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Exponential Bases of Cliffordian Polynomials in Fréchet Modules

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Abstract. This paper presents a generalization of the exponential base of special monogenic polynomials within the framework of Fréchet modules (F-modules). The study focuses on examining the convergence properties, specifically the effectiveness, of both the exponential simple base of special monogenic polynomials (ESBSMPs) and the exponential Cannon base of special monogenic polynomials (ECBSMPs) in Fréchet modules. These properties are investigated on hyper-closed and open balls, in open regions surrounding hyper-closed balls, for all entire special monogenic functions, as well as at the origin. Furthermore, an explicit upper bound for the order of the exponential simple base is established and shown to be attainable. Finally, we extend the discussion to equivalent and similar bases, verifying that the derived results remain valid under such transformations, which confirms the robustness and general applicability of the findings.

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Key Words and Phrases: Clifford analysis, special monogenic polynomials, Fréchet modules, bases of polynomials, growth of bases, effectiveness

1. Introduction

The development of bases theories in functional spaces has gained increasing importance in diverse mathematical areas including approximation theory, partial differential equations, and mathematical physics. Approximation theory has a crucial role providing tools for analyzing and solving problems arising in applied sciences and engineering. Recent developments demonstrate the diversity of its applications. The authors of [1] addressed approximate numerical solutions of time-fractional partial differential equations in three dimensions, achieving high accuracy even on irregular domains. In [2], the authors proposed a derivative-free iterative method with optimal fourth-order.convergence for finding multiple roots.

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Clifford analysis is considered as an elegant higher dimensional analogy to complex analysis. It allow an extension of the theory of holomorphic functions to higher dimensional setting by using Clifford algebra valued functions which are the solutions the generalized Cauchy–Riemann system (Dirac operator). These functions are known as Clifford holomorphic functions (monogenic functions).

In the context of single complex variable, the topic of basic sets was introduced by Whittaker [3, 4] and Cannon [5, 6]. In [7, 8], the authors proposed an extended adaptation of Whittaker-Cannon theory of bases of polynomial in the complex setting to the framework of Clifford algebra. They provided the effectiveness criteria in the convergence domain which means the action of approximating special monogenic functions (SMFs) via bases of special monogenic polynomials (SMPs) with or without some restrictions. Their study was concerned with the approximation of subclass of MFs which are generated by a particular special polynomials in axially symmetric domains. Subsequent papers proposed by several authors which expose the significant of this approach. The construction and effectiveness of certain derived bases of SMPs were studied in [9–11]. The convergence properties for the Bernoulli polynomials, Euler polynomials, Bessel polynomials, and Chebyshev in Clifford analysis [12–15].

Constructing bases of SMPs by the utilization the fundamentals of the functional analysis was proposed in [16] where the authors characterized the convergence of certain classes of bases for various F-modules. The effectiveness and the growth of the equivalent base constructed with SMPs in F-modules were studied in [17]. Recently, the authors in [18] deduced the extended Ruscheweyh differential operator and examined the representation of its derived bases of SMPs in different convergence regions. Recently in [19], the authors investigated the representation of a SMFs in terms of infinite series of Cliffordian Hasse derivative bases in hyper regions was introduced.

The representation of regular functions of several complex variables by means of exponential base of polynomials in hyperelliptical regions were discussed in [20]. In the Clifford context, the exponential function $\exp(x), x$ a Clifford variable was introduced in [21] as an extension of the classical complex function e^z. In [21], the author discussed the convergence properties of the exponential base of SMPs with the bases associated with the base of polynomials

$$\{\mathcal{Q}_n(x)\} = \{\sum_k q_k(x)\mathcal{Q}_{n,k}\},$$

where the $\mathcal{Q}_{n,k}$ are real Clifford coefficients. Precisely, the restriction $\mathcal{Q}_{n,n} = 1$ for $n \in \mathbb{N}$ was imposed on the diagonal of the matrix of entries of the matrix $\mathcal{Q} = (\mathcal{Q}_{n,k})$. In the current study, we relinquish the aforementioned condition and consider a more generalized form for these diagonal elements to be $\mathcal{Q}_{n,n} = \alpha_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a bounded sequence of the positive numbers. We assign certain conditions to the coefficients of the power associated infinite matrices of an original base. Consequently, the representation of a SMF is provided in terms of ESBSMPs. We show that exponential base satisfies the higher-dimensional effectiveness criteria for the F-module $\mathcal{W}^{\bar{B}(R)}$. Furthermore, we find that the order of the ESBSMPs is bounded above by an attainable upper bound. Moreover, we investigate the convergence properties of the ECBSMPs for the F-modules

$\mathcal{W}^{\bar{B}(R)}$, $\mathcal{W}^{B(R)}$, $\mathcal{W}^{B+(R)}$, $\mathcal{W}^{\infty}$, $\mathcal{W}^{0^+}$ which will be defined in Section 2.

2. Notations and basic results

The associated 2^m-dimensional real algebra $\mathcal{A}_m$ constructed from the space R^m with an orthogonal base $\{e_1, e_2, \ldots, e_m\}$ is identified such that $R^m \subset \mathcal{A}_m$. Suppose that $\{e_j\}_{j=1}^m$ is an orthonormal base of $\mathbb{R}^m$. Then the non-commutative multiplication in $\mathbb{R}_m$ is subjected to $e_k e_\ell + e_\ell e_k = -2\delta_{k\ell}$, where $k, \ell = 1, \ldots, m$ and $\delta_{k\ell}$ stands for the Kronecker symbol (for details on the main concepts of $\mathcal{A}_m$, see [22]). The set $\{e_S : S \subset \{0, 1, \ldots, m\}\}$ where $S = \{s_1, s_2, \ldots s_h\}$, $0 \leq s_1 < s_2 < \cdots < s_h \leq m$, $e_S = e_{s_1} e_{s_2} \ldots e_{s_h}$, $e_\phi, e_0 = 1$, creates a base of $\mathbb{R}_{0,m}$. Embedding $\mathbb{R}^{m+1}$ in $\mathbb{R}_{0,m}$. it is identified that $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ corresponds to $x = \sum_{i=0}^m x_i e_i$. Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_k = -e_k$ $(1 \leq k \leq m)$. The norm of an element $u \in \mathcal{A}_m$ is defined by $\left(\sum_A |u_A|^2\right)^{\frac{1}{2}}$ and we have always use the formula $|uv| \leq 2^{\frac{m}{2}} |u| \, |v|$ where the products $u\overline{u}$ or $v\overline{v}$ produce real numbers.

Clifford analysis provide a function theory that is a higher dimensional analog of functions theory of single complex variable (see e.g. [22, 23]). In this context, a monogenic functions is a null solution of the Dirac operator $D = \sum_{i=0}^m e_i \frac{\partial}{\partial x_i}$, in $\mathbb{R}^{m+1}$ (see [24]).

Consider a regular function φ having values in $\mathcal{A}_m$ defined in some open subset $\mathcal{M}$ of $\mathbb{R}^{m+1}$. If $D\varphi = 0$ or $\varphi D = 0$, then φ is called left-monogenic or right-monogenic respectively.

