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Fixed Point Theorems on Multiplicative Cone Bipolar Metric Space and Their Applications

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Abstract. In this article, the notion of a multiplicative cone bipolar metric space is introduced. Fixed point results in the setting of these spaces have been established and supported with suitable non-trivial examples. Our results generalize and extend proven results from previous studies. The main results are applied to obtain analytical solutions to integral equations and fractional differential equations. These findings contribute to the growing body of literature on fixed point theory and its applications in analysis.

2020 Mathematics Subject Classifications: 47H10, 54H25

Key Words and Phrases: Multiplicative cone bipolar metric space, fixed point, covariant map, contravariant map

1. Introduction

It would be fair to say that the concept of metric fixed point theory started with the famous contraction mapping theorem of S. Banach [1], that was generalised by Kannan [2], Reich [3], Junck[4], to name a few. This theory has seen rapid development in the past nineteenth and twentieth centuries. In the overlaps made in these centuries, while metric spaces and normed spaces developed, the domains were only taken as value regions with single variables and real positive numbers. In other words, new metric spaces are produced by taking the domains X, X^2, and X^3. However, bipolar metric space is defined as a new space by going beyond the conventional definition of metric spaces that have been defined for years. At the same time, this theory has been applied to real life and various fields of science, namely engineering, economics, medical sciences, and computer, etc.

Metric Fixed Point Theory has vast applications. The attraction of productive research activity in the fixed point theory has taken shape in the form of the search for fixed points of generalized contraction mappings. Not to mention, a lot of investigators have released many publications on fixed point theory in various ways. And the existence of fixed points

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of contraction functions has been recently one of the popular topics in the discipline of fixed point theory. One of the generalisation of the metric spaces was the cone metric space, introduced in 2007 by Huang and Zhang [5] who presented fixed point theorems of contractive mappings in the setting of cone metric space. Subsequently, fixed point results were reported by generalising the cone metric space such as dislocated cone metric space,cone b-metric space, rectangular cone metric space, rectangular cone b metric space etc were reported in literature, for example, [6–9].
In 2016, the concept of bipolar metric space has been established by Mutlu & U. Gürdal,[10] and the authors explored several basic fixed point and coupled fixed point theorems for co-variant and contra-variant mappings subject to contractive conditions; see [11]. Many important contributions by many authors have been made in bipolar metric spaces see, [12–19]. It was in 2021 that Gaba Aphane and Aydi, [20] showed fixed point theorems on bipolar metric space. Ampadu in 2007[21] introduced to the concept of multiplicative cone metric space & proved coupled version of the higher order Banach contracton principle.

In this article, the definition to the concept of multiplicative cone bipolar metric space is being introduced & fixed point theorems are established. The derived results extend / generalise proven results of the past and are supplemented with non-trivial examples. The results are applied to find solutions of integral equation and fractional differential equations. The rest of the paper is organised as follows: In Section 2, some basic definitions are given. In Section -3 fixed point results in the setting of MCBMS are established and the derived results are supplemented using non-trivial examples. Fractional calculus plays an important role in designing and analysis of various mathematical models that help in achieving sustainable development goals of the United nations and fixed point theory plays a vital role in analysing the existence of unique solutions to those systems, see [22–26]. Accordingly, in Section-4, the fixed point results are applied to find analytical solution to integral and fractional differential equations. Finally, the manuscript is concluded by placing some open problems for future research in Section-5.

2. Preliminaries

The following definitions and monograph are required in the sequel.
Let $\mathcal{A}$ always be a real Banach space & $\mathcal{Z} \subseteq \mathcal{A}$. $\mathcal{Z}$ is known as a cone iff

(i) $\mathcal{Z}$ is closed, non-void, & $\mathcal{Z} \neq \{0\}$;

(ii) $\mathfrak{a}, \mathfrak{c} \in \mathbb{R}$, $\mathfrak{a}, \mathfrak{c} \geq 0$, $\mathfrak{q}, \vartheta \in \mathcal{Z} \implies \mathfrak{a}\mathfrak{q} + \mathfrak{c}\vartheta \in \mathcal{Z}$;

(iii) $\mathfrak{q} \in \mathcal{Z}$ and $-\mathfrak{q} \in \mathcal{Z} \implies \mathfrak{q} = 0$.

Given a cone $\mathcal{Z} \subset \mathcal{A}$, we define a partial ordering $\leq$ with respect to $\mathcal{Z}$ as $\mathfrak{q} \leq \vartheta$ iff $\vartheta - \mathfrak{q} \in \mathcal{Z}$. We shall write $\mathfrak{q} < \vartheta$ to indicate that $\mathfrak{q} \leq \vartheta$ and $\mathfrak{q} \neq \vartheta$, while $\mathfrak{q} \ll \vartheta$ will stand for $\vartheta - \mathfrak{q} \in int\mathcal{Z}$, $int\mathcal{Z}$ is the interior of $\mathcal{Z}$.
The cone $\mathcal{Z}$ is referred to Normal if there is a number $\mathcal{W} > 0$ s.t for all $\mathfrak{q}, \vartheta \in \mathcal{A}$,

$$0 \leq \mathfrak{q} \leq \vartheta \ \Rightarrow \ ||\mathfrak{q}|| \leq ||\vartheta||.$$

The smallest non negative number that satisfies the above is known as normal constant $\mathcal{Z}$. The cone $\mathcal{Z}$ is regular, if every monotonically increasing and upper bounded sequence is convergent. To be specific, if $\{\mathfrak{q}_\ell\}$ is sequence for which

$$\mathfrak{q}_1 \leq \mathfrak{q}_2 \leq \cdots \leq \mathfrak{q}_\ell \leq \cdots \leq \vartheta$$

For some $\vartheta \in \mathcal{A}$, there is $\mathfrak{q} \in \mathcal{A}$ such that $||\mathfrak{q}_\ell - \mathfrak{q}|| \to 0(\ell \to \infty)$. In other words the cone $\mathcal{Z}$ is regular iff every sequence which is non-increasing and bounded from below is convergent. It is a common knowledge that a regular cone is a normal one. In what follows we always have in mind that $\mathcal{A}$ is a Banach space, $\mathcal{Z}$ is a cone in $\mathcal{A}$ with $int\mathcal{Z} \neq \emptyset$ and $\leq$ is partial ordering with respect to $\mathcal{Z}$.

Definition 2.1. *Consider $\mathcal{B}$ and $\mathcal{F}$ be non-void sets and $\alpha : \mathcal{B} \times \mathcal{F} \to \mathcal{A}$ be a function such that*

(i) If $\alpha(\mathfrak{q}, \vartheta) = 1$ then $\mathfrak{q} = \vartheta$, for all $(\mathfrak{q}, \vartheta) \in \mathcal{B} \times \mathcal{F}$., where 1 represents the unit element.

(ii) If $\mathfrak{q} = \vartheta$, then $\alpha(\mathfrak{q}, \vartheta) = 1$, for all $(\mathfrak{q}, \vartheta) \in \mathcal{B} \times \mathcal{F}$

(iii) $\alpha(\mathfrak{q}, \vartheta) = \alpha(\vartheta, \mathfrak{q})$, for all $\mathfrak{q}, \vartheta \in \mathcal{B} \cap \mathcal{F}$

(iv) $\alpha(\mathfrak{q}, \vartheta) \leq \alpha(\mathfrak{q}, \omega)\alpha(\beta, \omega)\alpha(\beta, \vartheta)$, for all $\mathfrak{q}, \beta \in \mathcal{B}$ and $\omega, \vartheta \in \mathcal{F}$.

Then the triplet $(\mathcal{B}, \mathcal{F}, \alpha)$ is called a multiplicative cone bipolar metric space(MCBMS).

Example 1. *Consider $\mathcal{A} = \mathbb{R}^2$, $\mathcal{Z} = \{(\mathfrak{q}, \vartheta) \in \mathcal{A} | \mathfrak{q}, \vartheta \geq 0\} \subset \mathbb{R}^2$, $\mathcal{B} = [0, 1]$, $\mathcal{F} = [1, 2]$ and $\alpha : \mathcal{B} \times \mathcal{F} \to \mathcal{A}$ such that $\alpha(\mathfrak{q}, \vartheta) = e^{(|\mathfrak{q}-\vartheta|, \beta|\mathfrak{q}-\vartheta|)}$, where $\beta \geq 0$ is a constant. Then $(\mathcal{B}, \mathcal{F}, \alpha)$ is a MCBMS.*

Definition 2.2. *(i) Let us consider a MCBMS $(\mathcal{B}, \mathcal{F}, \alpha)$. Then the sets points are $\mathcal{B}$, $\mathcal{F}$ and $\mathcal{B} \cap \mathcal{F}$ are named as left, right and central points, respectively, and any sequence, that is consisted of only left (or right, or central) points is called a left (or right, or central) sequence on $(\mathcal{B}, \mathcal{F}, \alpha)$.*

(ii) Let $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ & $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ be MCBMS & $\Omega : \mathcal{B}_1 \cup \mathcal{F}_1 \to \mathcal{B}_2 \cup \mathcal{F}_2$ be a function. If $\Omega(\mathcal{B}_1) \subseteq \mathcal{B}_2$ and $\Omega(\mathcal{F}_1) \subseteq \mathcal{F}_2$, then Ω is called a covariant map, or a map from $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ to $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ and this is written as $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightrightarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$. If $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightrightarrows (\mathcal{F}_2, \mathcal{B}_2, \overline{\alpha}_2)$ is a map, then Ω is called a contravariant map from $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ to $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ and this is denoted as $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightleftarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$.

