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Direct Product of Q-Algebras

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Abstract. In this work, the direct product of Q-algebras is discussed. We investigate some properties of this direct product related to concepts of subalgebra, ideal and G-part of Q-algebras. We obtain that the set G-part of the direct product of Q-algebras is an exactly the direct product of sets of G-part of Q-algebras. We also provide the characterization of two elements subsets of the direct product of Q-algebras to be subalgebras. Moreover, we show the connection between the direct product of Q-algebras and CI-algebras.

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Key Words and Phrases: Q-algebra, direct product, subalgebra, G-part, atom, CI-algebra

1. Introduction and Preliminaries

Back to the year 1966, Y. Imai and K. Iseki created a new logical algebra which is called a BCK-algebra [1]. In the same year K. Iseki announced a notion of a BCI-algebra which is a generalization of a BCK-algebra in [2]. For more insight of BCK, BCI-algebras see also [3–5]. Since then many algebraic structures are invented, most of them are generalization of BCK, BCI-algebras. Q. Hu and X. Li introduced the notion of BCH-algebra in 1983 [6], some basic properties of BCH-algebras are explored. In 1984, Y. Komori introduced the notion of BCC-algebras [7]. In [8], Y. B. Jun, E. H. Roh and H. S. Kim introduced another generalization of BCK, BCI, BCC-algebras in 1998, which called BH-algebra. The authors of [8] provided that every bounded BH-algebra contains a maximal ideal. In 2001, J. Neggers, S. S. Ahn and H. S. Kim introduced the notion of Q-algebra which is a generalization of BCK, BCI, BCH, BH-algebras [9]. A Q-algebra consists of a non-empty set X and a constant $0 \in X$, with a binary operation $*$ on X that satisfies the following three conditions: for any $x, y, z \in X$
(Q_1) $x * x = 0$,

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(Q_2) $x * 0 = x$,
(Q_3) $(x * y) * z = (x * z) * y$.

We will omit the symbol of a binary operation, we write xy instead of $x*y$ for convenient reason. There are various literature discussed the notion of Q-algebra. In [9], some basic properties of Q-algebra are investigated. The authors also presented the concepts of ideal and quadratic Q-algebras. A non-empty subset I of a Q-algebra X is an ideal if I satisfies the conditions (I_1) and (I_2) as following: for all $x, y \in X$,
(I_1) $0 \in I$,
(I_2) $xy \in I$ and $y \in I$ imply $x \in I$.

The authors in [9] showed that the subset $B(X) = \{x \in X | \; 0x = 0\}$ of a Q-algebra X is an ideal. They also studied the set $G(X) = \{ \; x \in X | \; 0x = x\}$ of X, which is called the G-part of X. It is easy to see that $0 \in G(X)$. Hence, $G(X) \neq \emptyset$. In general, the set $G(X)$ need not to be an ideal of X. The authors proved that the set $G(X)$ is an ideal whenever $|X| = 2$ and $|X| = 3$. In [10], [11] and [12], the authors explored more properties of Q-algebras concerning to the concepts of ideal, G-part and atom. The characterization of ideals and some properties of the set G-part were presented in [11]. The authors showed that if the set G-part is an ideal, it is an abelian group. In [10], the concepts of atom and strong atom in Q-algebras are offered. An element $a \in X$ is an atom of X if $xa = 0$ implies $x = a$ for $x \in X$. If $0 \neq a$ is an atom and $ax = a$ for all $x \in X \setminus \{a\}$, then we call that a is a strong atom. In [10], the authors obtained that the set of all strong atoms together with a constant 0 is a subalgebra. A non-empty subset S of a Q-algebra X is called a subalgebra if $xy \in S$ for any $x, y \in S$. Moreover, they showed that if $G(X)$ is an ideal, then all elements in $G(X)$ are atoms. In [12], the authors proved that if an ideal I of a Q-algebra X contains a strong atom, then $I = X$. There is discussion of another kind of ideals in [13]. S. M. Mootafa et al discussed the concepts of Q-ideal and fuzzy Q-ideal in 2012. They founded that any Q-ideal is an ideal. There are other discussions of Q-algebras. In [14, 15] H. K. Abdullah and M. Tach introduced the notion of prime ideal and fuzzy prime ideal. The authors showed that an ideal I of a Q-algebra X with $|I| = |X| - 1$ is a prime ideal. The other approaches of discussion in Q-algebras are morphisms and mappings, more details of these topics can be found in [16, 17]. The concept of direct products was discussed in various kind of algebras, for instant groups, rings and modules. The direct product forms by taking the Cartesian product of their base sets of algebras as the carrier set and defining operations component-wise. In 2016, J. A. V. Lingcong and J. C. Endam considered the direct product of B-algebras [18], some basic properties were investigated. They also studied isomorphisms among the direct product of B-algebras and obtained a necessary condition for a mapping to be an isomorphism. In 2019 the direct product in BG-algebras was discussed by S. Widianto et al.[19]. They showed that the direct product of commutative BG-algebras is again a commutative BG-algebra. Later, the direct product of BP-algebras and GK-algebras were presented in [20] and [21], respectively. In [22], C. Chanmanee et al. introduced the concept of the direct product of infinite family of B-algebras in 2022. The authors called this product by "the external direct product" as it is a generalization of the direct product. In 2023-2024, the concept of an external direct product was considered in IUP-algebras, dual UP-algebras and JU-algebras, for more details see [22], [23] and [24], respectively.