Definition 1. *Let $\mathcal{Q}(x)$ be MP. Then $\mathcal{Q}(x)$ is SMP if there exist $u_{k,\ell} \in \mathcal{A}_m$, and we have*

$$\mathcal{Q}(x) = \sum_{k,\ell}^{finite} \overline{x}^k x^\ell u_{k,\ell}.$$

Definition 2. *Let φ be a MF in $\mathcal{M}$ where $\mathcal{M} \subset \mathbb{R}^{m+1}$ is an open and connected. which contain 0. The function φ is SMF in $\mathcal{M}$ iff it has Taylor expansion near zero and can be written in the form $\varphi(x) = \sum_{\ell=0}^{\infty} \mathcal{Q}_\ell(x) u_\ell$ for some SMPs $\mathcal{Q}_\ell(x)$ where $u_\ell \in \mathcal{A}_m$.*

The right $\mathcal{A}_m$ module has the the following form:

$$\mathcal{A}_m[x] = \text{span}_{\mathcal{A}_m}\{q_\ell(x) : \ \ell \in \mathbb{N}\},$$

where $q_\ell(x)$ was first constructed in [7] as follows:

$$q_\ell(x) = \frac{\ell!}{(m)_\ell} \sum_{s+t=\ell} \frac{(\frac{m-1}{2})_s (\frac{m+1}{2})_t}{s!t!} \, \overline{x}^s x^t, \tag{1}$$

where $(a)_h = a(a+1)\ldots(a+h-1)$ is the Pochhamer symbol for $a \in \mathbb{R}$.

Remark 1. *If $\mathcal{Q}_\ell(x)$ is a SMP which is homogeneous and of degree ℓ and $\mathcal{Q}_\ell(x) = q_\ell(x)u$, for $u \in \mathcal{A}_m$ ([7]), then*

$$\|q_\ell\|_R = \sup_{\overline{B}(R)} |q_\ell(x)| = R^\ell.$$

Definition 3. *A space $\mathcal{F}$ is called an F-module over $\mathcal{A}_m$ if $\mathcal{F}$ is a Hausdorff space associated with a countable proper sets of seminorms $\mathfrak{N} = \{\|.\|_k\}_{k\geq 0}$ such that:*

(i) For $\varphi \in \mathcal{F}$, and $k < l$, we have $\|\varphi\|_k \leq \|\varphi\|_l$.

(ii) A subset $W \subset \mathcal{F}$ is open if for all $\varphi \in W$, there exists $\epsilon > 0$, $L \geq 0$ such that $\{\psi \in \mathcal{F} : \|\varphi - \psi\|_k) \leq \epsilon\} \subset W$, for all $k \leq L$.

(iii) $\mathcal{F}$ is complete regarding the topology determined by the family $\mathfrak{N}$.

Definition 4. *Let $\{u_n\}$ be sequence in an F-module $\mathcal{F}$. Then, the sequence $\{u_n\}$ converges to φ in $\mathcal{F}$ iff $\lim_{n\to\infty} \|u_n - \varphi\|_k = 0$ for all $\|.\|_k \in \mathfrak{N}$.*

Let $\varphi(x)$ where $x \in \mathbb{R}^{m+1}$ be a SMF. Each notation in Table 1 below expresses a class of SMFs in the indicated regions including: hyper open ball $B(R)$, hyper closed ball $\overline{B}(R)$, any hyper open ball enclosing hyper closed ball $B_+(R)$, for all entire special monogenic functions and at the origin. Note that all these spaces are actually F-modules associated with the corresponding semi-norms system.

Table 1: Types of F-Modules

F-Module	Notation	Associated Semi-Norms
Space containing SMFs in $B(R)$:	$\mathcal{W}^{B(R)}$	$\|\varphi\|_r = \sup_{\bar{B}(r)} \|\varphi(x)\|, \forall r < R$
Space containing SMFs $\overline{B}(R)$:	$\mathcal{W}^{\bar{B}(R)}$	$\|\varphi\|_R = \sup_{\bar{B}(R)} \|\varphi(x)\|$
Space containing SMFs in $B_+(R)$	$\mathcal{W}^{B_+(R)}$	$\|\varphi\|_r = \sup_{\bar{B}(r)} \|\varphi(x)\|, \forall R < r\}$
Space of entire SMFs	$\mathcal{W}^{\infty}$	$\|\varphi\|_n = \sup_{\overline{B(n)}} \|\varphi(x)\|, n < \infty$
Space of SMFs at 0	$\mathcal{W}^{0^+}$	$\|\varphi\|_\epsilon = \sup_{\bar{B}(\epsilon)} \|\varphi(x)\|,\ \epsilon > 0$

Definition 5. *Let $\{\mathcal{Q}_n(x)\}$ be a sequence of an F-module $\mathcal{F}$. Then $\{\mathcal{Q}_n(x)\}$ forms a base if $q_n(x)$ is given in the form:*

$$q_n(x) = \sum_{k=0}^{\infty} \mathcal{Q}_k(x)\ \tilde{\mathcal{Q}}_{n,k}, \quad \tilde{\mathcal{Q}}_{n,k} \in \mathcal{A}_m, \tag{2}$$

where $\tilde{\mathcal{Q}} = (\tilde{\mathcal{Q}}_{n,k})$ is the Clifford matrix of operators of the base $\{\mathcal{Q}_n(x)\}$ which has the form:

$$\mathcal{Q}_n(x) = \sum_{k=0}^{\infty} q_k(x)\ \mathcal{Q}_{n,k}, \quad \mathcal{Q}_{n,k} \in \mathcal{A}_m. \tag{3}$$

The matrix $\mathcal{Q} = (\mathcal{Q}_{n,k})$ is the Clifford matrix of coefficient of $\{\mathcal{Q}_n(x)\}$. As deduced in [7], the set $\{\mathcal{Q}_n(x)\}$ form a base iff

$$\mathcal{Q}\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}\mathcal{Q} = I, \tag{4}$$

where I is the identity matrix.

Suppose that $\varphi(x) = \sum_{n=0}^{\infty} q_n(x)\ u_n(g)$ is any SMF of $\mathcal{F}$. By using the formula of $q_n(x)$ as in (2), the basic series

$$g(x) \sim \sum_{n=0}^{\infty} \mathcal{Q}_n(x)\ \Pi_n(g), \tag{5}$$

follows, where

$$\Pi_n(g) = \sum_{k=0}^{\infty} \tilde{\mathcal{Q}}_{k,n}\ a_k(g)\ . \tag{6}$$

Definition 6. *If the associated basic series (5) is normally convergent to every $\varphi(x)$ in an F-module $\mathcal{F}$, then the base $\{\mathcal{Q}_n(x)\}$ is called effective for $\mathcal{F}$.*

More details on the convergence properties of bases $\{\mathcal{Q}_n(x)\}$ in the sense of F-modules can be found in [16, 17].

$$\|\mathcal{Q}_n\|_R = \sup_{\overline{B}(R)} |\mathcal{Q}_n(x)|, \tag{7}$$

where

$$\|\mathcal{Q}_k\ \tilde{\mathcal{Q}}_{n,k}\|_R = \sup_{\overline{B}(R)} |\mathcal{Q}_k(x)\ \tilde{\mathcal{Q}}_{n,k}|.$$

Examining the effectiveness of bases throughout this study is mainly conducted using the value of the Cannon function

$$\Omega(\mathcal{Q}, R) = \limsup_{n\to\infty} \{\Omega(\mathcal{Q}_n, R)\}^{\frac{1}{n}}, \tag{8}$$

where

$$\Omega(\mathcal{Q}_n, R) = \sum_k \|\mathcal{Q}_k\ \tilde{\mathcal{Q}}_{n,k}\|_R, \tag{9}$$

is called the Cannon sum as we shall see in the following theorem which determines the effectiveness criteria (see [16, 17]).

