \frac{1}{10} \\ 0.00000000241\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{1}{10} < x^\star \le \frac{1}{5} \\ 0.00000000145\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{1}{5} < x^\star \le \frac{3}{10} \\ 0.00000000108\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{3}{10} < x^\star \le \frac{2}{5} \\ 0.00000000101\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{2}{5} < x^\star \le \frac{1}{2} \\ 0.00000000113\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{1}{2} < x^\star \le \frac{3}{5} \\ 0.00000000139\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{3}{5} < x^\star \le \frac{7}{10} \\ 0.00000000179\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{7}{10} < x^\star \le \frac{4}{5} \\ 0.00000000232\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{4}{5} < x^\star \le \frac{9}{10} \\ 0.00000000298\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } \frac{9}{10} < x^\star \le 1 \end{cases}$$

$$\text{FCPCBS} = \begin{cases} \mathcal{B}_0^{\{C,3\}}(x^\star) = -0.00013566283\,x^{\star 3} + 0.00002004706\,x^{\star 2} + x^\star + 1 & \text{if } 0 < x^\star \le 1.0 \\ \mathcal{B}_1^{\{C,3\}}(x^\star) = -0.0000001223\,x^{\star 3} + 0.00002004706\,x^{\star 2} - 2x^\star + 1 & \text{if } 0 < x^\star \le 1.0 \\ \mathcal{B}_2^{\{C,3\}}(x^\star) = \quad 0.00000000652\,x^{\star 3} - 0.00000000096\,x^{\star 2} + 3x^\star - 1 & \text{if } 0 < x^\star \le 1.0 \end{cases}$$

$$\text{FFCPCBS} = \begin{cases} \mathcal{B}_0^{\{C,3\}}(x^\star) = 1 + x^\star + 0.00002004706\,x^{\star 2} - 0.00005495675\,x^{\star 3} & \text{if } 0 < x^\star \le 1.0 \\ \mathcal{B}_1^{\{C,3\}}(x^\star) = 1 - 2x^\star + 0.0000000181\,x^{\star 2} - 0.0000000494\,x^{\star 3} & \text{if } 0 < x^\star \le 1.0 \\ \mathcal{B}_2^{\{C,3\}}(x^\star) = -1 + 3x^\star - 0.00000000096\,x^{\star 2} + 0.00000000427\,x^{\star 3} & \text{if } 0 < x^\star \le 1.0 \end{cases}$$

Tables 2–4 present a comparison between the approximate and exact solutions of the functions $\mathcal{Y}_0(x)$, $\mathcal{Y}_1(x)$, and $\mathcal{Y}_2(x)$. The computations are carried out using $\mathcal{N} = 10$, step size $h = 0.1$, and sampling points defined by $x_r^\star = a + rh$, for $r = 0, 1, \ldots, \mathcal{N} - 1$. Three cubic $\mathcal{B}$-spline methods are considered: NFCBI, FCPCBS, and FFCPCBS, for Example 1 respectively.

Table 2: Compares the exact and approximate based on a least square error of $\mathcal{Y}_0(x)$ for Example 1

$x_r^\star$	Exact	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	1.0	1.0000000000	1.0000000000	1.0000000000
1/10	1.1	1.1000000648	1.1000001689	1.1000001255
1/5	1.2	1.2000002525	1.2000005490	1.2000002019
3/10	1.3	1.3000005556	1.3000009508	1.2999997791
2/5	1.4	1.4000009692	1.4000011846	1.3999984073
1/2	1.5	1.5000014905	1.5000010607	1.4999956363
3/5	1.6	1.6000021183	1.6000003895	1.5999910161
7/10	1.7	1.7000028525	1.6999989813	1.6999840968
4/5	1.8	1.8000036938	1.7999966465	1.7999744282
9/10	1.9	1.9000046433	1.8999931954	1.8999615604
1.0	2.0	2.0000057023	1.9999884384	1.9999450433
L.S.E		8.3881×10^{-11}	1.9617×10^{-10}	5.5071×10^{-9}
Runtime (sec)		2.7607	2.3051	2.2901

Table 3: Compares the exact and approximate based on a least square error of $\mathcal{Y}_1(x)$ for Example 1

$x_r^\star$	Exact	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	1.0	1.000000000000	1.000000000000	1.000000000000
1/10	0.8	0.800000000058	0.800000000058	0.800000000113
1/5	0.6	0.600000000222	0.599999999743	0.600000000181
3/10	0.4	0.400000000475	0.399999998322	0.399999999888
2/5	0.2	0.200000000796	0.199999995056	0.199999998564
1/2	0.0	1.16518×10^{-9}	-1.0782×10^{-8}	-3.931×10^{-9}
3/5	-0.2	-0.199999998438	-0.200000019929	-0.200000008090
7/10	-0.4	-0.399999998025	-0.400000033119	-0.400000014319
4/5	-0.6	-0.599999997604	-0.600000051085	-0.600000023023
9/10	-0.8	-0.799999997174	-0.800000074561	-0.800000034606
1.0	-1.0	-0.999999996730	-1.000000104288	-1.000000049477
L.S.E		3.3029×10^{-17}	2.0681×10^{-14}	4.4633×10^{-15}
Runtime (sec)		2.7607	2.3051	2.2901

