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Fixed Point Theorems for Mappings Contracting Perimeter of Triangles Embedded with F-Contractions in b-Metric Spaces

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Abstract. In this article, the concept of a mapping contracting perimeter of triangles embedded with F-contractions in the framework of b-metric spaces is introduced. Some related fixed point results are established. Banach Contraction Principle is derived as a corollary of main result. Additionally, we construct examples of mappings contracting perimeters of triangles embedded with F-contractions which are not contraction mappings in the framework of b-metric spaces. The results of this article are the extensions of some already established results in literature.

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Key Words and Phrases: Fixed point (FP), Banach contraction principle (BCP), metric space (MS), b-metric space (b-MS), mapping contracting perimeters of triangle (MCPT)

1. Introduction and Preliminaries

Fixed point (FP) theory is a significant and highly active area within functional analysis. It offers crucial methods for addressing problems encountered across multiple fields of mathematical analysis. This theory plays a key role in ensuring both the existence and uniqueness of solutions to integral and differential equations. For more details, see [1–9]. In 1922, the Polish mathematician Banach [10] introduced the contraction principle, which has become one of the most renowned and influential results in mathematics. In

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the existing literature, the Banach contraction principle (BCP) has been generalized in two main ways: either by modifying the contraction condition, or by altering the structure of the metric space (MS). Within FP theory, numerous types of contractions have been formulated in a MS, including Boyd and Wong nonlinear contraction [11], the Meir-Keeler contraction [12, 13], Suzuki contraction [14], Kannan contraction [15], Ćirić generalized contraction [16] and quasi contraction [17], weak contraction [18], Chatterjea contraction [19], Zamfirescu contraction [20], the F-Suzuki contraction [21], and among others [22, 23]. A MS is a vast concept, and even small modifications to its axioms can lead to the creation of different structures, such as a 2-MS [24], a cone MS [25], and many others. The notion of a b-metric space (b-MS) was pioneered by Bakhtin [26] in 1989, and later refined by Czerwik [27] in 1993. This innovation introduced a new coefficient in the triangular inequality of a MS, laying the groundwork for the development of b-MSs. Researchers have developed a wide range of FP results utilizing the framework of b-MS. Karapinar [28] discusses foundational aspects and key contributions in FP theory within the framework of b-MS. In 2022, Berinde and Păcurar [29] survey the early progress and key issues in FP theory within b-MS. Ma et al. [30] introduced the concept of C^*-algebra-valued contraction mappings. Building on this, Batul et al. [31] generalized the idea by relaxing the contraction condition initially proposed in [30]. In another development, Shehwar et al. [32] extended Caristi FP theorem to mappings defined on C^*-algebra-valued MSs. They demonstrated the existence of FPs by employing the concept of minimal elements within these spaces and introduced a partial order on the set $\mathcal{U}$. Recently, Pasicki [33] explores the characteristics of Cauchy sequences in b-MS, contributing to a deeper understanding of their convergence properties.

In 2012, Wardowski [34] introduced a novel type of contractions, known as an F-contraction, for real-valued functions defined on the set of positive real numbers and satisfying specific conditions. He also established a fixed point theorem for this class of contractions. Since then, numerous researchers have extended and explored F-contraction mappings within various types of MS. Fabiano et al. [35] present an in-depth overview of F-contractions, focusing on their origins, theoretical advancements and applications in generalized MS.

Petrov [36] obtained some FP theorems for "mappings contracting perimeters of triangles" (MCPTs) in the framework of MSs. In this paper, the author [36] introduced a new type of mappings in MSs which can be characterized as a MCPT.

Influenced by the work of Petrov, we bring to light some FP theorems for MCPTs embedded with F-contractions in the framework of b-MSs.

Standard contraction mappings represent a notable subclass within this broader framework, enabling us to recover Banach classical result as a straightforward corollary. Furthermore, we provide examples of mappings that contract the perimeters of triangles embedded with F-contractions in b-MSs, but do not qualify as contraction mappings in the traditional sense.

The following are some definitions and results which are useful for the proof of main theorems.

Definition 1. *[26] Let $\mathcal{U}$ be a nonempty set and let $\mathsf{s} \geq 1$ be a given real number. A*

function $\sigma_{\mathsf{b}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^{+}$ *is called a* b*-metric provided that, for all* $\eta,\ \xi,\ \zeta \in \mathcal{U}$*,*
($\mathsf{M}_{\mathsf{b}_1}$)*:* $\sigma_{\mathsf{b}}(\eta, \xi) \geq 0$,
($\mathsf{M}_{\mathsf{b}_2}$)*:* $\sigma_{\mathsf{b}}(\eta, \xi) = 0$ *if and only if* $\eta = \xi$*,*
($\mathsf{M}_{\mathsf{b}_3}$)*:* $\sigma_{\mathsf{b}}(\eta, \xi) = \sigma_{\mathsf{b}}(\xi, \eta)$*,*
($\mathsf{M}_{\mathsf{b}_4}$)*:* $\sigma_{\mathsf{b}}(\eta, \zeta) \leq \mathsf{s}\ [\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta)]$*.*
The pair $(\mathcal{U},\ \sigma_{\mathsf{b}})$ *is called a* b*-MS.*

In general, a b-metric is not a continuous function. However, throughout the article, we will assume that the b-metric is continuous.

Example 1. *Let* $\mathcal{U} = \mathbb{N}$*. Define* $\sigma_b : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ *by*

$$\sigma_b(\eta, \xi) = \begin{cases} 0, & \text{if } \eta = \xi, \\ 4\alpha, & \text{if } \eta, \xi \in \{1, 2\}, \\ \alpha, & \text{if } \eta \text{ or } \xi \notin \{1, 2\} \text{ and } \eta \neq \xi, \end{cases}$$

where $\alpha > 0$ *is a constant. Here* $(\mathcal{U}, \sigma_b)$ *is a* b*-MS with* $\mathsf{s} = 3$*.*

Definition 2. *[34] Suppose* $\mathsf{F} : \mathbb{R}^{+} \to \mathbb{R}$ *is a function that satisfies the following:*
*(**F**-1):* F *is increasing, i.e., for all* $\eta, \xi \in \mathbb{R}^{+}$ *such that* $\eta < \xi,\ \Rightarrow\ \mathsf{F}(\eta) < \mathsf{F}(\xi)$*.*
*(**F**-2): For any sequence* $\{\eta_n\}_{n=1}^{\infty}$ *of positive real numbers,* $\lim_{n \to +\infty} \eta_n = 0$ *if and only if* $\lim_{n \to +\infty} \mathsf{F}(\eta_n) = -\infty$*.*
*(**F**-3): There exists* $k \in (0, 1)$ *such that* $\lim_{\alpha \to 0^{+}} \alpha^{k} \mathsf{F}(\alpha) = 0$*.*

Definition 3. *[34] Let* $(\mathcal{U}, \sigma)$ *be a MS. A mapping* $\Upsilon : \mathcal{U} \to \mathcal{U}$ *is said to be a Wardowski* F*-contraction if there are* $\mathsf{F} \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\eta, \xi \in \mathcal{U},\ \sigma(\Upsilon\eta, \Upsilon\xi) > 0 \quad \Rightarrow \quad \tau + \mathsf{F}(\sigma(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}(\sigma(\eta, \xi)).$$

In 2015, Cosentine et al. [37] introduced a new condition in Definition 2 to derive certain fixed point results in b-MSs. In this article, we further extend this definition by incorporating an additional condition into Definition 2.
($\mathbf{F} - 4$): Let $\mathsf{s} \geq 1$ be a real number. For each sequence $\{\beta_n\}_{n \in \mathbb{N}}$ of positive real numbers such that

$$\tau + \mathsf{F}(\mathsf{s}^2 \beta_n) \leq \mathsf{F}(\beta_{n-1}) \tag{1}$$

for all $\mathsf{n} \in \mathbb{N}$ and some $\tau > 0$, then

$$\tau + \mathsf{F}(\mathsf{s}^{\mathsf{n}} \beta_n) \leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-2} \beta_{n-1}). \tag{2}$$

Throughout the paper, $\mathcal{F}$ denotes the collection of mappings that satisfy ($\mathbf{F} - 1$) to ($\mathbf{F} - 4$).

