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An Inverse Problem for a Parabolic Equation with Nonlocal Boundary Conditions and Two-Point Overdetermination

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Abstract. This paper is devoted to the study of an inverse boundary value problem for a parabolic equation with nonlocal boundary conditions and two-point overdetermination. To analyze the solvability of the problem, we first consider an associated auxiliary inverse boundary value problem. By applying the Fourier method, the solution of the auxiliary problem is reduced to a system of integral equations. The existence and uniqueness of the solution to the auxiliary problem are then established using the contraction mapping principle in an appropriate functional space. Finally, by employing the equivalence between the original and auxiliary formulations, the existence and uniqueness of the classical solution to the initial nonlinear inverse boundary value problem are proved.

2020 Mathematics Subject Classifications: 35R30, 35K10, 35A09, 35A01, 35A02

Key Words and Phrases: Inverse problem, parabolic equation, nonlocal conditions, classical solution, existence, uniqueness

1. Introduction

It is known that mathematical modeling of various real-world processes in the natural sciences often leads to the study of inverse problems. An inverse problem generally refers to a situation where the goal is to deduce the causes or parameters of a system based on observed effects or data. One of the most widely studied types of inverse problems is the class of inverse boundary value problems. In the theory of mathematical physics, an inverse boundary value problem involves the simultaneous determination of unknown coefficients and/or the right-hand side of partial differential equations, using additional measurements. Inverse problems arise when the characteristics of an object of interest cannot be observed

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directly. For example, this includes restoring the characteristics of field sources based on their known values at specific points, as well as recovering or interpreting the original signal from a known output signal.

The practical importance of inverse problems is substantial, as they arise in diverse fields including seismology, biology, medicine, mineral exploration, seawater desalination, fluid movement in porous media, and etc. As a result, they present some of the most pressing challenges in modern mathematics. The foundations of the theory and practice of investigating inverse problems of mathematical physics were established and developed in the seminal works of distinguished mathematicians such as Tikhonov [1], Lavrent'ev [2], Ivanov [3], and their followers. The practical significance of inverse problems has attracted considerable attention from researchers, resulting in the publication of numerous articles and monographs on the subject in recent decades (see, for example, [4–15] and references therein).

It should be noted that one of the most notable problems among inverse boundary value problems are inverse problems in which the boundary conditions include nonlocal conditions. In the literature, the term "nonlocal boundary value problems" refers to problems that involve conditions relating the values of the solution and/or its derivatives either at different points on the boundary or at boundary points and certain interior points. The term "nonlocal conditions" and their classification were introduced by A.A. Dezin in [16]. Nonlocal conditions arise in situations where the boundary of the domain of a real process is not accessible for direct measurements, but additional information about the phenomenon under study can be obtained at interior points of the domain. Often, this information is provided in the form of average values of the desired solution. In mathematical modeling, it is convenient to express such information as an integral. Note that problems with nonlocal integral conditions occur in the study of processes such as those in turbulent plasma, heat propagation, diffusion theory, and in the modeling of certain technological processes.

Now, let us examine the content of some relevant works dedicated to inverse boundary value problems for parabolic equations. In the article by Durdiev and Rashidov [17], the inverse problem of determining the multidimensional kernel of the integral term in a second-order parabolic equation is considered, and a local existence and uniqueness theorem for the inverse problem is proven. The studies [18], [19], [20], and [21] focus on investigating the classical solvability of inverse parabolic problems with various nonlocal boundary conditions. The existence and uniqueness conditions for the solution of the inverse problem of a parabolic equation with nonlocal boundary conditions and integral overdetermination are established in the paper by Ivanchov and Pabyrivs'ka [22]. Kamynin [23] investigated an inverse problem concerned with the simultaneous determination of the right-hand side and the coefficient of the lowest-order derivative in a parabolic equation, subject to an integral observation condition. In the article by Kerimov and Ismailov [24], under certain regularity and compatibility conditions on the given data, the existence, uniqueness, and continuous dependence of the solution on the data were established for an inverse problem with nonlocal boundary conditions and an integral overdetermination condition. A nonlocal inverse boundary value problems for mixed-type partial differential equations are studied, and a criterion for the uniqueness of the solution to the considered problem is established in the work of Martemyanova [25]. In the article by Prilepko, Kamynin, and Kostin [26], the inverse problem of determining the source in a non-uniform parabolic equation under the condition of integral observation is considered, and sufficient conditions for the unique solvability of the inverse problem are obtained.

The numerical aspects of inverse problems for parabolic equations with various boundary conditions were studied in [27–31], and the references therein.

The distinctive contribution of the present study lies in the application of a novel methodological approach to the analysis of an inverse problem for a parabolic equation with nonlocal boundary conditions that depend on both spatial and temporal variables.