Definition 2.3. *Let $(\mathcal{B}, \mathcal{F}, \alpha)$ be a MCBMS. A left sequence $\{\mathfrak{q}_\ell\}$ converges to a right point ϑ iff for every $\mathfrak{z} \in \mathcal{A}$ with $0 \ll \mathfrak{z}$ $\exists$ an $\ell_0 \in \mathbb{N}$ s.t $\alpha(\mathfrak{q}_\ell, \vartheta) \ll \mathfrak{z}$ for all $\ell \geq \ell_0$. Similarly, a right seqence $\{\vartheta_\ell\} \to \mathfrak{q}$ iff, for every $\mathfrak{z} \in \mathcal{A}$ with $0 \ll \mathfrak{z}$ $\exists$ an $\ell_0 \in \mathbb{N}$ s.t, whenever $\ell \geq \ell_0, \alpha(\mathfrak{q}, \vartheta_\ell) \ll \mathfrak{z}$.*

Definition 2.4. *Let $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ and $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ be a MCBMS.*

(i) *A map* $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightrightarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ *is known to be continuous at a point* $\mathfrak{q}_0 \in \mathcal{B}$, *if for every* $\mathfrak{z}, \delta \in \mathcal{A}$ *with* $0 \ll \mathfrak{z}$, $\exists$ *a* $0 \ll \delta$ *s.t whenever* $\vartheta \in \mathcal{F}_1$ & $\alpha_1(\mathfrak{q}_0, \vartheta) \ll \delta$, $\alpha_2(\Omega(\mathfrak{q}_0),\ \Omega(\vartheta)) \ll \mathfrak{z}$. *It is continuous at a point* $\vartheta_0 \in \mathcal{F}_1$ *if for every* $\mathfrak{z}, \delta \in \mathcal{A}$ *with* $0 \ll \mathfrak{z}$, $\exists$ *a* $0 \ll \delta$ *s.t whenever* $\mathfrak{q} \in \mathcal{B}_1$ *and* $\alpha_1(\mathfrak{q}, \vartheta_0) \ll \delta$, $\alpha_2(\Omega(\mathfrak{q}), \Omega(\vartheta_0)) \ll \mathfrak{z}$. *In the case where* f *is continuous at any point* $\mathfrak{q} \in \mathcal{B}_1$ & $\vartheta \in \mathcal{F}_1$, *then it is called continuous.*

(ii) *A contravariant function* $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \leftrightarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ *is continuous iff it is continuous as a covariant function* $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightrightarrows (\mathcal{B}_2, \mathcal{F}_2, \bar{\alpha_2})$.

This definition assumes that a covariant or a contravariant map Ω from $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ to $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ is continuous, if and only if $\{\mathfrak{x}_\ell\} \to \mathsf{z}$ on $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ implies $\{\Omega(\mathfrak{x}_\ell)\} \to \Omega(\mathsf{z})$ on $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$.

Definition 2.5. *Let* $(\mathcal{B}_1, \mathcal{F}_1, \alpha_1)$ *and* $(\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ *be MCBMS. A covariant map* $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \rightrightarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ *such that*

$$\alpha(\Omega(\mathfrak{q}), \Omega(\vartheta)) \leq (\alpha(\mathfrak{q}, \vartheta))^\lambda \text{ for all } \mathfrak{q} \in \mathcal{B}_1, \vartheta \in \mathcal{F}_1$$

or a contravariant map $\Omega : (\mathcal{B}_1, \mathcal{F}_1, \alpha_1) \leftrightarrows (\mathcal{B}_2, \mathcal{F}_2, \alpha_2)$ *such that*

$$\alpha(\Omega(\mathfrak{q}), \Omega(\vartheta)) \leq (\alpha(\mathfrak{q}, \vartheta))^\lambda \text{ for all } \mathfrak{q} \in \mathcal{B}_1, \vartheta \in \mathcal{F}_1$$

is known as Lipschitz continuous. If λ*=1, then the covariant or contravariant function is known as non-expansive, & it is termed as contraction if it is confirmed for* $\lambda \in (0, 1)$.

Definition 2.6. *Consider* $(\mathcal{B}, \mathcal{F}, \alpha)$ *be a MCBMS.*

(i) *A sequence* $(\{\mathfrak{q}_n\}, \{\vartheta_n\})$ *on the set* $\mathcal{B} \times \mathcal{F}$ *is known as bisequence on* $(\mathcal{B}, \mathcal{F}, \alpha)$.

(ii) *If both* $\{\mathfrak{q}_n\}$ & $\{\vartheta_n\}$ *converge, then the bisequence* $(\mathfrak{q}_n, \vartheta_n)$ *is known as convergent. If* $\{\mathfrak{q}_n\}$ *and* $\{\vartheta_n\}$ *both converge to a same point* $u \in \mathcal{B} \cap \mathcal{F}$, *then this bisequence is known as biconvergent.*

(iii) *A bisequence* $(\{\mathfrak{q}_n\}, \{\vartheta_n\})$ *on* $(\mathcal{B}, \mathcal{F}, \alpha)$ *is known as a Cauchy bisequence, if for each* $\epsilon > 0$, $\exists$ *a number* $\ell_0 \in \mathbb{N}$, *s.t for all positive integers* $\ell, \mathfrak{r} \geq \ell_0, \alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) < \epsilon$.

Definition 2.7. *A MCBMS is complete, if every Cauchy bisequence is convergent.*

In the next section fixed point results in the setting of MCBMS are presented.

3. Main Results

Now we present our first result.

Theorem 3.1. *Let* $(\mathcal{B}, \mathcal{F}, \alpha)$ *be a complete MCBMS,* $\mathcal{Z}$ *be a cone with constant* $\mathcal{W}$ &. *Given a contraction function* $\Omega : (\mathcal{B}, \mathcal{F}, \alpha) \rightrightarrows (\mathcal{B}, \mathcal{F}, \alpha)$, *the mapping* $\Omega : \mathcal{B} \cup \mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ *possesses a UFP(unique fixed point).*

Proof. Let $\mathfrak{q}_0 \in \mathcal{B}$ & $\vartheta_0 \in \mathcal{F}$. for all $\ell \in \mathbb{N}$, define $\Omega(\mathfrak{q}_\ell) = \mathfrak{q}_{\ell+1}$. Then $(\{\mathfrak{q}_\ell\}, \{\vartheta_\ell\})$ be a bisequence on $(\mathcal{B}, \mathcal{F}, \alpha)$. Consider $\mathcal{M} := \alpha(\mathfrak{q}_0, \vartheta_0)\alpha(\mathfrak{q}_0, \vartheta_1)$. Then, for all positive integer ℓ and $\mathfrak{p}$,

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_\ell) &= \alpha(\Omega(\mathfrak{q}_{\ell-1}), \Omega(\vartheta_{\ell-1})) \\ &\leq (\alpha(\mathfrak{q}_{\ell-1}, \vartheta_{\ell-1}))^\lambda \\ &\vdots \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\lambda^\ell}\end{aligned}$$

and also,

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_{\ell+1}) &= \alpha(\Omega(\mathfrak{q}_{\ell-1}), \Omega(\vartheta_\ell)) \\ &\leq (\alpha(\mathfrak{q}_{\ell-1}, \vartheta_\ell))^\lambda \\ &\vdots \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_1))^{\lambda^\ell}.\end{aligned}$$

$$\begin{aligned}\alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_\ell) &\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+1})\alpha(\mathfrak{q}_\ell, \vartheta_{\ell+1})\alpha(\mathfrak{q}_\ell, \vartheta_\ell) \\ &\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+1})\mathcal{M}^{\lambda^\ell} \\ &\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+2})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+2})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+1})\mathcal{M}^{\lambda^\ell} \\ &\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+2})\mathcal{M}^{(\lambda^{\ell+1}+\lambda^\ell)} \\ &\vdots \\ &\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+\mathfrak{p}})\mathcal{M}^{(\lambda^{\ell+\mathfrak{p}-1}+\cdots+\lambda^{\ell+1}+\lambda^\ell)} \\ &\leq \mathcal{M}^{(\lambda^{\ell+\mathfrak{p}}+\cdots+\lambda^{\ell+1}+\lambda^\ell)} \\ &\leq \mathcal{M}^{\frac{\lambda^\ell}{1-\lambda}}\end{aligned}$$

and similarly $\alpha(\mathfrak{q}_\ell, \vartheta_{\ell+\mathfrak{p}}) \leq \mathcal{M}^{\frac{\lambda^\ell}{1-\lambda}}$. Now,

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) &\leq \alpha(\mathfrak{q}_\ell, \vartheta_{\ell_0})\alpha(\mathfrak{q}_{\ell_0}, \vartheta_{\ell_0})\alpha(\mathfrak{q}_{\ell_0}, \vartheta_\mathfrak{r}) \\ &\leq 3\mathcal{M}^{\frac{\lambda^\ell}{1-\lambda}}.\end{aligned}$$

Therefore, $\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) \to 0(\ell, \mathfrak{r} \to ++\infty)$. Hence $(\{\mathfrak{q}_\ell\}, \{\vartheta_\ell\}))$ is a Cauchy bisequence. Since $(\mathcal{B}, \mathcal{F}, \alpha)$ is complete, $(\{\mathfrak{q}_\ell\}, \{\vartheta_\ell\})$ converges, & biconverges to a point $\mathfrak{x} \in \mathcal{B} \cap \mathcal{F}$,

$$\{\Omega(\vartheta_\ell)\} = \{\vartheta_{\ell+1}\} \to \mathfrak{x} \in \mathcal{B} \cap \mathcal{F}.$$

Since Ω is continuous, it follows that $\Omega(\vartheta_\ell) \to \Omega(\mathfrak{x})$, then $\Omega(\mathfrak{x}) = \mathfrak{x}$. Hence $\mathfrak{x}$ is a FP of Ω. If z is any FP of Ω, then $\Omega(\mathsf{z}) = \mathsf{z} \Rightarrow \mathsf{z} \in \mathcal{B} \cap \mathcal{F}$ &

$$\alpha(\mathfrak{x}, \mathsf{z}) = \alpha(\Omega(\mathfrak{x}), \Omega(\mathsf{z})) \leq (\alpha(\mathfrak{x}, \mathsf{z}))^\lambda$$

where $0 < \lambda < 1$, which implies $\alpha(\mathfrak{x}, \mathsf{z}) = 0$, and so $\mathfrak{x} = \mathsf{z}$.