2. Main Results

The following section is concerned with the principal results of this paper.

Definition 4. *Consider a* b*-MS* $(\mathcal{U}, \sigma_\mathsf{b})$ *and* $(\mathsf{s} \geq 1)$ *with at least three elements, i.e.,* $|\mathcal{U}| \geq 3$*. A mapping* $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ *is said to be a MCPT embedded with an* F*-contraction on* $\mathcal{U}$ *if there exist* $\mathsf{F} \in \mathcal{F}$ *and* $\tau > 0$ *such that the following inequality*

$$\tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi) + \sigma_\mathsf{b}(\Upsilon\xi, \Upsilon\zeta) + \sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\zeta)) \leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_\mathsf{b}(\eta, \xi) + \sigma_\mathsf{b}(\xi, \zeta) + \sigma_\mathsf{b}(\eta, \zeta))\right), \tag{3}$$

holds for all possible combinations of three pairwise distinct points η, ξ, ζ *in* $\mathcal{U}$*.*

Proposition 1. *Let* $(\mathcal{U}, \sigma_\mathsf{b})$ *be a complete* b*-MS and* $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ *be a MCPT embedded with an* F*-contraction. Then* Υ *is continuous.*

Proof. Suppose that $(\mathcal{U}, \sigma_\mathsf{b})$ is a b-MS with $|\mathcal{U}| \geq 3$, $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is a MCPT embedded with an F-contraction on $\mathcal{U}$ and let η_0 be an isolated point in $\mathcal{U}$. Then, clearly, Υ is continuous at η_0. Let suppose that η_0 be a limit point of $\mathcal{U}$. Now, we show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta) < \epsilon$ whenever $\sigma_\mathsf{b}(\eta_0, \eta) < \delta$.
Since η_0 is a limit point, for every $\delta > 0$ there exists $\xi \in \mathcal{U}$ such that $\sigma_\mathsf{b}(\eta_0, \xi) < \delta$. Using (3), we have

$$\begin{aligned}
\mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta)) &\leq \tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta)), \\
&\leq \tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta) + \sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\xi) + \sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi)) \\
&\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_\mathsf{b}(\eta_0, \eta) + \sigma_\mathsf{b}(\eta_0, \xi) + \sigma_\mathsf{b}(\eta, \xi))\right) \\
&\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_\mathsf{b}(\eta_0, \eta) + \sigma_\mathsf{b}(\eta_0, \xi) + \mathsf{s}(\sigma_\mathsf{b}(\eta_0, \eta) + \sigma_\mathsf{b}(\eta_0, \xi)))\right) \\
&\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})(\sigma_\mathsf{b}(\eta_0, \eta) + \sigma_\mathsf{b}(\eta_0, \xi))\right) \\
&< \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})(\delta + \delta)\right) \\
&= \mathsf{F}\left(2\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})\delta\right).
\end{aligned} \tag{4}$$

Setting $\delta = \dfrac{\epsilon \mathsf{s}^2}{2(1 + \mathsf{s})}$, then equation (4) becomes

$$\mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta)) < \mathsf{F}(\epsilon).$$

Since F in increasing, one has

$$\sigma_\mathsf{b}(\Upsilon\eta_0, \Upsilon\eta)) < \epsilon.$$

Hence, the MCPT embedded with an F-contraction is continuous.

Definition 5. *Consider a mapping Υ on the* b*-MS $\mathcal{U}$. A point $\eta \in \mathcal{U}$ is said to be a periodic point of period* n *if $\Upsilon^{\mathsf{n}}(\eta) = \eta$, where* n *is the least positive integer for which $\Upsilon^{\mathsf{n}}(\eta) = \eta$, such a positive integer* n *is called the prime period of η.*

Theorem 1. *Consider a complete* b*-MS $(\mathcal{U}, \sigma_{\mathsf{b}})$ with at least three elements, i.e., $|\mathcal{U}| \geq 3$. Assume that the mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ satisfies a MCPT embedded with* F*-contraction condition on $\mathcal{U}$. Then the following statements are true:*

i) The mapping Υ has a FP if and only if it does not have periodic points with a prime period 2.

ii) Υ possesses at most two FPs.

Proof. Suppose that no point is periodic with prime period 2 under the mapping Υ. Our objective is to show that Υ has a FP. Let $\eta_0 \in \Upsilon$ and, $\Upsilon\eta_0 = \eta_1, \Upsilon\eta_1 = \eta_2, \cdots, \Upsilon\eta_{\mathsf{n}} = \eta_{\mathsf{n}+1}, \cdots$.
Assume that, for all $i = 0, 1, 2, \cdots$, there are no FP of the mapping Υ among the points η_{i}. Our goal is to demonstrate the distinctness of every point η_{i}. We have $\eta_{\mathsf{i}} \neq \eta_{\mathsf{i}+1} = \Upsilon\eta_{\mathsf{i}}$ because η_{i} is not a FP. We also know that $\eta_{\mathsf{i}+2} = \Upsilon(\Upsilon(\eta_{\mathsf{i}})) \neq \eta_{\mathsf{i}}$ since Υ lacks any periodic points of prime period 2. Moreover, $\eta_{\mathsf{i}+1} \neq \eta_{\mathsf{i}+2} = \Upsilon\eta_{\mathsf{i}+1}$ since $\eta_{\mathsf{i}+1}$ is not a FP. As a result, pairwise distinct points are η_{i}, $\eta_{\mathsf{i}+1}$, and $\eta_{\mathsf{i}+2}$. Furthermore, suppose that

$$\begin{aligned}
\gamma_0 &= \sigma_{\mathsf{b}}(\eta_0, \eta_1) + \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_0), \\
\gamma_1 &= \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1), \\
&\vdots \\
\gamma_{\mathsf{n}} &= \sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{\mathsf{n}+1}) + \sigma_{\mathsf{b}}(\eta_{\mathsf{n}+1}, \eta_{\mathsf{n}+2}) + \sigma_{\mathsf{b}}(\eta_{\mathsf{n}+2}, \eta_{\mathsf{n}}), \\
&\vdots
\end{aligned}$$

Applying the contraction condition to the pairwise distinct points η_{i}, $\eta_{\mathsf{i}+1}$, and $\eta_{\mathsf{i}+2}$, we obtain

$$\begin{aligned}
\mathsf{F}\left(\sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_1, \eta_3)\right) &= \mathsf{F}\left(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta_1) + \sigma_{\mathsf{b}}(\Upsilon\eta_1, \Upsilon\eta_2) + \sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta_2)\right), \\
&\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}((\sigma_{\mathsf{b}}(\eta_0, \eta_1) + \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_0, \eta_2))\right) - \tau, \\
\mathsf{F}(\gamma_1) &\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\gamma_0)\right) - \tau.
\end{aligned}$$

Since F is increasing, we can write above equation as

$$\mathsf{F}(\mathsf{s}^2\gamma_1) \leq \mathsf{F}(\gamma_0) - \tau.$$

Similarly,

$$\begin{aligned}
\mathsf{F}(\mathsf{s}^2\gamma_2) &\leq \mathsf{F}(\gamma_1) - \tau, \\
\mathsf{F}(\mathsf{s}^2\gamma_3) &\leq \mathsf{F}(\gamma_2) - \tau, \\
&\vdots \\
\mathsf{F}(\mathsf{s}^2\gamma_n) &\leq \mathsf{F}(\gamma_{n-1}) - \tau,
\end{aligned}$$

$$\mathsf{F}(\mathsf{s}^2\gamma_{n+1}) \leq \mathsf{F}(\gamma_n) - \tau. \tag{5}$$