Table 4: Compares the exact and approximate based on a least square error of $\mathcal{Y}_2(x)$ for Example 1

$x_r^\star$	Exact	$\mathcal{B}_2^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_2^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_2^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	-1.0	-1.000000000000	-1.000000000000	-1.000000000000
1/10	-0.7	-0.700000000003	-0.700000000003	-0.700000000004
1/5	-0.4	-0.400000000012	-0.399999999986	-0.399999999996
3/10	-0.1	-0.100000000026	-0.099999999911	-0.099999999945
2/5	0.2	0.199999999956	0.200000000265	0.200000000182
1/2	0.5	0.499999999931	0.500000000576	0.500000000415
3/5	0.8	0.799999999903	0.800000001064	0.800000000786
7/10	1.1	1.099999999878	1.100000001777	1.100000001333
4/5	1.4	1.399999999844	1.400000002733	1.400000002077
9/10	1.7	1.699999999889	1.700000003988	1.700000003044
1.0	2.0	1.999999999756	2.000000005566	2.000000004277
L.S.E		1.6268×10^{-19}	5.8856×10^{-17}	3.4339×10^{-17}
Runtime (sec)		2.7607	2.3051	2.2901

Table 5: Least square error for $\mathcal{B}_0^{\{C,3\}}(x^\star)$, $\mathcal{B}_1^{\{C,3\}}(x^\star)$, and $\mathcal{B}_2^{\{C,3\}}(x^\star)$ in Example 1 using different step sizes.

Method	Step size h	L.S.E. $\mathcal{B}_0^{\{C,3\}}(x^\star)$	L.S.E. $\mathcal{B}_1^{\{C,3\}}(x^\star)$	L.S.E. $\mathcal{B}_2^{\{C,3\}}(x^\star)$
NFCBI	1/20	8.3882×10^{-13}	3.3029×10^{-19}	1.6271×10^{-21}
	1/50	8.3882×10^{-15}	3.3028×10^{-21}	1.6305×10^{-23}

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Method	Step size h	L.S.E. $\mathcal{B}_0^{\{C,3\}}(x^\star)$	L.S.E. $\mathcal{B}_1^{\{C,3\}}(x^\star)$	L.S.E. $\mathcal{B}_2^{\{C,3\}}(x^\star)$
	$1/100$	2.0676×10^{-15}	3.3021×10^{-22}	1.6875×10^{-25}
	$1/1000$	2.0677×10^{-19}	3.2342×10^{-27}	1.4463×10^{-28}
FCPCBS	$1/20$	1.0386×10^{-9}	8.4106×10^{-16}	2.4058×10^{-18}
	$1/50$	6.2009×10^{-11}	5.0151×10^{-17}	1.4232×10^{-19}
	$1/100$	1.8338×10^{-11}	2.3886×10^{-17}	6.6535×10^{-20}
	$1/1000$	4.5237×10^{-11}	3.0396×10^{-17}	1.0171×10^{-20}
FFCPCBS	$1/20$	2.1963×10^{-10}	1.4663×10^{-16}	8.7157×10^{-18}
	$1/50$	3.3795×10^{-11}	3.5882×10^{-17}	4.7505×10^{-18}
	$1/100$	4.0923×10^{-11}	4.1016×10^{-17}	4.6055×10^{-18}
	$1/1000$	3.8287×10^{-11}	4.1630×10^{-17}	4.5875×10^{-18}

The plots of the approximate solutions obtained by the NFCBI, FCPCBS, and FFCPCBS methods for $\mathcal{Y}_0(x)$, $\mathcal{Y}_1(x)$, and $\mathcal{Y}_2(x)$ are illustrated in Figures 1, 3, and 5, respectively, along with the corresponding exact solutions. The least square errors for each iteration are presented in Figures 2, 4, and 6. These results are based on Example 1 using $\mathcal{N} = 10$ and $h = 0.1$.

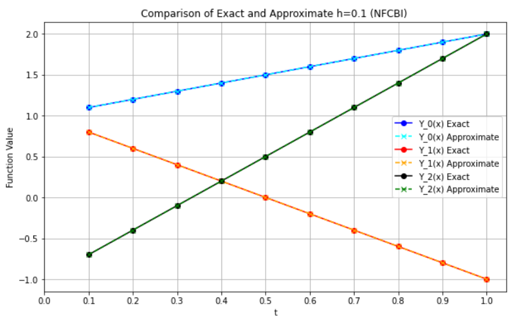

Figure 1: Approximate solution using NFCBI for Example 1.

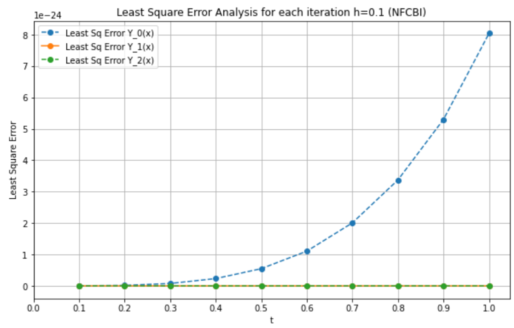

Figure 2: Least square error plot using NFCBI for Example 1.