Example 2. *Let $\mathcal{A} = \mathcal{M}_{\ell\times\ell}(\mathbb{R})$ be a set all real enteries and $\mathcal{Z} = \mathcal{M}_{\ell\times\ell}(\mathbb{R})$ be a set all non negative real enteries.*
Let $\mathcal{B} = \{\mathcal{J}_\ell(\mathbb{R}) : \mathcal{J}_\ell(\mathbb{R})$ is an upper triangular matrices over $\mathbb{R}\}$, $\mathcal{F} = \{\mathcal{L}_\ell(\mathbb{R}) : \mathcal{L}_\ell(\mathbb{R})$ is an upper triangular m and the function $\alpha : \mathcal{B} \times \mathcal{F} \to \mathcal{A}$ is defined as

$$\alpha(\mathcal{U}, \mathcal{V}) = e^{\sum_{\mathsf{i},\mathsf{j}=1}^{\ell} |\varsigma_{\mathsf{ij}} - \mathfrak{m}_{\mathsf{ij}}|}$$

for all $\mathcal{U} = (\varsigma_{\mathsf{ij}})_{\ell\times\ell} \in \mathcal{B}$ and $\mathcal{V} = (\mathfrak{m}_{\mathsf{ij}})_{\ell\times\ell} \in \mathcal{F}$. Then $(\mathcal{B}, \mathcal{F}, \alpha)$ is a complete MCBMS. Also define $\mathcal{T} : (\mathcal{B}, \mathcal{F}, \alpha) \rightrightarrows (\mathcal{B}, \mathcal{F}, \alpha)$ as

$$\mathcal{T}(\mathcal{U}) = \left(\frac{\varsigma_{\mathsf{ij}}}{4}\right)_{\ell\times\ell}$$

for all $\mathcal{U} = (\varsigma_{\mathsf{ij}})_{\ell\times\ell} \in \mathcal{J}_\ell(\mathbb{R}) \cup \mathcal{L}_\ell(\mathbb{R})$. Now,

$$\begin{aligned}
\alpha(\mathcal{T}(\mathcal{U}), \mathcal{T}(\mathcal{V})) &= e^{\frac{1}{4}\sum_{\mathsf{i},\mathsf{j}=1}^{\ell} |\varsigma_{\mathsf{ij}} - \mathfrak{m}_{\mathsf{ij}}|} \\
&\leq e^{\frac{1}{2}\sum_{\mathsf{i},\mathsf{j}=1}^{\ell} |\varsigma_{\mathsf{ij}} - \mathfrak{m}_{\mathsf{ij}}|} \\
&= \left(e^{\sum_{\mathsf{i},\mathsf{j}=1}^{\ell} |\varsigma_{\mathsf{ij}} - \mathfrak{m}_{\mathsf{ij}}|}\right)^{\frac{1}{2}} \\
&= (\alpha(\mathcal{U}, \mathcal{V}))^{\lambda}
\end{aligned}$$

for all $\mathcal{U} = (\varsigma_{\mathsf{ij}})_{\ell\times\ell} \in \mathcal{B}$ and $\mathcal{V} = (\mathfrak{m}_{\mathsf{ij}})_{\ell\times\ell} \in \mathcal{F}$. All the criteria of Theorem 3.1 are satisfied with $\lambda = \frac{1}{2}$ & $\mathcal{T}$ possesses a UFP $(0_{\ell\times\ell}, 0_{\ell\times\ell}) \in \mathcal{J}_\ell(\mathbb{R}) \cup \mathcal{L}_\ell(\mathbb{R})$ where $0_{\ell\times\ell}$ is the null matrix.

Example 3. *Let $\mathcal{A} = \mathbb{R}$, $\mathcal{Z} = \{\mathfrak{q} \in \mathcal{A} | \mathfrak{q} \geq 0\}$. Take $\mathcal{B} = [0, 1]$ & $\mathcal{F} = \{0\} \cup \mathbb{N} - \{1\}$ be equipped with $\alpha(\mathfrak{q}, \vartheta) = e^{|\mathfrak{q} - \vartheta|}$ for all $\mathfrak{q} \in \mathcal{B}$, $\vartheta \in \mathcal{F}$. Then, $(\mathcal{B}, \mathcal{F}, \alpha)$ is a complete MCBMS. Also define $\Omega : \mathcal{B} \cup \mathcal{F} \rightrightarrows \mathcal{B} \cup \mathcal{F}$ as*

$$\Omega(\mathfrak{q}) = \begin{cases} \frac{\mathfrak{q}}{5}, & \text{if } \mathfrak{q} \in (0, 1], \\ 0, & \text{if } \mathfrak{q} \in \{0\} \cup \mathbb{N} - \{1\}, \end{cases}$$

for all $\mathfrak{q} \in \mathcal{B} \cup \mathcal{F}$. Let $\mathfrak{q} \in \mathcal{B}$ and $\vartheta \in \mathcal{F}$, then

$$\begin{aligned}
\alpha(\Omega\mathfrak{q}, \Omega\vartheta) &= e^{|\frac{\mathfrak{q}}{5} - 0|} \\
&\leq e^{\frac{1}{2}|\mathfrak{q} - \vartheta|} \\
&= \left(e^{|\mathfrak{q} - \vartheta|}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, conditions of Theorem 3.1 are satisfied & Ω possesses a UFP $\mathfrak{q} = 0$.

Below we prove a similar result for contravariant maps.

Theorem 3.2. *Let $(\mathcal{B}, \mathcal{F}, \alpha)$ be a complete MCBMS, $\mathcal{Z}$ be a cone with constant $\mathcal{W}$ & given a contravariant contraction function $\Omega : (\mathcal{B}, \mathcal{F}, \alpha) \leftrightarrows (\mathcal{B}, \mathcal{F}, \alpha)$. Then the function $\Omega : \mathcal{B} \cup \mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ has a UFP.*

Proof. Let $\mathfrak{q}_0 \in \mathcal{B}$. for all $\ell \in \mathbb{N}$, define $\Omega(\mathfrak{q}_\ell) = \vartheta_\ell$ and $\Omega(\vartheta_\ell) = \mathfrak{q}_{\ell+1}$. Then $(\{\mathfrak{q}_\ell\}, \{\vartheta_\ell\})$ is a bisequence on $(\mathcal{B}, \mathcal{F}, \alpha)$. Then for all $\ell, \mathfrak{p} \in \mathbb{Z}^+$,

$$\begin{aligned}
\alpha(\mathfrak{q}_\ell, \vartheta_\ell) &= \alpha(\Omega(\vartheta_{\ell-1}), \Omega(\mathfrak{q}_\ell)) \\
&\leq \alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}) \\
&= (\alpha(\Omega(\vartheta_{\ell-1}), \Omega(\mathfrak{q}_{\ell-1})))^\lambda \\
&\leq (\alpha(\mathfrak{q}_{\ell-1}, \vartheta_{\ell-1}))^{\lambda^2} \\
&\vdots \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\lambda^{2\ell}} \\
\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\ell) &= \alpha(\Omega(\vartheta_\ell), \Omega(\mathfrak{q}_\ell)) \\
&\leq (\alpha(\mathfrak{q}_\ell, \vartheta_\ell))^\lambda \\
&\vdots \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\lambda^{2\ell+1}}.
\end{aligned}$$

$$\begin{aligned}
\alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_\ell) &\leq \alpha(\mathfrak{q}_{\ell\mathfrak{p}}, \vartheta_{\ell+1})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+1})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\ell) \\
&\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+1})(\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell+2}+\lambda^{2\ell+1})} \\
&\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+2})\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+2})\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+1}) \\
&(\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell+2}+\lambda^{2\ell+1})} \\
&\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+2})(\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell+4}+\lambda^{2\ell+3}+\lambda^{2\ell+2}+\lambda^{2\ell+1})} \\
&\vdots \\
&\leq \alpha(\mathfrak{q}_{\ell+\mathfrak{p}}, \vartheta_{\ell+\mathfrak{p}-1})(\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell+2\mathfrak{p}-2}+....+\lambda^{2\ell+1})} \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell+2\mathfrak{p}-1}+\lambda^{2\ell+2\mathfrak{p}-2}+\lambda^{2\ell+2\mathfrak{p}-3}+....+\lambda^{2\ell+1})} \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\frac{\lambda^{2\ell+1}}{1-\lambda}}.
\end{aligned}$$

$$\begin{aligned}
\alpha(\mathfrak{q}_\ell, \vartheta_{\ell+\mathfrak{p}}) &\leq \alpha(\mathfrak{q}_\ell, \vartheta_\ell)\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\ell)\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+\mathfrak{p}}) \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell}+\lambda^{2\ell+1})}\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+\mathfrak{p}}) \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell}+\lambda^{2\ell+1})}\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+1})\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+1}) \\
&\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+\mathfrak{p}}) \\
&\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\lambda^{2\ell}+\lambda^{2\ell+1}+\lambda^{2\ell+2}+\lambda^{2\ell+3})}\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+\mathfrak{p}})
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&\leq (\alpha(\mathfrak{q}_0\vartheta_0))^{(\lambda^{2\ell}+\lambda^{2\ell+1}+\cdots+\lambda^{2\ell+2\mathfrak{p}-1})}\alpha(\mathfrak{q}_{\ell+\mathfrak{p}},\vartheta_{\ell+\mathfrak{p}}) \\
&\leq \alpha(\mathfrak{q}_0,\vartheta_0)^{(\lambda^{2\ell}+\lambda^{2\ell+1}+\cdots+\lambda^{2\ell+2\mathfrak{p}-1}+\lambda^{2\ell+2\mathfrak{p}})} \\
&\leq (\alpha(\mathfrak{q}_0,\vartheta_0))^{\frac{\lambda^{2\ell}}{1-\lambda}}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\alpha(\mathfrak{q}_\ell,\vartheta_\mathfrak{r}) &\leq \alpha(\mathfrak{q}_\ell,\vartheta_{\ell_0})\alpha(\mathfrak{q}_{\ell_0},\vartheta_{\ell_0})\alpha(\mathfrak{q}_{\ell_0},\vartheta_\mathfrak{r}) \\
&\leq (\alpha(\mathfrak{q}_0,\vartheta_0))^{3\frac{\lambda^{2\ell}}{1-\lambda}}.
\end{aligned}
$$