In this work, we discuss a concept of the direct product of Q-algebras. We explore some properties of direct product of Q-algebras. The connection of the G-part of Q-algebras and the G-part of its direct product is investigated. Moreover, we provide a condition for the direct product of Q-algebras to be a CI-algebra. Now we recall some properties that we will use later.

Proposition 1. *[25] Every Q-algebra X satisfies the following property: $0(xy) = (0x)(0y)$ for all $x, y \in X$.*

Proposition 2. *[5] Let X be a Q-algebra and let a and b elements of $G(X)$. Then $ab = ba$.*

Corollary 1. *[5] Let X be a Q-algebra. A left cancellation law holds in $G(X)$, i.e. for all $a, b, c \in G(X)$, $ab = ac$ implies $b = c$.*

Proposition 3. *[11] Let X be a Q-algebra and let $0 \neq a, b, c \in G(X)$. Then the following three properties hold:*

(i) $a \neq b$ implies $ab \notin \{0, a, b\}$.

(ii) If $ab = c$, then $ac = b$ and $bc = a$.

(iii) For any $x \in X$, $xa \neq x$.

Proposition 4. *[10] Every element of a Q-algebra X is an atom if and only if $a(xb) = b(xa)$ for all $a, b, x \in X$.*

2. Direct Product Q-algebras

We will define a direct product on Q-algebras in a usual way.

Definition 1. *Let $\{(X_i; *_i, 0_i)|\ i \in I\}$ be a non-empty family of Q-algebras. Let $\prod_{i\in I} X_i$ be the cartesian product of the set $X_i, i \in I$:*

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} | x_i \in X_i, i \in I\}.$$

We define a binary operation $\circledast$ on $\prod_{i\in I} X_i$ as following: for $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$,

$$(x_i)_{i\in I} \circledast (y_i)_{i\in I} = (x_i * y_i)_{i\in I}.$$

Then we get that $(\prod_{i\in I} X_i; \circledast, (0_i)_{\in I})$ is a direct product Q-algebra as shown in the following proposition.

Proposition 5. *Let $\{(X_i; *_i, 0_i)|\ i \in I\}$ be a non-empty family of Q-algebras. Then $(\prod_{i\in I} X_i; \circledast, (0_i)_{\in I})$ is a Q-algebra.*