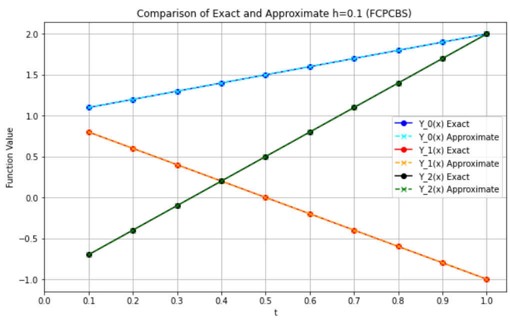

Figure 3: Approximate solution using FCPCBS for Example 1.

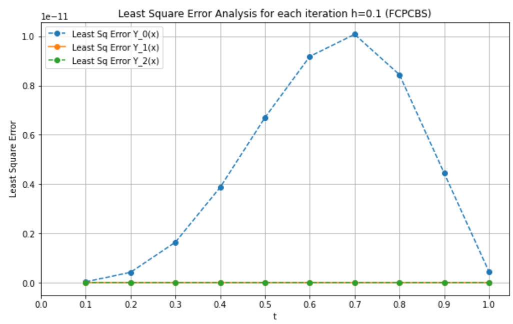

Figure 4: Least square error plot using FCPCBS for Example 1.

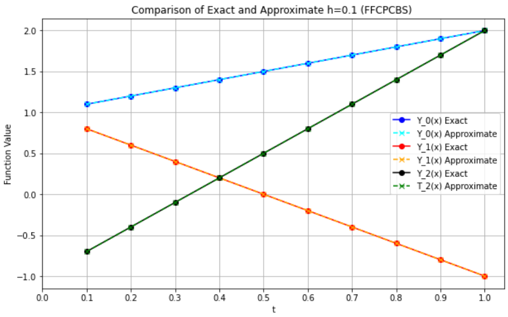

Figure 5: Approximate solution using FFCPCBS for Example 1.

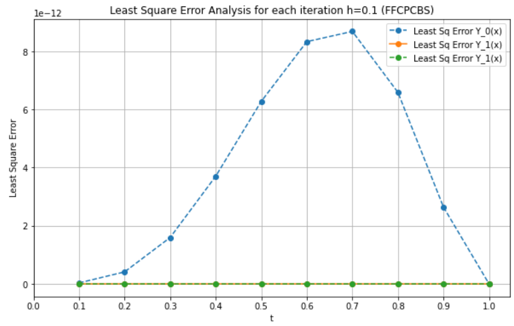

Figure 6: Least square error plot using FFCPCBS for Example 1.

Table 6: The values of control points of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$ and $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$ for three methods NFCBI, FCPCBS, and FFCPCBS, respectively, for Example 2

Method	Interval	CP of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$
	]0.0, 0.1]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = 1.20000000000011 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.666666666660888882$ $\vartheta_{23} = 1.55986334959834 \times 10^{-10}$
	]0.1, 0.2]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = 2.12500000000011 \times 10^{-11}$	$\vartheta_{20} = 1.00000000000000000000$ $\vartheta_{21} = 1.000000000000000000$ $\vartheta_{22} = 0.66666666666088888882$ $\vartheta_{23} = -5.5622173533720 \times 10^{-11}$
	]0.2, 0.3]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = 2.5185185185185765 \times 10^{-11}$	$\vartheta_{20} = 1.00000000000000000000$ $\vartheta_{21} = 1.00000000000000000000$ $\vartheta_{22} = 0.66666666666088888882$ $\vartheta_{23} = -3.35780785892 \times 10^{-11}$
	]0.3, 0.4]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666666$ $\vartheta_{03} = 2.18749999999993 \times 10^{-11}$	$\vartheta_{20} = 1.00000000000000000000$ $\vartheta_{21} = 1.00000000000000000000$ $\vartheta_{22} = 0.66666666666088888882$ $\vartheta_{23} = -2.08461720108132 \times 10^{-11}$
NFCBI	]0.4, 0.5]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = 1.15000000000002 \times 10^{-11}$	$\vartheta_{20} = 1.00000000000000000000$ $\vartheta_{21} = 1.00000000000000000000$ $\vartheta_{22} = 0.66666666666088888882$ $\vartheta_{23} = -3.06465963717528 \times 10^{-11}$
	]0.5, 0.6]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -3.20000000000103 \times 10^{-12}$	$\vartheta_{20} = 1.00000000000000000000$ $\vartheta_{21} = 1.00000000000000000000$ $\vartheta_{22} = 0.66666666666088888882$ $\vartheta_{23} = -3.00983517944444 \times 10^{-11}$