Therefore, $\alpha(\mathfrak{q}_\ell,\vartheta_\mathfrak{r}) \to 0(\ell,\mathfrak{r} \to ++\infty)$. Therefore, $(\{\mathfrak{q}_\ell\},\{\vartheta_\ell\})$ is a Cauchy bisequence. Since $(\mathcal{B},\mathcal{F},\alpha)$ is complete, $(\{\mathfrak{q}_\ell\},\{\vartheta_\ell\})$ converges, then $\{\mathfrak{q}_\ell\} \to \mathfrak{x}, \{\vartheta_\ell\} \to \mathfrak{x}$, where $\mathfrak{x} \in \mathcal{B}\cap\mathcal{F}$. Since the contravariant mapping Ω is continuous

$$\{\mathfrak{q}_n\} \to \mathfrak{x},$$

$\Rightarrow$

$$\{\vartheta_\ell\} = \{\Omega(\mathfrak{q}_\ell)\} \to \Omega(\mathfrak{x})$$

and combining this with $\{\vartheta_\ell\} \to \mathfrak{x}$ gives $\Omega(\mathfrak{x}) = \mathfrak{x}$. Suppose z is a FP of Ω, then $\Omega(\mathsf{z}) = \mathsf{z}$ implies $\mathsf{z} \in \mathcal{B}\cap\mathcal{F}$ so that

$$
\begin{aligned}
\alpha(\mathfrak{x},\mathsf{z}) &= \alpha(\Omega(\mathfrak{x}),\Omega(\mathsf{z})) \\
&\leq (\alpha(\mathfrak{x},\mathsf{z}))^\lambda,
\end{aligned}
$$

which gives $\alpha(\mathfrak{x},\mathsf{z}) = 1$. Hence $\mathfrak{x} = \mathsf{z}$.

Example 4. *Let $\mathcal{A} = \mathbb{R}$, $\mathcal{Z} = \{\mathfrak{q} \in \mathcal{A} | \mathfrak{q} \geq 0\}$. Let $\mathcal{B} = \{0,1,2,7\}$ and $\mathcal{F} = \{0,\frac{1}{4},\frac{1}{2},3\}$ be equipped with $\alpha(\mathfrak{q},\vartheta) = e^{|\mathfrak{q}-\vartheta|}$ for all $\mathfrak{q} \in \mathcal{B}$, $\vartheta \in \mathcal{F}$. Then, $(\mathcal{B},\mathcal{F},\alpha)$ is a complete MCBMS. Also define $\Omega : \mathcal{B}\cup\mathcal{F} \rightleftarrows \mathcal{B}\cup\mathcal{F}$ as*

$$
\Omega(\mathfrak{q}) = \begin{cases} \frac{1}{4}, & \text{if } \mathfrak{q} \in \{2,7\}, \\ 0, & \text{if } \mathfrak{q} \in \{0,\frac{1}{4},\frac{1}{2},1,3\}, \end{cases}
$$

for all $\mathfrak{q} \in \mathcal{B}\cup\mathcal{F}$. Let $\mathfrak{q} \in \mathcal{B}$ and $\vartheta \in \mathcal{F}$, then we get

$$\alpha(\Omega\mathfrak{q},\Omega\vartheta) \leq (\alpha(\mathfrak{q},\vartheta))^{\frac{1}{2}}.$$

Therefore, criteria of Theorem 3.2 are verified & Ω possesses a UFP $\mathfrak{q} = 0$.

Finally, we express a theorem based of Kannan's FP result [2].

Theorem 3.3. *Consider* $\Omega : (\mathcal{B}, \mathcal{F}, \alpha) \leftrightarrows (\mathcal{B}, \mathcal{F}, \alpha)$*, where* $(\mathcal{B}, \mathcal{F}, \alpha)$ *is a complete MCBMS,* $\mathcal{Z}$ *be a cone with constant* $\mathcal{W}$ *and let* $\beta \in (0, \frac{1}{2})$ *s.t the inequality*

$$\alpha(\Omega\vartheta, \Omega\mathfrak{q}) \leq (\alpha(\mathfrak{q}, \Omega\mathfrak{q})\alpha(\Omega\vartheta, \vartheta))^{\beta}$$

holds for all $\mathfrak{q} \in \mathcal{B}$ *and* $\vartheta \in \mathcal{F}$*. Then the function* $\Omega : \mathcal{B}\cup\mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ *possesses a UFP.*

Proof. Consider $\mathfrak{q}_0 \in \mathcal{B}$, for each positive integer ℓ, we define $\vartheta_\ell = \Omega\mathfrak{q}_\ell$ & $\mathfrak{q}_{\ell+1} = \Omega\vartheta_\ell$. Then we have

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_\ell) &= \alpha(\Omega\vartheta_{\ell-1}, \Omega\mathfrak{q}_\ell)\\ &\leq (\alpha(\mathfrak{q}_\ell, \Omega\mathfrak{q}_\ell)\alpha(\Omega\vartheta_{\ell-1}, \vartheta_{\ell-1}))^{\beta}\\ &= (\alpha(\mathfrak{q}_\ell, \vartheta_\ell)\alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}))^{\beta}\end{aligned}$$

for all integers $\ell \geq 1$. Then,

$$\alpha(\mathfrak{q}_\ell, \vartheta_\ell) \leq (\alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}))^{\frac{\beta}{1-\beta}},$$

and

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}) &= \alpha(\Omega\vartheta_{\ell-1}, \Omega\mathfrak{q}_{\ell-1})\\ &\leq (\alpha(\mathfrak{q}_{\ell-1}, \Omega\mathfrak{q}_{\ell-1})\alpha(\Omega\vartheta_{\ell-1}, \vartheta_{\ell-1}))^{\beta}\\ &= (\alpha(\mathfrak{q}_{\ell-1}, \vartheta_{\ell-1})\alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}))^{\beta},\end{aligned}$$

so

$$\alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}) \leq (\alpha(\mathfrak{q}_{\ell-1}, \vartheta_{\ell-1}))^{\frac{\beta}{1-\beta}}.$$

Take $\jmath := \frac{\beta}{1-\beta}$, then we have $\jmath \in (0, 1)$ since $\beta \in (0, \frac{1}{2})$.

$$\begin{gathered}\alpha(\mathfrak{q}_\ell, \vartheta_\ell) \leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\jmath^{2\ell}},\\ \alpha(\mathfrak{q}_\ell, \vartheta_{\ell-1}) \leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\jmath^{2\ell-1}}.\end{gathered}$$

For each $\mathfrak{r} > \ell$,

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) &\leq \alpha(\mathfrak{q}_\ell, \vartheta_\ell)\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\ell)\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\mathfrak{r})\\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0)\alpha(\mathfrak{q}_{\ell+1}, \vartheta_\mathfrak{r}))^{(\jmath^{2\ell}+\jmath^{2\ell+1})}\\ &\ \ \vdots\\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\ell}+\jmath^{2\ell+1}+\cdots+\jmath^{2\mathfrak{r}-1}+\jmath^{2\mathfrak{r}})}\\ &= (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\ell}+\cdots+\jmath^{2\mathfrak{r}})}\\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\frac{\jmath^{2\ell}}{1-\jmath}}.\end{aligned}$$

As $\ell, \mathfrak{r} \to ++\infty$, we get

$$\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) \to 1.$$

Consequently, for each $\mathfrak{r} < \ell$

$$\begin{aligned}\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) &\leq \alpha(\mathfrak{q}_{\mathfrak{r}+1}, \vartheta_\mathfrak{r})\alpha(\mathfrak{q}_{\mathfrak{r}+1}, \vartheta_{\mathfrak{r}+1})\alpha(\mathfrak{q}_\ell, \vartheta_{\mathfrak{r}+1}) \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\mathfrak{r}+1}+\jmath^{2\mathfrak{r}+2})}\alpha(\mathfrak{q}_\ell, \vartheta_{\mathfrak{r}+1}) \\ &\vdots \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\mathfrak{r}+1}+\jmath^{2\mathfrak{r}+2}+\cdots+\jmath^{2\ell})}\alpha(\mathfrak{q}_\ell, \vartheta_\ell) \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\mathfrak{r}+1}+\jmath^{2\mathfrak{r}+2}+\cdots+\jmath^{2\ell+1})} \\ &= (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\jmath^{2\mathfrak{r}+1}+\cdots+\jmath^{2\ell+1})} \\ &\leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\frac{\jmath^{2\mathfrak{r}+1}}{1-\jmath}}.\end{aligned}$$