Proof. Since $0_i \in X_i$ for all $i \in I$, then $(0_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $(x_i)_{i\in I} \circledast (0_i)_{i\in I} = (x_i *_i 0_i)_{i\in I} = (x_i)_{i\in I}$ then the condition (Q_2) is satisfied. Moreover, $(x_i)_{i\in I} \circledast (x_i)_{i\in I} = (x_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$ there follows (Q_1) is fulfilled. Let consider

$$\begin{aligned} ((x_i)_{i\in I} \circledast (y_i)_{i\in I}) \circledast (z_i)_{i\in I} &= (x_i *_i y_i)_{i\in I} \circledast (z_i)_{i\in I} \\ &= ((x_i *_i y_i) *_i z_i)_{i\in I} \\ &= ((x_i *_i z_i) *_i y_i)_{i\in I} \\ &= (x_i *_i z_i)_{i\in I} \circledast (y_i)_{i\in I} \\ &= ((x_i)_{i\in I} \circledast (z_i)_{i\in I}) \circledast (y_i)_{i\in I}. \end{aligned}$$

Thus, the condition (Q_3) is satisfied. Hence, $(\prod_{i\in I} X_i; \circledast, (0_i)_{i\in I})$ is a Q-algebra.

Example 1. *Let $X_1 = \{0_1, a\}$ and $X_2 = \{0_2, x, y, z\}$ be the sets with the binary operations $*_1$ and $*_2$ defined on X_1 and X_2, respectively.*

$*_1$	0_1	a
0_1	0_1	a
a	a	0_1

$*_2$	0_2	x	y	z
0_2	0_2	x	z	y
x	x	0_2	y	z
y	y	z	0_2	x
z	z	y	x	0_2

.

Then $(X_1; *_1, 0_1)$ and $(X_2; *_2, 0_2)$ are Q-algebras. From Proposition 5, $(X_1 \times X_2; \circledast, (0_1, 0_2))$ is a Q-algebra, illustrated as the following table.

$\circledast$	$(0_1,0_2)$	$(0_1,x)$	$(0_1,y)$	$(0_1,z)$	$(a,0_2)$	(a,x)	(a,y)	(a,z)
$(0_1,0_2)$	$(0_1,0_2)$	$(0_1,x)$	$(0_1,z)$	$(0_1,y)$	$(a,0_2)$	(a,x)	(a,z)	(a,y)
$(0_1,x)$	$(0_1,x)$	$(0_1,0_2)$	$(0_1,y)$	$(0_1,z)$	(a,x)	$(a,0_2)$	(a,y)	(a,z)
$(0_1,y)$	$(0_1,y)$	$(0_1,z)$	$(0_1,0_2)$	$(0_1,x)$	(a,y)	(a,z)	$(a,0_2)$	(a,x)
$(0_1,z)$	$(0_1,z)$	$(0_1,y)$	$(0_1,x)$	$(0_1,0_2)$	(a,z)	(a,y)	(a,x)	$(a,0_2)$
$(a,0_2)$	$(a,0_2)$	(a,x)	(a,z)	(a,y)	$(0_1,0_2)$	$(0_1,x)$	$(0_1,z)$	$(0_1,y)$
(a,x)	(a,x)	$(a,0_2)$	(a,y)	(a,z)	$(0_1,x)$	$(0_1,0_2)$	$(0_1,y)$	$(0_1,z)$
(a,y)	(a,y)	(a,z)	$(a,0_2)$	(a,x)	$(0_1,y)$	$(0_1,z)$	$(0_1,0_2)$	$(0_1,x)$
(a,z)	(a,z)	(a,y)	(a,x)	$(a,0_2)$	$(0_1,z)$	$(0_1,y)$	$(0_1,x)$	$(0_1,0_2)$.

It is easy to see that $G(X_1 \times X_2) = \{(0_1, 0_2), (0_1, x), (a, 0_2), (a, x)\}$. It is a routine to verify that the set $S = \{(0_1, 0_2), (0_1, x), (a, 0_2), (a, x)\}$ is a subalgebra of $X_1 \times X_2$. Moreover, we get that $B_1 = \{0_1, a\}$ is a subalgebra of X_1, $B_2 = \{0_2, x\}$ is a subalgebra of X_2 and $S = B_1 \times B_2$. In general, a sub-direct product $\prod_{i\in I} B_i$ of a direct product $\prod_{i\in I} X_i$ of Q-algebras is a subalgebra whenever B_i is a subalgebra of X_i for all $i \in I$. This fact can be seen in the following proposition.