Continued on next page

Method	Interval	CP of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$
	]0.6, 0.7]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -2.28435374149652 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = -3.046413137949 \times 10^{-11}$
	]0.7, 0.8]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -4.572916666667 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = -3.08179046518142 \times 10^{-11}$
	]0.8, 0.9]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -8.17622410546180 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = -2.96219001164554 \times 10^{-11}$
	]0.9, 1.0]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -1.389125188695603 \times 10^{-10}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = -5.600000000000 \times 10^{-11}$
FCPCBS	]0.0, 1.0]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $b_{03} = 1.20000000000011 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $b_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = 1.559863349598 \times 10^{-10}$
FFCPCBS	]0.0, 1.0]	$\vartheta_{00} = -1.0000000000000000000$ $\vartheta_{01} = -1.0000000000000000000$ $\vartheta_{02} = -0.6666666666666666667$ $\vartheta_{03} = -6.345625943477962 \times 10^{-11}$	$\vartheta_{20} = 1.0000000000000000000$ $\vartheta_{21} = 1.0000000000000000000$ $\vartheta_{22} = 0.6666666666660888882$ $\vartheta_{23} = 4.999316747991 \times 10^{-11}$

From (21), we obtain the approximate solution for the classical and fractional order linear system of Volterra integro-differential equations (SVIDEs-CF) with variable coefficients, for Example 2, as shown below:

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} 1.55986334959834 \times 10^{-10}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & 0 < x^\star \leq \frac{1}{10} \\ -5.56221735337203 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{1}{10} < x^\star \leq \frac{1}{5} \\ -3.35780785892186 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{1}{5} < x^\star \leq \frac{3}{10} \\ -2.08461720108132 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{3}{10} < x^\star \leq \frac{2}{5} \\ -3.06465963717528 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{2}{5} < x^\star \leq \frac{1}{2} \\ -3.00983517944444 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{1}{2} < x^\star \leq \frac{3}{5} \\ -3.04641313794957 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{3}{5} < x^\star \leq \frac{7}{10} \\ -3.08179046518142 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{7}{10} < x^\star \leq \frac{4}{5} \\ -2.96219001164554 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{4}{5} < x^\star \leq \frac{9}{10} \\ -5.60000000000000 \times 10^{-11}{x^\star}^3 + 0.666666666660882{x^\star}^2 + 1, & \frac{9}{10} < x^\star \leq 1 \end{cases}$$

$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ \text{(NFCBI)} = \begin{cases} 1.73339564923936 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & 0 < x^\star \le \frac{1}{10} \\ -3.82689435696193 \times 10^{-11} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{1}{10} < x^\star \le \frac{1}{5} \\ -1.6224799281872 \times 10^{-12} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{1}{5} < x^\star \le \frac{3}{10} \\ -3.49298368007567 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{3}{10} < x^\star \le \frac{2}{5} \\ -1.32933664076518 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{2}{5} < x^\star \le \frac{1}{2} \\ -1.27451382780919 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{1}{2} < x^\star \le \frac{3}{5} \\ -1.31108457424034 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{3}{5} < x^\star \le \frac{7}{10} \\ -1.34647848426539 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{7}{10} < x^\star \le \frac{4}{5} \\ -1.22686305559228 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{4}{5} < x^\star \le \frac{9}{10} \\ -3.86468634872017 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, & \frac{9}{10} < x^\star \le 1 \end{cases}$$

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ \text{(FCPCBS)} = 1.199929045014870 \times 10^{-11} x^{\star 3} + 1.000000 x^{\star 2} - 1, \quad 0 < x^\star \le 1$$
$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ \text{(FCPCBS)} = 1.73339564923936 \times 10^{-10} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, \quad 0 < x^\star \le 1$$

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ \text{(FFCPCBS)} = -6.34567953738951 \times 10^{-11} x^{\star 3} + 1.000000 x^{\star 2} - 1, \quad 0 < x^\star \le 1$$
$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ \text{(FFCPCBS)} = 6.7346128673762 \times 10^{-11} x^{\star 3} - 1.00000000001735 x^{\star 2} + 1, \quad 0 < x^\star \le 1$$

Table 7: Compares the exact and approximate based on a least square error of $\mathcal{Y}_0(x)$ for Example 2.

$x_r^\star$	Exact	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	-1.00	-1.00	-1.00	-1.00
1/10	-0.99	-0.99	-0.99	-0.99
1/5	-0.96	-0.96	-0.96	-0.960000000001
3/10	-0.91	-0.909999999999	-0.910000000000	-0.910000000002
2/5	-0.84	-0.839999999998	-0.839999999999	-0.840000000004
1/2	-0.75	-0.749999999995	-0.749999999998	-0.750000000008
3/5	-0.64	-0.639999999992	-0.639999999997	-0.640000000014
7/10	-0.51	-0.509999999986	-0.509999999996	-0.510000000022
4/5	-0.36	-0.359999999980	-0.359999999994	-0.360000000032
9/10	-0.19	-0.189999999972	-0.189999999991	-0.190000000046
1.0	0.00	3.61087481×10^{-11}	1.2000000×10^{-11}	-6.3000000×10^{-11}
L.S.E		8.3881×10^{-11}	2.7712×10^{-21}	2.8489×10^{-22}
Run Time (s)		2.7607	0.9465	1.1776

Table 8: Compares the exact and approximate based on a least square error of $\mathcal{Y}_1(x)$ for Example 2.