Since $\mathsf{s} \geq 1$, one has

$$\gamma_0 > \gamma_1 > \cdots > \gamma_n > \cdots . \tag{6}$$

Assume that $\mathsf{j} \geq 3$ is the smallest natural number such that $\eta_\mathsf{j} = \eta_\mathsf{i}$ for some i satisfying $0 \leq \mathsf{i} < \mathsf{j} - 2$. Then, we have $\eta_{\mathsf{j}+1} = \eta_{\mathsf{i}+1}$ and $\eta_{\mathsf{j}+2} = \eta_{\mathsf{i}+2}$. Consequently, $\gamma_i = \gamma_j$ which contradicts (6), This shows that all η_i's are distinct. Further, we have to show that $\{\eta_\mathsf{n}\}$ is a Cauchy sequence. It is clear that

$$\mathsf{F}(\mathsf{s}^2\gamma_{n+1}) \leq \mathsf{F}(\gamma_n) - \tau.$$

Using $(\mathbf{F} - 2)$,

$$\mathsf{F}(\mathsf{s}^{\mathsf{n}+2}\gamma_{n+1}) \leq \mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_n) - \tau. \tag{7}$$

It follows by induction that

$$\begin{aligned}\mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_n) &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-2}\gamma_{n-1}) - \tau,\\ &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-4}\gamma_{n-2}) - 2\tau,\\ &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-6}\gamma_{n-3}) - 3\tau.\end{aligned}$$

By continuing this process, one can obtain

$$\mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_n) \leq \mathsf{F}(\gamma_0) - \mathsf{n}\tau. \tag{8}$$

Taking limit as $\mathsf{n} \to +\infty$ in (8) to obtain

$$\lim_{\mathsf{n}\to+\infty} \mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_\mathsf{n}) \to - + \infty$$

which together with $(\mathbf{F} - 2)$ yield that

$$\lim_{\mathsf{n}\to+\infty} \mathsf{s}^\mathsf{n}\gamma_\mathsf{n} = 0.$$

According to $(\mathbf{F} - 3)$, there is $\mathsf{k} \in (0,1)$ such that

$$\lim_{\mathsf{n}\to+\infty} (\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_n) = 0.$$

Multiplying (8) by $(\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}$ leads to

$$0 \leq (\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{F}(\mathsf{s}^\mathsf{n}\gamma_\mathsf{n}) + (\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{n}\tau \leq (\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{F}(\gamma_0).$$

Taking limit as $\mathsf{n} \to +\infty$, we get

$$\lim_{\mathsf{n}\to+\infty} (\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{n} = 0.$$

As a result, it can be concluded that there is $\mathsf{n}_1 \in \mathbb{N}$ such that

$$(\mathsf{s}^\mathsf{n}\gamma_n)^\mathsf{k}\mathsf{n} \leq 1, \qquad forall \ \ \mathsf{n} \geq \mathsf{n}_1.$$

Therefore,

$$(\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}} \leq \frac{1}{\mathsf{n}}, \qquad forall \ \ \mathsf{n} \geq \mathsf{n}_1.$$

It implies

$$\mathsf{s}^{\mathsf{n}}\gamma_n \leq \frac{1}{\mathsf{n}^{\frac{1}{\mathsf{k}}}}, \qquad forall \ \ \mathsf{n} \geq \mathsf{n}_1. \tag{9}$$

This implies that the series $\sum_{i=1}^{+\infty} \mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}$ converges. Now, by the triangular inequality, for all $\mathsf{n}, p \in \mathbb{N}$,

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+p}) \leq \mathsf{s}(\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+1}) + \sigma_{\mathsf{b}}(\eta_{n+1}, \eta_{n+p})).$$

That is,

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+p}) \leq \mathsf{s}\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+1}) + \mathsf{s}^2(\sigma_{\mathsf{b}}(\eta_{n+1}, \eta_{n+2}) + \sigma_{\mathsf{b}}(\eta_{n+2}, \eta_{n+p})).$$

Continuing in this way,

$$\sigma_{\mathsf{b}}(\eta_n, \eta_{n+p}) \leq \mathsf{s}\sigma_{\mathsf{b}}(\eta_n, \eta_{n+1}) + \mathsf{s}^2\sigma_{\mathsf{b}}(\eta_{n+1}, \eta_{n+2}) + \mathsf{s}^3\sigma_{\mathsf{b}}(\eta_{n+2}, \eta_{n+3}) + \cdots + \mathsf{s}^{\mathsf{p}}\sigma_{\mathsf{b}}(\eta_{n+p-1}, \eta_{n+p}). \tag{10}$$

Putting $\mathsf{n} = 0$ in (7),

$$\mathsf{F}(\mathsf{s}^2\gamma_1) \leq \mathsf{F}(\gamma_0) - \tau. \tag{11}$$

As $\gamma_1 = \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1)$, (11) becomes,

$$\mathsf{F}(\mathsf{s}^2(\sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1))) \leq \mathsf{F}(\gamma_0) - \tau.$$

That is,

$$\mathsf{F}(\mathsf{s}^2(\sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1))) \leq \mathsf{F}(\gamma_0).$$

Since F is increasing and $\mathsf{s} \geq 1$,

$$\sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1)) \leq \gamma_0.$$

Thus,

$$\sigma_{\mathsf{b}}(\eta_1, \eta_2) \leq \gamma_0.$$

Continuing the same process, we obtain

$$\begin{aligned}
\sigma_{\mathsf{b}}(\eta_2, \eta_3) &\leq \gamma_1, \\
\sigma_{\mathsf{b}}(\eta_3, \eta_4) &\leq \gamma_2, \\
&\vdots \\
\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{\mathsf{n}+1}) &\leq \gamma_{n-1}, \\
\sigma_{\mathsf{b}}(\eta_{\mathsf{n}+1}, \eta_{\mathsf{n}+2}) &\leq \gamma_n, \\
&\vdots
\end{aligned}$$

Substituting in (10),

$$\begin{aligned}\sigma_{\mathsf{b}}(\eta_n,\eta_{n+p}) &\leq \mathsf{s}\gamma_{n-1}+\mathsf{s}^2\gamma_n+\mathsf{s}^3\gamma_{n+1}+\cdots+\mathsf{s}^{\mathsf{p}}\gamma_{n+p-2},\\ &\leq \mathsf{s}(\gamma_{n-1}+\mathsf{s}\gamma_n+\mathsf{s}^2\gamma_{n+1}+\cdots+\mathsf{s}^{p-1}\gamma_{n+p-2}),\\ &\leq \frac{\mathsf{s}}{\mathsf{s}^{n-1}}(\mathsf{s}^{n-1}\gamma_{n-1}+\mathsf{s}^{n}\gamma_n+\mathsf{s}^{n+1}\gamma_{n+1}+\cdots+\mathsf{s}^{n+p-2}\gamma_{n+p-2}),\\ &\leq \frac{1}{\mathsf{s}^{n-2}}\left(\sum_{i=n-1}^{n+p-2}\mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}\right),\\ &\leq \frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{+\infty}\mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}\right).\end{aligned}$$

Therefore, for all $\mathsf{n}\geq\mathsf{n}_1$ and $p\in\mathbb{N}$, (9) implies that

$$\sigma_{\mathsf{b}}(\eta_n,\eta_{n+p})\leq\frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{+\infty}\mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}\right)\leq\frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{+\infty}\frac{1}{\mathsf{i}^{\frac{1}{k}}}\right).$$

