EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6774
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Efficient Treatment of Some Important Fractal-Fractional Models: Theoretical and Numerical Study

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Abstract. In this study, we investigate two fundamental fractal-fractional (FF) models: the competitive dynamics among Egyptian banks and the Brusselator system. For the banking model, optimal control strategies are proposed to mitigate profit downturns during crises, such as the COVID-19 pandemic, through a system of four fractional differential equations. Recognizing the slow convergence of traditional numerical methods, an efficient integration technique is developed to simulate both models with enhanced accuracy and computational efficiency. The simulation results reveal the dynamic behaviors of the studied systems for various FF-operator values, confirming the robustness and precision of the proposed approach when compared with the classical fourth-order Runge–Kutta method (RK4M). The presented technique offers a simple yet powerful framework for modeling and analyzing complex FF-based dynamical systems.

2020 Mathematics Subject Classifications: 34A12, 41A30, 47H10, 65N20

Key Words and Phrases: Banks' competition model, optimal control, Brusselator system, fractal-fractional derivative, numerical integration, RK4M

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6774

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1. Introduction

Mathematical models are not only instrumental in addressing scientific challenges but are also gaining increasing prominence in sectors that drive economic growth, such as banking and finance. A mathematical competition model, which is typically represented by a system of ordinary differential equations (ODEs), is commonly employed to simulate competitive interactions among banks ([1], [2]). The Lotka Volterra system is frequently utilized in the financial domain to compare and analyze the profit dynamics of banks ([3], [4]). Previous studies, such as [5] and [6], have extended this framework by incorporating various definitions of fractional derivatives to capture the complex competitive dynamics between Indonesian commercial and rural banks. According to the Central Bank of Egypt (CBE), the Egyptian banking system comprises 33 banks that are categorized into four distinct groups [7], and the regulations issued by the CBE apply uniformly to all these institutions.

The CBE performs the dual roles of national monetary authority and central bank for the nation. Like all other central banks, its primary function is to oversee both domestic and international institutions. Additionally, its responsibilities include creating and implementing Egypt's financial laws, printing banknotes, overseeing foreign exchange reserves, regulating the usage of the country's currency, and overseeing the management of both private and public loans. Since there is no obvious distinction in their business sectors, which will be our emphasis in this study, there may be competition amongst the four bank groups previously described in [8].

Several scholars have lately examined the fractional Brusselator system ([9], [10]). In [9], authors studied the stability of this model, where in [10], the authors demonstrated the existence of the system's solutions numerically.

Since most of the fractional differential equations lack exact solutions, one must resort to approximate and numerical methods ([11],[12]). In the case of the Egyptian banks model and Brusselator system, it was shown that by offering a great deal more alternatives for the optimal data fitting setting [13].

A lot of researchers have been interested in the FF-derivatives because it lets them look at the fractal dimension while still analyzing the phenomenon in fractional items. This category of FF derivatives has been employed in various significant investigations, publications, and scientific reports; these can be found there and in the references therein ([14],[15]). Also, the The operational research ([16],[17]) is a very important tool to make the best decision.

The outline of the paper is as follows: Section 2 presents some preliminaries concerning the fractional calculus. Section 3 gives the description of the two fractal-fractional models: the banks' competition model and the Brusselator system. Section 4 investigates the numerical implementation of the proposed method for solving the models under study. Section 5 outlines the numerical simulation for these models. Finally, Section 6 ends the paper with conclusions and future work.

2. Preliminaries

Fractional models define the fractional derivative using power law memory kernels (PLMKs). This makes it easier for the system to describe memory and global correlation. This is the paramount definition employed in the formulation of fractional calculus theory ([18]-[19]). Thus, we will quickly go over some fundamental definitions of fractional calculus with the PLMKs and fractal notions in this brief part.

Definition 1. *[20]*

Let $\Omega(t)$ belong to $C(0,b)$ and be fractal differentiable on $(0,b)$ with order ϱ. The fractal-fractional derivative of order ξ, with Riemann-Liouville's derivative including the PLMK, is given as follows:

$$^{FFP}D^{\xi,\varrho}\,\Omega(t) = \frac{1}{\Gamma(n-\xi)}\frac{d}{dt^{\varrho}}\int_0^t (t-y)^{n-\xi-1}\,\Omega(y)\,dy,$$

where $n-1<\xi,\ \varrho \leq n,\ n\in\ \mathbb{N}$, and $\frac{d\,\Omega(y)}{d\,y^{\varrho}} = \lim_{t\to y}\frac{\Omega(t)-\Omega(y)}{t^{\varrho}-y^{\varrho}}$.

Definition 2. *[20]*

Let $\Upsilon(t)$ belongs to $C(0,b)$ and fractal differentiable on $(0,b)$ with order ϱ. The fractal-fractional integral of order ξ, with including the PLMK is given as follows:

$$^{FFP}I^{\xi,\varrho}\,\Upsilon(t) = \frac{\varrho}{\Gamma(\xi)}\int_0^t (t-y)^{\xi-1}\,y^{\varrho-1}\,\Upsilon(y)\,dy.$$

3. Description of the fractal-fractional models

Here, we will try to describe and introduce the fractal-fractional form of the studied models: The FF banks' competition model and the FF Brusselator system.

3.1. Fractal-fractional banks' competition model

We suppose that, at any given period t, the profits gained by each of the four classes that make up Egypt's banking system public, private, Arab, foreign, and investment collaboration banks are $P(t)$, $A(t)$, $F(t)$, and $I(t)$. Here, we will use the symbols $[P(t),\ A(t),\ F(t),\ I(t)] = [\Upsilon_s(t),\ s=1,2,3,4]$. As a result, the suggested competition model between them is given as follows:

$$\frac{d\Upsilon_s(t)}{dt} = \alpha_s\,\Upsilon_s(t)\left(1-\frac{\Upsilon_s(t)}{\beta_s}\right) - \delta_s\,\bar{\mathbb{N}}[\Upsilon_1,\ \Upsilon_2,\ \Upsilon_3,\ \Upsilon_4], \qquad \Upsilon_s(0) = \hat{\hat{\Upsilon}}_s, \qquad s=1,2,3,4. \tag{1}$$

The nonlinear term $\bar{\mathbb{N}}[\Upsilon_1,\ \Upsilon_2,\ \Upsilon_3,\ \Upsilon_4] = \Upsilon_1(t)\,\Upsilon_2(t)\,\Upsilon_3(t)\,\Upsilon_4(t)$ in the system (1) is the area of the market where the four banks engage with one another [21]. Where

(i) α_k, $k = 1, 2, 3, 4$ are the rates of growth of banks that collaborate on investments, international, Arabic, and private sectors, respectively.