Proposition 6. *Let* $\prod_{i\in I} X_i$ *be the direct product of* Q*-algebras and let* $\emptyset \neq B_i \subseteq X_i$ *for all* $i \in I$*. Then* B_i *is a subalgebra of* X_i *for all* $i \in I$ *if and only if* $\prod_{i\in I} B_i$ *is a subalgebra of* $\prod_{i\in I} X_i$*.*

Proof. Assume B_i is a subalgebra of X_i for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} B_i$. Then $x_i * y_i \in B_i$ for all $i \in I$, there follows that $(x_i)_{i\in I} \circledast (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} B_i$. Thus, $\prod_{i\in I} B_i$ is a subalgebra. Conversely, let $x_i, y_i \in B_i$ for each $i \in I$. Since $\prod_{i\in I} B_i$ is a subalgebra then $(x_i *_i y_i)_{i\in I} = (x_i)_{i\in I} \circledast (y_i)_{i\in I} \in \prod_{i\in I} B_i$. Therefore, $x_i *_i y_i \in B_i$ and there follows B_i is a subalgebra of X_i for all $i \in I$.

Example 2. *Consider* Q*-algebras* $(X_1; *_1, 0_1)$ *and* $(X_2; *_2, 0_2)$ *from Example 1. We see that* $G(X_1) = \{0_1, a\}$*,* $G(X_2) = \{0_2, x\}$ *and* $G(X_1) \times G(X_2) = \{0_1, a\} \times \{0_2, x\} = \{(0_1, 0_2), (0_1, x), (a, 0_2), (a, x)\} = G(X_1 \times X_2)$*.*

Example 2 shows a relation of the set G-part of the direct product of Q-algebras, $G(\prod_{i\in I} X_i)$, and a direct product of the set G-part of X_i, $i \in I$, $\prod_{i\in I} G(X_i)$. Both sets are coincide as shown in the following proposition.

Proposition 7. $G(\prod_{i\in I} X_i) = \prod_{i\in I} G(X_i)$*.*

Proof. Let $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$. Then $(a_i)_{i\in I} = (0_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i *_i a_i)_{i\in I}$. It follows that $a_i = 0_i *_i a_i$ for all $i \in I$. Hence, $a_i \in G(X_i)$ for all $i \in I$. Therefore, $(a_i)_{i\in I} \in \prod_{i\in I} G(X_i)$, i.e. $G(\prod_{i\in I} X_i) \subseteq \prod_{i\in I} G(X_i)$. For the opposite inclusion, let $(c_i)_{i\in I} \in \prod_{i\in I} G(X_i)$. Then for all $i \in I$, $c_i \in G(X_i)$. It follows that $0_i *_i c_i = c_i$ and then $(c_i)_{i\in I} = (0_i *_i c_i)_{i\in I} = (0_i)_{i\in I} \circledast (c_i)_{i\in I}$. Thus, $(c_i)_{i\in I} \in G(\prod_{i\in I} X_i)$. Hence, $\prod_{i\in I} G(X_i) \subseteq G(\prod_{i\in I} X_i)$.

For any element $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$, $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (a_i)_{i\in I}$. Combining this fact and conditions $(Q_1), (Q_2)$ we get the following proposition.

Proposition 8. *A subset* $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ *is a subalgebra of* $\prod_{i\in I} X_i$ *for all* $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$*.*

Proof. Let $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$ and let $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$. Properties (Q_1) and (Q_2) imply that $(0_i)_{i\in I} \circledast (0_i)_{i\in I} = (0_i)_{i\in I} \in S$, $(a_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I} \in S)$ and $(a_i)_{i\in I} \circledast (0_i)_{i\in I} = (a_i)_{i\in I} \in S$. Since $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$, then $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (a_i)_{i\in I}$. Therefore, $(0_i)_{i\in I} \circledast (a_i)_{i\in I} \in S$. Altogether, $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ is closed. Hence, $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ is a subalgebra of $\prod_{i\in I} X_i$.

Next proposition shows necessary and sufficient conditions for a two-element subset of $\prod_{i\in I} X_i$ containing a constant $(0_i)_{i\in I}$ to be a subalgebra of $\prod_{i\in I} X_i$.