The paper is organized as follows. Section 1 establishes the relevance of the article's topic, formulates its purpose, and offers a comprehensive review of the relevant literature with rigorous comparisons to previous works. In Section 2, the mathematical formulation of the inverse problem is presented and a lemma on the reduction of the original problem to an auxiliary one are introduced. Section 3 presents some auxiliary results from spectral theory and introduces special functional spaces. Section 4 examines the existence and uniqueness of the classical solution to the considered inverse boundary value problem. Section 5 summarizes the key results of the investigation.

2. Mathematical formulation of the problem

Let $T > 0$ be a fixed time moment, and let D_T denote the rectangular domain in the xt-plane bounded by the inequalities $0 \le x \le 1$ and $0 \le t \le T$. Consider the problem of determining the unknown functions $u(x,t) \in C^{2,1}(D_T)$, $a(t) \in C[0,T]$, and $b(t) \in C[0,T]$ such that the triple $\{u(x,t), a(t), b(t)\}$ satisfies the following parabolic equation

$$a_1(t)u_t(x,t) + a(t)u(x,t) = u_{xx}(x,t) + b(t)g(x,t) + f(x,t) \quad (x,t) \in D_T, \tag{1}$$

with the time-nonlocal condition

$$u(x,0) + \delta u(x,T) = \varphi(x), \quad 0 \le x \le 1, \tag{2}$$

the Neumann boundary condition

$$u_x(0,t) = 0, \quad 0 \le t \le T, \tag{3}$$

nonlocal integral condition of the first kind

$$\int_0^1 (x-1)u(x,t)dx = 0, \quad 0 \le t \le T, \tag{4}$$

and the overdetermination conditions

$$u(x_i,t) = h_i(t), \quad i = 1,2; \quad 0 < x_1, x_2 < 1, \quad x_1 \ne x_2, \quad 0 \le t \le T, \tag{5}$$

where $\delta \ge 0$ is given number, $a_1(t) > 0, f(x,t), g(x,t), \varphi(x), h_1(t)$, and $h_2(t)$ are known functions.

Definition 1. *The triple* $\{u(x,t), a(t), b(t)\}$ *is said to be a classical solution to problem* (1)–(5) *if all three of the following conditions are satisfied for all functions* $u(x,t), a(t)$, *and* $b(t)$:

i) *The function* $u(x,t)$ *and its derivatives* $u_t(x,t)$ *and* $u_{xx}(x,t)$ *are continuous in the rectangle* D_T.

ii) *The functions* $a(t)$ *and* $b(t)$ *are continuous on the interval* $[0,T]$.

iii) *Equation* (1) *and conditions* (2)–(5) *are satisfied in the classical (usual) sense.*

The following theorem is proved using a method similar to that presented in [32].

Theorem 1. *Suppose that* $\delta \geq 0,\ \ 0 < a_1(t) \in C[0,T],\ \ f(x,t), g(x,t) \in C(D_T),\ \ \varphi(x) \in C[0,1],\ \int\limits_0^1 (x-1)f(x,t)dx = \int\limits_0^1 (x-1)g(x,t)dx = 0,\ 0 \leq t \leq T,$
$h_i(t) \in C^1[0,T]\ (i=1,2),\ h(t) \equiv h_2(t)g(x_1,t) - h_1(t)g(x_2,t) \neq 0,\ 0 \leq t \leq T,$
and the compatibility conditions

$$\int\limits_0^1 (x-1)\varphi(x)dx = 0, \quad \varphi(x_i) = h_i(0) + \delta h_i(T),\ i=1,2,$$

hold. Then the problem of finding a classical solution of (1)–(5) *is equivalent to the problem of determining the functions* $u(x,t) \in C^{2,1}(D_T),\ a(t) \in C[0,T]$, *and* $b(t) \in C[0,T]$ *satisfying* (1)–(3), *and the conditions*

$$u(0,t) = u(1,t),\ 0 \leq t \leq T, \tag{6}$$

$$a_1(t)h_i'(t) + a(t)h_i(t) = u_{xx}(x_i,t) + a(t)g(x_i,t) + f(x_i,t),\ i=1,2;\ 0 \leq t \leq T. \tag{7}$$

3. Some auxiliary results from spectral theory and the introduction of special functional spaces

Let us consider the following sequences of functions

$$X_0(x) = 2, ..., X_{2k-1}(x) = 4x\sin\lambda_k x,\ X_{2k}(x) = 4\cos\lambda_k x, \tag{8}$$

$$Y_0(x) = 1-x, ..., Y_{2k-1}(x) = \sin\lambda_k x,\ Y_{2k}(x) = (1-x)\cos\lambda_k x. \tag{9}$$

According to the Keldysh theorem [33], the system of root functions (9) is complete in $L_2(0,1)$. Moreover, the system of functions (8) and (9) form a biorthogonal system and satisfy the necessary and sufficient conditions to be a basis in the space $L_2(0,1)$, for $\lambda_k = 2k\pi\ (k=1,2,...)$ as first established by V.A. Il'in [34].