(ii) β_k, $k = 1, 2, 3, 4$ are the highest earnings recorded by the four categories mentioned above.

(iii) δ_k, $k = 1, 2, 3, 4$ are the competition parameters.

Let us now reinterpret, in the Caputo sense, the banks' competitiveness model under fractal-fractional derivative which is formulated as follows:

$$ {}^{FFP}D^{\xi,\varrho}\Upsilon_s(t) = \alpha_s\,\Upsilon_s(t)\left(1 - \frac{\Upsilon_s(t)}{\beta_s}\right) - \delta_s\,\bar{\mathbb{N}}[\Upsilon_1,\,\Upsilon_2,\,\Upsilon_3,\,\Upsilon_4]. \tag{2} $$

To examine numerically the system with the FF derivative (2), we can rewrite (2) as:

$$ {}^{RL}D^{\xi}\Upsilon_s(t) = \varrho\,t^{\varrho-1}\left[\alpha_s\,\Upsilon_s(t)\left(1 - \frac{\Upsilon_s(t)}{\beta_s}\right) - \delta_s\,\bar{\mathbb{N}}[\Upsilon_1,\,\Upsilon_2,\,\Upsilon_3,\,\Upsilon_4]\right]. \tag{3} $$

Upon substituting the derivative ${}^{RL}D$ with ${}^{C}D$ and utilizing the fractional integral, the following is the obtained solution:

$$ \bar{\Upsilon}(t) = \bar{\Upsilon}(0) + \frac{\varrho}{\Gamma(\xi)}\int_0^t y^{\varrho-1}(t-y)^{\xi-1}\mathbb{F}(\bar{\Upsilon}(y), y)\,dy, \tag{4} $$

the vector function $\bar{\Upsilon}(t)$ is given by $\bar{\Upsilon}(t) = [\Upsilon_1(t), \Upsilon_2(t), \Upsilon_3(t), \Upsilon_4(t)]^T$ and

$$ \mathbb{F}(\bar{\Upsilon}(t), t) = (\mathbf{f}_1(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, t), \mathbf{f}_2(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, t), \mathbf{f}_3(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, t), \mathbf{f}_4(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, t))^T, \tag{5} $$

where

$$ \mathbf{f}_s(\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, t) = \alpha_s\,\Upsilon_s(t)\left(1 - \frac{\Upsilon_s(t)}{\beta_s}\right) - \delta_s\,\Upsilon_1(t)\,\Upsilon_2(t)\,\Upsilon_3(t)\,\Upsilon_4(t). $$

We can implement the Banach fixed point (BFP) theorem to prove adequate constraints on the existence of a unique solution of the model (1). Let $\mathbb{B} = \mathbb{R}^4$ be a Banach space with the norm:

$$ \|\bar{\Upsilon}(t)\| = \max_{t\in I}\|\Upsilon_1(t) + \Upsilon_2(t) + \Upsilon_3(t) + \Upsilon_4(t)\|, $$

and $I = (t_0 - \omega, t_0 + \omega)$. To accomplish our aim, we define the mapping $\mathbb{T} : \mathbb{B} \to \mathbb{B}$ by:

$$ \mathbb{T}(\bar{\Upsilon}(t)) = \bar{\Upsilon}(t) = \bar{\Upsilon}(0) + \frac{\varrho}{\Gamma(\xi)}\int_0^t y^{\varrho-1}\,(t-y)^{\xi-1}\mathbb{F}(\bar{\Upsilon}(y), y)\,dy. \tag{6} $$

We can follow the same manner as in [22] to prove the existence by showing that $\mathbb{T}$ is a contraction mapping.

3.2. Fractal-fractional Brusselator system

Here, we will examine the Brusselator system, which is expressed as follows ([9], [10]):

$$\dot{\theta}_1(t) = \delta - (\gamma + 1)\,\theta_1 + \theta_1^2\theta_2, \tag{7}$$

$$\dot{\theta}_2(t) = \gamma\,\theta_1 - \theta_1^2\theta_2, \tag{8}$$

$$\theta_k(0) = \theta_{k,0}, \qquad k = 1, 2. \tag{9}$$

The parameters $\delta > 0$, $\gamma > 0$, and $\theta_{k,0}$, $k = 1, 2$ are constants [18].

Now, we formulate the fractal-fractional Brusselator system in the following form:

$$\begin{aligned} {}^{FFP}D^{\xi,\varrho}\theta_1(t) &= \delta - (\gamma + 1)\,\theta_1 + \theta_1^2(t)\theta_2, \\ {}^{FFP}D^{\xi,\varrho}\theta_2(t) &= \gamma\,\theta_1 - \theta_1^2\theta_2. \end{aligned} \tag{10}$$

The system (10) can be written as:

$$\begin{aligned} {}^{RL}D^{\xi}\theta_1(t) &= \varrho\, t^{\varrho-1}\,(\delta - (\gamma + 1)\,\theta_1 + \theta_1^2\theta_2), \\ {}^{RL}D^{\xi}\theta_2(t) &= \varrho\, t^{\varrho-1}\,(\gamma\,\theta_1 - \theta_1^2\theta_2). \end{aligned} \tag{11}$$