Theorem 1. *(i) A base $\{\mathcal{Q}_n(x)\}$ is effective for $\mathcal{W}^{\bar{B}(R)}$, iff $\Omega(\mathcal{Q}, R) = R$;*

(ii) A base $\{\mathcal{Q}_n(x)\}$ is effective for $\mathcal{W}^{B_+(R)}$, iff $\Omega(\mathcal{Q}, R^+) = R$;

(iii) A base $\{\mathcal{Q}_n(x)\}$ *is effective for* $\mathcal{W}^{B(R)}$ *iff* $\Omega(\mathcal{Q}, r) < R \ \ \forall\ r < R$;

(iv) A base $\{\mathcal{Q}_n(x)\}$ *is effective for* $\mathcal{W}^{\infty}$ *iff* $\Omega(\mathcal{Q}, R) < \infty\ \forall\ R < \infty$;

(v) A base $\{\mathcal{Q}_n(x)\}$ *is effective for* $\mathcal{W}^{0^+}$ *iff* $\Omega(\mathcal{Q}, 0^+) = 0$.

For the base defined in (3), we have the Clifford version Cauchy inequality [16]:

$$|\mathcal{Q}_{n,k}| \leq \frac{\|\mathcal{Q}_n\|_R}{R^k}. \tag{10}$$

Definition 7. *If* $\{\mathcal{Q}_n(x)\}$ *be a base of SMPs, then the series in (2) is finite. Let* $N(n)$ *denote the number of terms in (2) for which* $\mathcal{Q}_{n,k} \neq 0$ *and* $\limsup_{n\to\infty}\{N(n)\}^{\frac{1}{n}} = 1$. *Then, the base* $\{\mathcal{Q}_n(x)\}$ *is said to be a Cannon base of SMPs (CBSMPs).*

Definition 8. *Any base* $\mathcal{Q}_n(x)$ *is simple it has degree* n. *In a particular case when* $\mathcal{Q}_{n,n} = 1$ *for all* $n \in \mathbb{N}$, *then* $\mathcal{Q}_n(x)$ *i a simple monic base*

Now, we characterize the growth of a base as introduced in [7, 8] where the order in Clifford setting was given by:

$$\rho = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log\ \Omega(\mathcal{Q}_n, R)}{n\ \log\ n}. \tag{11}$$

Remark 2. *A significant relation between the order of base and the entire functions be represented in the sense that if the base* $\{\mathcal{Q}_n(x)\}$ *is of order* ρ, *then it will represent every entire SMF which has order* $\rho' < \frac{1}{\rho}$ *in any finite ball.*

3. Effectiveness and order of the exponential simple base of special monogenic polynomials

In this section, we investigate the convergence properties of the ESBSMPs for the F-module $\mathcal{W}^{\bar{B}(R)}$ as follows.

3.1. Effectiveness of the exponential simple base of special monogenic polynomials for the F-module $\mathcal{W}^{\bar{B}(R)}$

Let $\{\mathcal{Q}_n(x)\}$ be a simple base of SMPs (SBSMPs) of a Clifford variable $x \in \mathbb{R}^{m+1}$, whose Clifford matrix of coefficients is $\mathcal{Q}$. Then the set $\{{}_{\mathcal{Q}}E_n(x)\}$, whose Clifford matrices of coefficients and operators are $E = e^{\mathcal{Q}}$ and $\tilde{E} = e^{-\mathcal{Q}}$ is a base. This base was defined in [21] as the ESBSMP. Suppose that the elements $\{\mathcal{Q}_{n,k}\}$ of the Clifford matrix of coefficients of the SBSMPs $\{\mathcal{Q}_n(x)\}$ satisfy

$$\begin{cases} |\mathcal{Q}_{n,k}| & \leq M\frac{b^{n-k}}{n+1}\alpha_k,\ 0 \leq k < n,\ M \geq 1 \\ \mathcal{Q}_{n,n} & = \alpha_n, \end{cases} \tag{12}$$

where $\{\alpha_n\}$ is a bounded sequence of positive numbers, k and b are finite positive numbers.

Before we proceed with our results, we state and prove the following important lemma.

Lemma 1. *If* $\mathcal{Q}_{n,k}^{(j)} \in \mathcal{A}_m$ *are the elements of the power Clifford matrix* $\mathcal{Q}^{(j)}, j \geq 1$ *and satisfying (12), then*

$$\left|\mathcal{Q}_{n,x}^{(j)}\right| \leq 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} MC\right)^{j} b^{n-k}, 0 \leq k < n, \tag{13}$$

where C *is a real number such that* $\alpha_k \leq C$, $0 \leq k < n$.

Proof. The elements $\mathcal{Q}_{n,k}^{(j)}$ of the power Clifford matrix $Q^j, j \geq 1$ are given as

$$\mathcal{Q}_{n,k}^{(j)} = \sum_{t=k}^{n} \mathcal{Q}_{t,k}^{(j-1)} \mathcal{Q}_{n,t} \tag{14}$$

Assume that $j = 2$ in (14) and by using (12), we obtain that

$$\begin{aligned}
\left|\mathcal{Q}_{n,k}^{(2)}\right| &= \left|\sum_{t=k}^{n} \mathcal{Q}_{t,k} \mathcal{Q}_{n,t}\right| \\
&\leq 2^{\frac{m}{2}} \sum_{t=k}^{n} |\mathcal{Q}_{t,k}| \, |\mathcal{Q}_{n,t}| \\
&\leq 2^{\frac{m}{2}} \sum_{t=k}^{n} M \frac{b^{t-k}}{t+1} \alpha_k \cdot M \frac{b^{n-t}}{n+1} \alpha_t \\
&= \frac{2^{\frac{m}{2}} (MC)^2 \cdot b^{n-k}}{n+1} \sum_{t=k}^{n} \frac{1}{t+1} \\
&\leq 2^{\frac{m}{2}} (MC)^2 b^{n-k} \\
&= 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} MC\right)^2 b^{n-k},
\end{aligned}$$

where $0 \leq k < n$. We assume that

$$\left|\mathcal{Q}_{n,k}^{(j)}\right| \leqslant 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} M\alpha_k\right)^{j} b^{n-k} \tag{15}$$

Applying (12), (14) and (15) implies that

$$\begin{aligned}
\left|\mathcal{Q}_{n,k}^{(j+1)}\right| &\leq 2^{\frac{m}{2}} \sum_{t=k}^{n} \left|Q_{t,k}^{(j)}\right| |\mathcal{Q}_{n,t}| \\
&\leqslant 2^{\frac{m}{2}} \sum_{t=k}^{n} 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} MC\right)^{j} b^{t-k} \cdot \frac{Mb^{n-t}}{n+1} \alpha_t \\
&\leqslant 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} MC\right)^{j+1} b^{n-k}.
\end{aligned}$$

which gives the desire result by mathematical induction.