Taking limit as $\mathsf{n}\longrightarrow+\infty$,

$$\sigma_{\mathsf{b}}(\eta_n,\eta_{n+p})\longrightarrow 0.$$

It follows that $\{\eta_n\}$ is a Cauchy sequence in $\mathcal{U}$. As $(\mathcal{U},\sigma_{\mathsf{b}})$ is complete, $\{\eta_n\}$ has a limit η^* in $\mathcal{U}$. To show that $\Upsilon\eta^*=\eta^*$, we apply the triangular inequality and inequality (3). For this,

$$\begin{aligned}\sigma_{\mathsf{b}}(\eta^*,\Upsilon\eta^*) &\leq \mathsf{s}(\sigma_{\mathsf{b}}(\eta^*,\eta_n)+\sigma_{\mathsf{b}}(\eta_n,\Upsilon\eta^*)),\\ &= \mathsf{s}\sigma_{\mathsf{b}}(\eta^*,\eta_n)+\mathsf{s}\sigma_{\mathsf{b}}(\Upsilon\eta_{n-1},\Upsilon\eta^*),\\ &\leq \mathsf{s}\sigma_{\mathsf{b}}(\eta^*,\eta_n)+\mathsf{s}(\sigma_{\mathsf{b}}(\Upsilon\eta_{n-1},\Upsilon\eta^*)+\sigma_{\mathsf{b}}(\Upsilon\eta_{n-1},\Upsilon\eta_n)+\sigma_{\mathsf{b}}(\Upsilon\eta_n,\Upsilon\eta^*)),\\ &\leq \mathsf{s}\sigma_{\mathsf{b}}(\eta^*,\eta_n)+\mathsf{s}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_{n-1},\eta^*)+\sigma_{\mathsf{b}}(\eta_{n-1},\eta_n)+\sigma_{\mathsf{b}}(\eta_n,\eta^*))),\\ &\leq \mathsf{s}\sigma_{\mathsf{b}}(\eta^*,\eta_n)+\mathsf{s}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_{n-1},\eta^*)+\sigma_{\mathsf{b}}(\eta_{n-1},\eta_n)+\sigma_{\mathsf{b}}(\eta_n,\eta^*)),)\\ &\leq \mathsf{s}\sigma_{\mathsf{b}}(\eta^*,\eta_n)+(\frac{1}{\mathsf{s}}(\sigma_{\mathsf{b}}(\eta_{n-1},\eta^*)+\sigma_{\mathsf{b}}(\eta_{n-1},\eta_n)+\sigma_{\mathsf{b}}(\eta_n,\eta^*))).\end{aligned}$$

Taking the limit as $n\to+\infty$, we note that each term in the preceding sum vanishes, and so

$$\sigma_{\mathsf{b}}(\eta^*,\Upsilon\eta^*)=0.$$

Therefore, we conclude that $\Upsilon\eta^*=\eta^*$. In order to prove there exists at most two FPs. Assume by contradiction that Υ has at least three pairwise distinct FPs, say η, ξ, and ζ. That is, $\Upsilon\eta=\eta$, $\Upsilon\xi=\xi$ and $\Upsilon\zeta=\zeta$.
Then by contraction condition,

$$\begin{aligned}\tau+\mathsf{F}\left(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta)\right) &= \tau+\mathsf{F}\left(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta)\right),\\ &\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta))\right).\end{aligned}$$

Since F is increasing,

$$\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta)\leq\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta)),$$

which is a contradiction, since $\mathsf{s}^2\geq 1$. Thus, we conclude that Υ possesses at most two FPs. Conversely, Suppose that Υ possesses a FP η^*. We have to prove that there are no periodic point in Υ with a prime period 2. For this, suppose by contradiction that Υ has a periodic point η of prime period 2, that is, $\Upsilon(\Upsilon\eta)=\eta$. Define $\xi=\Upsilon\eta$ and $\eta=\Upsilon\xi$. Then

$$\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\eta^*)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\eta^*))=\mathsf{F}(\sigma_{\mathsf{b}}(\xi,\eta)+\sigma_{\mathsf{b}}(\eta,\eta^*)+\sigma_{\mathsf{b}}(\xi,\eta^*)),$$

which contradicts (3). Thus, Υ does not have periodic points with a prime period 2.

Remark 1. *In the assumption of Theorem 1, if we add an extra condition that the FP η^* is the limit of some iterative sequence, then Υ has a unique FP. Now, we show that $\eta_{\mathsf{n}}\neq\eta^* for all \mathsf{n}=1,2,\cdots$. For this, let η_0 be an initial point, and iterative sequence will be $\eta_1=\Upsilon\eta_0,\eta_2=\Upsilon\eta_1,\cdots$. In that case, there is only one FP η^*. Assume in fact that Υ has another FP x^{**}. For every $\mathsf{n}=1,2,\cdots$, it is evident that $\mathsf{x}_{\mathsf{n}}\neq\eta^{**}$. Thus, for every $\mathsf{n}=1,2,\cdots$, we have that the points η^*, η^{**}, and η_{n} are pairwise distinct. Now, by contraction condition,*

$$\begin{aligned}\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\eta^*,\eta^{**})+\sigma_{\mathsf{b}}(\eta^*,\eta_{\mathsf{n}+1})+\sigma_{\mathsf{b}}(\eta^{**},\eta_{\mathsf{n}+1}))&=\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta^*,\Upsilon\eta^{**})+\sigma_{\mathsf{b}}(\Upsilon\eta^*,\Upsilon\eta_{\mathsf{n}})+\sigma_{\mathsf{b}}(\Upsilon\eta^{**},\Upsilon\eta_{\mathsf{n}}))\\&\leq\mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta^*,\eta^{**})+\sigma_{\mathsf{b}}(\eta^*,\eta_{\mathsf{n}})+\sigma_{\mathsf{b}}(\eta^{**},\eta_{\mathsf{n}}))\right).\end{aligned}$$

As $\mathsf{n}\longrightarrow 0$, we get $\sigma_{\mathsf{b}}(\eta^*,\eta_{\mathsf{n}+1})\longrightarrow 0$, $\sigma_{\mathsf{b}}(\eta^*,\eta_{\mathsf{n}})\longrightarrow 0$, $\sigma_{\mathsf{b}}(\eta^{**},\eta_{\mathsf{n}+1})\longrightarrow\sigma_{\mathsf{b}}(\mathsf{x}^{**},\mathsf{x}^*)$ and $\sigma_{\mathsf{b}}(\eta^{**},\eta_{\mathsf{n}})\longrightarrow\sigma_{\mathsf{b}}(\eta^{**},\eta^*)$. Hence,

$$\tau+\mathsf{F}(2\sigma_{\mathsf{b}}(\eta^*,\eta^{**}))\leq\mathsf{F}(\frac{1}{\mathsf{s}^2}2\sigma_{\mathsf{b}}(\eta^*,\eta^{**})),$$

implies

$$\mathsf{F}(2\sigma_{\mathsf{b}}(\eta^*,\eta^{**}))\leq\mathsf{F}(\frac{2}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta^*,\eta^{**})).$$

Since F is increasing,

$$2\sigma_{\mathsf{b}}(\eta^*,\eta^{**})\leq\frac{2}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta^*,\eta^{**}).$$

which is a contradiction as $\mathsf{s}\geq 1$. This shows that $\eta^*=\eta^{**}$. Therefore, Υ has a unique FP.

The followings are examples of a MCPT embedded with F-contraction with exactly two FPs.