$x_r^\star$	Exact	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	1.00	1.00	1.00	1.00
1/10	0.99	0.99	0.99	0.99
1/5	0.96	0.959999999999	0.960000000001	0.96
3/10	0.91	0.909999999998	0.910000000003	0.91
2/5	0.84	0.839999999997	0.840000000008	0.840000000002
1/2	0.75	0.749999999994	0.750000000017	0.750000000004
3/5	0.64	0.639999999991	0.640000000031	0.640000000008
7/10	0.51	0.509999999987	0.510000000051	0.510000000015
4/5	0.36	0.359999999982	0.360000000078	0.360000000023
9/10	0.19	0.189999999977	0.190000000112	0.190000000035
1.0	0.00	-2.8274724×10^{-11}	1.560000×10^{-10}	5.000000×10^{-11}
L.S.E		2.7712×10^{-21}	1.9151×10^{-21}	4.6922×10^{-21}
Run Time (s)		0.9465	0.9465	1.1776

Table 9: Least square error for $\mathcal{Y}_0(x)$, and $\mathcal{Y}_1(x)$, in Example 2 using different step sizes.

Method	Step Size h	LSE of $\mathcal{B}_0^{\{C,3\}}(x^\star)$	LSE of $\mathcal{B}_1^{\{C,3\}}(x^\star)$
NFCBI	1/10	2.7712×10^{-21}	1.9151×10^{-21}
	1/40	2.1379×10^{-23}	2.1333×10^{-23}
	1/80	5.8327×10^{-25}	1.1559×10^{-24}
	1/100	1.4706×10^{-25}	1.0112×10^{-24}
FCPCBS	1/10	2.8489×10^{-22}	4.6922×10^{-21}
	1/40	2.8487×10^{-24}	4.8138×10^{-22}
	1/80	2.8303×10^{-28}	4.8167×10^{-26}
	1/100	2.8490×10^{-30}	2.2284×10^{-28}
FFCPCBS	1/10	7.9664×10^{-21}	4.5745×10^{-21}
	1/40	3.7361×10^{-21}	1.7798×10^{-21}
	1/80	5.6535×10^{-23}	3.1644×10^{-23}
	1/100	3.5825×10^{-24}	7.9086×10^{-24}

The plots of the approximate solutions obtained by NFCBI, FCPCBS, and FFCPCBS for $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ are shown in Figures 7, 9, and 11, respectively, along with the exact solution plots. The corresponding least square errors for each iteration are illustrated in Figures 8, 10, and 12, respectively, for Example 2.

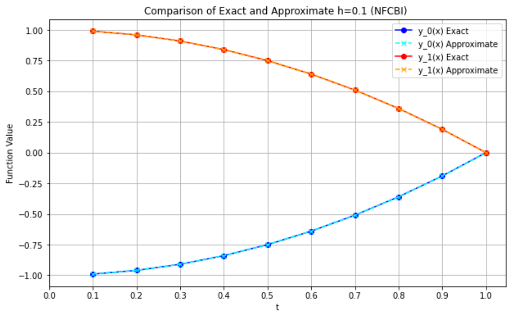

Figure 7: Approximation solution of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *NFCBI* for example 2.

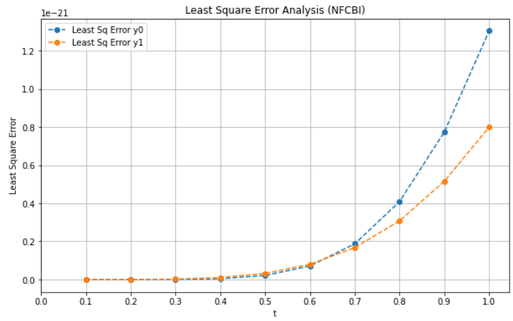

Figure 8: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *NFCBI*for example 2.

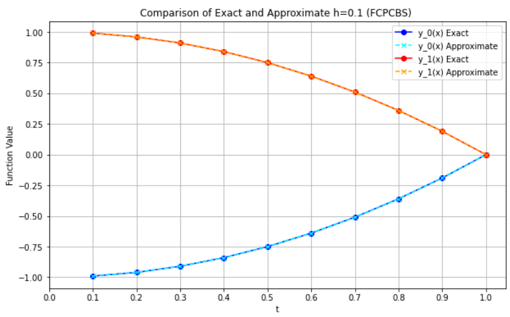

Figure 9: Approximation solution of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FCPCBS* for example 2.

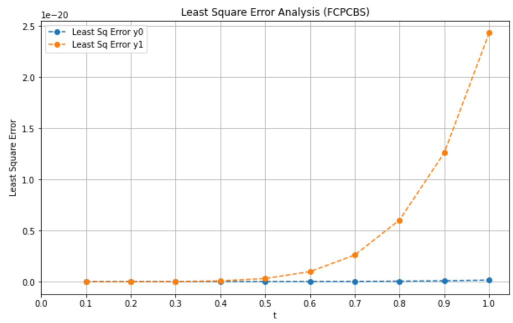

Figure 10: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FCPCBS*for example 6.2.

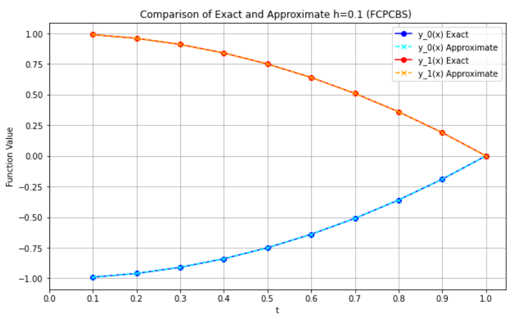

Figure 11: Approximation solution of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FFCPCBS* for example 2.