As $\ell, \mathfrak{r} \to ++\infty$, we get

$$\alpha(\mathfrak{q}_\ell, \vartheta_\mathfrak{r}) \to 1.$$

Therefore, $(\{\mathfrak{q}_\ell\}, \{\vartheta_m\})$ is a Cauchy bisequence. Since $(\mathcal{B}, \mathcal{F}, \alpha)$ is complete, $\{\mathfrak{q}_\ell\} \to \mathfrak{x}, \{\vartheta_\mathfrak{r}\} \to \mathfrak{x}$, where $\mathfrak{x} \in \mathcal{B} \cup \mathcal{F}$. Since

$$\{\Omega\mathfrak{q}_\ell\} = \{\vartheta_\ell\} \to \mathfrak{x}.$$

On the other hand,

$$\alpha(\Omega\mathfrak{x}, \Omega\mathfrak{q}_\ell) \leq (\alpha(\mathfrak{q}_\ell, \Omega\mathfrak{q}_\ell)\alpha(\Omega\mathfrak{x}, \mathfrak{x}))^\beta = (\alpha(\mathfrak{q}_\ell, \vartheta_\ell)\alpha(\Omega\mathfrak{x}, \mathfrak{x}))^\beta.$$

As $\ell \to ++\infty$,

$$\alpha(\Omega\mathfrak{x}, \mathfrak{x}) \leq (\alpha(\Omega\mathfrak{x}, \mathfrak{x}))^\beta$$

Therefore, $\alpha(\Omega\mathfrak{x}, \mathfrak{x}) = 1$. Hence $\Omega\mathfrak{x} = \mathfrak{x}$. If z is any FP of Ω, then $\Omega\mathsf{z} = \mathsf{z} \Rightarrow \mathsf{z} \in \mathcal{B} \cap \mathcal{F}$. Then

$$\begin{aligned}\alpha(\mathfrak{x}, \mathsf{z}) = \alpha(\Omega\mathfrak{x}, \Omega\mathsf{z}) &\leq (\alpha(\mathfrak{x}, \Omega\mathfrak{x})\alpha(\Omega\mathsf{z}, \mathsf{z}))^\beta \\ &= (\alpha(\mathfrak{x}, \mathfrak{x})\alpha(v, \mathsf{z}))^\beta = 1.\end{aligned}$$

Consequently $\mathfrak{x} = \mathsf{z}$.

Theorem 3.4. *Let $(\mathcal{B}, \mathcal{F}, \alpha)$ be a complete MCBMS, $\mathcal{Z}$ be a cone with constant $\mathcal{W}$ & $\Omega, \mathcal{S} : (\mathcal{B}, \mathcal{F}, \alpha) \rightleftarrows (\mathcal{B}, \mathcal{F}, \alpha)$ be a contravariant mapping satisfying*

$$\alpha(\mathcal{S}\vartheta, \Omega\mathfrak{q}) \leq \left(\frac{\alpha(\mathfrak{q}, \Omega\mathfrak{q})\alpha(\mathcal{S}\vartheta, \vartheta)}{\alpha(\mathfrak{q}, \vartheta)}\right)^\beta (\alpha(\mathfrak{q}, \vartheta))^\gamma (\alpha(\mathfrak{q}, \Omega\mathfrak{q})\alpha(\mathcal{S}\vartheta, \vartheta))^\imath, \tag{1}$$

for all $(\mathfrak{q},\vartheta) \in \mathcal{B} \times \mathcal{F}$*, with* $\mathfrak{q} \neq \vartheta$*,* $\beta,\gamma,\imath \geq 0$*,* $\gamma < \beta$ *&* $0 \leq \beta+\gamma+2\imath < 1$*. Then* $\Omega, \mathcal{S} : \mathcal{B} \cup \mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ *have a unique CFP.*

Proof. Consider $\mathfrak{q}_0 \in \mathcal{B}$ *&* $\vartheta_0 \in \mathcal{F}$ *then for each* $\ell \in \mathbb{N} \cup \{0\}$*, we define*

$$\mathcal{S}\mathfrak{q}_{2\ell} = \vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell+1} = \vartheta_{2\ell+1}, \mathcal{S}\vartheta_{2\ell} = \mathfrak{q}_{2\ell+1}, \Omega\vartheta_{2\ell+1} = \mathfrak{q}_{2\ell+2}.$$

Now by (1), we get

$$\begin{aligned}
\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1}) =&\alpha(\mathcal{S}\vartheta_{2\ell},\Omega\mathfrak{q}_{2\ell+1})\\
\leq&\left(\frac{\alpha(\mathfrak{q}_{2\ell+1},\Omega\mathfrak{q}_{2\ell+1})\alpha(\mathcal{S}\vartheta_{2\ell+1},\vartheta_{2\ell})}{\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})}\right)^{\beta}(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\gamma}\\
&(\alpha(\mathfrak{q}_{2\ell+1},\Omega\mathfrak{q}_{2\ell+1})+\alpha(\mathcal{S}\vartheta_{2\ell},\vartheta_{2\ell}))^{\imath}\\
=&\left(\frac{\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})}{\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})}\right)^{\beta}(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\gamma}\\
&(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\imath}\\
=&(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1}))^{\beta}(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\gamma}\\
&(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\imath},
\end{aligned}$$

which implies that

$$\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1}) \leq (\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\frac{\gamma+\imath}{1-\beta-\imath}}. \tag{2}$$

Also, we have

$$\begin{aligned}
\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}) =&\alpha(\mathcal{S}\vartheta_{2\ell},\Omega\mathfrak{q}_{2\ell})\\
\leq&\left(\frac{\alpha(\mathfrak{q}_{2\ell},\Omega\mathfrak{q}_{2\ell})\alpha(\mathcal{S}\vartheta_{2\ell},\vartheta_{2\ell})}{\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell})}\right)^{\beta}(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\gamma}\\
&(\alpha(\mathfrak{q}_{2\ell},\Omega\mathfrak{q}_{2\ell})+\alpha(\mathcal{S}\vartheta_{2\ell},\vartheta_{2\ell}))^{\imath}\\
=&\left(\frac{\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})}{\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell})}\right)^{\beta}(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\gamma}\\
&(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\imath}\\
=&(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\beta}(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\gamma}(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\imath}\\
&(\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}))^{\imath},
\end{aligned}$$

which implies that

$$\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}) \leq (\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\frac{\gamma+\imath}{1-\beta-\imath}}. \tag{3}$$

Since $\beta+\gamma+2\imath \in [0,1)$ *and* $\frac{\gamma+\imath}{1-\beta-\imath} = \wp$ *(say),* $\wp \in [0,1)$*. Hence, from (2) and (3), we get*

$$\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1}) \leq (\alpha(\mathfrak{q}_0,\vartheta_0))^{\wp^{4\ell+2}} \quad \text{and} \quad \alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell}) \leq (\alpha(\mathfrak{q}_0,\vartheta_0))^{\wp^{4\ell+1}}. \tag{4}$$

Now, for any $\ell \in \mathbb{N}$,

$$\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+1}) \leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell+2}}, \quad \alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell}) \leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell+1}}$$

and

$$\alpha(\mathfrak{q}_{\ell}, \vartheta_{\ell}) \leq (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell}}.$$

For all $\mathfrak{r}, \ell \in \mathbb{N}$,
Case 1. If $\mathfrak{r} > \ell$,

$$\begin{aligned}
\alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}}) \leq & \alpha(\mathfrak{q}_{\ell}, \vartheta_{\ell})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell})\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\mathfrak{r}}) \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell}}(\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell+1}}\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\mathfrak{r}}) \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\wp^{2\ell}+\wp^{2\ell+1})}\alpha(\mathfrak{q}_{\ell+1}, \vartheta_{\ell+1}) \\
& \alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\ell+1})\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\mathfrak{r}}) \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\wp^{2\ell}+\wp^{2\ell+1})}(\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell+2}} \\
& (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell+3}}\alpha(\mathfrak{q}_{\ell+2}, \vartheta_{\mathfrak{r}}) \\
& \vdots \\
\leq & \alpha(\mathfrak{q}_0, \vartheta_0)^{\wp^{2\ell}(1+\wp+\wp^2+\wp^3+\ldots)} \\
= & (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\ell}(\frac{1}{1-\wp})}.
\end{aligned}$$

Since $\wp < 1$, $\lim_{\ell,\mathfrak{r}\to++\infty} \alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}}) = 1$.
Case 2. If $\mathfrak{r} < \ell$, *we have*

$$\begin{aligned}
\alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}}) \leq & \alpha(\mathfrak{q}_{\mathfrak{r}+1}, \vartheta_{\mathfrak{r}})\alpha(\mathfrak{q}_{\mathfrak{r}+1}, \vartheta_{\mathfrak{r}+1})\alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}+1}) \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\mathfrak{r}+1}}(\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\mathfrak{r}+2}}\alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}+1}) \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\wp^{2\mathfrak{r}+1}+\wp^{2\mathfrak{r}+2})}\alpha(\mathfrak{q}_{\mathfrak{r}+2}, \vartheta_{\mathfrak{r}+1}) \\
& \alpha(\mathfrak{q}_{\mathfrak{r}+2}, \vartheta_{\mathfrak{r}+2})\alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}+2}) \\
& \vdots \\
\leq & (\alpha(\mathfrak{q}_0, \vartheta_0))^{(\wp^{2\mathfrak{r}+1}+\wp^{2\mathfrak{r}+2}+\wp^{2\mathfrak{r}+3}+\wp^{2\mathfrak{r}+4}+\ldots)} \\
= & (\alpha(\mathfrak{q}_0, \vartheta_0))^{\wp^{2\mathfrak{r}+1}(\frac{1}{1-\wp})}.
\end{aligned}$$