Upon substituting the derivative ${}^{RL}D$ with ${}^{C}D$ and utilizing the fractional integral, the following is the obtained solution:

$$\bar{\theta}(t) = \bar{\theta}(0) + \frac{\varrho}{\Gamma(\xi)} \int_0^t (t - s)^{\xi-1} s^{\varrho-1} \mathbb{F}(\bar{\theta}(s), s)\, ds, \tag{12}$$

where

$$\begin{aligned} &\bar{\theta}(t) = [\theta_1(t), \theta_2(t)]^T, \qquad \bar{\theta}(0) = [\theta_1(0), \theta_2(0)]^T, \\ &\mathbb{F}(\bar{\theta}(t), t) = \begin{pmatrix} \mathbf{f}_1(\theta_1, \theta_2, t) \\ \mathbf{f}_2(\theta_1, \theta_2, t) \end{pmatrix} = \begin{pmatrix} \delta - (\gamma + 1)\,\theta_1 + \theta_1^2\theta_2 \\ \gamma\,\theta_1 - \theta_1^2\theta_2 \end{pmatrix}. \end{aligned} \tag{13}$$

Using the BFP theorem, we can now establish adequate constraints on the existence and uniqueness of the solution of the system (11). Let $\mathbb{B} = \mathbb{R}^2$ with the norm:

$$\|\bar{\theta}(t)\| = \max_{t \in I} \|\theta_1(t) + \theta_2(t)\|.$$

After that, we proceed as we did in the preceding section.

4. Numerical implementation on the proposed models

To demonstrate the numerical simulation of the models under study, we will set up numerical schemes for them in their fractional form of the FF type in this section.

4.1. The Banks' competition model

To do this, let us modify the model found in (3) as follows:

$$ {}^{C}D^{\xi}\Upsilon_i(t) = \varrho\, t^{\varrho-1}\left[\alpha_i\,\Upsilon_i(t)\left(1-\beta_i^{-1}\Upsilon_i(t)\right) - \delta_i\,\bar{\mathbb{N}}[\Upsilon_1,\,\Upsilon_2,\,\Upsilon_3,\,\Upsilon_4]\right], \quad i=1,2,3,4. \tag{14} $$

By utilizing the integral on the system (14), we obtain:

$$ \Upsilon_i(t) = \Upsilon_i(0) + \frac{\varrho}{\Gamma(\xi)}\int_0^t (t-y)^{\xi-1}\, y^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1(y),\Upsilon_2(y),\Upsilon_3(y),\Upsilon_4(y),y)\,dy, \quad i=1,2,3,4, \tag{15} $$

where $\mathbf{f}_i(\Upsilon_1(t),\Upsilon_2(t),\Upsilon_3(t),\Upsilon_4(t),t)$, are given in (5). Thus, at $t=t_{n+1}$

$$ \Upsilon_i^{n+1} = \hat{\hat{\Upsilon}}_i + \frac{\varrho}{\Gamma(\xi)}\int_0^{t_{n+1}} (t_{n+1}-y)^{\xi-1}\; y^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1(y),\Upsilon_2(y),\Upsilon_3(y),\Upsilon_4(y),y)\,dy. \tag{16} $$

Let us approximate $y^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,y)$, as follows:

$$ \begin{aligned} L_j(y) &\approx y^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,y) \\ &= \frac{y-t_{j-1}}{t_j-t_{j-1}} t_j^{\varrho-1}\,\mathbf{f}_i\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j\right) - \frac{y-t_j}{t_j-t_{j-1}} t_{j-1}^{\varrho-1}\,\mathbf{f}_i\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1}\right). \end{aligned} \tag{17} $$

When we use the approximation (17) in the first equation of (16), respectively, we get:

$$ \begin{aligned} \Upsilon_1^{n+1} &= \hat{\hat{\Upsilon}}_1 + \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} (t_{n+1}-y)^{\xi-1}\; y^{\varrho-1}\,\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,y)\,dy \\ &= \hat{\hat{\Upsilon}}_1 + \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} (t_{n+1}-y)^{\xi-1}\left[\frac{y-t_{j-1}}{t_j-t_{j-1}} t_j^{\varrho-1}\,\mathbf{f}_1\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j\right)\right. \\ &\qquad\qquad \left. - \frac{y-t_j}{t_j-t_{j-1}} t_{j-1}^{\varrho-1}\,\mathbf{f}_1\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1}\right)\right] dy \\ &= \hat{\hat{\Upsilon}}_1 + \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} (t_{n+1}-y)^{\xi-1}\left[\frac{y-t_{j-1}}{t_j-t_{j-1}} t_j^{\varrho-1}\,\mathbf{f}_1\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j\right)\right] dy \\ &\qquad - \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}} (t_{n+1}-y)^{\xi-1}\left[\frac{y-t_j}{t_j-t_{j-1}} t_{j-1}^{\varrho-1}\,\mathbf{f}_1\left(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1}\right)\right] dy, \end{aligned} \tag{18} $$

and after some arrangements, we can write:

$$\begin{aligned}\Upsilon_1^{n+1} =&\hat{\hat{\Upsilon}}_1 + \frac{\varrho}{\Gamma(\xi)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{(t_{n+1}-y)^{\xi-1}(y-t_{j-1})}{t_j - t_{j-1}} \left[t_j^{\varrho-1}\,\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j)\right] dy \\ &- \frac{\varrho}{\Gamma(\xi)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \frac{(t_{n+1}-y)^{\xi-1}(y-t_{j})}{t_j - t_{j-1}} \left[t_{j-1}^{\varrho-1}\,\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1})\right] dy \\ &= \hat{\hat{\Upsilon}}_1 + \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n} I_1 - \frac{\varrho}{\Gamma(\xi)}\sum_{j=0}^{n} I_2,\end{aligned} \tag{19}$$

where

$$\begin{aligned}I_1 &= \int_{t_j}^{t_{j+1}} \frac{(t_{n+1}-y)^{\xi-1}(y-t_{j-1})}{t_j - t_{j-1}} \left[t_j^{\varrho-1}\,\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j)\right] dy, \\ &= \frac{t_j^{\varrho-1}\,h^{\xi}}{\xi(\xi+1)}\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j)\left\{(n-j+1)^{\xi}(\xi+n-j+2)-(n-j)^{\xi}(2\xi+n-j+2)\right\}, \\ I_2 &= \int_{t_j}^{t_{j+1}} \frac{(t_{n+1}-y)^{\xi-1}(y-t_{j})}{t_j - t_{j-1}} \left[t_{j-1}^{\varrho-1}\,\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1})\right] dy \\ &= \frac{t_{j-1}^{\varrho-1}\,h^{\xi}}{\xi(\xi+1)}\mathbf{f}_1(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1})\left\{(n-j+1)^{\xi+1}-(n-j)^{\xi}(\xi+n-j+1)\right\}.\end{aligned} \tag{20}$$