Then an arbitrary function $v(x) \in L_2(0,1)$ can be expanded into a biorthogonal series:

$$v(x) = v_0X_0(x) + \sum_{k=1}^{\infty} v_{2k-1}X_{2k-1}(x) + \sum_{k=1}^{\infty} v_{2k}X_{2k}(x),$$

where the coefficients v_0, v_{2k-1}, and v_{2k} are computed according to the formulas

$$v_0 = \int_0^1 v(x)Y_0(x)dx,\ v_{2k-1} = \int_0^1 v(x)Y_{2k-1}(x)dx,\ v_{2k} = \int_0^1 v(x)Y_{2k}(x)dx.$$

It is easy to see that

$$\begin{gathered} |v_0| \le \|v(x)(1-x)\|_{L_2(0,1)}, \\ \left(\sum_{k=1}^{\infty} |v_{2k-1}|^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v(x)\|_{L_2(0,1)},\quad \left(\sum_{k=1}^{\infty} |v_{2k}|^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v(x)(1-x)\|_{L_2(0,1)}. \end{gathered} \tag{10}$$

The following items are true:

1. For certain function $v(x)$ with the properties

$$v(x) \in C[0,1],\ v'(x) \in L_2(0,1),\ v(0) = v(1)$$

the following estimates are valid

$$\begin{gathered} \left(\sum_{k=1}^{\infty} (\lambda_k |v_{2k-1}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v'(x)\|_{L_2(0,1)}, \\ \left(\sum_{k=1}^{\infty} (\lambda_k |v_{2k}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v'(x)(1-x) - v(x)\|_{L_2(0,1)}. \end{gathered} \tag{11}$$

2. If the function $v(x)$ satisfies conditions

$$v(x), v'(x) \in C[0,1],\ v''(x) \in L_2(0,1),\ v(0) = v(1),\ v'(0) = 0,$$

then

$$\begin{gathered} \left(\sum_{k=1}^{\infty} (\lambda_k^2 |v_{2k-1}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v''(x)\|_{L_2(0,1)}, \\ \left(\sum_{k=1}^{\infty} (\lambda_k^2 |v_{2k}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v''(x)(1-x) - 2v'(x)\|_{L_2(0,1)}. \end{gathered} \tag{12}$$

3. Under conditions

$$v(x), v'(x) \in C[0,1],\ v''(x) \in L_2(0,1),\ v(0) = v(1),\ v'(0) = 0,\ v''(0) = v''(1),$$

it can be stated that the following estimates are valid:

$$\begin{gathered} \left(\sum_{k=1}^{\infty} (\lambda_k^3 |v_{2k-1}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v'''(x)\|_{L_2(0,1)}, \\ \left(\sum_{k=1}^{\infty} (\lambda_k^3 |v_{2k}|)^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}} \|v'''(x)(1-x) - 3v''(x)\|_{L_2(0,1)}. \end{gathered} \tag{13}$$

In order to study the problem (1)–(3), (6), (7), we consider the following special functional spaces: Let $B^3_{2,T}$ [35] denote the set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x),$$

considered in domain D_T. Moreover, the functions $u_k(t)$ $(k=0,1,...)$ contained in last sum are continuous on the interval $[0,T]$, and

$$J_T(u)\equiv\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k-1}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_{2k}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<\infty.$$

The norm in the space $B^3_{2,T}$ is defined as follows:

$$\|u(x,t)\|_{B^3_{2,T}}=J_T(u).$$

Furthermore, let E^3_T denote the space consisting of the topological product $B^3_{2,T}\times C[0,T]\times C[0,T]$ which is the norm of the element $z=\{u,a,b\}$ defined by the formula

$$\|z\|_{E^3_T}=\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}\,.$$

It is clear that the spaces $B^3_{2,T}$ and E^3_T are Banach spaces [35].

4. Existence and uniqueness of the classical solution

Since the system (8) forms a Riesz basis in $L_2(0,1)$, each solution to problem (1)–(3), (6), (7) can be sought in the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)X_k(x), \tag{14}$$

where

$$u_k(t)=\int_0^1 u(x,t)Y_k(x)dx\ (k=0,1,...), \tag{15}$$

and the functions $X_k(x)$, $Y_k(x)$ $(k=0,1,...)$ are correspondingly defined by relations (8) and (9).

