EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6811
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Solving Fractional Differential Equations in $\mathfrak{b}$-Metric Spaces Tailored with a Directed Graph

Dur-e-Shehwar Sagheer[1], Haitham Qawaqneh[2], Samina Batul[1], Isma Urooj[1], Zainab Rahman[1], Hassen Aydi[3,4,*]

[1] *Department of Mathematics, Capital University of Science and Technology, Islamabad, Pakistan*
[2] *Al-Zaytoonah University of Jordan, Amman 11733, Jordan*
[3] *Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia*
[4] *Department of Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*

Abstract. This study introduces an extended class of contractions incorporating auxiliary functions. We examine the existence of solutions for Caputo fractional differential equations with integral boundary conditions within the framework of $\mathfrak{b}$-metric spaces endowed with a directed graph. The established findings not only generalize but also unify a number of significant results in the existing literature, as demonstrated by supporting theoretical developments and illustrative examples.

2020 Mathematics Subject Classifications: 34A08, 47H10, 54H25

Key Words and Phrases: $\mathfrak{b}$-metric space ($\mathfrak{b}$MS), fractional differential equation, auxiliary function, directed graph ($\mathcal{DG}$), coincidence point ($\mathcal{CP}$), Caputo fractional differential equation ($\mathcal{CFDE}$)

1. Introduction and Preliminaries

Fixed point theory constitutes a dynamic and significant area of mathematical research, with its principles providing foundational tools for a wide array of scientific disciplines. Its application to fractional differential equations, in particular, has been instrumental in advancing analytical techniques for solving complex nonlinear problems. The field has evolved considerably [1, 2] since Banach's seminal contraction principle [3], leading to numerous extensions in generalized metric spaces. A prominent such generalization is the $\mathfrak{b}$-metric space ($\mathfrak{b}$MS), introduced independently by Czerwik [4] and Bakhtin [5] , which

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6811

Email addresses: d.e.shehwar@cust.edu.pk (D. e. Shehwar Sagheer), h.alqawaqneh@zuj.edu.jo (H. Qawaqneh), samina.batul@cust.edu.pk (S. Batul), mmt203011@cust.pk (I. Urooj), mmt193039@cust.pk (Z. Rahman), hassen.aydi@isima.rnu.tn (H. Aydi)

has since become a fertile ground for research. Notable contributions include the work of Afshari et al. [6] and Aydi et al. [7] on multivalued mappings, among other significant developments [8–16].

A pivotal expansion of this framework occurred when Jachymski [Jachymski] incorporated the concept of directed graphs into metric fixed point theory, inspiring subsequent research in graph-based frameworks [17–21]. More recently, the integration of auxiliary functions has further enriched the theory. Karapinar et al. [22] demonstrated their efficacy in establishing existence and uniqueness results for fractional differential equations, an approach extended by Wongsaijai et al. [23] to metric spaces with graphs.

Motivated by these advancements, this research aims to develop a unified approach within $\mathfrak{b}$-metric spaces endowed with a directed graph. The primary contributions of this work are fourfold:

(i) We introduce a broader class of contractions integrated with auxiliary functions, extending the results of the aforementioned studies to the more general setting of $\mathfrak{b}$MSs. Previous research has established foundational results in this area. Karapinar et al. [22] demonstrated the existence of fixed points in the context of metric spaces; however, their work did not incorporate a directed graph framework. Similarly, Wongsaijai et al. [23] extended this line of inquiry by considering a directed graph, but their analysis was confined to a context that did not address the challenges posed by the triangular inequality in $\mathfrak{b}$-metric spaces.

(ii) We establish significant theoretical results, including theorems on the existence of coincidence points, leveraging the properties of these auxiliary functions.

(iii) We pioneer the analysis of fixed points in a context involving two distinct metrics on a non-empty set.