Theorem 2. *Suppose that $\{\mathcal{Q}_n(x)\}$ is a SBSMPs satisfying* (12). *Then the ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is effective for $\mathcal{W}^{\bar{B}(R)}$, where $R \geq b$, however, it may not be effective for $\mathcal{W}^{\bar{B}(R)}$ when $R < b$.*

Proof. Since ${}_{\mathcal{Q}}E = e^{\mathcal{Q}}$ is the Clifford matrix of coefficients of the ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$, then it follows that

$$\begin{aligned} {}_{\mathcal{Q}}E_n(x) &= \sum_{k=0}^{n} q_k(x)\ {}_{\mathcal{Q}}E_{n,k} \\ &= \sum_{k=0}^{n} q_k(x) \left(\sum_{j=0}^{\infty} \frac{\mathcal{Q}_{n,k}^{(j)}}{j!} \right) \end{aligned} \tag{16}$$

Applying (13) in (16), it follows that

$$\begin{aligned} \|{}_{\mathcal{Q}}E_n\|_R &= \sup_{\bar{B}(R)} |{}_{\mathcal{Q}}E_n(x)| \\ &\leq 2^{\frac{m}{2}} \sum_{k=0}^{n} R^k \left(\sum_{j=0}^{\infty} \frac{\left|\mathcal{Q}_{n,k}^{(j)}\right|}{j!} \right) \\ &\leqslant \sum_{k=0}^{n} R^k b^{n-k} \sum_{j=0}^{\infty} \frac{(2^{\frac{m}{2}} MC)^j}{j!} \\ &\leqslant R^n e^{2^{\frac{m}{2}} MC} \sum_{k=0}^{n} \left(\frac{b}{R}\right)^{n-k} \\ &\leq e^{2^{\frac{m}{2}} MC} (n+1) R^n, \qquad \text{for all } R \geq b. \end{aligned} \tag{17}$$

Moreover, since the Clifford matrix of operators of the ESBSMPs is $\{{}_{\mathcal{Q}}E_n(x)\}$ is ${}_{\mathcal{Q}}\bar{E} = e^{-\mathcal{Q}}$, hence by using (13) and (17) in the Cannon sum of $\{{}_{\mathcal{Q}}E_n(x)\}$ yields:

$$\begin{aligned} \Omega\left({}_{\mathcal{Q}}E_n, \bar{B}(R)\right) &= \sum_{k=0}^{n} \left\| {}_{\mathcal{Q}}E_k\, {}_{\mathcal{Q}}\bar{E}_{n,k} \right\|_R \\ &= \sum_{k=0}^{n} \sup_{\bar{B}(R)} \left| {}_{\mathcal{Q}}E_k(x)\, {}_{\mathcal{Q}}\bar{E}_{n,k} \right| \\ &\leq 2^{\frac{m}{2}} \sum_{k=0}^{n} \|{}_{\mathcal{Q}}E_k\|_R \left| {}_{\mathcal{Q}}\bar{E}_{n,k} \right| \\ &= 2^{\frac{m}{2}} \sum_{k=0}^{n} \|{}_{\mathcal{Q}}E_k\|_R \left| \sum_{j=0}^{\infty} (-1)^j \frac{\mathcal{Q}_{n,k}^{(j)}}{j!} \right| \\ &\leq 2^{\frac{m}{2}} \sum_{k=0}^{n} \|{}_{\mathcal{Q}}E_k\|_R \sum_{j=0}^{\infty} \frac{\left|\mathcal{Q}_{n,k}^{(j)}\right|}{j!} \end{aligned}$$

$$\begin{aligned}
&\leq e^{2^{\frac{m}{2}}MC} \quad \sum_{k=0}^{n}(k+1)R^k\sum_{j=0}^{\infty} b^{n-k}\frac{(2^{\frac{m}{2}}MC)^j}{j!}\\
&\leq e^{2^{(\frac{m}{2}+1)}MC}\sum_{k=0}^{n}(k+1)R^k b^{n-k}\\
&= e^{2^{(\frac{m}{2}+1)}MC}R^n\sum_{k=0}^{n}(k+1)\left(\frac{b}{R}\right)^{n-k}\\
&\leq e^{2^{(\frac{m}{2}+1)}MC}(n+1)^2R^n \quad \text{for all } R \geq b.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) &= \limsup_{n\to\infty}\left\{\Omega\left({}_{\mathcal{Q}}E_n, \bar{B}(R)\right)\right\}^{\frac{1}{n}}\\
&\leq R \text{ for all } R \geq b.
\end{aligned}$$

As $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) \geq R$, then $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) = R$. and therefore the ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ will be effective for $\mathcal{W}^{\bar{B}(R)}$ for all $R \geq b$.

When $R < b$ the ESBSHPs $\{{}_{\mathcal{Q}}E_n(x)\}$ may not be effective for $\mathcal{W}^{\bar{B}(R)}$ where $R < b$. To show this fact, we illustrate the following example

Example 1. *Consider the SBSMPs $\{\mathcal{Q}_n(x)\}$ for which*

$$Q_{n,k} = \begin{cases}\alpha_n, & k = n,\\ \frac{b^{n-k}}{n+1}\alpha_k, & 0 \leq k < n,\end{cases} \tag{18}$$

where $\alpha_i = \frac{1}{i+1}$ for $i = k, n$.

As in (13) we can apply (18) to prove by mathematical induction that

$$\mathcal{Q}_{n,0}^{(j)} \geq b^n\alpha_n^j. \tag{19}$$

Now, the Cannon sum for ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is given as

$$\begin{aligned}
\Omega_n\left({}_{\mathcal{Q}}E_n, \bar{B}(R)\right) &= \sum_{k=0}^{n}\left\|{}_{\mathcal{Q}}E_k \ {}_{\mathcal{Q}}\tilde{E}_{n,k}\right\|_R\\
&\geq \left\|{}_{\mathcal{Q}}E_n {}_{\mathcal{Q}}\tilde{E}_{n,n}\right\|_R\\
&= e^{\alpha_n}\left\|{}_{\mathcal{Q}}E_n\right\|_R\\
&\geq e^{\alpha_n}|{}_{\mathcal{Q}}E_{n,0}|\\
&= e^{\alpha_n}\sum_{j=0}^{\infty}\frac{\mathcal{Q}_{n,0}^{(j)}}{j!}\\
&\geq e^{\alpha_n}b^n\sum_{j=0}^{\infty}\frac{\alpha_n^j}{j!} = e^{2\alpha_n}b^n \geq b^n.
\end{aligned} \tag{20}$$

Thus $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) \geq b > R$ for $R < b$, and the ESBSMPs is not effective for $\mathcal{W}^{\bar{B}(R)}$ for $R < b$ as required.

Remark 3. *In Theorem 2, if $\{\mathcal{Q}_n(x)\}$ is taken to be a simple monic base ($\mathcal{Q}_{n,n} = 1$ for $n \in \mathbb{N}$), then the results in [21] becomes a particular case of our Theorem 2.*

3.2. The Order of the exponential simple base of special monogenic polynomials

Let $\{\mathcal{Q}_n(x)\}$ be a SBSMP for which

$$\begin{cases} |\mathcal{Q}_{n,k}| & \leq M\frac{n^{\lambda(n-k)}}{n+1}\alpha_k \quad 0 \leqslant k < n \\ \mathcal{Q}_{n,n} & = \alpha_n, n \in \mathbb{N} \end{cases} \tag{21}$$

where λ is positive constant and $1 \leq M < \infty$. To justify the main result of this section, we first introduce the following lemma.