Using the method of separation of variables, from (1), (2), we have

$$a_1(t)u_0'(t)+a(t)u_0(t)=f_0(t)+b(t)g_0(t), \tag{16}$$

$$a_1(t)u_{2k-1}'(t)+a(t)u_{2k-1}(t)+\lambda_k^2u_{2k-1}(t)=f_{2k-1}(t)+b(t)g_{2k-1}(t),\ k=1,2,..., \tag{17}$$

$$a_1(t)u_{2k}'(t)+a(t)u_{2k}(t)+\lambda_k^2u_{2k}(t)=f_{2k}(t)+b(t)g_{2k}(t)-2\lambda_ku_{2k-1}(t),\ k=1,2,..., \tag{18}$$

$$u_k(0)+\delta u_k(T)=\varphi_k,\ \ k=0,1,..., \tag{19}$$

where

$$f_k(t)=\int_0^1 f(x,t)Y_k(x)dx,\ \ g_k(t)=\int_0^1 g(x,t)Y_k(x)dx,$$

$$\varphi_k=\int_0^1 \varphi(x)Y_k(x)dx,\ \ k=0,1,....,\ \ \lambda_k=2k\pi,\ \ k=1,2,....$$

Solving problem (16)–(19), we get

$$u_0(t)=(1+\delta)^{-1}\left(\varphi_0-\delta\int_0^T\frac{1}{a_1(\tau)}F_0(\tau;u,a,b)d\tau\right)+\int_0^t\frac{1}{a_1(\tau)}F_0(\tau;u,a,b)d\tau, \tag{20}$$

$$u_{2k-1}(t)=\frac{e^{-\int_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k-1}+\int_0^t\frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T\frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau,\ \ k=1,2,..., \tag{21}$$

$$u_{2k}(t)=\frac{e^{-\int_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k}+\int_0^t\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$+\frac{2\lambda_k e^{-\int_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\left[\int_0^t\frac{d\tau}{a_1(\tau)}-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T\frac{d\tau}{a_1(\tau)}\right]\varphi_{2k-1}$$

$$+2\lambda_k\left[\int_0^t\frac{1}{a_1(\tau)}\left(\int_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right.$$

$$\left.-\frac{\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T\frac{1}{a_1(\tau)}\left(\int_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right]$$

$$-\frac{2\lambda_k\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T \frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\xi$$

$$\times\left[\int_0^t \frac{1}{a_1(\tau)}d\tau-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T \frac{1}{a_1(\tau)}d\tau\right],\ k=1,2,..., \tag{22}$$

where

$$F_k(t;u,a,b)=-a(t)u_k(t)+b(t)g_k(t)+f_k(t),\ k=0,1,....$$

Substituting the expressions of (20), (21), and (22) into (14), we get the following formula for the component $u(x,t)$ of the classical solution to problem (1)–(3), (6), (7):

$$u(x,t)=\left[(1+\delta)^{-1}\left(\varphi_0-\delta\int_0^T \frac{1}{a_1(\tau)}F_0(\tau;u,a,b)d\tau\right)+\int_0^t \frac{1}{a_1(\tau)}F_0(\tau;u,a,b)d\tau\right]X_0(x)$$

$$+\sum_{k=1}^\infty\left\{\frac{e^{-\int_0^t \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k-1}+\int_0^t \frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$\left.-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T \frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau\right\}X_{2k-1}(x)$$

$$+\sum_{k=1}^\infty\left\{\frac{e^{-\int_0^t \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k}+\int_0^t \frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T \frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$-\frac{2\lambda_k e^{-\int_0^t \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\left[\int_0^t \frac{d\tau}{a_1(\tau)}-\frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int_0^T \frac{d\tau}{a_1(\tau)}\right]\varphi_{2k-1}$$

$$+2\lambda_k\left[\int_0^t \frac{1}{a_1(\tau)}\left(\int_0^\tau \frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int_\xi^t \frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right.$$

$$-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T \frac{1}{a_1(\tau)}\left(\int\limits_0^\tau \frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t \frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\Bigg]$$

$$-\frac{2\lambda_k\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T \frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t \frac{\lambda_k^2}{a_1(s)}ds}d\xi$$

$$\times\left[\int\limits_0^t \frac{1}{a_1(\tau)}d\tau-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T \frac{1}{a_1(\tau)}d\tau\right]\Bigg\}X_{2k}(x). \tag{23}$$