Again, since $\wp < 1$, $\lim_{\ell,\mathfrak{r}\to++\infty} \alpha(\mathfrak{q}_{\ell}, \vartheta_{\mathfrak{r}}) = 1$.
Therefore, $(\{\mathfrak{q}_{\ell}\}, \{\vartheta_m\})$ *is a Cauchy bisequence. Since* $(\mathcal{B}, \mathcal{F}, \alpha)$ *is complete,* $\{\mathfrak{q}_{\ell}\} \to \mathfrak{q}^*, \{\vartheta_{\mathfrak{r}}\} \to \mathfrak{q}^*$, *where* $\mathfrak{q}^* \in \mathcal{B} \cup \mathcal{F}$. *Also,* $\{\mathcal{S}(\mathfrak{q}_{2\ell})\} = \{\vartheta_{2\ell}\} \to \mathfrak{q}^* \in \mathcal{B} \cap \mathcal{F} \Rightarrow \mathcal{S}(\mathfrak{q}_{2\ell})$ *has a unique limit* $\mathfrak{q}^*$, *and* $\{\mathfrak{q}_{\ell}\} \to \mathfrak{q}^* \Rightarrow \{\mathfrak{q}_{2\ell}\} \to \mathfrak{q}^*$. *Since* $\mathcal{S}$ *is a continuous,* $\mathcal{S}(\mathfrak{q}_{2\ell}) \to \mathcal{S}\mathfrak{q}^*$. *(i.e)* $\mathcal{S}\mathfrak{q}^* = \mathfrak{q}^*$.
Correspondingly, $\{\Omega(\vartheta_{2\ell+1})\} = \{\mathfrak{q}_{2\ell+2}\} \to \mathfrak{q}^* \in \mathcal{B} \cap \mathcal{F} \Rightarrow \Omega(\vartheta_{2\ell+1})$ *has a unique limit* $\mathfrak{q}^*$, *and* $\{\vartheta_{\ell}\} \to \mathfrak{q}^* \Rightarrow \{\vartheta_{2\ell+1}\} \to \mathfrak{q}^*$. *Now, the stability of* $\Omega \Rightarrow \{\Omega(\vartheta_{2\ell+1})\} \to \Omega\mathfrak{q}^*$. *Therefore,*

$\Omega\mathfrak{q}^* = \mathfrak{q}^*$.
Let $\vartheta^* \in \mathcal{B} \cap \mathcal{F}$ *such that* $\mathcal{S}\vartheta^* = \Omega\vartheta^* = \vartheta^* \in \mathcal{B} \cap \mathcal{F}$. *Then, we get*

$$\begin{aligned}\alpha(\vartheta^*, \mathfrak{q}^*) =& \alpha(\mathcal{S}\vartheta^*, \Omega\mathfrak{q}^*)\\ \leq& \left(\frac{\alpha(\mathfrak{q}^*, \Omega\mathfrak{q}^*)\alpha(\mathcal{S}\vartheta^*, \vartheta^*)}{\alpha(\mathfrak{q}^*, \vartheta^*)}\right)^{\beta} (\alpha(\mathfrak{q}^*, \vartheta^*))^{\gamma} (\alpha(\mathfrak{q}^*, \Omega\mathfrak{q}^*)\alpha(\mathcal{S}\vartheta^*, \vartheta^*))^{\imath}\\ =& \left(\frac{\alpha(\mathfrak{q}^*, \mathfrak{q}^*)\alpha(\vartheta^*, \vartheta^*)}{\alpha(\mathfrak{q}^*, \vartheta^*)}\right)^{\beta} (\alpha(\mathfrak{q}^*, \vartheta^*))^{\gamma} (\alpha(\mathfrak{q}^*, \mathfrak{q}^*)\alpha(\vartheta^*, \vartheta^*))^{\imath}\\ =& (\alpha(\mathfrak{q}^*, \vartheta^*))^{\gamma-\beta}.\end{aligned}$$

Hence, $\mathfrak{q}^* = \vartheta^*$.

Theorem 3.5. *Let* $(\mathcal{B}, \mathcal{F}, \alpha)$ *be a complete MCBMS,* $\mathcal{Z}$ *be a cone with constant* $\mathcal{W}$ *and* $\Omega, \mathcal{S} : (\mathcal{B}, \mathcal{F}, \alpha) \rightleftarrows (\mathcal{B}, \mathcal{F}, \alpha)$ *be a contravariant function satisfying*

$$\alpha(\mathcal{S}\vartheta, \Omega\mathfrak{q}) \leq \left(\frac{\alpha(\mathfrak{q}, \Omega\mathfrak{q})\alpha(\mathfrak{q}, \mathcal{S}\vartheta)\alpha(\mathcal{S}\vartheta, \vartheta)\alpha(\vartheta, \Omega\mathfrak{q})}{\alpha(\mathfrak{q}, \mathcal{S}\vartheta)\alpha(\vartheta, \Omega\mathfrak{q})}\right)^{\beta}, \tag{5}$$

for all $(\mathfrak{q}, \vartheta) \in \mathcal{B} \times \mathcal{F}$, *with* $\mathfrak{q} \neq \vartheta$ *and* $\beta \in (0, \frac{1}{2})$. *Then* $\Omega, \mathcal{S} : \mathcal{B} \cup \mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ *have a unique CFP.*

Proof. Let $\mathfrak{q}_0 \in \mathcal{B}$ *&* $\vartheta_0 \in \mathcal{F}$ *then for each* $\ell \in \mathbb{N} \cup \{0\}$, *define*

$$\mathcal{S}\mathfrak{q}_{2\ell} = \vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell+1} = \vartheta_{2\ell+1}, \mathcal{S}\vartheta_{2\ell} = \mathfrak{q}_{2\ell+1}, \Omega\vartheta_{2\ell+1} = \mathfrak{q}_{2\ell+2}.$$

Now by (5), we get

$$\begin{aligned}\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell+1}) =& \alpha(\mathcal{S}\vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell+1})\\ \leq& \left(\frac{\alpha(\mathfrak{q}_{2\ell+1}, \Omega\mathfrak{q}_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1}, \mathcal{S}\vartheta_{2\ell})\alpha(\mathcal{S}\vartheta_{2\ell}, \vartheta_{2\ell})\alpha(\vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell+1})}{\alpha(\mathfrak{q}_{2\ell+1}, \mathcal{S}\vartheta_{2\ell})\alpha(\vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell+1})}\right)^{\beta}\\ =& \left(\frac{\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1}, \mathfrak{q}_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell})\alpha(\vartheta_{2\ell}, \vartheta_{2\ell+1})}{\alpha(\mathfrak{q}_{2\ell+1}, \mathfrak{q}_{2\ell+1})\alpha(\vartheta_{2\ell}, \vartheta_{2\ell+1})}\right)^{\beta}\\ =& (\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell}))^{\frac{\beta}{1-\beta}},\end{aligned}$$

which implies that

$$\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell+1}) \leq (\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell}))^{\frac{\beta}{1-\beta}}. \tag{6}$$

Also, we have

$$\begin{aligned}\alpha(\mathfrak{q}_{2\ell+1}, \vartheta_{2\ell}) =& \alpha(\mathcal{S}\vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell})\\ \leq& \left(\frac{\alpha(\mathfrak{q}_{2\ell}, \Omega\mathfrak{q}_{2\ell})\alpha(\mathfrak{q}_{2\ell}, \mathcal{S}\vartheta_{2\ell})\alpha(\mathcal{S}\vartheta_{2\ell}, \vartheta_{2\ell})\alpha(\vartheta_{2\ell}, \Omega\mathfrak{q}_{2\ell})}{\alpha(\mathfrak{q}_{2\ell}, \mathcal{S}\vartheta_{2\ell})\alpha(\vartheta_{2\ell}, \mathcal{S}\mathfrak{q}_{2\ell})}\right)^{\beta}\end{aligned}$$

$$
\begin{aligned}
=&\left(\frac{\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell})\alpha(\mathfrak{q}_{2\ell},\mathfrak{q}_{2\ell+1})\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})\alpha(\vartheta_{2\ell},\vartheta_{2\ell})}{\alpha(\mathfrak{q}_{2\ell},\mathfrak{q}_{2\ell+1})\alpha(\vartheta_{2\ell},\vartheta_{2\ell})}\right)^{\beta}\\
=&(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\frac{\beta}{1-\beta}},
\end{aligned}
$$

which implies that

$$
\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})\leq(\alpha(\mathfrak{q}_{2\ell},\vartheta_{2\ell}))^{\frac{\beta}{1-\beta}}. \tag{7}
$$

If we say $\jmath:=\frac{\beta}{1-\beta}$, *then we have* $\jmath\in(0,1)$ *since* $\beta\in(0,\frac{1}{2})$. *Hence, from the previous two inequalities (6) and (7), we get*

$$
\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell+1})\leq(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{4\ell+2}}\quad\text{and}\quad\alpha(\mathfrak{q}_{2\ell+1},\vartheta_{2\ell})\leq(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{4\ell+1}}. \tag{8}
$$

Now, we can get that for any $\ell\in\mathbb{N}$,

$$
\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\ell+1})\leq(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell+2}},\quad\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\ell})\leq(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell+1}}
$$

and

$$
\alpha(\mathfrak{q}_{\ell},\vartheta_{\ell})\leq(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell}}.
$$