(iv) The proposed framework is not merely theoretical; it provides an effective method for predicting solutions to fractional differential equations, which are critical in modeling phenomena across physics and engineering [24–26]. A key advantage of our results is their capacity to yield precise estimates for solutions, enhancing their practical utility.

2. Preliminaries

In this work, X denotes a non-empty set and $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$ will represent a directed graph ($\mathcal{DG}$). Let us examine some fundamental principles and implications that will serve as the stage for our primary findings.

Definition 1. *Let $X \neq \emptyset$ and $\mathfrak{b} \in \mathbb{R}$ such that $\mathfrak{b} \geq 1$. A mapping $d_{\mathfrak{b}} : X \times X \to [0, \infty)$ satisfying the following axioms is called a $\mathfrak{b}$-metric on X:*
$\acute{d}b(1)$*:* $d_{\mathfrak{b}}(x, y) = 0 \Leftrightarrow x = y$*;*
$\acute{d}b(2)$*:* $d_{\mathfrak{b}}(x, y) = d_{\mathfrak{b}}(y, x)$*;*

$\acute{d}b(3)$: $d_{\mathfrak{b}}(x,y) \leq \mathfrak{b}\{d_{\mathfrak{b}}(x,z) + d_{\mathfrak{b}}(z,y)\}$, for all $x, y, z \in X$. The pair $(X, d_{\mathfrak{b}})$ is said to be a $\mathfrak{b}$MS.

Example 1. *[27]*
Let $X = l_p(\mathbb{R})$ (with $p \in (0,1)$) be the space of all real sequences having the property $l_p(\mathbb{R}) = \{x = x_k \subset \mathbb{R}\}$, such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$ and define a function $d_{\mathfrak{b}} : X \times X \to \mathbb{R}^+$ as

$$d_{\mathfrak{b}}(x,y) = \Big(\sum_{k=1}^{\infty} |x_k - y_k|^p\Big)^{1/p},$$

then $(X, d_{\mathfrak{b}})$ is a $\mathfrak{b}$MS with $\mathfrak{b} = 2^{\frac{1}{p}}$.

Encouraged by the ideas of Charoensawan et al. [28], we present some key concepts, particularly regarding common fixed points and coincidence points in $\mathfrak{b}$MSs endowed with a $\mathcal{DG}$.

Definition 2. *Let $(X, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS and $d_{\mathfrak{b}}$ be a continuous metric. Let Δ denote the diagonal of $X \times X$, then $(X, d_{\mathfrak{b}})$ is considered to be endowed with a directed graph $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$ if $V(\bar{\mathcal{G}})$ is the set of all the elements of X, and the set $E(\bar{\mathcal{G}})$ contains all elements of Δ, excluding parallel edges.*

Definition 3. *Let $(X, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS together with $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$, and $d_{\mathfrak{b}}$ be a continuous metric, then for $T, S : X \to X$,*

$$\mathcal{C}(T,S) = \{x \in X : Tx = Sx\},$$

is called the set of $\mathcal{CP}$s of T and S. We also have

$$\mathcal{C}_m(T,S) = \{x \in X : Tx = Sx = x\},$$

denoting the set of common fixed points for the two mappings. Also, we define

$$V(T,S) = \{x \in X : (Tx, Sx) \in E(\bar{\mathcal{G}})\}.$$

Remark 1. *The set $E(\bar{\mathcal{G}})$ possesses the transitive property*

$$\text{if } \ (x,y),(y,z) \in E(\bar{\mathcal{G}}), \quad \text{then} \quad (x,z) \in E(\bar{\mathcal{G}}).$$

Definition 4. *Let $(X, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS together with a $\mathcal{DG}$, where $d_{\mathfrak{b}}$ is a continuous metric functional. A self-mapping T on X is called $\mathcal{G}_{\mathfrak{b}}$-continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(\bar{\mathcal{G}})$, we have*

$$\text{if } \ x_n \longrightarrow x \qquad Tx_n \longrightarrow Tx \ \ \forall \ \ n \in \mathbb{N},$$

and if this holds for all $x \in X$, then T is called $\mathcal{G}_{\mathfrak{b}}$-continuous.