Now from (7), taking into account (14), we have

$$a(t)=[h(t)]^{-1}\{g(x_1,t)(f(x_2,t)-a_1(t)h_2'(t))-g(x_2,t)(f(x_1,t)-a_1(t)h_1'(t))$$
$$-\sum_{k=1}^{\infty}\lambda_k^2u_{2k-1}(t)(g(x_1,t)X_{2k-1}(x_2)-g(x_2,t)X_{2k-1}(x_1))$$
$$-\sum_{k=1}^{\infty}\lambda_k^2u_{2k}(t)(g(x_1,t)X_{2k}(x_2)-g(x_2,t)X_{2k}(x_1))\Bigg\}, \tag{24}$$

$$b(t)=[h(t)]^{-1}\{h_1(t)(f(x_2,t)-a_1(t)h_2'(t))-h_2(t)(f(x_1,t)-a_1(t)h_1'(t))$$
$$-\sum_{k=1}^{\infty}\lambda_k^2u_{2k-1}(t)(h_1(t)X_{2k-1}(x_2)-h_2(t)X_{2k-1}(x_1))$$
$$-\sum_{k=1}^{\infty}\lambda_k^2u_{2k}(t)(h_1(t)X_{2k}(x_2)-h_2(t)cX_{2k}(x_1))\Bigg\}. \tag{25}$$

Next, substituting the expressions for $u_0(t), u_{2k-1}(t)$, and $u_{2k}(t)$ from (20), (21), and (22) into (24) and (25), respectively, we obtain:

$$a(t)=[h(t)]^{-1}\{g(x_1,t)(f(x_2,t)-a_1(t)h_2'(t))-g(x_2,t)(f(x_1,t)-a_1(t)h_1'(t))$$

$$-\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int\limits_0^t \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k-1}+\int\limits_0^t \frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int\limits_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T \frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T \frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int\limits_\tau^t \frac{\lambda_k^2}{a_1(s)}ds}d\tau\right]$$

$$\times(g(x_1,t)X_{2k-1}(x_2)-g(x_2,t)X_{2k-1}(x_1))$$

$$-\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int\limits_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k}+\int\limits_0^t\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$+\frac{2\lambda_k e^{-\int\limits_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\left[\int\limits_0^t\frac{d\tau}{a_1(\tau)}-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{d\tau}{a_1(\tau)}\right]\varphi_{2k-1}$$

$$+2\lambda_k\left[\int\limits_0^t\frac{1}{a_1(\tau)}\left(\int\limits_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}\left(\int\limits_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right]$$

$$-\frac{2\lambda_k\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi$$

$$\left.\times\left[\int\limits_0^t\frac{1}{a_1(\tau)}d\tau-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}d\tau\right]\right]$$

$$\times(g(x_1,t)X_{2k}(x_2)-g(x_2,t)X_{2k}(x_1))\}, \tag{26}$$

$$b(t)=[h(t)]^{-1}\{h_1(t)(f(x_2,t)-a_1(t)h_2'(t))-h_2(t)(f(x_1,t)-a_1(t)h_1'(t))$$

$$-\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int\limits_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k-1}+\int\limits_0^t\frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}F_{2k-1}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau\right]$$

$$\times(h_1(t)X_{2k-1}(x_2)-h_2(t)cX_{2k-1}(x_1))$$

$$-\sum_{k=1}^{\infty}\lambda_k^2\left[\frac{e^{-\int\limits_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\varphi_{2k}+\int\limits_0^t\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau\right.$$

$$-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}F_{2k}(\tau;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\tau$$

$$+\frac{2\lambda_k e^{-\int\limits_0^t\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\left[\int\limits_0^t\frac{d\tau}{a_1(\tau)}-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{d\tau}{a_1(\tau)}\right]\varphi_{2k-1}$$

$$+2\lambda_k\left[\int\limits_0^t\frac{1}{a_1(\tau)}\left(\int\limits_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right.$$

$$\left.-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}\left(\int\limits_0^\tau\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\xi^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi\right)d\tau\right]$$

$$-\frac{2\lambda_k\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\xi)}F_{2k-1}(\xi;u,a,b)e^{-\int\limits_\tau^t\frac{\lambda_k^2}{a_1(s)}ds}d\xi$$

$$\left.\times\left[\int\limits_0^t\frac{1}{a_1(\tau)}d\tau-\frac{\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}{1+\delta e^{-\int\limits_0^T\frac{\lambda_k^2}{a_1(s)}ds}}\int\limits_0^T\frac{1}{a_1(\tau)}d\tau\right]\right]$$

$$\times(h_1(t)X_{2k}(x_2)-h_2(t)cX_{2k}(x_1))\Big\}, \tag{27}$$

where

$$h(t)\equiv h_2(t)g(x_1,t)-h_1(t)g(x_2,t)\neq 0.$$

Thus the solution of problem (1)–(3), (6), (7) was reduced to the solution of systems (23), (26), (27) with respect to unknown functions $u(x,t), a(t)$ and $b(t)$.