Example 2. *Let $\mathcal{U}=\{2,3,10\}$. Let $\sigma_{\mathsf{b}}:\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}$ be defined as:*

$$\sigma_{\mathsf{b}}(\eta,\xi)=(\eta-\xi)^2,\qquad \text{for all}\quad \eta,\xi\in\mathcal{U}.$$

Then $(\mathcal{U}, \sigma_{\mathsf{b}})$ *is a* b-*MS with* $\mathsf{s} = 2$. *Define* $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ *by*

$$\Upsilon\eta = \eta, \Upsilon\xi = \xi \text{ and } \Upsilon\zeta = \eta.$$

Consider

$$\begin{aligned}\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) &= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \eta)),\\ &= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \eta) + \sigma_{\mathsf{b}}(\eta, \eta)),\\ &= \tau + \mathsf{F}(2\sigma_{\mathsf{b}}(\eta, \xi)),\\ &= \tau + \mathsf{F}(2(\eta - \xi)^2),\\ &= \tau + \mathsf{F}(2(2 - 3)^2),\end{aligned}$$

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) = \tau + \mathsf{F}(2). \tag{12}$$

Now,

$$\begin{aligned}\mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right) =&\mathsf{F}\left(\frac{1}{\mathsf{s}^2}((\eta - \xi)^2 + (\xi - \zeta)^2 + (\eta - \zeta)^2)\right),\\ =&\mathsf{F}\left(\frac{1}{2^2}((2 - 3)^2 + (3 - 10)^2 + (2 - 10)^2\right).\end{aligned}$$

By simplifying, one can get

$$\mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right) = \mathsf{F}(28.5). \tag{13}$$

Hence, by (12) and (13), we conclude that

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) \le \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right),$$

with $\mathsf{F}(\eta) = \ln(\eta)$ *and* $\tau = 1$. *Also,* Υ *has no periodic point of prime period 2. Take*

$$\begin{aligned}\Upsilon\zeta &= \eta,\\ \Upsilon(\Upsilon\zeta) &= \Upsilon(\eta),\\ \Upsilon^2(\zeta) &= \eta.\end{aligned}$$

Hence, Υ *has no periodic point with prime period 2. Therefore, all the assumptions of Theorem (1) are true. Thus,* Υ *has exactly two FPs, namely* η *and* ξ.

Note: In the previous example, we verified the conditions of Theorem (1) using a continuous b-MS. Now, in the next example, we will use a discontinuous b-MS to verify Theorem 1.

Example 3. *Let $\mathcal{U} = \{2, 3, 10\}$. Let $\sigma_{\mathsf{b}} : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}$ be defined for all $\eta, \xi \in \mathcal{U}$ as follows:*

$$\sigma_{\mathsf{b}} = \begin{cases} 0 & \text{if } \eta = \xi, \\ 1 & \text{if } |\eta - \xi| = 1, \\ 5 & \text{if } |\eta - \xi| > 1. \end{cases}$$

Then, one can easily verify that $(\mathcal{U}, \sigma_{\mathsf{b}})$ is a b*-MS with $\mathsf{s} = 2$. Define $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ by*

$$\Upsilon\eta = \eta, \Upsilon\xi = \xi \text{ and } \Upsilon\zeta = \eta.$$

Consider

$$\begin{aligned} \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) &= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \eta)), \\ &= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \eta) + \sigma_{\mathsf{b}}(\eta, \eta)), \\ &= \tau + \mathsf{F}(2\sigma_{\mathsf{b}}(\eta, \xi)), \\ &= \tau + \mathsf{F}(2(1)), \end{aligned}$$

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) = \tau + \mathsf{F}(2). \tag{14}$$

Now,

$$\begin{aligned} \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right) &= \mathsf{F}\left(\frac{1}{2^2}((1) + (5) + (5)\right), \\ &= \mathsf{F}\left(\frac{11}{4}\right). \end{aligned}$$

By simplifying, one can get

$$\mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right) = \mathsf{F}(2.75). \tag{15}$$

Hence, by (14) and (15), we conclude that

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) \leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right),$$

with $\mathsf{F}(\eta) = \ln(\eta)$ and $\tau = 0.01$. Also, Υ has no periodic point of prime period 2. Take

$$\begin{aligned} \Upsilon\zeta &= \eta, \\ \Upsilon(\Upsilon\zeta) &= \Upsilon(\eta), \\ \Upsilon^2(\zeta) &= \eta. \end{aligned}$$

Hence, Υ has no periodic point with prime period 2. Therefore, all the assumptions of Theorem (1) are true. Thus, Υ has exactly two FPs, namely η and ξ.

In next example, we prove that if Υ has periodic points of prime period 2, then Υ has no FP.

Example 4. *Let* $\mathcal{U} = \{\eta, \xi, \zeta\}$. *Define* $\sigma_{\mathsf{b}} : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}$ *as*

$$\sigma_{\mathsf{b}}(\eta, \xi) = (\eta - \xi)^2, \qquad \text{for all} \quad \eta, \xi \in \mathcal{U}.$$

Then one can prove that $(\mathcal{U}, \sigma_{\mathsf{b}})$ *is a* b*-MS with* s= *2. Now, define* $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ *by*

$$\Upsilon\eta = \xi, \Upsilon\xi = \eta \text{ and } \Upsilon\zeta = \eta.$$

Then Υ *has no FP. Here,* η *and* ξ *are periodic points with prime period 2. Consider*

$$\begin{aligned} \Upsilon\eta &= \xi, \\ \Upsilon(\Upsilon\eta) &= \Upsilon(\xi), \\ \Upsilon^2(\eta) &= \eta. \end{aligned}$$

Also,

$$\begin{aligned} \Upsilon\xi &= \eta, \\ \Upsilon(\Upsilon\xi) &= \Upsilon(\eta), \\ \Upsilon^2(\xi) &= \xi. \end{aligned}$$

Definition 6. *Let* $(\mathcal{U}, \sigma_{\mathsf{b}})$ *be a* b*-MS. Then a* F*-mapping* $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ *is called an* F*-contraction mapping on* $\mathcal{U}$ *if there exist a positive real number* $\mathsf{s} \geq 1$ *and* $\tau > 0$ *such that*

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)\right), \qquad \text{where } \mathsf{F} \in \mathcal{F} \text{ and for all} \quad \eta, \xi \in \mathcal{U}. \tag{16}$$

The following corollary provides a simple and direct proof of Banach FP theorem, in the framework of a b-MS.

Corollary 1. *Suppose that* $(\mathcal{U}, \sigma_{\mathsf{b}})$ *is a complete* b*-MS, where* $\mathcal{U} \neq \emptyset$. *Then the* F*-contraction mapping* $\Upsilon : \mathcal{U} \to \mathcal{U}$ *guarantees that* Υ *has a unique FP.*

Proof. Suppose that $\mathcal{U}$ is a complete b-MS and $|\mathcal{U}| = 1$. Let $\mathcal{U} = \{u\}$. In this case, since $\mathcal{U}$ has only one element η, the mapping Υ must map η to itself. That is, $\Upsilon\eta = \eta$. This is because there are no other element in $\mathcal{U}$ for $\Upsilon\eta$ to map. So, we can see that η is indeed a FP of Υ, and it is unique. Therefore, the Banach FP theorem holds trivially for a set $\mathcal{U}$ of order 1.
Now, if $|\mathcal{U}| = 2$, suppose that $\mathcal{U} = \{u, v\}$ and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is an F-contraction mapping. Assume, if possible, that Υ has two distinct FPs η and ξ. Then, $\Upsilon\eta = \eta$ and $\Upsilon\xi = \xi$. By the definition of an F-contraction mapping, we have

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi))$$

That is,

$$\mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi)) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)) - \tau.$$

Hence,

$$\mathsf{F}(\sigma_{\mathsf{b}}(\eta,\xi)) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi)).$$

F is increasing, so

$$\sigma_{\mathsf{b}}(\eta,\xi) \leq \frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi).$$

This is a contradiction since $\mathsf{s} \geq 1$. Therefore, our assumption that Υ has two distinct FPs is false. Hence, Υ can have at most one FP. So, for $|\mathcal{U}| = 1$, 2 the proof is complete. Assume $\mathcal{U}$ has at least three elements, i.e., $|\mathcal{U}| \geq 3$, if Υ has some $\eta \in \mathcal{U}$ with prime period 2, i.e., $\Upsilon(\Upsilon(\eta)) = \eta$, then

$$\sigma_{\mathsf{b}}(\eta,\Upsilon\eta) = \sigma_{\mathsf{b}}(\Upsilon\eta,\eta) = \sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon(\Upsilon\eta)).$$

which is contradiction with (16). It follows that Υ has no periodic point with prime period 2. Considering pairwise distinct elements η, ξ, $\zeta \in \mathcal{U}$, and applying (16), we have