Remark 2. *In a* $\mathfrak{b}$*MS equipped with* $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$*, we say that* $(X, d_{\mathfrak{b}}, \bar{\mathcal{G}})$ *has property* $\mathcal{A}$ *if for any sequence* $\{x_n\}$ *in* X *with* $x_n \to x \in X$*, we have*

$$(x_n, x_{n+1}) \in E(\bar{\mathcal{G}}) \quad (x_n, x) \in E(\bar{\mathcal{G}}) \quad \forall \;\; n \in \mathbb{N}. \tag{1}$$

Definition 5. *Let* $(X, d_{\mathfrak{b}})$ *be a* $\mathfrak{b}$*MS endowed with a* $\mathcal{DG}$ *where* $d_{\mathfrak{b}}$ *be a continuous metric functional. Also let* $T, S : X \to X$ *be two functions then* T *is called* S*-edge preserving with respect to* $\bar{\mathcal{G}}$*, if following condition holds,*

$$(Sx, Sy) \in E(\bar{\mathcal{G}}) \quad \Rightarrow \quad (Tx, Ty) \in E(\bar{\mathcal{G}}) \quad \forall \;\; x \in X.$$

Agarwal et al. [29] demonstrated some excellent results for local and global fixed points. They used generalized contractions in the domain of metric spaces having two distance functions simultaneously. Inspired by his work, we present the following ideas, which are going to facilitate us in proving our main results.
Let X be a non-empty set and we define $d_{\mathfrak{b}}$ and $d'_{\mathfrak{b}}$ such that $(X, d_{\mathfrak{b}})$ and $(X, d'_{\mathfrak{b}})$ both are $\mathfrak{b}$MSs. Then if we say $d_{\mathfrak{b}} \leq d'_{\mathfrak{b}}$ then it means $d_{\mathfrak{b}}(x, y) \leq d'_{\mathfrak{b}}(x, y)$ for all $x, y \in X$. If there are two functions $T, S : X \to X$ then T is said to be S-non-decreasing if

$$Sx \leq Sy \;\; Tx \leq Ty \;\; \forall \quad x, y \in X,$$

and if S is an identity map then we say T is called a non-decreasing map.

Definition 6. *Let* $(X, d_{\mathfrak{b}})$ *be a* $\mathfrak{b}$*MS endowed with a directed graph* $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$*, where* $d_{\mathfrak{b}}$ *is a continuous metric.consider a sequence* $\{x_n\}$ *be a sequence in* $(X, d_{\mathfrak{b}})$*. The mappings* $T, S : X \to X$ *are* $d_{\mathfrak{b}}$ *are called compatible, if we have* $\lim_{n\to\infty} d_{\mathfrak{b}}(STx_n, TSx_n) = 0$*, whenever* $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$*.*

Definition 7. *Let* $(X, d_{\mathfrak{b}})$ *and* $(\mathcal{N}, d'_{\mathfrak{b}})$ *be two* $\mathfrak{b}$*MSs and* $T : X \to \mathcal{N}$ *and* $S : X \to X$ *be two functions. Then* T *is* S*-Cauchy on* X *if for any sequence* $\{x_n\}$ *in* X *the sequence* $\{Sx_n\}$ *is Cauchy in* X *implies that the sequence* $\{Tx_n\}$ *is Cauchy in* $(\mathcal{N}, d'_{\mathfrak{b}})$*.*

The following definition is analogous to those given in [23] and [22] in $\mathfrak{b}$MS.