We state the following lemma without proof.

Lemma 1. *If* $\{u(x,t),a(t),b(t)\}$ *is any solution to problem* (1)–(3), (6), (7), *then the functions*

$$u_k(t)=\int\limits_0^1 u(x,t)Y_k(x)dx\ (k=0,1,...),$$

satisfy the system (20)–(22) *on an interval* $[0, T]$.

To study the uniqueness of the solution to problem (1)–(3), (6), (7), the following corollary plays an important role.

Corollary 1. *Assume that the system* (23), (26), (27) *has a unique solution. Then the problem* (1)–(3), (6), (7) *has at most one solution, i.e., if the problem* (1)–(3), (6), (7) *has a solution, then it is unique.*

Let us now consider the operator

$$\Phi(u, a, b) = \{\Phi_1(u, a, b), \Phi_2(u, a, b), \Phi_3(u, a, b)\},$$

in the space E_T^3, where

$$\Phi_1(u, a, b) = \tilde{u}(x, t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) X_k(x),\ \Phi_2(u, a, b) = \tilde{a}(t),\ \Phi_3(u, a, b) = \tilde{b}(t),$$

and the functions $\tilde{u}_0(t), \tilde{u}_{2k-1}(t), \tilde{u}_{2k}(t)$ $(k = 1, 2, ...), \tilde{a}(t)$, and $\tilde{b}(t)$ are equal to the right-hand sides of (20), (21), (22), (26), and (27), respectively.

Assume that the data for the problem (1)–(3), (6), (7) satisfy the following conditions:

C_1) $\varphi(x) \in C^2[0, 1],\ \varphi'''(x) \in L_2(0, 1)$, and $\varphi(0) = \varphi(1),\ \varphi'(0) = 0,\ \varphi''(0) = \varphi''(1)$;

C_2) $f(x, t), f_x(x, t), f_{xx}(x, t) \in C(D_T), f_{xxx}(x, t) \in L_2(D_T)$, and
$f(0, t) = f(1, t), f_x(0, t) = 0, f_{xx}(0, t) = f_{xx}(1, t),\ 0 \le t \le T$;

C_3) $g(x, t), g_x(x, t), g_{xx}(x, t) \in C(D_T), g_{xxx}(x, t) \in L_2(D_T)$, and
$g(0, t) = g(1, t), g_x(0, t) = 0, g_{xx}(0, t) = g_{xx}(1, t),\ 0 \le t \le T$;

C_4) $\delta \ge 0,\ 0 < a_1(t) \in C[0, T],\ h_i(t) \in C^1[0, T]\ (i = 1, 2)$,
$h(t) \equiv h_2(t)g(x_1, t) - h_1(t)g(x_2, t) \ne 0,\ 0 \le t \le T$.

Then, by applying simple transformations, we obtain:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^3} \le A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} + D_1(T) \|b(t)\|_{C[0,T]}, \tag{28}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} + D_2(T) \|b(t)\|_{C[0,T]}, \tag{29}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le A_3(T) + B_3(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} + D_3(T) \|b(t)\|_{C[0,T]}, \tag{30}$$

where

$$A_1(T) = (1 + \delta)^{-1} \|\varphi(x)(1 - x)\|_{L_2(0,1)}$$
$$+\frac{1}{m}(1 + \delta(1 + \delta)^{-1})[\sqrt{T} \|f(x, t)(1 - x)\|_{L_2(D_T)}$$
$$+\sqrt{2}\rho(T) \left\|\varphi'''(x)\right\|_{L_2(0,1)} + \frac{\sqrt{2T}}{m}(1 + \delta\rho(T)) \|f_{xxx}(x, t)\|_{L_2(D_T)}$$

$$+\frac{3}{\sqrt{2}}\rho(T)\left\|\varphi'''(x)(1-x)-3\varphi''(x)\right\|_{L_2(0,1)}$$

$$+\frac{3\sqrt{T}}{m\sqrt{2}}(1+\delta\rho(T))\left\|f_{xxx}(x,t)(1-x)-3f_{xx}(x,t)\right\|_{L_2(D_T)}$$

$$+\frac{2\sqrt{2MT}}{m}(1+\delta\rho(T))\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$

$$+\frac{2\sqrt{2M}}{m^2}(1+\delta\rho(T))^2T\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_1(T)=\frac{1}{m}(1+\delta(1+\delta)^{-1})T$$

$$+\frac{5T}{m\sqrt{2}}(1+\delta\rho(T))+\frac{2\sqrt{2M}}{m^2}(1+\delta\rho(T))^2T\sqrt{T},$$