Proposition 9. *Let $(a_i)_{i\in I} \in \prod_{i\in I} X_i$ and let $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$. Then S is a subalgebra of $\prod_{i\in I} X_i$ if and only if $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$ or $(a_i)_{i\in I} \in B(\prod_{i\in I} X_i)$.*

Proof. ($\Rightarrow$) Assume $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ is a subalgebra of $\prod_{i\in I} X_i$. Then by closure property of S, $(0_i)_{i\in I} \circledast (a_i)_{i\in I} \in S$, there follows $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (a_i)_{i\in I}$ or $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$. Hence, $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$ or $(a_i)_{i\in I} \in B(\prod_{i\in I} X_i)$.

($\Leftarrow$) If $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$, then $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ is a subalgebra by Proposition 8. If $(a_i)_{i\in I} \in B(\prod_{i\in I} X_i)$, then $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$. From this fact and $(Q_1), (Q_2)$ we can conclude that $S = \{(0_i)_{i\in I}, (a_i)_{i\in I}\}$ is closed. Therefore, S is a subalgebra of $\prod_{i\in I} X_i$.

Moreover, the set $G(\prod_{i\in I} X_i)$ itself is also a subalgebra.

Proposition 10. *$G(\prod_{i\in I} X_i)$ is a subalgebra of $\prod_{i\in I} X_i$.*

Proof. Since $0_i \in G(X_i)$ for all $i \in I$ and by Proposition 7, then $(0)_{i\in I} \in G(\prod_{i\in I} X_i)$. Therefore, $G(\prod_{i\in I} X_i) \neq \emptyset$. Let $(a_i)_{i\in I}, (b_i)_{i\in I} \in G(\prod_{i\in I} X_i)$. Then $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (a_i)_{i\in I}$ and $(0_i)_{i\in I} \circledast (b_i)_{i\in I} = (b_i)_{i\in I}$. By Proposition 1 we get $(0_i)_{i\in I} \circledast [(a_i)_{i\in I} \circledast (b_i)_{i\in I}] = [(0_i)_{i\in I} \circledast (a_i)_{i\in I}] \circledast [(0_i)_{i\in I} \circledast (b_i)_{i\in I}] = (a_i)_{i\in I} \circledast (b_i)_{i\in I}$. It follows that $(a_i)_{i\in I} \circledast (b_i)_{i\in I} \in G(\prod_{i\in I} X_i)$. Therefore, $G(\prod_{i\in I} X_i)$ is closed.

Proposition 11. *$G(\prod_{i\in I} X_i)$ is an abelian group.*

Proof. Let $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in G(\prod_{i\in I} X_i)$. Then $(a_i)_{i\in I} \circledast (b_i)_{i\in I} \in G(\prod_{i\in I} X_i)$ by Proposition 10. The commutative property follows from Proposition 2. By (Q_3) and the commutative property, there follows

$$\begin{aligned}
[(a_i)_{i\in I} \circledast (b_i)_{i\in I}] \circledast (c_i)_{i\in I} &= [(a_i)_{i\in I} \circledast (c_i)_{i\in I}] \circledast (b_i)_{i\in I} \\
&= [(c_i)_{i\in I} \circledast (a_i)_{i\in I}] \circledast (b_i)_{i\in I} \\
&= [(c_i)_{i\in I} \circledast (b_i)_{i\in I}] \circledast (a_i)_{i\in I} \\
&= [(b_i)_{i\in I} \circledast (c_i)_{i\in I}] \circledast (a_i)_{i\in I} \\
&= (a_i)_{i\in I} \circledast [(b_i)_{i\in I} \circledast (c_i)_{i\in I}].
\end{aligned}$$

Therefore, an associative law is fulfilled. Since $(a_i)_{i\in I} \in G(\prod_{i\in I} X_i)$ and by (Q_2), then $(0_i)_{i\in I} \circledast (a_i)_{i\in I} = (a_i)_{i\in I} = (a_i)_{i\in I} \circledast (0_i)_{i\in I}$. Therefore, $(0_i)_{i\in I}$ is an identity element.