Definition 8. *Let* $(X, d_{\mathfrak{b}})$ *be a* $\mathfrak{b}$*MS a function and* $\{x_n\}, \{y_n\}$ *be two sequences in* X *such that the sequence* $\{d_{\mathfrak{b}}(x_n, y_n)\}$ *is decreasing and convergent. A function* $\psi : X \times X \to [0, 1]$ *satisfying the condition,*

$$\text{if} \quad \lim_{n\to\infty} \psi(x_n, y_n) = 1 \quad \text{then} \quad \lim_{n\to\infty} d_{\mathfrak{b}}(x_n, y_n) = 0,$$

is called an auxiliary function. Throughout the article, such a family of functions will be represented by $\Psi = \Psi(X)$*. Moreover, let the function* $\phi : [0, \infty) \to [0, \infty)$ *be increasing and continuous satisfying the property* $\phi(x) = 0x = 0$*.*

In the upcoming discussion, denote the collection of all such functions by Φ.

3. Main Results

In the following section, we derive several fixed point results in the framework of $\mathfrak{b}$MS implementing rational type contractions. In addition, an example and an application are provided to help readers understand our conclusion more thoroughly.

Definition 9. *Let $(X, d_\mathfrak{b})$ be a $\mathfrak{b}$MS together with a $\mathcal{DG}$, $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$ and let $T, S : X \to X$ be two functions, where T is S-edge preserving with respect to $\bar{\mathcal{G}}$, then (T, S) is called a $\psi - \phi$-contraction if for all $x, y \in X$ with $(Sx, Sy) \in E(\bar{\mathcal{G}})$ there exist two functions $\phi \in \Phi$ and $\psi \in \Psi$, such that*

$$\phi(d_\mathfrak{b}(Tx, Ty)) \leq \psi(Sx, Sy)\phi(\delta(\mathbb{M}((Sx, Sx))), \tag{2}$$

where $\delta \leq \frac{1}{\mathfrak{b}^\nu}$ for all $\nu > 3$ and $\mathbb{M} : X \times X \to [0, \infty)$ for any $x, y \in X$ is given as:

$$\mathbb{M}(Sx, Sy) = \max\left\{\frac{d_\mathfrak{b}(Sx, Tx)d_\mathfrak{b}(Ty, Sy)}{d_\mathfrak{b}(Sx, Sy)}, d_\mathfrak{b}(Sx, Sy), d_\mathfrak{b}(Sx, Tx), d_\mathfrak{b}(Sy, Ty), \frac{d_\mathfrak{b}(Sx, Ty) + d_\mathfrak{b}(Sy, Tx)}{2\mathfrak{b}}\right\}. \tag{3}$$

Lemma 1. [30] Let $(X, d_\mathfrak{b})$ be a $\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, and the sequences $\{x_n\}, \{y_n\} \in X$ converge respectively to a and b. Then we have

$$\frac{1}{\mathfrak{b}^2} d_\mathfrak{b}(a, b) \leq \liminf_{n\to\infty} d_\mathfrak{b}(x_n, y_n) \leq \limsup_{n\to\infty} d_\mathfrak{b}(x_n, y_n) \leq \mathfrak{b} d_\mathfrak{b}(a, b).$$

In particular, if $a = b$, then we have $\lim_{n\to\infty} d_\mathfrak{b}(x_n, y_n) = 0$. Moreover, for each $c \in X$, we have

$$\frac{1}{\mathfrak{b}} d_\mathfrak{b}(a, c) \leq \liminf_{n\to\infty} d_\mathfrak{b}(x_n, c) \leq \limsup_{n\to\infty} d_\mathfrak{b}(x_n, c) \leq \mathfrak{b} d_\mathfrak{b}(a, c).$$

Lemma 2. Let $(X, d_\mathfrak{b})$ be a complete $\mathfrak{b}$MS equipped with a $\mathcal{DG}$, and $d_\mathfrak{b}$ be continuous metric. Also consider the pair (T, S) is a $\psi - \phi$-contraction satisfying the following axioms:

(1) $T(X) \subseteq S(X)$;

(2) $E(\bar{\mathcal{G}})$ is transitive;

(3) $\lim_{n\to\infty} d_\mathfrak{b}(Sx_n, Sx_{n+1}) = 0$,

then $\{Sx_n\}$ is a Cauchy sequence in $(X, d_\mathfrak{b})$.