Lemma 2. *If $\mathcal{Q}_{n,k}^{(j)} \in \mathcal{A}_m$ are the elements of the power Clifford matrix $\mathcal{Q}^{(j)}, j \geq 1$ satisfying (21), then*

$$\left|\mathcal{Q}_{n,k}^{(j)}\right| \leqslant 2^{-\frac{m}{2}} \left(2^{\frac{m}{2}} MC\right)^j n^{\lambda(n-k)}, 0 \leqslant k < n. \tag{22}$$

Proof. As proceed in the proof of Lemma (1), the inequality (22). can be verified.

Theorem 3. *Let $\{\mathcal{Q}_n(x)\}$ be a SBSMP satisfies* (21). *Then the ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is of order $\Gamma \leq \lambda$. Moreover, the value λ is attainable.*

Proof. Applying (22) in (16), we get

$$\begin{aligned} \|{}_{\mathcal{Q}}E_n\|_R &\leq 2^{\frac{m}{2}} \sum_{k=0}^{n} R^k \left(\sum_{j=0}^{\infty} \frac{\left|\mathcal{Q}_{n,k}^{(j)}\right|}{j!} \right) \\ &\leq \sum_{k=0}^{n} R^k \left(\sum_{j=0}^{\infty} \frac{\left(2^{\frac{m}{2}} MC\right)^j}{j!} n^{\lambda(n-k)} \right) \\ &\leq e^{2^{\frac{m}{2}} MC} n^{\lambda n} \sum_{k=0}^{n} \left(\frac{R}{n^{\lambda}}\right)^k \\ &\leq e^{2^{\frac{m}{2}} MC} n^{\lambda_n} (n+1) = K_1(m)\ n^{\lambda_n}(n+1), \quad \text{for } n^{\lambda} > R. \end{aligned} \tag{23}$$

Where $K_1(m) = e^{2^{\frac{m}{2}} MC}$.
Introducing (22) and (23) in the Cannon sum of the ESBSMP $\{{}_{\mathcal{Q}}E_n(x)\}$ we have

$$\begin{aligned}\Omega\left({}_{\mathcal{Q}}E_n,\bar{B}(R)\right) &\leq 2^{\frac{m}{2}}\sum_{k=0}^{n}\|{}_{\mathcal{Q}}E_k\|_R\sum_{j=0}^{\infty}\frac{\left|\mathcal{Q}_{n,k}^{(j)}\right|}{j!}\\ &\leq 2^{\frac{m}{2}}\sum_{k=0}^{n}k_1(m)k^{\lambda k}(k+1)\sum_{j=0}^{\infty}2^{-m/2}\left(\frac{\left(2^{\frac{m}{2}}MC\right)^j}{j!}n^{\lambda(n-k)}\right)\\ &\leq K_1^2(m)(n+1)^2n^{\lambda n} \qquad (24)\end{aligned}$$

When n approaches infinity, and using the definition of the order, it follows that the order Γ of ESBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is at most λ.

The following example demonstrates the fact that the bound λ is attainable.

Example 2. *Consider the SBSMPS $\{\mathcal{Q}_n(x)\}$ given by*

$$\mathcal{Q}_{n,k}=\begin{cases}M\frac{n^{\lambda(n-k)}}{n+1}\alpha_k, & 0\leqslant k<n\\ \alpha_n, & k=n\end{cases} \qquad (25)$$

where $\alpha_k=\frac{1}{k+1}$ and $\alpha_n=\frac{1}{n+1}$.

As in (19) we can use (25) to prove by mathematical induction that

$$\mathcal{Q}_{n,0}^{(j)}\geqslant n^{\lambda n}\left(M\alpha_n\right)^j.$$

Now, it can be verified, as in (20) that

$$\begin{aligned}\Omega\left({}_{\mathcal{Q}}E_n,\bar{B}(R)\right) &\geqslant e^{\alpha_n}\sum_{j=0}\frac{\mathcal{Q}_{n,0}^{(j)}}{j!}\\ &\geq e^{\alpha_n}n^{\lambda n}\sum_{j=0}^{\infty}\frac{\left(M\alpha_n\right)^j}{j!}\\ &\geq e^{\frac{1+M}{1+n}}n^{\lambda n}.\end{aligned}$$

Thus, $\Gamma\geq\lambda$, but $\Gamma\leq\lambda$ which means that $\Gamma=\lambda$ and the bound is λ is attainable. This completes the proof.

Remark 4. *In Theorem 3, if $\{\mathcal{Q}_n(x)\}$ is a simple monic base ($\mathcal{Q}_{n,n}=1$ for $n\in\mathbb{N}$), then the result in [21] becomes a particular case of our Theorem 3.*

4. Effectiveness of the the exponential Cannon base of special monogenic polynomials

In the current section, we study the convergence properties of the ECBSMPs for the F-modules $\mathcal{W}^{\bar{B}(R)}$, $\mathcal{W}^{B(R)}$, $\mathcal{W}^{B_+(R)}$, $\mathcal{W}^{0^+}$ and $\mathcal{W}^{\infty}$.

Theorem 4. *The ECBSMPs $\{_{\mathcal{Q}}E_n(x)\}$ is effective for $\mathcal{W}^{\bar{B}(R)}$ for all $b \leqslant R < \frac{b}{a}$ if the Cannon base $\{Q_n(x)\}$ satisfy*

$$|\mathcal{Q}_{n,k}| \leq M \quad b^{n-k} a^k, \quad 0 < a < 1 \tag{26}$$

and may not be effective for $\mathcal{W}^{\bar{B}(R)}$ for all $R < b$ or $R \geqslant \frac{b}{a}$.

Proof. Suppose that $\{\mathcal{Q}_n(x)\}$ is a Cannon base, the relation (14) has the form

$$\mathcal{Q}_{n,k}^{(j)} = \sum_{t=0}^{\infty} \mathcal{Q}_{t,k}^{(j-1)} \mathcal{Q}_{n,t} \tag{27}$$

Thus, as in (13), applying (26), we obtain the following inequality

$$\left|\mathcal{Q}_{n,k}^{(j)}\right| \leqslant M^j (1-a)^{-(j-1)} b^{n-k} a^k. \tag{28}$$

Using (28), we get

$$\begin{aligned}
\|_{\mathcal{Q}}E_n\|_R &\leq 2^{\frac{m}{2}} \sum_k R^k \left(\sum_{j=0}^{\infty} \frac{\left|\mathcal{Q}_{n,k}^{(j)}\right|}{j!} \right) \\
&\leq 2^{\frac{m}{2}} e^{M(1-a)^{-1}} (1-a) \sum_k R^k b^{n-k} a^k \\
&= 2^{\frac{m}{2}} e^{M(1-a)^{-1}} (1-a) \left[\sum_{k \leq n} \left(\frac{b}{R}\right)^{n-k} a^k + \sum_{k>n} \left(\frac{b}{R}\right)^n \left(\frac{Ra}{b}\right)^k \right] R^n \\
&= K_2 R^n, \text{for all} \quad b \leq R < \frac{b}{a}.
\end{aligned} \tag{29}$$

where

$$K_2 = 2^{\frac{m}{2}} e^{M(1-a)^{-1}} (1-a) \left[\sum_{k \leq n} \left(\frac{b}{R}\right)^{n-k} a^k + \sum_{k>n} \left(\frac{b}{R}\right)^n \left(\frac{Ra}{b}\right)^k \right].$$