Since $(a_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$ by (Q_2), then $(a_i)_{i\in I}$ is an inverse of itself. Altogether, $G(\prod_{i\in I} X_i)$ is an abelian group.

It is known that an abelian group G such that every element (except an identity element e) has order 2, G has an order 2^k for some positive integer k. By this fact and Proposition 11 we obtain the following proposition.

Proposition 12. *If* $1 \neq |\prod_{i\in I} X_i| = k$ *for some odd number* k*, then* $G(\prod_{i\in I} X_i) \neq \prod_{i\in I} X_i$.

Proof. Assume that $|\prod_{i\in I} X_i| = k$ for some odd number $k > 1$. Suppose $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$. Then $|G(\prod_{i\in I} X_i)| = k$. There follows $|G(\prod_{i\in I} X_i)|$ is odd, this is impossible since $G(\prod_{i\in I} X_i)$ is an abelian group by Proposition 11. Hence, $G(\prod_{i\in I} X_i) \neq \prod_{i\in I} X_i$.

Proposition 13. *If* $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$*, then every element of* $\prod_{i\in I} X_i$ *is an atom.*

Proof. Assume $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$. Then by Proposition 11, $\prod_{i\in I} X_i$ is an abelian group. Let $(a_i)_{i\in I} \in \prod_{i\in I} X_i$. Assume $(x_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ is an abelian group, then $(a_i)_{i\in I} \circledast (x_i)_{i\in I} = (x_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$. From $(a_i)_{i\in I} \circledast (x_i)_{i\in I} = (0_i)_{i\in I}$ and by (Q_1) $(a_i)_{i\in I} \circledast (a_i)_{i\in I} = (0_i)_{i\in I}$, applying Corollary 1 we get $(a_i)_{i\in I} = (x_i)_{i\in I}$. Therefore, $(a_i)_{i\in I}$ is an atom for all $(a_i)_{i\in I} \in \prod_{i\in I} X_i$.

The converse of Proposition 13 is not true. Let consider a Q-algebra $(X_1 \times X_2; \circledast, (0_1, 0_2))$ from Example 1.

$\circledast$	$(0_1,0_2)$	$(0_1,x)$	$(0_1,y)$	$(0_1,z)$	$(a,0_2)$	(a,x)	(a,y)	(a,z)
$(0_1,0_2)$	$(0_1,0_2)$	$(0_1,x)$	$(0_1,z)$	$(0_1,y)$	$(a,0_2)$	(a,x)	(a,z)	(a,y)
$(0_1,x)$	$(0_1,x)$	$(0_1,0_2)$	$(0_1,y)$	$(0_1,z)$	(a,x)	$(a,0_2)$	(a,y)	(a,z)
$(0_1,y)$	$(0_1,y)$	$(0_1,z)$	$(0_1,0_2)$	$(0_1,x)$	(a,y)	(a,z)	$(a,0_2)$	(a,x)
$(0_1,z)$	$(0_1,z)$	$(0_1,y)$	$(0_1,x)$	$(0_1,0_2)$	(a,z)	(a,y)	(a,x)	$(a,0_2)$
$(a,0_2)$	$(a,0_2)$	(a,x)	(a,z)	(a,y)	$(0_1,0_2)$	$(0_1,x)$	$(0_1,z)$	$(0_1,y)$
(a,x)	(a,x)	$(a,0_2)$	(a,y)	(a,z)	$(0_1,x)$	$(0_1,0_2)$	$(0_1,y)$	$(0_1,z)$
(a,y)	(a,y)	(a,z)	$(a,0_2)$	(a,x)	$(0_1,y)$	$(0_1,z)$	$(0_1,0_2)$	$(0_1,x)$
(a,z)	(a,z)	(a,y)	(a,x)	$(a,0_2)$	$(0_1,z)$	$(0_1,y)$	$(0_1,x)$	$(0_1,0_2)$.

It is not difficult to verify that all elements of $X_1 \times X_2$ are atoms and $G(X_1 \times X_2) = \{(0_1, 0_2), (0_1, x), (a, 0_2), (a, x)\}$. Therefore, $G(X_1 \times X_2) \neq X_1 \times X_2$.