Proof. Suppose on contrary that $\{Sx_n\}$ is not Cauchy and the sequences $\{j_n\}$, $\{k_n\} \in \mathbb{N}$ are such that $j_n > k_n > n$,

$$d_\mathfrak{b}(Sx_{j_n}, Sx_{k_n}) \geq \epsilon,$$

and

$$d_\mathfrak{b}(Sx_{j_n-1}, Sx_{k_n}) < \epsilon. \tag{4}$$

Using the triangular inequality,

$$\epsilon \leq d_{\mathfrak{b}}(x_{j_n}, x_{k_n}) \leq b[d_{\mathfrak{b}}(x_{j_n}, x_{j_n-1}) + d_{\mathfrak{b}}(x_{j_n-1}, x_{k_n})] < bd_{\mathfrak{b}}(x_{j_n}, x_{j_n-1}) + b\epsilon.$$

By letting limit $n \to \infty$,

$$\epsilon \leq \liminf_{n\to\infty} d_{\mathfrak{b}}(x_{j_n}, x_{k_n}) \leq \limsup_{n\to\infty} d_{\mathfrak{b}}(x_{j_n}, x_{k_n}) \leq b\epsilon. \tag{5}$$

Also, one can easily get

$$\frac{\epsilon}{\mathfrak{b}} \leq \liminf_{n\to\infty} d_{\mathfrak{b}}(x_{j_n}, x_{k_n+1}) \leq \limsup_{n\to\infty} d_{\mathfrak{b}}(x_{j_n}, x_{k_n+1}) \leq \mathfrak{b}^2\epsilon,$$

$$\frac{\epsilon}{\mathfrak{b}} \leq \liminf_{n\to\infty} d_{\mathfrak{b}}(x_{j_n+1}, x_{k_n}) \leq \limsup_{n\to\infty} d_{\mathfrak{b}}(x_{j_n+1}, x_{k_n}) \leq \mathfrak{b}^2\epsilon,$$

$$\frac{\epsilon}{\mathfrak{b}^2} \leq \liminf_{n\to\infty} d_{\mathfrak{b}}(x_{j_n+1}, x_{k_n+1}) \leq \limsup_{n\to\infty} d_{\mathfrak{b}}(x_{j_n+1}, x_{k_n+1}) \leq \mathfrak{b}^3\epsilon.$$

Now, as $T(X) \subseteq S(X)$ holds, then we can construct a sequence $x_n \in X$ such that $Sx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ then $(Sx_0, Tx_0) = (Sx_0, Sx_1) \in E(\bar{\mathcal{G}})$, and inductively we may write $(Sx_{n-1}, Sx_n) \in E(\bar{\mathcal{G}})$, for any n. For $(Sx_{j_n}, Sx_{k_n}) \in E(\bar{\mathcal{G}})$, we may write