The details of the derivation the formula given in (20) can be found in [22].
Now, using I_1 and I_2 in (19), and general for all components of the system (Υ_i^{n+1}, $i = 1,2,3,4$), the model defined in (16) converts to the following system of algebraic equations (for $i = 1,2,3,4$):

$$\begin{aligned}\Upsilon_i^{n+1} =& \frac{\varrho\,h^{\xi}}{\Gamma(\xi+2)}\sum_{j=0}^{n} t_j^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_j)\left\{(n-j+1)^{\xi}(\xi+n-j+2)-(n-j)^{\xi}(2\xi+n-j+2)\right\} \\ &- \frac{\varrho\,h^{\xi}}{\Gamma(\xi+2)}\sum_{j=0}^{n} t_{j-1}^{\varrho-1}\,\mathbf{f}_i(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4,t_{j-1})\left\{(n-j+1)^{\xi+1}-(n-j)^{\xi}(\xi+n-j+1)\right\} + \hat{\hat{\Upsilon}}_i.\end{aligned} \tag{21}$$

4.1.1. Optimal control of banks' profits

Recently, a wide variety of issues have been successfully addressed by the optimal control theory in practical applications [23]. An intriguing question that arises in our present BCM is how to affect the profits of the given competition model (2) by selecting an effective mechanism to guide the model from ($\Upsilon_1(0)$, $\Upsilon_2(0)$, $\Upsilon_3(0)$, $\Upsilon_4(0)$) to a desired final state ($\Upsilon_1(T_f)$, $\Upsilon_2(T_f)$, $\Upsilon_3(T_f)$, $\Upsilon_4(T_f)$) in time T_f. This prefix objective is profit maximization. To do this, let's look at the system below:

$$^{FFP}D^{\xi,\varrho}\,\Upsilon_i(t) = \alpha_i\,\Upsilon_i(t)\left(1-\beta_i^{-1}\Upsilon_i(t)\right) - \delta_i\,\bar{\mathrm{N}}[\Upsilon_1,\,\Upsilon_2,\,\Upsilon_3,\,\Upsilon_4] - \varepsilon_i\,O_c(t)\,\Upsilon_i(t). \tag{22}$$

with the same previous initial conditions. Where $\varepsilon_i = 1,\ i = 1, 2, 3, 4$ indicates that there are no investment limits because the constants $\varepsilon_k,\ k = 1, 2, 3, 4$ reflect the maximum level of investment by the four bank categories. For that matter, $O_c(t) \in L^2[0, T_f]$ is the control parameter that indicates how much profit from each population is either retained by the bank for market reinvestment. The best choice of this function can be derived and obtained through Theorem 4 in [8]. For this type of problem, the majority of the current numerical approaches converge slowly, leading to imprecise approximations [24].

4.2. The Brusselator system

In the same manner as in the previous subsection, we can construct the numerical scheme for solving the Brusselator system (10). For this purpose, let us recall the model given in (10):

$$\begin{aligned} {}^{C}D^{\xi}\theta_1(t) &= \varrho\, t^{\varrho-1}(\delta - (\gamma+1)\,\theta_1(t) + \theta_1^2(t)\theta_2(t)),\\ {}^{C}D^{\xi}\theta_2(t) &= \varrho\, t^{\varrho-1}(\gamma\,\theta_1(t) - \theta_1^2(t)\theta_2(t)). \end{aligned} \tag{23}$$

By utilizing the integrals on the system (23), we obtain:

$$\begin{aligned} \theta_1(t) &= \theta_1(0) + \frac{\varrho}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1}\, s^{\varrho-1}\, \mathbf{f}_1(\theta_1(s), \theta_2(s), s)\, ds,\\ \theta_2(t) &= \theta_2(0) + \frac{\varrho}{\Gamma(\xi)} \int_0^t (t-s)^{\xi-1}\, s^{\varrho-1}\, \mathbf{f}_2(\theta_1(s), \theta_2(s), s)\, ds, \end{aligned} \tag{24}$$

where the functions $\mathbf{f}_k(\theta_1(t), \theta_2(t), t),\ k = 1, 2$, are given in (13). Thus, at time $t = t_{n+1}$

$$\begin{aligned} \theta_1^{n+1} &= \theta_1(0) + \frac{\varrho}{\Gamma(\xi)} \int_0^{t_{n+1}} (t_{n+1}-s)^{\xi-1}\ s^{\varrho-1}\, \mathbf{f}_1(\theta_1(s), \theta_2(s), s)\, ds,\\ \theta_2^{n+1} &= \theta_2(0) + \frac{\varrho}{\Gamma(\xi)} \int_0^{t_{n+1}} (t_{n+1}-s)^{\xi-1}\ s^{\varrho-1}\, \mathbf{f}_2(\theta_1(s), \theta_2(s), s)\, ds. \end{aligned} \tag{25}$$

Thus, the following algebraic equation system is transformed by the system provided in (25) as follows ($k = 1, 2$):

$$\begin{aligned} \theta_k^{n+1} &= \theta_{k,0} + \frac{\varrho\, h^{\xi}}{\Gamma(\xi+2)}.\\ &\sum_{j=0}^{n} \Big[t_j^{\varrho-1} \Big\{ (n-j+1)^{\xi}(\xi+n-j+2) - (n-j)^{\xi}(2\xi+n-j+2) \Big\}\, \mathbf{f}_k\left(\theta_1, \theta_2, t_j\right)\\ &\quad - t_{j-1}^{\varrho-1} \Big\{ (n-j+1)^{\xi+1} - (n-j)^{\xi}(\xi+n-j+1) \Big\}\, \mathbf{f}_k\left(\theta_1, \theta_2, t_{j-1}\right) \Big]. \end{aligned} \tag{26}$$