Corollary 2. *If* $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$*, then* $(x_i)_{i\in I} \circledast [(y_i)_{i\in I} \circledast (z_i)_{i\in I}] = (z_i)_{i\in I} \circledast [(y_i)_{i\in I} \circledast (x_i)_{i\in I}]$ *for all* $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$.

Proof. Assume that $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$. Then by Proposition 13 all elements of $\prod_{i\in I} X_i$ are atoms. By Proposition 4 we get $(x_i)_{i\in I} \circledast [(y_i)_{i\in I} \circledast (z_i)_{i\in I}] = (z_i)_{i\in I}) \circledast [(y_i)_{i\in I} \circledast (x_i)_{i\in I}]$ for all $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$.

Next proposition shows a condition for the direct product of Q-algebras to be a CI-algebra. A sysytem $(X; *, 0)$ consists of a non-empty set X, together with a binary operation $*$ defined on X and a constant $0 \in X$ is called a CI-algebra if (CI_1) $xx = 0$, (CI_2) $0x = x$ and (CI_3) $x(yz) = y(xz)$ for all $x, y, z \in X$, are satisfied.

Proposition 14. *$G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$ if and only if $\prod_{i\in I} X_i$ is a CI-algebra.*

Proof. Assume that $G(\prod_{i\in I} X_i) = \prod_{i\in I} X_i$. Then $\prod_{i\in I} X_i$ is an abelian group by Proposition 11. We want to show that $\prod_{i\in I} X_i$ is a CI-algebra. It is enough to show only the condition (CI_3). Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. From a commutative property and (Q_3) we can calculate that $(x_i)_{i\in I} \circledast [(y_i)_{i\in I} \circledast (z_i)_{i\in I}] = (x_i)_{i\in I} \circledast [(z_i)_{i\in I} \circledast (y_i)_{i\in I}] = [(z_i)_{i\in I} \circledast (y_i)_{i\in I}] \circledast (x_i)_{i\in I} = [(z_i)_{i\in I} \circledast (x_i)_{i\in I}] \circledast (y_i)_{i\in I} = (y_i)_{i\in I} \circledast [(z_i)_{i\in I} \circledast (x_i)_{i\in I}] = (y_i)_{i\in I} \circledast [(x_i)_{i\in I} \circledast (z_i)_{i\in I}]$. Hence, (CI_3) is satisfied. Therefore, $\prod_{i\in I} X_i$ is a CI-algebra. The converse direction follows from the condition (CI_2).

3. Conclusion

We discussed the direct product of Q-algebras. We obtained that the direct product of Q-algebras is again a Q-algebra. Several basic properties of the direct product of Q-algebras are presented. We also studied this topic related to many concepts in Q-algebras, for instant, subalgebra, G-part and atom. We showed necessary and sufficient conditions for a two-element subset of the direct product of Q-algebras, $\prod_{i\in I} X_i$, containing a constant $(0_i)_{i\in I}$ to be a subalgebra of $\prod_{i\in I} X_i$. We also provided some properties of the set G-part, $G(\prod_{i\in I} X_i)$, of a Q-algebra $\prod_{i\in I} X_i$. We proved that the set $G(\prod_{i\in I} X_i)$ is a subalgebra, moreover, it is an abelian group. We also showed that every element of $\prod_{i\in I} X_i$ is an atom whenever the set $G(\prod_{i\in I} X_i)$ is equal to $\prod_{i\in I} X_i$. For further study, one can consider the direct product of Q-algebras in the direction that related to the following topics:

- ideals and filters;
- fuzzy subalgebras, fuzzy ideals;
- homomorphisms and isomorphisms;
- more insight of atoms and strong atoms.

Another direction of study is a concept of a generalization of the direct product, i.e. the external direct product of Q-algebras.

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References

[1] Y. Imai and K. Iseki. On axiom system of propositional calculi xiv. *Proceeding of Japan Academy*, 42(1):19–22, 1966.