$$D_1(T)=\frac{1}{m}(1+\delta(1+\delta)^{-1})T\left\|g(x,t)(1-x)\right\|_{L_2(D_T)}$$

$$+\frac{\sqrt{2}}{m}(1+\delta\rho(T))\sqrt{T}\left(1+\frac{2\sqrt{TM}}{m}(1+\delta\rho(T))^2\right)\left\|g_{xxx}(x,t)\right\|_{L_2(D_T)}$$

$$+\frac{3}{m\sqrt{2}}(1+\delta\rho(T))\sqrt{T}\left\|g_{xxx}(x,t)(1-x)-3g_{xx}(x,t)\right\|_{L_2(D_T)},$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times\{\left\|h_1(t)(f(x_2,t)-a_1(t)h_2'(t))-h_2(t)(f(x_1,t)-a_1(t)h_1'(t))\right\|_{C[0,T]}$$

$$+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\||g(x_1,t)|+|g(x_2,t)|\right\|_{C[0,T]}[\sqrt{2}\rho(T)\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$

$$+\frac{2}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}+\frac{3}{\sqrt{2}}\rho(T)\left\|\varphi'''(x)(1-x)-3\varphi''(x)\right\|_{L_2(0,1)}$$

$$+\frac{3}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\left\|f_{xxx}(x,t)(1-x)-3f_{xx}(x,t)\right\|_{L_2(D_T)}$$

$$+\frac{4\sqrt{MT}}{m}(1+\delta\rho(T))\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\frac{4\sqrt{M}}{m^2}(1+\delta\rho(T))^2T\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},$$

$$B_2(T)=4\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\||g(x_1,t)|+|g(x_2,t)|\right\|_{C[0,T]}T$$

$$\times\left(1+\frac{(1+\delta\rho(T))}{m}\left(3+\frac{4\sqrt{TM}}{m}(1+\delta\rho(T))\right)\right),$$

$$D_2(T)=4\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left\||g(x_1,t)|+|g(x_2,t)|\right\|_{C[0,T]}$$

$$\times\left[\frac{2}{m}(1+\delta\rho(T))\sqrt{T}\left(\frac{1}{\sqrt{2}}+\frac{2\sqrt{TM}}{m^2}(1+\delta\rho(T))\right)\|g_{xxx}(x,t)\|_{L_2(D_T)}\right.$$
$$\left.+\frac{3}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\|g_{xxx}(x,t)(1-x)-3g_{xx}(x,t)\|_{L_2(D_T)}\right],$$
$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times\{\left\|h_1(t)\left(f(x_2,t)-a_1(t)h_2'(t)\right)-h_2(t)\left(f(x_1,t)-a_1(t)h_1'(t)\right)\right\|_{C[0,T]}$$
$$+4\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}[\sqrt{2}\rho(T)\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$
$$+\frac{2}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\|f_{xxx}(x,t)\|_{L_2(D_T)}+\frac{3}{\sqrt{2}}\rho(T)\left\|\varphi'''(x)(1-x)-3\varphi''(x)\right\|_{L_2(0,1)}$$
$$+\frac{3}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\|f_{xxx}(x,t)(1-x)-3f_{xx}(x,t)\|_{L_2(D_T)}$$
$$+\frac{4\sqrt{MT}}{m}(1+\delta\rho(T))\left\|\varphi'''(x)\right\|_{L_2(0,1)}+\frac{4\sqrt{M}}{m^2}(1+\delta\rho(T))^2T\|f_{xxx}(x,t)\|_{L_2(D_T)},$$
$$B_3(T)=4\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}$$
$$\times\||h_1(t)|+|h_2(t)|\|_{C[0,T]}T\left(1+\frac{(1+\delta\rho(T))}{m}\left(3+\frac{4\sqrt{TM}}{m}(1+\delta\rho(T))\right)\right),$$
$$D_2(T)=4\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}$$
$$\times\left[\frac{2}{m}(1+\delta\rho(T))\sqrt{T}\left(\frac{1}{\sqrt{2}}+\frac{2\sqrt{TM}}{m^2}(1+\delta\rho(T))\right)\|g_{xxx}(x,t)\|_{L_2(D_T)}\right.$$
$$\left.+\frac{3}{m}(1+\delta\rho(T))\sqrt{\frac{T}{2}}\|g_{xxx}(x,t)(1-x)-3g_{xx}(x,t)\|_{L_2(D_T)}\right]$$

From inequalities (28)–(30) we conclude

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}+\left\|\tilde{b}(t)\right\|_{C[0,T]}$$
$$\leq A(T)+B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}}+D(T)\|b(t)\|_{C[0,T]}, \tag{31}$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T),\quad B(T)=B_1(T)+B_2(T)+B_3(T),$$
$$D(T)=D_1(T)+D_2(T)+D_3(T).$$

Now, let us prove the following theorem.