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\xi))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi)).$$

Let $\mathsf{F}(\eta) = \ln(\eta)$, so

$$\tau + \ln(\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\xi))) \leq \ln(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi)).$$

That is,

$$e^{\tau}\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\xi)) \leq \frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi).$$

That is,

$$\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\xi)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi). \tag{17}$$

Similarly, one can get

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\xi),\Upsilon(\zeta))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\xi,\zeta))$$

implying

$$\sigma_{\mathsf{b}}(\Upsilon(\xi),\Upsilon(\zeta)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\xi,\zeta). \tag{18}$$

and

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\zeta))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\zeta)).$$

Thus,

$$\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\zeta)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\zeta). \tag{19}$$

Adding (17), (18) and (19), one has

$$\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\xi))+\sigma_{\mathsf{b}}(\Upsilon(\xi),\Upsilon(\zeta))+\sigma_{\mathsf{b}}(\Upsilon(\eta),\Upsilon(\zeta))\le\frac{e^{-\tau}}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta)),$$

with $\alpha=e^{-\tau}$. Hence, Υ is a MCPT embedded with an F-contraction on $\mathcal{U}$. By Theorem 1, a FP exists for the mapping Υ. For the uniqueness, suppose that Υ has two FPs η and η^*, i.e, $\Upsilon\eta=\eta$ and $\Upsilon\eta^*=\eta^*$. Now, by the definition of a b-metric and the given assumption, one writes

$$\begin{aligned}0<\mathsf{F}(\sigma_{\mathsf{b}}(\eta,\eta^*))&=\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\eta^*)),\\&<\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\eta^*)),\\\mathsf{F}(\sigma_{\mathsf{b}}(\eta,\eta^*))&\le\mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\eta^*)),\end{aligned}$$

since F is increasing,

$$\sigma_{\mathsf{b}}(\eta,\eta^*)\le(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\eta^*)),$$

where $\mathsf{s}\ge 1$, which is only possible when

$$\sigma_{\mathsf{b}}(\eta,\eta^*)=0.$$

Thus,

$$\eta=\eta^*.$$

Hence, Υ has a unique FP.

Proposition 2. *Consider a b-MS $(\mathcal{U},\sigma_{\mathsf{b}})$ with at least three elements, i.e, $|\mathcal{U}|\ge 3$, and $\Upsilon:\mathcal{U}\to\mathcal{U}$ is a MCPT embedded with an F-contraction. Then, for all points $\xi\in\mathcal{U}$, Υ is an F-contraction mapping if η is a limit point of $\mathcal{U}$.*

Proof. Consider an accumulation point $\eta\in\mathcal{U}$ and any point $\xi\in\mathcal{U}$. If $\xi=\eta$, then (16) is obviously satisfied. Now, consider the case where $\xi\ne\eta$. As η is a limit point, which implies the existence of a sequence $\{\zeta_{\mathsf{n}}\}$ converging to η, satisfying $\zeta_{\mathsf{n}}\ne\mathsf{x}$, $\zeta_{\mathsf{n}}\ne\xi$ with all distinct elements ζ_{n}. Consequently, applying (3) establishes the following

$$\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta_{\mathsf{n}})+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta_{\mathsf{n}}))\le\mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta_{\mathsf{n}})+\sigma_{\mathsf{b}}(\eta,\zeta_{\mathsf{n}}))),\tag{20}$$

which is satisfied for all $\mathsf{n}\in\mathbb{N}$. As $\sigma_{\mathsf{b}}(\eta,\zeta_{\mathsf{n}})\longrightarrow 0$, $\zeta_{\mathsf{n}}\longrightarrow\mathsf{x}$ and the continuity of b-MS implies $\sigma_{\mathsf{b}}(\xi,\zeta_{\mathsf{n}})\to\sigma_{\mathsf{b}}(\eta,\xi)$. By continuity of Υ, $\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta_{\mathsf{n}})\to\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\eta)=0$ and $\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta_{\mathsf{n}})\to\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)$. Taking limit $\mathsf{n}\to+\infty$ in (20) gives

$$\begin{aligned}\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi))&\le\mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\eta,\xi))),\\\tau+\mathsf{F}(2\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi))&\le\mathsf{F}(\frac{2}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi)),\\\tau+\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi))&\le\mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta,\xi)).\end{aligned}$$

Hence, if η is a limit point of $\mathcal{U}$, then Υ is an F-contraction.

Corollary 2. *Consider $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is a MCPT embedded with an* F*-contraction, and* $(\mathcal{U}, \sigma_{\mathsf{b}})$ *is a* b*-MS with at least three points, i.e,* $|U| \geq 3$. *Then Υ is an* F*-contracting mapping whenever every element of $\mathcal{U}$ is an accumulation point of $\mathcal{U}$.*
In a MS $(\mathcal{U}, \sigma_{\mathsf{b}})$, ξ is an intermediate point for η and ζ, whenever

$$\sigma_{\mathsf{b}}(\eta, \zeta) = \sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) \qquad \textit{where} \quad \eta, \xi, \zeta \in \mathcal{U}. \tag{21}$$

Let us develop an example demonstrating the distinction between a MCPT embedded with an F-contraction and an F-contraction in the framework of a b-MS.

Example 5. *Suppose $\mathcal{U}$ has countably infinite elements, $|\mathcal{U}| = \aleph_0$, specifically $\mathcal{U} = \{\eta^*, \eta_0, \eta_1, \ldots\}$. Consider a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$, that is, a MCPT embedded with an* F*-contraction, but exhibits non-contracting behavior in a* b*-MS $\mathcal{U}$.*

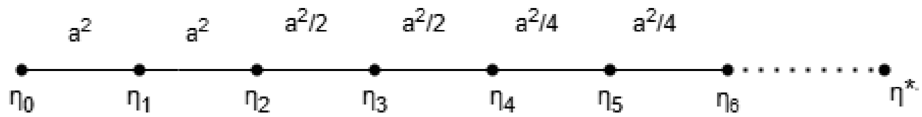

Figure 1: The points of the space ($\mathcal{U}$, σ_{b}) with consecutive distances between them

Let a *be a positive real number. Define* σ_{b} *on $\mathcal{U} \times \mathcal{U}$ as follows:*

$$\sigma_{\mathsf{b}}(\eta, \xi) = \begin{cases} \mathsf{a}^2/2^{\lfloor i/2 \rfloor}, & \textit{if} \quad \eta = \eta_{\mathsf{i}}, \xi = \eta_{\mathsf{i}+1}, i = 1, 2, 3, \cdots ., \\ \sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta_{\mathsf{i}+1}) + \cdots + \sigma_{\mathsf{b}}(\eta_{\mathsf{j}-1}, \eta_{\mathsf{j}}), & \textit{if} \quad \eta = \eta_{\mathsf{i}}, \xi = \eta_{\mathsf{j}}, i+1 < j, \\ 4\mathsf{a}^2 - \sigma_{\mathsf{b}}(\eta_0, \eta_{\mathsf{i}}), & \textit{if} \quad \eta = \eta_{\mathsf{i}}, \xi = \eta^*, \\ 0, & \textit{if} \quad \eta = \xi, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function defined as the greatest integer less than or equal to a given number.
Clearly, for every triplet of distinct points in $\mathcal{U}$, one point is situated between the remaining two, we can see in Fig.1. Furthermore, the space has a single accumulation point η^*, and so it is complete.
Define the mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ given by $\Upsilon(\eta_{\mathsf{i}}) = \eta_{\mathsf{i}+1}$ for all $\mathsf{i} \in \mathbb{N} \cup \{0\}$, and $\Upsilon(\eta^*) = \eta^*$. We can see that Υ is not an F*-contraction mapping. Indeed, $\sigma_{\mathsf{b}}(\eta_{2\mathsf{n}}, \eta_{2\mathsf{n}+1}) = \sigma_{\mathsf{b}}(\Upsilon\eta_{2\mathsf{n}}, \Upsilon\eta_{2\mathsf{n}+1})$, for all $\mathsf{n} = 0,1,2,\cdots$.*
Now, we prove that Υ is a MCPT embedded with an F*-contraction . Consider the first triplets of points $\eta_{\mathsf{i}}, \eta_{\mathsf{j}}, \eta^* \in \mathcal{U}$ with $0 \leq \mathsf{i} < \mathsf{j}$. According to the definition of the σ_{b} given above*