[2] K. Iseki. An algebra related with a propositional calculus. *Proceeding of Japan Academy*, 42(1):26–29, 1966.

[3] Y. Arai, K.Iseki, and S. Tanaka. Characterizations of bci, bck-algebra. *Proceeding of Japan Academy*, 42:105–107, 1966.

[4] K. Iseki and S. Tanaka. An introduction to theory of bck-algebra. *Mathematics Japonica.*, 23:1–26, 1978.

[5] Y. B. Jun, M. Hong, E. H. Roh, and J. Meng. On the bci-g part of bci-algebras (iii). *Communications of the Korean Mathematical Society*, 9(3):531–538, 1994.

[6] Q. Hu and X. Li. On bch-algebras. *Mathematics Seminar Note Kobe University*, 2(11):313–320, 1983.

[7] Y. Komori. The class of bcc-algebras is not a variety. *Mathematica Japonica*, 29:391–394, 1984.

[8] Y. B. Jun, E. H. Roh, and H. S. Kim. On bh-algebras. *Scientiae Mathematicae*, 1(3):347–354, 1998.

[9] J. Neggers, S. Ahn, and H.S. Kim. On q-algebras. *International Journal of Mathematics and Mathematical Sciences*, 27(12):749–757, 2001.

[10] A. Anantayasethi, T. Kunawat, and P. Moonnipa. Relation between g-part and atoms in q-algebras. *European Journal of Pure and Applied Mathematics*, 17(4):3268–3276, 2024.

[11] A. Anantayasethi and J. Koppitz. Characterization of ideals of q-algebras related to its g-part. *Journal of Discrete Mathematical Sciences and Cryptography*, 28(1):131–141, 2025.

[12] K. Saengsura, N. Sarasit, and A. Anantayasethi. On the strong atoms of q-algebra. *Axioms*, 14:271, 2005.

[13] S. M. Mostafa, M. A. Naby, and O. R. Elgendy. Fuzzy q-ideals in q-algebras. *World Applied Programming*, 2(2):69–80, 2012.

[14] H. K. Abdullah and M. Tach. Intuitionistic fuzzy prime ideal on q-algebras. *International Journal of Academic and Applied Research*, 4(10):66–78, 2020.

[15] H. K. Abdullah and M. Tach. Prime ideal in q-algebra. *International Journal of Academic and Applied Research*, 4(10):79–87, 2020.

[16] S. S. Ahn, H. S. Kim, and H. D. Lee. R-maps and l-map in q-algebras. *IJPAM.*, 12(4):419–425, 2004.

[17] S. M. Lee and K. H. Kim. On right fixed maps of q-algebras. *Internayinal Mathematical Forum*, 6(1):31–37, 2011.

[18] J. A. V. Lingcong and J. C. Endam. Direct product of b-algebras. *International of Algebra*, 10(1):33–40, 2016.

[19] S. Widianto, S. Gemawati, and Kartini. Direct product of bg-algebras. *International Journal of Algebra*, 13(5):239–247, 2019.

[20] A. Setian, S. Gemawati, and L. Deswita. Direct product of bp-algebras. *International Journal of Mathematics Trends and Tcehnology*, 66(10):63–66, 2020.

[21] J. Kavitha and R. Gowri. Direct product of gk-algebras. *Indian Journal of Science and Tchnology*, 14(35):2802–2805, 2021.

[22] C. Chanmanee, R. Prasertpong, P. Julatha, N. Lekkoksung, and A. Iampan. On external direct products of iup-algebras. *Journal of Innovative Computing, Information and Control*, 19(3):775–787, 2023.

[23] C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha, and A. Iampan. External direct products on dual up (bcc)-algebras. *Journal of Mathematics and Computer Science*, 29(2):175–191, 2023.

[24] C. Chanmanee, P. Julatha, W. Nakkhasen, R. Prasertpong, , and A. Iampan. External direct products ju-algebras. *International Journal of Analysis and Applications*, 22:183, 2024.

[25] S. Ahn and S. E. Kang. The role of t(x) in the ideal theory of q-algebras. *Honam Mathematical J.*, 32(3):515–523, 2010.