5. Numerical simulation

5.1. The Banks' competition model

We will show a numerical simulation in $[0, 9]$, for the model (2) and the modified system (22) with optimal control, with different values of ϱ, ξ, n, h, to show the precision and caliber of the proposed scheme. We utilize the same values for the parameters in [8] in all figures:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.7, 0.5, 0.45, 0.3), \qquad (\beta_1, \beta_2, \beta_3, \beta_4) = (33696, 31162, 11679, 3710),$$

$$(\delta_1, \delta_2, \delta_3, \delta_4) = (1.9, 2.3, 1.02, 5.0)\times 10^{-18}, \qquad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (0.392, 0.327, 0.245, 0.295).$$

Use the following initial conditions [8]:

$$\hat{\hat{\Upsilon}}_1 = 2.25, \qquad \hat{\hat{\Upsilon}}_2 = 2.3, \qquad \hat{\hat{\Upsilon}}_3 = 0.74, \qquad \hat{\hat{\Upsilon}}_4 = 0.6250.$$

We also give a comparison of the outcomes obtained by the suggested technique with those acquired by employing the RK4M.

Figures 1-4 give the approximate solution that was achieved for the system under investigation by using the suggested technique.

(i) The behavior of the numerical solution is given in Figure 1 for $n = 50$, using different values of $\varrho = \xi = 1.0, 0.95, 0.9, 0.85$.

(ii) In Figure 2, we show a comparison between the results using the RK4M at ($\varrho = \xi = 1$) with $n = 90$ and the results using the suggested technique.

(iii) In Figure 3, the approximate solution of the modified system with the optimal control $O_c(t) = -2\,e^{-2t}$ by using the proposed method at ($\varrho = \xi = 0.95$) and RK4M ($\varrho = \xi = 1$), at $n = 90$.

(iv) In Figure 4, we study the influence of the approximation order, n, on the approximate solution of the proposed model with and without optimal control.

These findings indicate that the presented technique is suitable for solving the proposed system, as the characteristics of the numerical solution derived from this technique are contingent upon the values of ϱ, ξ, n. Furthermore, we established a reinvestment-control mechanism as a proposal to implement a pre-reinvestment-control system that can absorb this unexpected fall in profits during crises, using the optimal function to recommend remedies. Simulations were run again after taking the controlling function into account, and the outcomes demonstrate how the banks' earnings are managed.

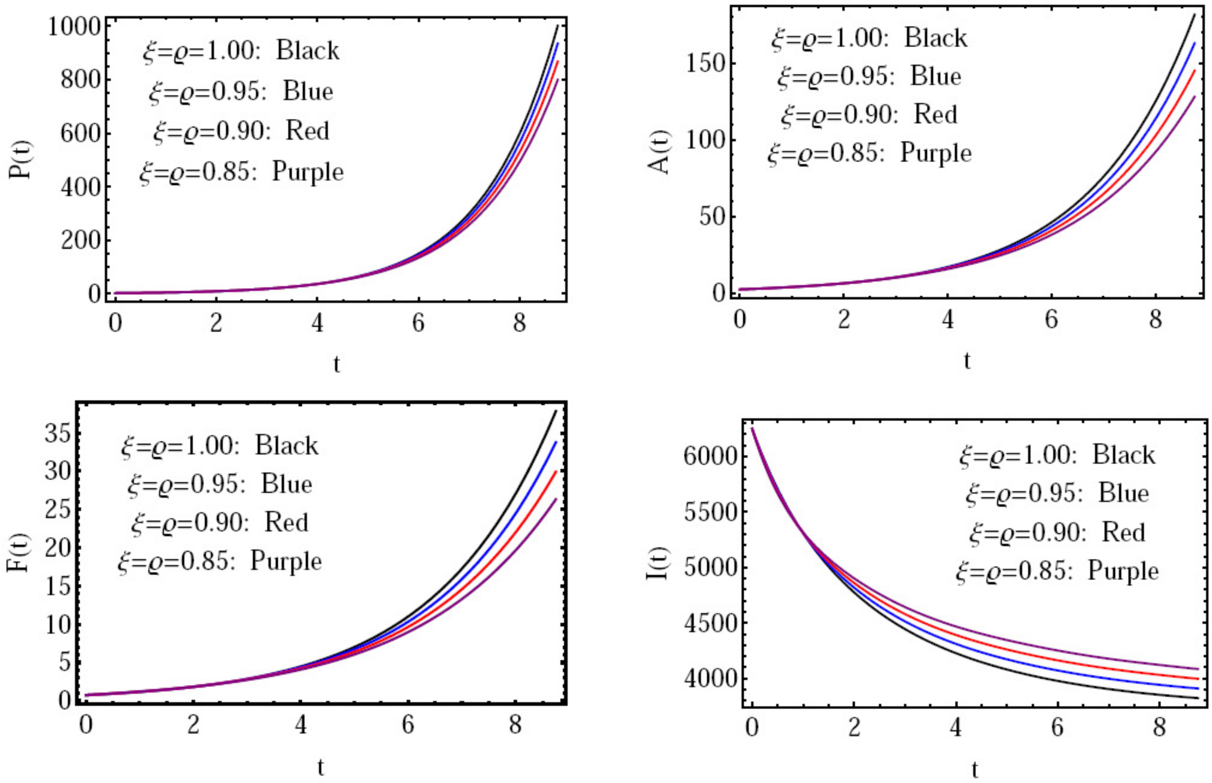

Fig. 1: The approximate solution with various values of ϱ, ξ.

5.2. The Brusselator model

We will show a numerical simulation on a test case of the system, to show the quality and correctness of the given numerical method (26) in the interval $[0, 4]$, $h = 0.01$ where fractional order $\varrho \in (0, 1]$, and fractal dimension $\xi \in (0, 1]$, with different values of δ, γ, with initial conditions $\theta_{1,0} = \theta_{2,0} = 1$. Figures 5-8 give the numerical solutions for the model under consideration by implementing the given method.