Hence, making use of of (28) and (29) in $\Omega_n\left(_{\mathcal{Q}}E_n, \bar{B}(R)\right)$ of the exponential base $\{_{\mathcal{Q}}E_n(x)\}$ associated with the Cannon base $\{\mathcal{Q}_n(x)\}$ yields.

$$\Omega_n\left(_{\mathcal{Q}}E_n, \bar{B}(R)\right) \leq 2^{\frac{m}{2}} \sum_k \|_{\mathcal{Q}}E_k\|_R \left|_{\mathcal{Q}}\bar{E}_{n,k}\right|$$

$$\leq 2^{\frac{m}{2}} \sum_k \|{}_{\mathcal{Q}}E_k\|_R \sum_{j=0}^{\infty} \frac{\left|Q_{n,k}^{(j)}\right|}{j!}$$
$$\leq K_2 \, e^{M(1-a)^{-1}} (1-a) \sum_k R^k b^{n-k} a^k$$
$$\leq K_2^2 \, R^n$$

for all $b \leq R < \frac{b}{a}$. Thus, the Cannon function for $\{{}_{\mathcal{Q}}E_n(x)\}$ satisfies $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) \leq R$, but $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) \geq R$. Therefore, $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) = R$, for $b \leq R < \frac{b}{a}$, and the ECBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is effective for $\mathcal{W}^{\bar{B}(R)}$ for $b \leq R < \frac{b}{a}$.

When $R < b$ or $R \geqslant \frac{b}{a}$, the ECBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ may not be effective in for $\mathcal{W}^{\bar{B}(R)}$. To show this fact consider the CBSMPs $\{Q_n(x)\}$ for which

$$\mathcal{Q}_{n,k} = \begin{cases} 1, & k = 0 \\ M b^{n-k} a^k, & k \neq 0. \end{cases}$$

Similar steps as in (20) can be obtained $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) > R$ for $R < b$, and the ECBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ is not effective for $\mathcal{W}^{\bar{B}(R)}$. Also, $\Omega\left({}_{\mathcal{Q}}E, \bar{B}(R)\right) = \infty$ for $R \geq \frac{b}{a}$ and the ECBSMPs $\{\mathcal{Q}_n(x)\}$ is not effective for $\mathcal{W}^{\bar{B}(R)}$ for $R \geq \frac{b}{a}$.

In examining the effectiveness of the ECBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ for the F-module $\mathcal{W}^{B_+(R)}$, we assume that the base $\mathcal{Q}_n(x)$ satisfies the prescribed condition.

$$\mu(\mathcal{Q}, R^+) \leq R, \tag{30}$$

where

$$\mu(\mathcal{Q}, R^+) = \limsup_{n \to \infty} \{\|\mathcal{Q}_n\|_{R^+}\}^{\frac{1}{n}}.$$

and

$$\|\mathcal{Q}_n\|_{R^+} = \sup_{B_+(R)} |\mathcal{Q}_n(x)|\,.$$

In the sequel, we present a proof for the following theorem

Theorem 5. *If the CBSMPs $\{\mathcal{Q}_n(x)\}$ satisfying (30), then the ECBSMPs $\{{}_{\mathcal{Q}}E_n(x)\}$ associated with $\{\mathcal{Q}_n(x)\}$ is effectiveness for $\mathcal{W}^{B_+(R)}$.*

Proof. Suppose that ρ_1, ρ_2 and r are chosen such that $R < \rho_1 < \rho_2 < r$. Then one can construct a sequence (R_j) of positive numbers satisfying $R < \rho_1 < \rho_2 < R_1 < R_2 < \cdots < R_j < \cdots < r$ and

$$\|\mathcal{Q}_n\|_{R_i} < K R_{i+1}^n ; n \geq 0 \tag{31}$$

where K is a positive finite constant. For the power base $\left\{\mathcal{Q}_n^{(j)}(x)\right\}$ corresponding to the original base $\{\mathcal{Q}_n(x)\}$, it follows that

$$\mathcal{Q}_n^{(j)}(x) = \sum_k \mathcal{Q}_k^{(j-1)}(x)\mathcal{Q}_{n,k}, \tag{32}$$

where $\{\mathcal{Q}_{n,k}\}$ represent the elements of the matrix $\mathcal{Q}$, and by substituting $s = 2$ into (32), then applying Cauchy's inequality together with relation (31), one obtains

$$\begin{aligned}
\left\|\mathcal{Q}_n^{(2)}\right\|_{R_1} &= \sup_{\bar{B}(R_1)} \left|\mathcal{Q}_n^{(2)}(x)\right| \\
&\leq 2^{\frac{m}{2}} \|\mathcal{Q}_n\|_{R_3} \sum_k \frac{\|\mathcal{Q}_k\|_{R_1}}{R_3^k} \\
&< K 2^{\frac{m}{2}} \sum_k \left(\frac{R_2}{R_3}\right)^k \|\mathcal{Q}_n\|_{R_3} \\
&= K 2^{\frac{m}{2}} S\left(R_2, R_3\right) \|\mathcal{Q}_n\|_{R_3}.
\end{aligned}$$

Throughout the following discussion, we assume that

$$\left\|\mathcal{Q}_n^{(j)}\right\|_{R_1} < \left(K\ 2^{\frac{m}{2}}\right)^{j-1} \prod_{N=1}^{j-1} S\left(R_{2N}, R_{2N+1}\right) \|\mathcal{Q}_n\|_{R_{2j-1}} \tag{33}$$

From (31), (32), and (33), together with an application of Cauchy's inequality, it follows that

$$\begin{aligned}
\left\|\mathcal{Q}_n^{(j+1)}\right\|_{R_1} &\leq 2^{\frac{m}{2}} \|\mathcal{Q}_n\|_{R_{2j+1}} \| \sum_k \frac{\left\|\mathcal{Q}_k^{(j)}\right\|_{R_1}}{R_{2j+1}^k} \\
&< 2^{\frac{m}{2}} \left(K\ 2^{\frac{m}{2}}\right)^{j-1} \prod_{N=1}^{j-1} S\left(R_{2N}, R_{2N+1}\right) \|\mathcal{Q}_n\|_{R_{2j+1}} \sum_k \frac{\|\mathcal{Q}_k\|_{R_{2j-1}}}{R_{2j-1}^k} \\
&< \left(K\ 2^{\frac{m}{2}}\right)^{j} \prod_{N=1}^{j} S\left(R_{2N}, R_{2N+1}\right) \|\mathcal{Q}_n\|_{R_{2j+1}}
\end{aligned}$$

Hence, the validity of assumption (33) is established by induction. Defining $k_1 = \max_{1\leq N\leq j-1} K\ 2^{\frac{m}{2}} S\left(R_{2N}, R_{2N+1}\right)$, relation (33) takes the form

$$\left\|\mathcal{Q}_n^{(j)}\right\|_{R_1} < K_1^{j-1} \|\mathcal{Q}_n\|_{R_{2j-1}} \tag{34}$$