For all $\mathfrak{r},\ell\in\mathbb{N}$,
Case 1. If $\mathfrak{r}>\ell$,

$$
\begin{aligned}
\alpha(\mathfrak{q}_{\ell},\vartheta_{\mathfrak{r}})\leq&\alpha(\mathfrak{q}_{\ell},\vartheta_{\ell})\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\ell})\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\mathfrak{r}})\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell}}(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell+1}}\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\mathfrak{r}})\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{(\jmath^{2\ell}+\jmath^{2\ell+1})}\alpha(\mathfrak{q}_{\ell+1},\vartheta_{\ell+1})\\
&\alpha(\mathfrak{q}_{\ell+2},\vartheta_{\ell+1})\alpha(\mathfrak{q}_{\ell+2},\vartheta_{\mathfrak{r}})\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{(\jmath^{2\ell}+\jmath^{2\ell+1})}(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell+2}}\\
&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell+3}}\alpha(\mathfrak{q}_{\ell+2},\vartheta_{\mathfrak{r}})\\
&\vdots\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{(\jmath^{2\ell}+\jmath^{2\ell+1}+\jmath^{2\ell+2}+\jmath^{2\ell+3}+\cdots)}\\
=&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\ell}(\frac{1}{1-\jmath})}.
\end{aligned}
$$

Since $\jmath<1$, $\lim_{\ell,\mathfrak{r}\to++\infty}\alpha(\mathfrak{q}_{\ell},\vartheta_{\mathfrak{r}})=0$.
Case 2. If $\mathfrak{r}<\ell$,

$$
\begin{aligned}
\alpha(\mathfrak{q}_{\ell},\vartheta_{\mathfrak{r}})\leq&\alpha(\mathfrak{q}_{\mathfrak{r}+1},\vartheta_{\mathfrak{r}})\alpha(\mathfrak{q}_{\mathfrak{r}+1},\vartheta_{\mathfrak{r}+1})\alpha(\mathfrak{q}_{\ell},\vartheta_{\mathfrak{r}+1})\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\mathfrak{r}+1}}(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{\jmath^{2\mathfrak{r}+2}}\alpha(\mathfrak{q}_{\ell},\vartheta_{\mathfrak{r}+1})\\
\leq&(\alpha(\mathfrak{q}_{0},\vartheta_{0}))^{(\jmath^{2\mathfrak{r}+1}+\jmath^{2\mathfrak{r}+2})}\alpha(\mathfrak{q}_{\mathfrak{r}+2},\vartheta_{\mathfrak{r}+1})
\end{aligned}
$$

$$\begin{aligned}
&\alpha(\mathfrak{q}_{\mathfrak{r}+2},\vartheta_{\mathfrak{r}+2})\alpha(\mathfrak{q}_\ell,\vartheta_{\mathfrak{r}+2})\\
&\vdots\\
\leq&(\alpha(\mathfrak{q}_0,\vartheta_0))^{(\jmath^{2\mathfrak{r}+1}+\jmath^{2\mathfrak{r}+2}+\jmath^{2\mathfrak{r}+3}+\jmath^{2\mathfrak{r}+4}+\dots)}\\
=&(\alpha(\mathfrak{q}_0,\vartheta_0))^{\jmath^{2\mathfrak{r}+1}(\frac{1}{1-\jmath})}.
\end{aligned}$$

Again, since $\jmath<1$, $\lim_{\ell,\mathfrak{r}\to++\infty}\alpha(\mathfrak{q}_\ell,\vartheta_\mathfrak{r})=0$.
Therefore, $(\{\mathfrak{q}_\ell\},\{\vartheta_m\})$ *is a Cauchy bisequence. Since* $(\mathcal{B},\mathcal{F},\alpha)$ *is complete,* $\{\mathfrak{q}_\ell\}\to\mathfrak{q}^*,\{\vartheta_\mathfrak{r}\}\to\mathfrak{q}^*$, *where* $\mathfrak{q}^*\in\mathcal{B}\cup\mathcal{F}$. *Also,* $\{\mathcal{S}(\mathfrak{q}_{2\ell})\}=\{\vartheta_{2\ell}\}\to\mathfrak{q}^*\in\mathcal{B}\cap\mathcal{F}\Rightarrow\mathcal{S}(\mathfrak{q}_{2\ell})$ *has a unique limit* $\mathfrak{q}^*$, *and* $\{\mathfrak{q}_\ell\}\to\mathfrak{q}^*\Rightarrow\{\mathfrak{q}_{2\ell}\}\to\mathfrak{q}^*$. *Since* $\mathcal{S}$ *is a continuous,* $\{\mathcal{S}(\mathfrak{q}_{2\ell})\}\to\mathcal{S}\mathfrak{q}^*$. *(i.e)* $\mathcal{S}\mathfrak{q}^*=\mathfrak{q}^*$. *Similarly,* $\{\Omega(\vartheta_{2\ell+1})\}=\{\mathfrak{q}_{2\ell+2}\}\to\mathfrak{q}^*\in\mathcal{B}\cap\mathcal{F}\Rightarrow\Omega(\vartheta_{2\ell+1})$ *has a unique limit* $\mathfrak{q}^*$, *and* $\{\vartheta_\ell\}\to\mathfrak{q}^*\Rightarrow\{\vartheta_{2\ell+1}\}\to\mathfrak{q}^*$. *Now,* $\Omega\Rightarrow\{\Omega(\vartheta_{2\ell+1})\}\to\Omega\mathfrak{q}^*$. *Therefore,* $\Omega\mathfrak{q}^*=\mathfrak{q}^*$. *Hence,* Ω & $\mathcal{S}$ *have a CFP. Let* $\vartheta^*\in\mathcal{B}\cap\mathcal{F}$ *s.t* $\mathcal{S}\vartheta^*=\Omega\vartheta^*=\vartheta^*\in\mathcal{B}\cap\mathcal{F}$. *Then, we get*

$$\begin{aligned}
\alpha(\vartheta^*,\varpi^*)=&\alpha(\mathcal{S}\vartheta^*,\Omega\varpi^*)\\
\leq&\left(\frac{\alpha(\varpi^*,\Omega\varpi^*)\alpha(\varpi^*,\mathcal{S}\vartheta^*)\alpha(\mathcal{S}\vartheta^*,\vartheta^*)\alpha(\vartheta^*,\Omega\varpi^*)}{\alpha(\varpi^*,\mathcal{S}\vartheta^*)\alpha(\vartheta^*,\Omega\varpi^*)}\right)^\jmath\\
=&\left(\frac{\alpha(\varpi^*,\varpi^*)\alpha(\varpi^*,\vartheta^*)\alpha(\vartheta^*,\vartheta^*)\alpha(\vartheta^*,\varpi^*)}{\alpha(\varpi^*,\vartheta^*)\alpha(\vartheta^*,\varpi^*)}\right)^\jmath\\
=&1.
\end{aligned}$$

Therefore, $\mathfrak{q}^*=\vartheta^*$.

4. Application

In this section, the presence & the uniqueness of the solution to an integral equations is revealed as an application of Theorem 3.1.

Theorem 4.1. *Let us consider the integral equation*

$$\mathfrak{q}(\varphi)=\mathfrak{b}(\varphi)+\int_{\mathcal{E}_1\mathcal{E}_2}\mathcal{G}(\varphi,\mathfrak{s},\mathfrak{q}(\mathfrak{s}))d\mathfrak{s},\ \varphi\in\mathcal{E}_1\cup\mathcal{E}_2, \tag{9}$$

where $\mathcal{E}_1\cup\mathcal{E}_2$ *is a Lebesgue measurable set. Assume*

(i) $\mathcal{G}:(\mathcal{E}_1^2\cup\mathcal{E}_2^2)\times[0,+\infty)\to[0,+\infty)$ & $b\in L^{+\infty}(\mathcal{E}_1)\cup L^{+\infty}(\mathcal{E}_2)$,

(ii) $\exists$ *continuous function* $\theta:\mathcal{E}_1^2\cup\mathcal{E}_2^2\to[0,+\infty)$ & $\lambda\in(0,1)$ *s.t*

$$|\mathcal{G}(\varphi,\mathfrak{s},\mathfrak{q}(\mathfrak{s}))-\mathcal{G}(\varphi,\mathfrak{s},\vartheta(\mathfrak{s})|\leq\lambda\theta(\varphi,\mathfrak{s})(|\mathfrak{q}(\mathfrak{s})-\vartheta(\mathfrak{s})|,$$

for $\varphi,\mathfrak{s}\in\mathcal{E}_1^2\cup\mathcal{E}_2^2$,

(iii) $\sup_{\varphi\in\mathcal{E}_1\cup\mathcal{E}_2}\int_{\mathcal{E}_1\cup\mathcal{E}_2}\theta(\varphi,\mathfrak{s})d\mathfrak{s}\leq 1$.

Then the Integral Equation 9 has a unique solution in $L^{+\infty}(\mathcal{E}_1) \cup L^{+\infty}(\mathcal{E}_2)$.