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta_{\mathsf{j}}) + \sigma_{\mathsf{b}}(\eta_{\mathsf{j}}, \eta^*) = \sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta^*).$$

and adding $\sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta^*)$ in both sides, one writes

$$\begin{aligned} \sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta_{\mathsf{j}}) + \sigma_{\mathsf{b}}(\eta_{\mathsf{j}}, \eta^*) + \sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta^*) &= 2\sigma_{\mathsf{b}}(\eta_{\mathsf{i}}, \eta^*), \\ &= 2(4\mathsf{a}^2 - \sigma_{\mathsf{b}}(\eta_0, \eta_{\mathsf{i}})), \\ &= 8\mathsf{a}^2 - 2\sigma_{\mathsf{b}}(\eta_0, \eta_{\mathsf{i}}). \end{aligned}$$

Also,

$$\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta^{*})=\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta^{*}).$$

Adding $\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta^{*})$ in both sides,

$$\begin{aligned}\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta^{*})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta^{*}) =&2\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta^{*}),\\ =&2(\sigma_{\mathsf{b}}(\eta_{\mathsf{i+1}},\eta^{*})),\\ =&2(4\mathsf{a}^2-\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i+1}})),\\ =&8\mathsf{a}^2-2\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i+1}}).\end{aligned}$$

Using the formula for the sum of a geometric series with n *terms, we get*

$$\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})=\begin{cases}4\mathsf{a}^2\left(1-\left(\frac{1}{2}\right)^n\right), & \textit{if}\quad \mathsf{i}=2\mathsf{n},\\ 4\mathsf{a}^2\left(1-\left(\frac{1}{2}\right)^n\right)-\frac{\mathsf{a}^2}{2^{n-1}}, & \textit{if}\quad \mathsf{i}=2\mathsf{n}-1.\end{cases}\qquad \mathsf{n}=1,2,\cdots.$$

By the equation (21),

$$\begin{aligned}\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i+1}})&=\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{i+1}}),\\ &=\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})+\frac{\mathsf{a}^2}{(2^{\lfloor i/2\rfloor})}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta^{*})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta^{*})}{\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta^{*})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta^{*})}&=\frac{8\mathsf{a}^2-2\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i+1}})}{8\mathsf{a}^2-2\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})},\\ &=\frac{4\mathsf{a}^2-\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})-\frac{\mathsf{a}^2}{(2^{\lfloor i/2\rfloor})}}{4\mathsf{a}-\sigma_{\mathsf{b}}(\eta_0,\eta_{\mathsf{i}})},\end{aligned}$$

$$=\begin{cases}\dfrac{4\mathsf{a}^2-4\mathsf{a}^2\left(1-\left(\frac{1}{2}\right)^n\right)-\frac{\mathsf{a}^2}{(2^{\lfloor i/2\rfloor})}}{4\mathsf{a}^2-4\mathsf{a}^2(1-(\frac{1}{2})^n)}, & \textit{if}\ \ \mathsf{i}=2\mathsf{n},\\[2ex] \dfrac{4\mathsf{a}^2-4\mathsf{a}^2\left(1-\left(\frac{1}{2}\right)^n\right)+\frac{\mathsf{a}^2}{2^{n-1}}-\frac{\mathsf{a}^2}{(2^{\lfloor i/2\rfloor})}}{4\mathsf{a}^2-4\mathsf{a}^2(1-(\frac{1}{2})^n)+\frac{\mathsf{a}^2}{2^{n-1}}}, & \textit{if}\ \ \mathsf{i}=2\mathsf{n}-1,\end{cases}$$

$$=\begin{cases}\frac{3}{4}, & \textit{if}\quad \mathsf{i}=2\mathsf{n},\\ \frac{2}{3}, & \textit{if}\quad \mathsf{i}=2\mathsf{n}-1.\end{cases}\tag{22}$$

Considering η_i, η_j, $\eta_k \in \mathcal{U}$ *with* $0 \leq i < j < k$*, Fig.1 illustrates that*

$$\sigma_b(\eta_i, \eta_j) = \sigma_b(\eta_i, \eta_{i+1}) + \sigma_b(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma_b(\eta_{j-1}, \eta_j), \tag{23}$$

$$\sigma_b(\eta_j, \eta_k) = \sigma_b(\eta_j, \eta_{j+1}) + \sigma_b(\eta_{j+1}, \eta_{j+2}) + \cdots + \sigma_b(\eta_{k-1}, \eta_k), \tag{24}$$

and

$$\sigma_b(\eta_i, \eta_k) = \sigma_b(\eta_i, \eta_{i+1}) + \cdots + \sigma_b(\eta_{j-1}, \eta_j) + \cdots + \sigma_b(\eta_{k-1}, \eta_k). \tag{25}$$

Adding (23), (24) and (25) yields that

$$\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) = 2(\sigma_b(\eta_i, \eta_{i+1}) + \sigma_b(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma_b(\eta_{k-1}, \eta_k)). \tag{26}$$

Now, by the definition of σ_b*,*

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) = \sigma_b(\eta_{i+1}, \eta_{j+1}) = \sigma_b(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma_b(\eta_{j-1}, \eta_j), \tag{27}$$

$$\sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) = \sigma_b(\eta_{j+1}, \eta_{k+1}) = \sigma_b(\eta_{j+1}, \eta_{j+2}) + \cdots + \sigma_b(\eta_k, \eta_{k+1}), \tag{28}$$

and

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) = \sigma_b(\eta_{i+1}, \eta_{k+1}) = \sigma_b(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma_b(\eta_k, \eta_{k+1}). \tag{29}$$

Adding (27), (28) and (29) leads to

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) = 2(\sigma_b(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma_b(\eta_{k-1}, \eta_k) + \sigma_b(\eta_k, \eta_{k+1})). \tag{30}$$

By subtracting (26) and (30), one gets

$$\begin{aligned} \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) - (\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k)), \\ = 2(\sigma_b(\eta_i, \eta_{i+1}) + \sigma_b(\eta_k, \eta_{k+1})), \\ = 2\left(\frac{a^2}{(2^{\lfloor i/2 \rfloor})} - \frac{a^2}{(2^{\lfloor k/2 \rfloor})}\right). \end{aligned}$$

Rearranging this equation,

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) = \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) - 2\left(\frac{a^2}{(2^{\lfloor i/2 \rfloor})} - \frac{a^2}{(2^{\lfloor k/2 \rfloor})}\right). \tag{31}$$

We can see that $i + 1 < k$ *, which implies that*

$$\begin{aligned} 2^{\lfloor i/2+1 \rfloor} &< 2^{\lfloor k/2 \rfloor}, \\ \Rightarrow 2.2^{\lfloor i/2 \rfloor} &< 2^{\lfloor k/2 \rfloor}, \\ \Rightarrow \frac{a^2}{2^{\lfloor k/2 \rfloor}} &\leq \frac{a^2}{(2.2^{\lfloor i/2 \rfloor})}. \end{aligned}$$

Using this in (31) to obtain

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) \leq \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) - \frac{2a^2}{2^{\lfloor i/2 \rfloor}} + \frac{a^2}{2^{\lfloor i/2 \rfloor}}$$

$$\Rightarrow \sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{k}})\leq \sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}})-\frac{\mathsf{a}^2}{2^{\lfloor i/2\rfloor}}. \tag{32}$$