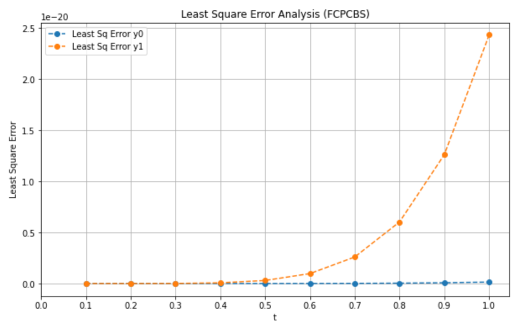

Figure 12: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FFCPCBS* for example 2.

Table 10: The values of control points of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$ and $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$ for three methods NFCBI, FCPCBS, and FFCPCBS, respectively, for Example 3.

Method	Interval	CP of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$
NFCBI	]0.0, 0.1]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999991827737578$	$\vartheta_{23} = 2.999999999999988454$
	]0.1, 0.2]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999996593810891$	$\vartheta_{23} = 2.999999999999992006$
	]0.2, 0.3]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999998175375771$	$\vartheta_{23} = 2.999999999999995115$
	]0.3, 0.4]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999998962879832$	$\vartheta_{23} = 2.999999999999994227$
	]0.4, 0.5]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999999433926590$	$\vartheta_{23} = 2.999999999999989342$
	]0.5, 0.6]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999999747462898$	$\vartheta_{23} = 2.999999999999985345$
	]0.6, 0.7]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999999971441733$	$\vartheta_{23} = 2.999999999999981348$
	]0.7, 0.8]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 2.000000000139726009$	$\vartheta_{23} = 2.999999999999974687$

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Method	Interval	CP of $\mathcal{B}_0^{\{C,3\}}(x^\star_{r+1})$	CP of $\mathcal{B}_1^{\{C,3\}}(x^\star_{r+1})$
	]0.8, 0.9]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 2.000000000271051181$	$\vartheta_{23} = 2.999999999999969358$
	]0.9, 1.0]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 2.000000000376601861$	$\vartheta_{23} = 2.999999999999964917$
FCPCBS	]0.0, 1.0]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999991827737578$	$\vartheta_{23} = 2.999999999999988454$
FFCPCBS	]0.0, 1.0]	$\vartheta_{00} = 1.000000000000000000$	$\vartheta_{20} = 1.000000000000000000$
		$\vartheta_{01} = 1.000000000000000000$	$\vartheta_{21} = 1.666666666666666666$
		$\vartheta_{02} = 1.000000000478453520$	$\vartheta_{22} = 2.333333333333333335$
		$\vartheta_{03} = 1.999999996102169719$	$\vartheta_{23} = 2.999999999999976907$

From (21), we obtain the approximate solution for the classical and fractional order linear system of Volterra integro-differential equations (SVIDEs-CF) with variable coefficients, for Example 3, as shown below:

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} 0.999999990392377\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & 0 < x^\star \le \frac{1}{10} \\ 0.999999995158451\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{1}{10} < x^\star \le \frac{1}{5} \\ 0.999999996740016\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{1}{5} < x^\star \le \frac{3}{10} \\ 0.999999997527520\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{3}{10} < x^\star \le \frac{2}{5} \\ 0.999999997998566\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{2}{5} < x^\star \le \frac{1}{2} \\ 0.999999998312103\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{1}{2} < x^\star \le \frac{3}{5} \\ 0.999999998536081\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{3}{5} < x^\star \le \frac{7}{10} \\ 0.999999998704366\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{7}{10} < x^\star \le \frac{4}{5} \\ 0.999999998835690\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{4}{5} < x^\star \le \frac{9}{10} \\ 0.999999998941242\, {x^\star}^3 + 1.43536027508162 \times 10^{-9}\, {x^\star}^2 + 1, & \frac{9}{10} < x^\star \le 1 \end{cases}$$

$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ (\text{NFCBI}) = \begin{cases} -1.15463194561016 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & 0 < x^\star \leq \frac{1}{10} \\ -7.99360577730113 \times 10^{-15}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{1}{10} < x^\star \leq \frac{1}{5} \\ -5.32907051820075 \times 10^{-15}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{1}{5} < x^\star \leq \frac{3}{10} \\ -5.32907051820075 \times 10^{-15}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{3}{10} < x^\star \leq \frac{2}{5} \\ -1.06581410364015 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{2}{5} < x^\star \leq \frac{1}{2} \\ -1.42108547152020 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{1}{2} < x^\star \leq \frac{3}{5} \\ -1.86517468137026 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{3}{5} < x^\star \leq \frac{7}{10} \\ -2.48689957516035 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{7}{10} < x^\star \leq \frac{4}{5} \\ -3.01980662698043 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{4}{5} < x^\star \leq \frac{9}{10} \\ -3.55271367880050 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, & \frac{9}{10} < x^\star \leq 1 \end{cases}$$