Proof. *Let* $\mathcal{A} = \mathbb{R}$, $\mathcal{Z} = \{(\mathfrak{q}, \vartheta) \in \mathcal{A} | \mathfrak{q}, \vartheta \geq 0\}$. *Let* $\mathcal{B} = L^{+\infty}(\mathcal{E}_1)$ *and* $\mathcal{F} = L^{+\infty}(\mathcal{E}_2)$ *be two normed linear spaces, where* $\mathcal{E}_1, \mathcal{E}_2$ *are Lebesgue measurable sets* & $m(\mathcal{E}_1 \cup \mathcal{E}_2) < +\infty$. *Consider* $\alpha : \mathcal{B} \times \mathcal{F} \to \mathcal{A}$ *to be defined by* $\alpha(\mathfrak{q}, \vartheta) = e^{\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} |\mathfrak{q} - \vartheta|}$ *for all* $(\mathfrak{q}, \vartheta) \in \mathcal{B} \times \mathcal{F}$. *Then* $(\mathcal{B}, \mathcal{F}, \alpha)$ *is a complete MCBMS. Also define the covariant function* $\Omega : L^{+\infty}(\mathcal{E}_1) \cup L^{+\infty}(\mathcal{E}_2) \to L^{+\infty}(\mathcal{E}_1) \cup L^{+\infty}(\mathcal{E}_2)$ *by*

$$\Omega(\mathfrak{q}(\varphi)) = \mathfrak{b}(\varphi) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\varphi, \mathfrak{s}, \mathfrak{q}(\mathfrak{s})) d\mathfrak{s}, \ \varphi \in \mathcal{E}_1 \cup \mathcal{E}_2.$$

Now,

$$\begin{aligned}
\alpha(\Omega\mathfrak{q}(\varphi), \Omega\vartheta(\varphi)) &= e^{\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} |\Omega\mathfrak{q}(\varphi) - \Omega\vartheta(\varphi)|} \\
&= e^{\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} \left| \mathfrak{b}(\varphi) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\varphi, \mathfrak{s}, \mathfrak{q}(\mathfrak{s})) d\mathfrak{s} - \left(\mathfrak{b}(\varphi) + \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \mathcal{G}(\varphi, \mathfrak{s}, \mathfrak{q}(\mathfrak{s})) d\mathfrak{s} \right) \right|} \\
&\leq e^{\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} |\mathcal{G}(\varphi, \mathfrak{s}, \mathfrak{q}(\mathfrak{s})) - \mathcal{G}(\varphi, \mathfrak{s}, \vartheta(\mathfrak{s}))| d\mathfrak{s}} \\
&\leq e^{\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \lambda\theta(\varphi, \mathfrak{s})(|\mathfrak{q}(\mathfrak{s}) - \vartheta(\mathfrak{s})|) d\mathfrak{s}} \\
&\leq e^{\lambda(\sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} |\mathfrak{q}(\mathfrak{s}) - \vartheta(\mathfrak{s})| \sup_{\varphi \in \mathcal{E}_1 \cup \mathcal{E}_2} \int_{\mathcal{E}_1 \cup \mathcal{E}_2} \theta(\varphi, \mathfrak{s}) d\mathfrak{s})} \\
&= (\alpha(\mathfrak{q}, \vartheta))^{\lambda}
\end{aligned}$$

Therefore, all the hypothesis of Theorem 3.1 are satisfied & *as a result, the integral equation possesses a unique solution.*

4.1. Application to fractional differential equations

We recall many important definitions from fractional calculus theory [**?**]-[**?**]. The Reiman-Liouville fraction derivative of a function $\mathfrak{s} \in \mathcal{C}[0, 1]$, of order $\delta > 0$ is as follows:

$$\frac{1}{\Gamma(\ell - \delta)} \frac{d^{\ell}}{d\eta^{\ell}} \int_0^{\eta} \frac{\mathfrak{s}(\mathfrak{e}) d\mathfrak{e}}{(\eta - \mathfrak{e})^{\delta - \ell + 1}} = \mathcal{D}^{\delta}\mathfrak{s}(\eta),$$

taking the right hand side as defined point-wise on $[0, 1]$ where, $[\delta]$ denotes the integer part of number δ, Γ is Euler gamma function.
Let us consider the fractional differential equation:

$$\begin{aligned}
&{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{q}}\mathfrak{s}(\eta) + \mathfrak{f}(\eta, \mathfrak{s}(\eta)) = 0, \quad 1 \leq \eta \leq 0, \quad 2 \leq \mathfrak{q} > 1; \\
&\mathfrak{s}(0) = \mathfrak{s}(1) = 0,
\end{aligned} \tag{10}$$

where $\mathfrak{f}$ is a continuous mapping from $[0, 1] \times \mathbb{R}$ to $\mathbb{R}$ and ${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{q}}$ represents the Caputo fractional derivative of order $\mathfrak{q}$ & it is defined as

$${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{q}} = \frac{1}{\Gamma(\ell - \mathfrak{q})} \int_0^{\zeta} \frac{\mathfrak{s}^{\ell}(\mathfrak{e}) d\mathfrak{e}}{(\eta - \mathfrak{e})^{\mathfrak{q} - \ell + 1}}$$

Let $\mathcal{A} = (\mathcal{C}[0,1],[0,+\infty))$, $\mathcal{Z} = \{\mathfrak{g} \in \mathcal{A} | \mathfrak{g}(\eta) \text{ on } [0,1]\}$. Suppose $\mathcal{B} = (\mathcal{C}[0,1],(-+\infty,0])$ is all continuous functions defined on $[0,1]$ that have their values in $(-+,0]$, where as $\mathcal{F} = (\mathcal{C}[0,1],[0,+\infty))$ is all continuous functions defined on $[0,+\infty))$. Consider $\alpha : \mathcal{B} \times \mathcal{F} \to \mathcal{A}$ to be defined as

$$\alpha(\mathfrak{s},\mathfrak{s}^{'}) = e^{\sup_{\eta\in[0,1]} |\mathfrak{s}(\eta)-\mathfrak{s}^{'}(\eta)|}$$

for all $(\mathfrak{s},\mathfrak{s}^{'}) \in \mathcal{B} \times \mathcal{F}$. Then $(\mathcal{B},\mathcal{F},\alpha)$ is a complete MCBMS.

Theorem 4.2. *Consider the nonlinear fractional differential equation* (10). *Suppose that the following conditions are satisfies:*

[label=(ii)]

(i) $\exists\ \eta \in [0,1]$, $\lambda \in (0,1)$ & $(\mathfrak{s},\mathfrak{s}^{'}) \in \mathcal{B} \times \mathcal{F}$ *such that*

$$|\mathfrak{f}(\eta,\mathfrak{s}) - \mathfrak{f}(\eta,\mathfrak{s}^{'})| \leq \lambda|\mathfrak{s}(\eta) - \mathfrak{s}^{'}(\eta)|;$$

(ii)

$$\sup_{\eta\in[0,1]} \int_0^1 |\mathcal{G}(\eta,\mathfrak{e})| d\mathfrak{m} \leq 1.$$

Then the fractional differential equation (10) *possesses a unique solution in* $\mathcal{B} \cup \mathcal{F}$.

Proof. The given fractional differential equation (10) *is equivalent to the succeeding integral equation*

$$\mathfrak{s}(\eta) = \int_0^1 \mathcal{G}(\eta,\mathfrak{e})\mathfrak{f}(\mathfrak{m},\mathfrak{s}(\mathfrak{e}))d\mathfrak{e},$$

where

$$\mathcal{G}(\eta,\mathfrak{e}) = \begin{cases} \frac{[\eta(1-\mathfrak{e})]^{\mathfrak{q}-1}-(\eta-\mathfrak{e})^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}, & 0 \leq \mathfrak{e} \leq \eta \leq 1, \\ \frac{[\eta(1-\mathfrak{e})]^{\mathfrak{q}-1}}{\Gamma(\mathfrak{q})}, & 0 \leq \eta \leq \mathfrak{e} \leq 1. \end{cases}$$

Define the covariant function $\Omega\colon \mathcal{B} \cup \mathcal{F} \to \mathcal{B} \cup \mathcal{F}$ *defined by*

$$\Omega\mathfrak{s}(\eta) = \int_0^1 \mathcal{G}(\eta,\mathfrak{e})\mathfrak{f}(\mathfrak{m},\mathfrak{s}(\mathfrak{e}))d\mathfrak{e}.$$

Now

$$\begin{aligned} |\Omega\mathfrak{s}(\eta) - \Omega\mathfrak{s}^{'}(\eta)| &= \left| \int_0^1 \mathcal{G}(\eta,\mathfrak{e})\mathfrak{f}(\mathfrak{m},\mathfrak{s}(\mathfrak{e}))d\mathfrak{e} - \int_o^1 \mathcal{G}(\eta,\mathfrak{e})\mathfrak{f}(\mathfrak{m},\mathfrak{s}^{'}(\mathfrak{e}))d\mathfrak{e} \right| \\ &\leq \int_0^1 |\mathcal{G}(\eta,\mathfrak{e})|d\mathfrak{e} \cdot \int_0^1 \left| \mathfrak{f}(\mathfrak{m},\mathfrak{s}(\mathfrak{e})) - \mathfrak{f}(\mathfrak{m},\mathfrak{s}^{'}(\mathfrak{e})) \right| d\mathfrak{e} \end{aligned}$$

$$\leq \lambda \big| \mathfrak{s}(\eta) - \mathfrak{s}^{'}(\eta) \big|.$$

Therefore,

$$\alpha(\Omega\mathfrak{s}, \Omega\mathfrak{s}^{'}) = e^{\sup_{\eta\in[0,1]} |\Omega\mathfrak{s}(\eta) - \Omega\mathfrak{s}^{'}(\eta)|} \leq e^{\lambda \sup_{\eta\in[0,1]} |\mathfrak{s}(\eta) - \mathfrak{s}^{'}(\eta)|} = (\alpha(\mathfrak{s}, \mathfrak{s}^{'}))^{\lambda}.$$

Therefore, all the criteria of Theorem 3.1 are satisfied & *as a result, the fractional differential equation* (10) *possesses a unique solution.*

5. Conclusion

In this article, the concept of MCBMS is introduced and analogues of fixed point theorems of Banach and Kannan are proved. The derived results have been supplemented with non trivial example. The practical applicability for finding solution to integral and fractional differential equations is also presented. It will be an open problem to extend the fixed point results using other contractive conditions such as Ciric, Reich, Junck etc. It is also an open problem to examine whether the space can be generalised in the form of Multiplicative Cone Bipoloar b-metric, dislocated MCBS, retangular MCBS etc.

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Conflict of interests

The author declare no conflicts of interest.

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