(i) The behavior of the approximate solution is shown in Figure 5 using different values of $\varrho = \xi = 1$, 0.9, 0.8, 0.7, with $\delta = 0.25$, $\gamma = 1$, and $n = 50$.

(ii) In Figure 6, we compare the outcomes obtained by the proposed strategy with the outcomes produced by utilizing the RK4M at ($\varrho = \xi = 1$), and $\delta = 0.0$, $\gamma = 1$ with $n = 90$.

(iii) In Figure 7, we demonstrate the approximate solution's behavior using several values of $\gamma = 0.5$, 1.0, 1.5, and $h = 0.1$, $\delta = 0.2$ at ($\varrho = \xi = 0.95$) with $n = 90$.

(iv) In Figure 8, the approximation solution's behavior using various values $\delta = 0.0$, 0.5, 1.0, and $\gamma = 0.2$ is presented with $n = 75$.

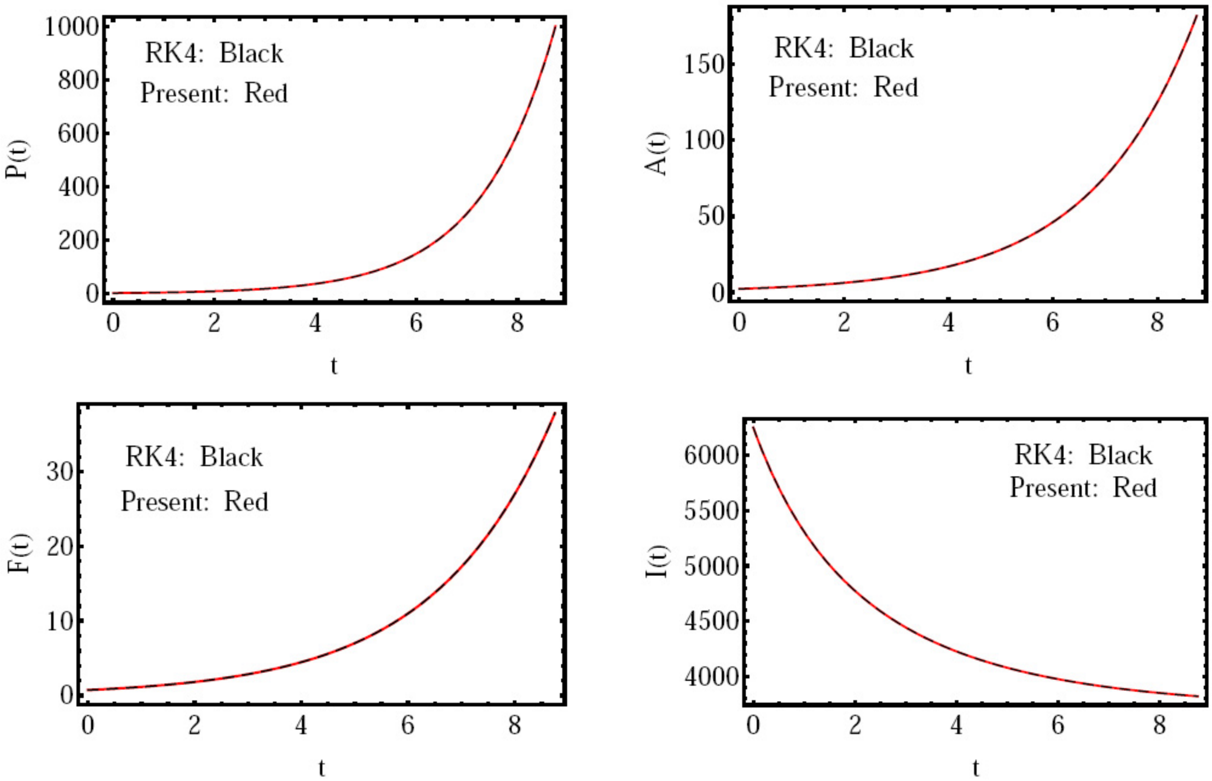

Fig. 2: The numerical solution by present method and RK4M at $\varrho = \xi = 1$.

The results shown in Figures 5-8 demonstrate that the behavior of the approximate solution derived from the specified technique is contingent upon the values of γ, δ, fractional order ϱ, and fractal dimension ξ. This signifies that the given scheme is suitable for addressing the studied system.

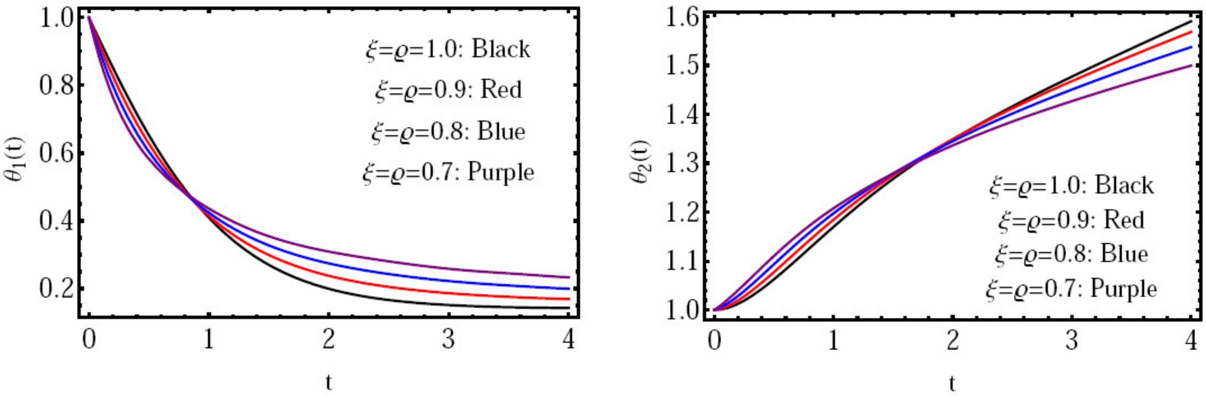

Fig. 5: The solution $\theta_1(t)$, $\theta_2(t)$ against distinct values of $\xi = \varrho$.