Observing that ${}_{\mathcal{Q}}E = e^{\mathcal{Q}}$ serves as the coefficient matrix associated with the exponential basis $\{{}_{\mathcal{Q}}E_n(x)\}$, we deduce that

$$
\begin{aligned}
{}_{\mathcal{Q}}E_n(x) &= \sum_k q_k(x)_{\mathcal{Q}}E_{n,k} \\
&= \sum_k q_k(x)\left(\sum_{j=0}^{\infty}\frac{\mathcal{Q}_{n,k}^{(j)}}{u!}\right)
\end{aligned} \tag{35}
$$

From (31), (34), together with Cauchy's inequality applied in (35), we obtain

$$
\begin{aligned}
\|{}_{\mathcal{Q}}E_n\|_{\rho_1} &= \sup_{\bar{B}(\rho_1)} |{}_{Q}E_n(x)| \\
&\le 2^{\frac{m}{2}} \sum_k \rho_1^k \left|\sum_{j=0}^{\infty}\frac{\mathcal{Q}_{n,k}^{(j)}}{j!}\right| \\
&< 2^{\frac{m}{2}} \sum_k \sum_{j=0}^{\infty} \frac{\|\mathcal{Q}_n\| R_1}{j!}\left(\frac{\rho_1}{R_1}\right)^k \\
&< 2^{\frac{m}{2}} \sum_k \sum_{j=0}^{\infty} \frac{K_1^{j-1}\|\mathcal{Q}_n\|_{R_{2j-1}}}{j!}\left(\frac{\rho_1}{R_1}\right)^k \\
&< 2^{\frac{m}{2}}\ r^n \sum_k \left\{\sum_{j=0}^{\infty}\frac{K_1^j}{j!}\right\}\left(\frac{\rho_1}{R}\right)^k \\
&= 2^{\frac{m}{2}}\ e^{K_1} S\left(\rho_1, R_1\right) r^n.
\end{aligned}
$$

It can be concluded that

$$
\|{}_{Q}E\|_{\rho_1} = \limsup_{n\to\infty}\left\{\|{}_{Q}E_n\|_{\rho_1}\right\}^{\frac{1}{n}} \le r
$$

Taking the limit as $r \to R^+$, we obtain

$$
\|{}_{Q}E\|_{R^+} \le R.
$$

Then, for any number $\rho_1 > R$, one can choose $\rho_2 > \rho_1 > R$ such that $\|{}_{Q}E\|_{\rho_1} < \rho_2$. Consequently, it follows that $\|{}_{Q}E_n\|_{\rho_1} < K_2\rho_2^n$, $\forall\ n \ge 0$, where $K_2 \ge 1$.

From the definition of the exponential base $\{{}_{Q}E_n(x)\}$, its Cannon sum is given by

$$
\begin{aligned}
\Omega\left({}_{Q}E_n, \bar{B}\left(\rho_1\right)\right) &= \sum_k \left\|{}_{Q}E_k\ {}_{Q}\bar{E}_{n,k}\right\|_{\rho_1} \\
&\le 2^{\frac{m}{2}} \sum_k \|{}_{Q}E_k\|_{\rho_1} \sum_{j=0}^{\infty}\frac{\left|Q_{n,k}^{(j)}\right|}{j!}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2^{\frac{m}{2}} \sum_{k} \|_{Q}E_k\|_{\rho_1} \sum_{j=0}^{\infty} \frac{\left\|\mathcal{Q}_n^{(j)}\right\|_{R_1}}{j! R_1^k} \\
&< K_2\ 2^{\frac{m}{2}} \sum_{k} \sum_{j=0}^{\infty} \|\mathcal{Q}_n\|_{R_{2j-1}} \frac{h_1^{j-1}}{j!} \left(\frac{\rho_2}{R_1}\right)^k \\
&< K_2\ 2^{\frac{m}{2}} \cdot e^{k_1} S\left(\rho_2, R_1\right) r^n.
\end{aligned}
$$

Taking the limit as n tend to infinity, we obtain

$$
\begin{aligned}
\Omega\left({}_{Q}E, \bar{B}\left(\rho_1\right)\right) &= \limsup_{n\to\infty} \left\{\Omega\left({}_{Q}E_n, \bar{B}\left(\rho_1\right)\right)\right\} \\
&\leq r
\end{aligned}
$$

which yields $\Omega\left({}_{Q}E, R^{+}\right) \leq R$ as $r \to R^{+}$. However, since $\Omega\left(\mathcal{Q}_E, R^{+}\right) \geq R$, it follows that $\Omega\left({}_{Q}E, R^{+}\right) = R$. R. Consequently, the exponential base $\{{}_{Q}E_n(x)\}$ is effective for $\mathcal{W}^{B_{+}(R)}$.

Remark 5. *When $R \to 0$ in condition (30) we obtain the effectiveness of the ECBSMPs, $\{{}_{Q}E_n(x)\}$ for the space $\mathcal{W}^{0^+}$.*

In order to examine the effectiveness of the ECBSMPs $\{{}_{Q}E_n(x)\}$ for the F-module $\mathcal{W}^{B(R)}$, we assume that the base $\{\mathcal{Q}_n(x)\}$ fulfills the condition

$$
\mu(\mathcal{Q}, r) < R, \quad \forall r < R, \tag{36}
$$

with

$$
\mu(\mathcal{Q}, r) = \limsup_{n\to\infty} \{\|\mathcal{Q}_n\|_r\}^{\frac{1}{n}}.
$$

Theorem 6. *If the CBSMPs $\{\mathcal{Q}_n(x)\}$ satisfying (36), then the ECBSMPs $\{{}_{Q}E_n(x)\}$ associated with $\{\mathcal{Q}_n(x)\}$ is effectiveness for $\mathcal{W}^{B(R)}$.*

Proof. The proof of the following theorem is omitted, as it can be established by arguments analogous to those employed in Theorem 4, together with the aid of Theorem 1 (see [16]).

Remark 6. *If $R \to \infty$ then condition (36) will be replaced by the condition*

$$
\mu(\mathcal{Q}, r) < \infty, \quad \forall r < \infty, \tag{37}
$$

and we have the effectiveness for the ECBSMPs $\{{}_{Q}E_n(x)\}$ for $\mathcal{W}^{\infty}$.

Now, let $\{\mathcal{Q}_{s,n}(x)\}$, $s = 1, 2, 3$ be three bases of SMPs where $x \in \mathbb{R}^{m+1}$. the equivalent base $\{T_n(x)\}$ defined by

$$\{T_n(x)\} = \{\tilde{\mathcal{Q}}_{3,n}(x)\}\{\mathcal{Q}_{2,n}(x)\}\{\mathcal{Q}_{1,n}(x)\} \tag{38}$$

where $\{\tilde{\mathcal{Q}}_{3,n}(x)\}$ is the inverse base of $\{\mathcal{Q}_{3,n}(x)\}$.

In a recent paper [17] the authors prove that the equivalent base $\{T_n(x)\}$ satisfy the conditions

$$\mu(T, r) < R, \quad \forall r < R, \tag{39}$$

and

$$\mu(T, R^+) \leq R, \tag{40}$$

when the bases $\{\mathcal{Q}_{s,n}(x)\}$, $s = 1, 2, 3$ is algebraic according to [10] and satisfying the following conditions, respectively

$$\mu(\mathcal{Q}_s, r) < R, \quad \forall r < R, \tag{41}$$

and

$$\mu(\mathcal{Q}_s, R^+) \leq R, \tag{42}$$

Applying Theorems 5 and 6, we conclude the following result.