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ (\text{FCPCBS}) = 0.999999990392377\, x^{\star 3} + 1.43536027508162 \times 10^{-9}\, x^{\star 2} + 1, \quad 0 < x^\star \leq 1.0$$
$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ (\text{FCPCBS}) = -1.15463194561016 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, \quad 0 < x^\star \leq 1.0$$

$$\mathcal{B}_0^{\{C,3\}}(x^\star)\ (\text{FFCPCBS}) = 0.999999994666809\, x^{\star 3} + 1.43536027508162 \times 10^{-9}\, x^{\star 2} + 1, \quad 0 < x^\star \leq 1.0$$
$$\mathcal{B}_1^{\{C,3\}}(x^\star)\ (\text{FFCPCBS}) = -2.30926389122033 \times 10^{-14}\, x^{\star 3} + 2.00000000000000\, x^\star + 1, \quad 0 < x^\star \leq 1.0$$

Tables 11 and 12 demonstrate a comparison between the approximate and exact solutions of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$, respectively. The computations are carried out by setting $\mathcal{N} = 10$, $h = 0.1$, and $x_r^\star = a + rh$ for $r = 0, 1, \ldots, \mathcal{N} - 1$, using three cubic $\mathcal{B}$-spline methods: NFCBI, FCPCBS, and FFCPCBS.

Table 11: Comparison of the exact and approximate solutions based on least square error for $\mathcal{Y}_0(x)$ for Example 3.

$x_r^\star$	Exact	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_0^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	1.000	1.000	1.000	1.000
1/10	1.001	1.00100000000475	1.001000000005	1.001000000009
1/5	1.008	1.00800000001868	1.007999999981	1.008000000015
3/10	1.027	1.02700000004116	1.026999999870	1.026999999985
2/5	1.064	1.06400000007142	1.063999999615	1.063999999888
1/2	1.125	1.12500000010866	1.124999999158	1.124999999692
3/5	1.216	1.21600000015215	1.215999998441	1.215999999365
7/10	1.343	1.34300000020120	1.342999997408	1.342999998874
4/5	1.512	1.51200000025526	1.511999996000	1.511999998188
9/10	1.729	1.72900000031386	1.728999994159	1.728999997275
1.0	2.000	2.00000000037660	1.999999991828	1.999999996102
L.S.E		3.881×10^{-19}	1.269×10^{-16}	2.768×10^{-17}
Runtime (s)		1.2066	1.2065	1.2085

Table 12: Comparison of the exact and approximate solutions based on least square error for $\mathcal{Y}_1(x)$ for Example 3.

$x_r^\star$	Exact	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (NFCBI)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FCPCBS)	$\mathcal{B}_1^{\{C,3\}}(x_r^\star)$ (FFCPCBS)
0	1.0	1.0	1.0	1.0
1/10	1.2	1.2	1.2	1.2
1/5	1.4	1.4	1.4	1.4
3/10	1.6	1.6	1.6	1.6
2/5	1.8	1.8	1.8	1.80000000000009
1/2	2.0	2.0	2.0000000000008	2.0000000000007
3/5	2.2	2.2	2.2000000000007	2.2000000000017
7/10	2.4	2.4	2.4000000000099	2.4000000000058
4/5	2.6	2.6	2.6000000000023	2.6000000000014
9/10	2.8	2.79999999999999	2.7999999999998	2.7000000000238
1.0	3.0	3.00000000000001	3.00000000000013	3.00000000000013
L.S.E		8.239×10^{-29}	2.598×10^{-28}	1.060×10^{-27}
Runtime (s)		1.2066	1.2065	1.2085

Table 13: Least square error for $\mathcal{Y}_0(x)$, and $\mathcal{Y}_1(x)$, in Example 3 using different step sizes.

Method	Step Size h	LSE of $\mathcal{B}_0^{\{C,3\}}(x^\star)$	LSE of $\mathcal{B}_1^{\{C,3\}}(x^\star)$
NFCBI	1/10	$3.88101892329981 \times 10^{-19}$	$8.23866607890194 \times 10^{-29}$
	1/50	$2.43602430852715 \times 10^{-22}$	$7.13328499648335 \times 10^{-29}$
	1/1000	$1.38050658413677 \times 10^{-29}$	$2.23866607890190 \times 10^{-29}$
FCPCBS	1/10	$1.2693341933429862 \times 10^{-16}$	$2.5983106065717076 \times 10^{-28}$
	1/50	$7.966390404945247 \times 10^{-20}$	$1.2232274411583314 \times 10^{-28}$
	1/1000	$3.944304526105059 \times 10^{-31}$	$3.4512664603419266 \times 10^{-31}$
FFCPCBS	1/10	$2.768234509326171 \times 10^{-17}$	$1.0601304490038872 \times 10^{-27}$
	1/50	$6.488786754097752 \times 10^{-19}$	$1.0597574191466658 \times 10^{-27}$
	1/1000	$1.7749370367472766 \times 10^{-29}$	$1.0601304490038872 \times 10^{-28}$

The plot of these approximate solutions obtained by NFCBI, FCPCBS and FFCPCBS for $\mathcal{Y}_0(x)$, and $\mathcal{Y}_1(x)$ are demonstrate in figures 13,15 and 17 with the exact solution plots, respectively. The least square error of each iteration are shows in figures 14, 16 and 18 for example 3.