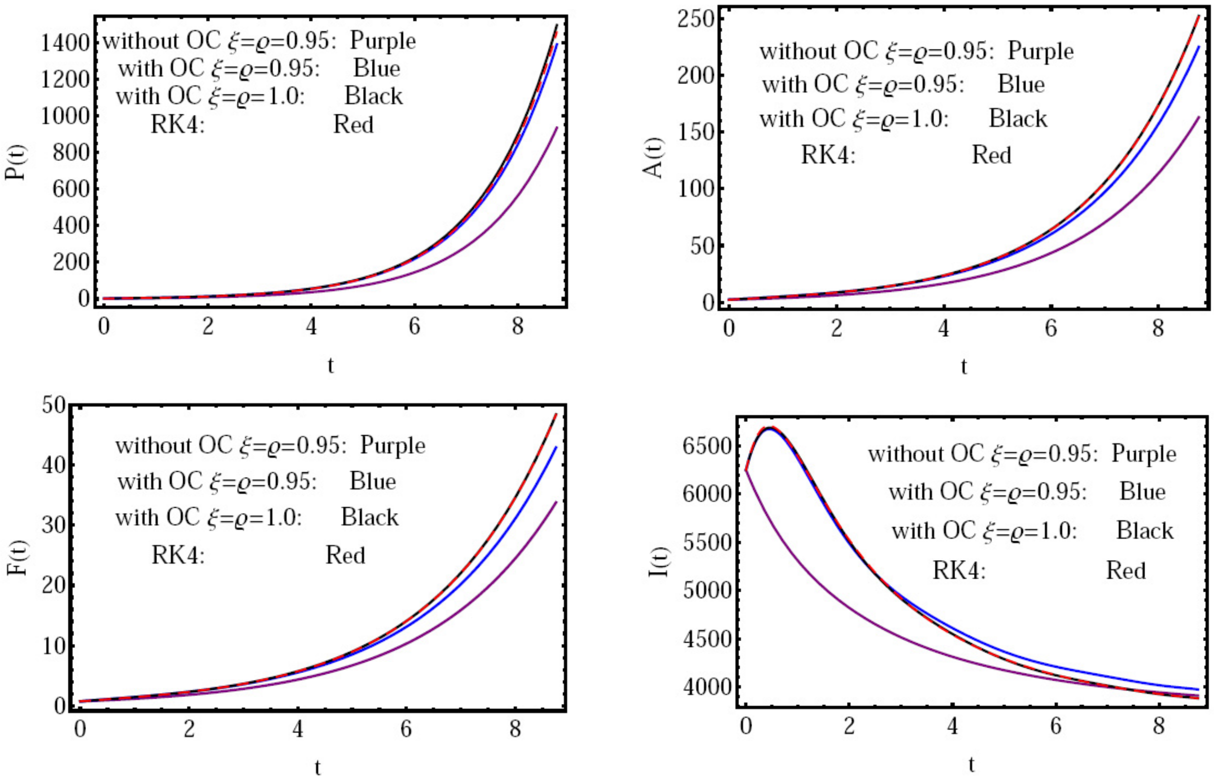

Fig. 3: The numerical solution with optimal control by present method ($\varrho = 0.95$) and RK4M ($\varrho = 1$).

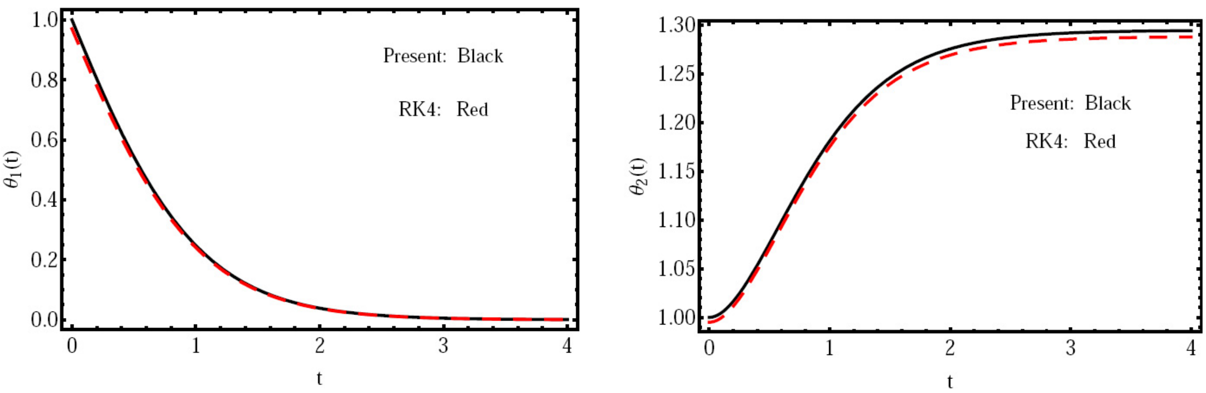

Fig. 6: The solution $\theta_1(t)$, $\theta_2(t)$ by the present method and the RK4M.