Corollary 1. *Let $\{\mathcal{Q}_{s,n}(x)\}$, $s = 1, 2, 3$ be three algebraic bases of SMPs satisfying the conditions* (41) *and* (42)*, then the ECBSMPs $\{_TE_n(x)\}$ associated with the equivalent base $\{T_n(x)\}$ are effectiveness for the spaces $\mathcal{W}^{B(R)}$, $\mathcal{W}^{B_+(R)}$, $\mathcal{W}^{0^+}$ and $\mathcal{W}^{\infty}$.*

Looking back to the equivalent base $\{T_n(x)\}$ and by taking $\{\mathcal{Q}_{3,n}(x)\} = \{\mathcal{Q}_{1,n}(x)\}$, we get the similar base $\{S_n(x)\}$ as studied in [25]. Therefore all the results of Corollary 1 will be satisfied but for the similar base $\{S_n(x)\}$ instead of equivalent base $\{T_n(x)\}$.

5. Conclusions

The current study discusses the representation of the exponential base of special monogenic polynomials (EBSMPs) in Fréchet modules. The coefficients of the power associated infinite matrices of an original base have some restriction for which certain classes of SMFs can be represented by ESBSMPs in hyber closed ball. The upper bound of the order of the ESBSMPs is is determined. Moreover, the effectiveness of the exponential base associated with Cannon sets is studied in different hyper regions . In the previous studies [9–11, 20, 25–27] in complex and Clifford analysis the convergence properties (effectiveness, order, type, and the T_ρ-property) of some associated bases of polynomials have been examined, such as, Hadamard product base, product base, inverse base, square root base, similar base and similar transposed base. Theses constituents are simple monic bases. It is interesting to explore more extended associated bases from those obtained in the existing by considering the constituents to be general bases. As far as we know from the literature that one of the untouched problems in Clifford context, is to represent any entire monogenic functions by a set of polynomials, in non-spherical regions, such as Faber regions. Having such tools would pave the way to explore these problems and may be fruitful to enrich the approximation theory in Clifford setting.

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References

[1] Fuzhang Wang, Imtiaz Ahmad, Hijaz Ahmad, M.D. Alsulami, K.S. Alimgeer, Clemente Cesarano, and Taher A. Nofal. Meshless method based on rbfs for solving three-dimensional multi-term time fractional pdes arising in engineering phenomenons. *Journal of King Saud University - Science*, 33(8):101604, 2021.

[2] Sunil Kumar, Deepak Kumar, Janak Raj Sharma, Clemente Cesarano, Praveen Agarwal, and Yu-Ming Chu. An optimal fourth order derivative-free numerical algorithm for multiple roots. *Symmetry*, 12(6), 2020.

[3] J. M. Whittaker. On the uniqueness of expansion in polynomials. *Journal of the London Mathematical Society*, Volume S1-10:108–111, 1935.

[4] J. M. Whittaker and C. Gattegno. *Sur les s'eries de base de polynômes quelconques.* Gauthier-Villars, Paris, 1949.

[5] B. Cannon. On the convergence of series of polynomials. *Proceedings of the London Mathematical Society*, 43:348–364, 1937.

[6] B. Cannon. On the representation of integral functions by general basic series. *Mathematische Zeitschrift*, 45:185–208, 1939.

[7] M. Abul-Ez and D. Constales. Basic sets of polynomials in Clifford analysis. *Complex Variables*, 14:177–185, 1990.

[8] M. Abul-Ez and D. Constales. Linear substitution for basic sets of polynomials in Clifford analysis. *Portugaliae Mathematica*, 48:143–154, 1990.

[9] M. Abul-Ez. Inverse sets of polynomials in Clifford analysis. *Archiv der Mathematik*, 58:561–567, 1992.

[10] M. Abul-Ez. Hadamard product of bases of polynomials in Clifford analysis. *Complex Variables and Elliptic Equations*, 43(2):109–128, 2000.

[11] M. Abul-Ez and D. Constales. The square root base of polynomials in Clifford analysis. *Archiv der Mathematik*, 80(5):486–495, 2003.

[12] M. Abdalla, M. Abul-Ez, and J. Morais. On the construction of generalized monogenic Bessel polynomials. *Mathematical Methods in the Applied Sciences*, 41:9335–9348, 2018.

[13] M. Abul-Ez and M. Zayed. Criteria in nuclear Fréchet spaces and Silva spaces with refinement of the Cannon–Whittaker theory. *Journal of Function Spaces*, 2020:15, 2020.

[14] M. Abul-Ez. Bessel polynomial expansions in spaces of holomorphic functions. *Journal of Mathematical Analysis and Applications*, 221:177–190, 1998.

[15] G. F. Hassan and L. Aloui. Bernoulli and Euler polynomials in Clifford analysis. *Advances in Applied Clifford Algebras*, 25:351–376, 2015.

[16] G. F. Hassan, L. Aloui, and A. Bakali. Basic sets of special monogenic polynomials in Fréchet modules. *Journal of Complex Analysis*, pages Article ID 2075938, 11 pages, 2017.

[17] M. Zayed and G. Hassan. Equivalent base expansions in the space of Cliffordian functions. *Axioms*, 12(6):544, 2023.

[18] G. Hassan and M. Zayed. Approximation of monogenic functions by hypercomplex Ruscheweyh derivative bases. *Complex Variables and Elliptic Equations*, 68:2073–2092, 2023.

[19] M. Zayed and G. Hassan. Expansions of generalized bases constructed via Hasse derivative operator in Clifford analysis. *AIMS Mathematics*, 8(11):26115–26133, 2023.

[20] Z. G. Kishka, M. A. Saleem, and M. A. Abul-Dahab. On simple exponential sets of polynomials. *Mediterranean Journal of Mathematics*, 11:337–347, 2014.

[21] M. A. Abul-Ez. Exponential base of special monogenic polynomials. *Pure Mathematics and Applications*, 8(2-4):137–146, 1997.

[22] F. Brackx, R. Delanghe, and F. Sommen. *Clifford Analysis*, volume 76 of *Research Notes in Mathematics*. Pitman, London, 1982.

[23] K. Gürlebeck, K. Habetha, and W. Sprößig. *Holomorphic functions in the plane and n-dimensional space*. Birkh"auser, Basel, 2008.

[24] E. M. Stein and G. I. Weiss. Generalization of the Cauchy–Riemann equations and representation of the rotation group. *American Journal of Mathematics*, 90:163–196, 1968.

[25] M. Abul-Ez and D. Constales. Similar functions and similar bases of polynomials in Clifford setting. *Complex Variables, Theory and Application: An International Journal*, 48(12):1055–1070, 2003.

[26] M. Abul-Ez. Product simple sets of polynomials in Clifford analysis. *Rivista di Matematica della Università di Parma*, 3(5):283–293, 1994.

[27] M. Abul-Ez and M. Zayed. Similar transposed bases of polynomials in Clifford analysis. *Applied Mathematics and Information Sciences*, 4:63–78, 2010.