One can show that $\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta^*)\leq 4\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{i+1}})$. *We have*

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}})\leq \sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta^*).$$

consequently, we obtain

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}})\leq 4\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{i+1}}).$$

Using equality (23) and the preceding inequality, we obtain

$$\begin{aligned}\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}}) &= 2\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}}),\\ &\leq 8\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{i+1}}),\\ &= 8\frac{\mathsf{a}^2}{(2^{\lfloor i/2\rfloor})}.\end{aligned}$$

By putting this inequality in (32), yields $\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{k}})\leq \sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}})-\frac{1}{8}(\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}}))$

$$\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{j}},\Upsilon\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\Upsilon\eta_{\mathsf{i}},\Upsilon\eta_{\mathsf{k}})\leq \frac{7}{8}(\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{j}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{j}},\eta_{\mathsf{k}})+\sigma_{\mathsf{b}}(\eta_{\mathsf{i}},\eta_{\mathsf{k}})). \tag{33}$$

By equations (22) and (33), inequality (3) satisfies for any three pairwise distinct points from the space $\mathcal{U}$ *with* $\mathsf{F}(\eta)=\ln(\eta)$ *and*

$$\frac{e^{-\tau}}{s^2}=\frac{7}{8}=\max\left\{\frac{2}{3},\frac{3}{4},\frac{7}{8}\right\}.$$

It should be noted that the sequence of iterates of any two points, η_{i} and η_{j}, in the preceding example overlap sets. Let's create an example of a mapping $\Upsilon:\mathcal{U}\longrightarrow\mathcal{U}$ that is a MCPT embedded with an F-contraction and is not a an F-contraction mapping. It has the feature that there are an infinite number of points such that the iteration sequences of these points are disjoint sets.

Example 6.
Consider the subset $\mathcal{U}\subseteq\mathbb{R}$ *consisting of* $\{\eta_0,\eta_1,...\}\cup[0,1]$, *where* $\eta_{2\mathsf{k}}=\dfrac{-4}{2^k}$ *and* $\eta_{2\mathsf{k}+1}=\dfrac{-3}{2^k}$ *for* $k\geq 0$, *illustrated in Fig.2.*

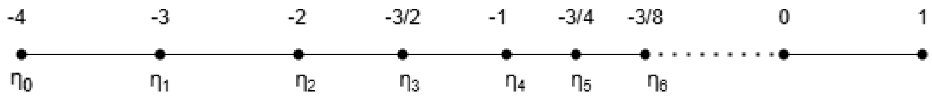

Figure 2: The b-MS ($\mathcal{U}$, σ_{b})

.

Let $\Upsilon:\mathcal{U}\longrightarrow\mathcal{U}$ *be defined by* $\Upsilon\eta_{\mathsf{i}}=\eta_{\mathsf{i+1}}$ *for all* $\mathsf{i}\in\{0\}\cup\mathbb{N}$ *and* $\Upsilon\eta=\frac{\eta}{2}$ *for* $\eta\in[0,1]$.

The mapping Υ *satisfies the required condition for sequences of iterates of points in [0,1] of the form* $\frac{\mathsf{p}}{2^k}$*, where* p *is a prime number greater than or equal to 3 and* k *is the smallest natural number ensuring* $\frac{\mathsf{p}}{2^k} \subseteq [0,1]$.

Setting $\mathsf{s} = 1$ *establishes an isometry between the previous example's* b*-MS and the subspace* $(\{0, \eta_0, \eta_1, \cdots\}, \sigma_{\mathsf{b}})$ *within* $(\mathcal{U}, \sigma_{\mathsf{b}})$*. The* F*-mapping* Υ *is defined in a similar manner for this subspace, and it follows that* Υ *is not an* F*-contraction mapping. We will demonstrate that for each of the three pairwise distinct points from the space* $(\mathcal{U}, \sigma_{\mathsf{b}})$*, inequality (3) is satisfied. The validity of this property for all distinct triplets in* $(\{0, \eta_0, \eta_1, \cdots\}, \sigma_{\mathsf{b}})$ *has been previously established. Since the* b*-metric* σ_{b} *is contractive on* $([0,1], \sigma_{\mathsf{b}})$*, and every* F*-contraction reduces to triangle perimeters, we only need to prove inequality (3) for three pairwise distinct points* $\eta, \xi, \zeta \in \mathcal{U}$ *satisfying:* $\eta < \xi < \zeta$ *where,* $\eta \in \{\eta_0, \eta_1, \cdots\}$ *and* $\zeta \in (0,1]$.
We begin by considering $\eta = \eta_{2\mathsf{k}} = \frac{-4}{2^{\mathsf{k}}}$*. Subsequently,*

$$\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta) = 2\sigma_{\mathsf{b}}(\eta, \zeta) = 2\left(\frac{4}{2^k} + \zeta\right)^2. \tag{34}$$

From $\Upsilon\eta = \Upsilon\eta_{2\mathsf{k}}$*, it follows that* $\Upsilon\eta_{2\mathsf{k}} = \eta_{2\mathsf{k}+1} = \frac{-3}{2^k}$*. which implies that*

$$\begin{aligned}
\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta) = 2\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta) &= 2\left(\frac{3}{2^k} + \frac{\zeta}{2}\right)^2, \\
&= 3^2 \times 2\left(\frac{1}{2^k} + \frac{\zeta}{6}\right)^2, \\
&= \frac{9}{16} \times 2\left(\frac{4}{2^k} + \frac{4\zeta}{6}\right)^2, \\
&\leq \frac{9}{16} \times 2\left(\frac{4}{2^k} + \zeta\right)^2.
\end{aligned}$$

By equation (34), one writes

$$\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta) \leq \frac{9}{16}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta)).$$

We can see that it satisfies the inequality (3). Similarly, for $\eta = \eta_{2\mathsf{k}+1} = \frac{-3}{2^k}$*, we have*

$$\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta) = 2\sigma_{\mathsf{b}}(\eta, \zeta) = 2\left(\frac{3}{2^k} + \zeta\right)^2. \tag{35}$$

Applying Υ *to* $\eta_{\mathsf{2k+1}}$ *yields* $\Upsilon\eta_{\mathsf{2k+1}} = \eta_{\mathsf{2(k+1)}} = \frac{-4}{2^{k+1}}$. *We get*

$$\begin{aligned}\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta) &= 2\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta),\\ &= 2\left(\frac{4}{2^{k+1}}+\frac{\zeta}{2}\right)^2,\\ &= 2\left(\frac{4}{2.2^{k}}+\frac{\zeta}{2}\right)^2,\\ &= 4\times 2\left(\frac{1}{2^{k}}+\frac{\zeta}{4}\right)^2,\\ &= \frac{4}{9}\times 2\left(\frac{3}{2^{k}}+\frac{3\zeta}{4}\right)^2,\\ &\leq \frac{4}{9}\times 2\left(\frac{3}{2^{k}}+2\zeta\right)^2.\end{aligned}$$

Now, by equation (35), we obtain

$$\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta)\leq\frac{2}{3}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma_{\mathsf{b}}(\eta,\zeta)),$$

which implies that inequality (3) holds with $\mathsf{F}(\eta)=\ln(\eta)$.

3. Conclusion

Using some well established results for “mappings contracting perimeter of triangles” the notion of a “mapping contracting perimeters of triangles (MCPT) embedded with an F-contraction in a b-metric space (b-MS)” has been introduced. The FP theorem has been proved and classical Banach FP theorem is derived as a simple corollary. Examples of a MCPT embedded with an F-contraction which are not contraction mappings in the framework of a b-MS have been established. These results open avenues for further research. Future work may explore the application of this framework in controlled, double controlled, partial and cone b-MSs.

Authors’ Contributions

All authors contribute equally in this paper.

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Conflict of interest

The authors declare that they have no conflict of interest.

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