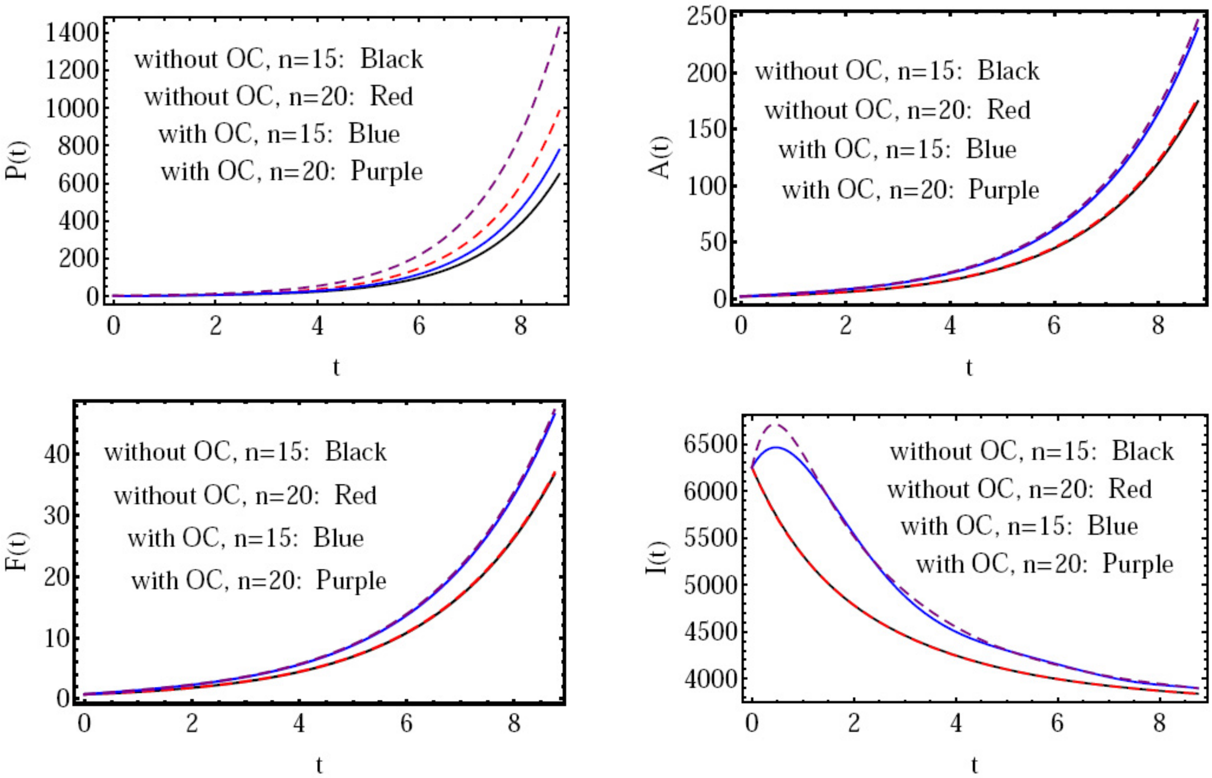

Fig. 4: The approximate solution with and without optimal control against distinct values of n.

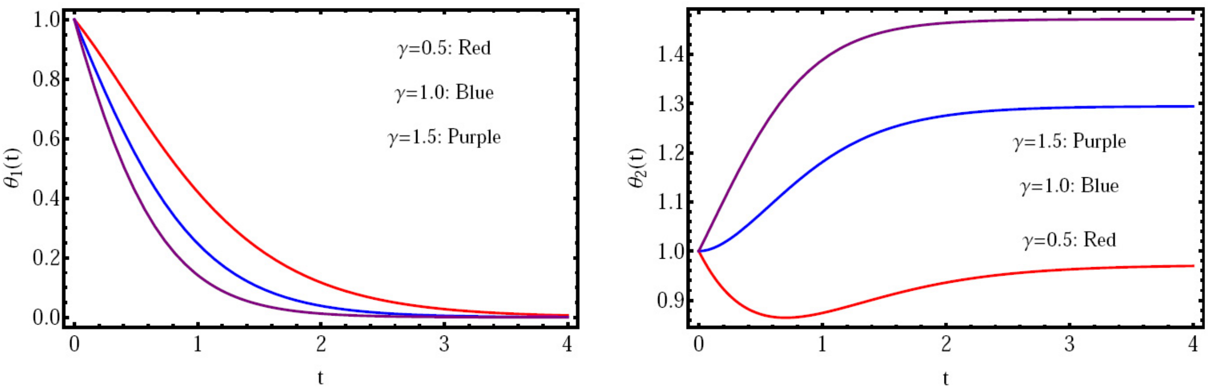

Fig. 7: The numerical solution $\theta_1(t)$, $\theta_2(t)$ against different values of γ.

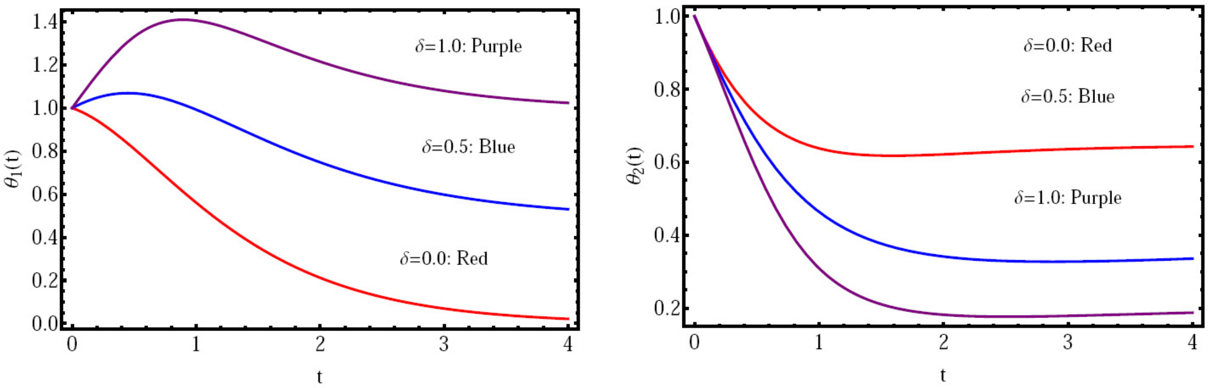

Fig. 8: The numerical solution $\theta_1(t)$, $\theta_2(t)$ against different values of δ.

6. Conclusions

This study delved into two significant fractal-fractional models: the competition among Egyptian banks and the Brusselator system employing the fractal-fractional derivative operator and fractional calculus methodologies. We investigated several choices of fractional order (ϱ), fractal dimension (ξ), and step size (h) to address these models. Our method proves remarkably successful in modeling these systems, with precision regulated by reducing h. Notably, numerical simulations using the FF operator outperform the RK4 approach for the models considered. We proposed a reinvestment-control function to manage profit loss during crises, demonstrating its efficacy through simulation outcomes. Numerical simulations are conducted using the Mathematica software package. Future endeavors may involve exploring alternative fractional derivatives. Extending the model to two-dimensional systems and using fractional variable orders. Exploring real experimental data validation monitoring stations. Developing efficient numerical solvers tailored for non-singular fractional kernels.

Acknowledgements

The researchers wish to extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post-Publishing Program.

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