Theorem 2. *Let the conditions $C_1) - C_4)$ and the condition*

$$(B(T)(A(T)+2)+D(T))(A(T)+2)<1, \tag{32}$$

be fulfilled. Then, problem (1)–(3), (6), (7) *has a unique solution in the ball* $K = K_R$ $(\|z\|_{E_T^3} \le R = A(T)+2)$ *of space* E_T^3.

Remark 1. *Inequality* (32) *is satisfied for sufficiently small values of* T.

Proof. Let's consider in the space E_T^3, the operator equation

$$z = \Phi z, \tag{33}$$

where $z = \{u, a, b\}$. The components $\Phi_i(u, a, b)$ $(i = 1, 2, 3)$ of operator $\Phi(u, a, b)$ defined by the right side of equations (23), (26), and (27), respectively.

Now, consider the operator $\Phi(u, a, b)$ in the ball $K = K_R$ of the space E_T^3.

Analogously to (31), we obtain that for any $z, z_1, z_2 \in K_R$, the following estimates hold:

$$\begin{aligned}\|\Phi z\|_{E_T^3} &\le A(T) + B(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3} + D(T)\|b(t)\|_{C[0,T]} \\ &\le A(T) + B(T)R^2 + D(T)R \\ &\le A(T) + B(T)(A(T)+2)^2 + D(T)(A(T)+2),\end{aligned} \tag{34}$$

$$\begin{aligned}\|\Phi z_1 - \Phi z_2\|_{E_T^3} &\le (B(T)R + D(T)) \\ &\times (\|a_1(t)-a_2(t)\|_{C[0,T]} + \|b_1(t)-b_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^3}) \\ &\le (B(T)(A(T)+2) + D(T))\|z_1 - z_2\|_{E_T^3}.\end{aligned} \tag{35}$$

Then by (32), from estimates (34) and (35) it is clear that the operator Φ acts in a ball $K = K_R$ and satisfy the assertion of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{u, a, b\}$ in the ball $K = K_R$, which is a unique solution of equation (33); i.e. $\{u, a, b\}$ is a unique solution of the systems (23), (26), (27) in the ball $K = K_R$.

Thus, we obtain that the function $u(x,t)$ as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Analogously [20] it can be show that the derivative $u_t(x,t)$ is also continuous in the region D_T.

It is easy to verify that Equation (1) and conditions (2), (3), (6), and (7) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution of problem (1)–(3), (6), (7) and by Lemma 1 this solution is unique in the ball $K = K_R$.

Hence, from Theorem 2, by virtue of Theorem 1, it follows that the original problem (1)–(5) has a unique classical solution, i.e. the following theorem is valid.

Theorem 3. *Assume that all the conditions of Theorem 2 are satisfied and*

$$\int_0^1 (x-1)f(x,t)dx = \int_0^1 (x-1)g(x,t)dx = 0, \ 0 \le t \le T,$$

$$\int_0^1 (x-1)\varphi(x)dx = 0, \ \varphi(x_i) = h_i(0) + \delta h_i(T), \ i = 1, 2.$$

Then problem (1)–(5) *has a unique classical solution in the ball* $K = K_R$ *of the space* E_T^3.

5. Conclusions

In this work, we have investigated the classical solvability of a nonlinear inverse boundary value problem for a parabolic equation subject to nonlocal boundary conditions. The analysis begins with a transformation of the original inverse problem into an equivalent auxiliary inverse boundary value problem with trivial data, which simplifies the subsequent analytical treatment. By employing the Fourier method, the auxiliary problem is reduced to a system of nonlinear integral equations whose properties can be rigorously analyzed within an appropriate functional framework.

The existence and uniqueness of the solution to the auxiliary problem have been established by means of the contraction mapping (Banach fixed-point) principle in a suitably defined Banach space. This approach provides a constructive framework for demonstrating the well-posedness of the problem and ensures the stability of the solution with respect to the given data. Owing to the established equivalence between the original and auxiliary formulations, the existence and uniqueness of a classical solution to the initial nonlinear inverse boundary value problem have consequently been proved.

The theoretical results obtained in this study contribute to the broader theory of inverse problems for parabolic equations with nonlocal conditions, which are often encountered in mathematical models of diffusion and heat conduction processes with memory or spatial interaction effects. Future research may extend the present analysis to fractional-order parabolic operators, multidimensional domains, or problems involving noisy data and regularization techniques.

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