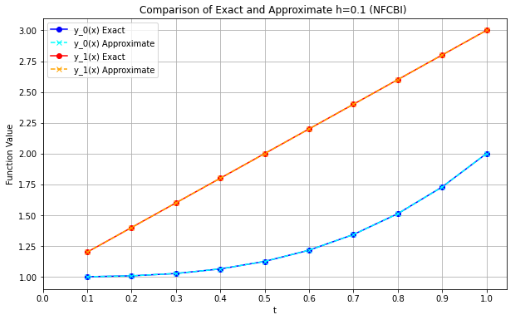

Figure 13: Approximation solution of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *NFCBI* for example 3.

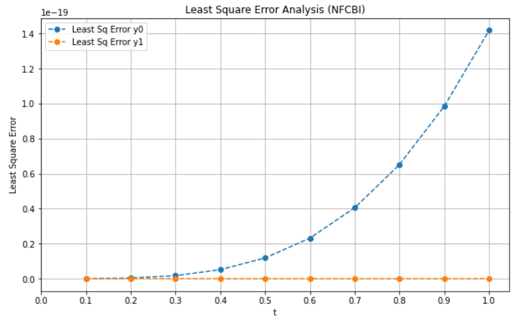

Figure 14: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *NFCBI*for example 5.3.

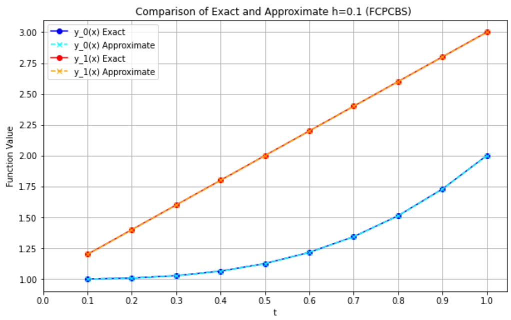

Figure 15: Approximation solution of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FCPCBS* for example 3.

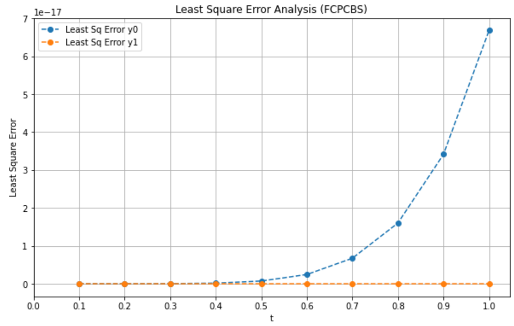

Figure 16: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FCPCBS*for example 3.

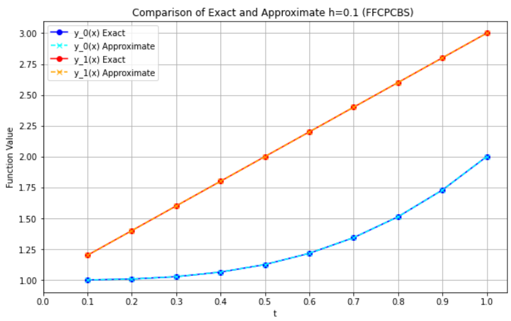

Figure 17: Approximation solution of $\mathcal{Y}_0(x)$and $\mathcal{Y}_1(x)$ method *FFCPCBS* for example 3.

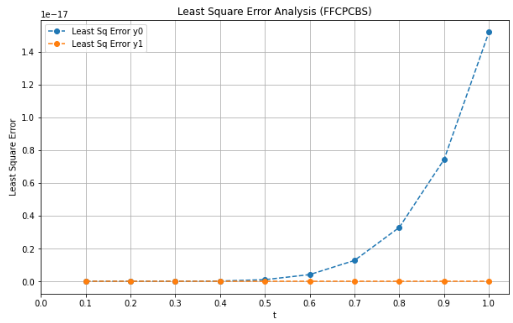

Figure 18: Least square error plot of $\mathcal{Y}_0(x)$ and $\mathcal{Y}_1(x)$ method *FFCPCBS* for example 3.

7. Conclusion

In this work, a high-accuracy cubic $\mathcal{B}$-spline collocation method is presented for solving a system of Volterra integro-differential equations with classical and fractional Caputo derivatives (SVIDE's-CF). Unlike previous studies that primarily focus on single equations [29, 32–34], the proposed approach efficiently handles systems of equations. The method was implemented in Python and validated through several benchmark examples (Tables 5, 9, and 13). The resulting least-squares errors demonstrate improvements by several orders of magnitude, highlighting the method's superior numerical accuracy and stable convergence properties across a variety of test problems. To further develop the method, future research can explore its generalization to non-linear systems, time-dependent parameters, and non-local operators. Additionally, incorporating adaptive mesh generation and refinement algorithms can render it more versatile and efficient. Indeed, this method is a powerful computational technique with applications across many disciplines of applied mathematics, physics, and engineering.

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Conflict of Interest

The authors declare no conflict of interest.

Availability of data material

All the data of the study are included.

Authors' Contributions

D.Kh.A., Sh.Sh.A. and K. H.F. wrote the main manuscript. All authors reviewed the manuscript.

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