$$\begin{aligned}\phi(d_{\mathfrak{b}}(Sx_{j_n+1}, Sx_{k_n+1})) &= \phi(d_{\mathfrak{b}}(Tx_{j_n}, Tx_{k_n})) \\ &\leq \psi(d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n}))\phi(\delta(\mathbb{M}(Sx_{j_n}, Sx_{k_n}))) \\ &\leq \phi(\delta(\mathbb{M}(Sx_{j_n}, Sx_{k_n}))),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathbb{M}(Sx_{j_n}, Sx_{k_n}) = \max\Bigg\{&\frac{d_{\mathfrak{b}}(Sx_{j_n}, Tx_{j_n})d_{\mathfrak{b}}(Tx_{k_n}, Sx_{k_n})}{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n})}, d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n}), \\ &d_{\mathfrak{b}}(Sx_{j_n}, Tx_{j_n}), d_{\mathfrak{b}}(Sx_{k_n}, Tx_{k_n}), \frac{d_{\mathfrak{b}}(Sx_{j_n}, Tx_{k_n}) + d_{\mathfrak{b}}(Sx_{k_n}, Tx_{j_n})}{2\mathfrak{b}}\Bigg\} \\ = \max\Bigg\{&\frac{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{j_n+1})d_{\mathfrak{b}}(Sx_{k_n+1}, Sx_{k_n})}{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n})}, d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n}), \\ &d_{\mathfrak{b}}(Sx_{j_n}, Sx_{j_n+1}), d_{\mathfrak{b}}(Sx_{k_n}, Sx_{k_n+1}), \\ &\frac{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n+1}) + d_{\mathfrak{b}}(Sx_{k_n}, Sx_{j_n+1})}{2\mathfrak{b}}\Bigg\}. \\ \leq \max\Bigg\{&\frac{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{j_n+1})d_{\mathfrak{b}}(Sx_{k_n+1}, Sx_{k_n})}{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n})}, d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n}), \\ &d_{\mathfrak{b}}(Sx_{j_n}, Sx_{j_n+1}), d_{\mathfrak{b}}(Sx_{k_n}, Sx_{k_n+1}), \\ &\frac{d_{\mathfrak{b}}(Sx_{j_n}, Sx_{k_n}) + d_{\mathfrak{b}}(Sx_{k_n}, Sx_{k_n+1}) + d_{\mathfrak{b}}(Sx_{k_n}, Sx_{j_n}) + d_{\mathfrak{b}}(Sx_{j_n}, Sx_{j_n+1})}{2}\Bigg\}.\end{aligned}$$

After doing simple calculations and using the assumption (3), we get

$$\limsup_{k\to\infty} \mathbb{M}(Sx_{k_n}, Sx_{j_n}) \leq \limsup d_{\mathfrak{b}}(Sx_{k_n}, Sx_{j_n}).$$

From (6), *we obtain*

$$\frac{\epsilon}{\mathfrak{b}^2} \leq \limsup_{k\to\infty} d_{\mathfrak{b}}(Sx_{j_n+1}, Sx_{k_n+1})) \leq \limsup_{k\to\infty} \delta(d_{\mathfrak{b}}(Sx_{k_n}, Sx_{j_n})).$$

Also, by using property of ϕ, *we get*

$$\phi(\frac{\epsilon}{\mathfrak{b}^2}) \leq \limsup_{k\to\infty} \phi(d_{\mathfrak{b}}(Sx_{j_n+1}, Sx_{k_n+1})) \leq \limsup_{k\to\infty} \phi(\delta(d_{\mathfrak{b}}(Sx_{k_n}, Sx_{j_n}))).$$

Now, by the continuity of ϕ *and using* $\delta \leq \frac{1}{\mathfrak{b}^\nu}$ *we get*

$$\phi(\frac{\epsilon}{\mathfrak{b}^2}) \leq \phi(\frac{1}{\mathfrak{b}^\nu}(\mathfrak{b}\epsilon)),$$

which is a contradiction as $\nu > 3$. *Hence,* $\{Sx_n\}$ *is a Cauchy sequence in* $(X, d_{\mathfrak{b}})$.

Theorem 1. *Let* $(X, d_{\mathfrak{b}})$ *be a complete* $\mathfrak{b}MS$ *with* $\bar{\mathcal{G}} = (V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}}))$ *a* $\mathcal{DG}$ *and* $d_{\mathfrak{b}}$ *be continuous. Let* $d'_{\mathfrak{b}}$ *be another continuous function and the pair* (T, S) *be an* $\psi - \phi$*-contraction with respect to* $d_{\mathfrak{b}}$ *together with the following:*

(i) $S : (X, d'_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ *is continuous, and* $S(X)$ *is closed w.r.t* $d'_{\mathfrak{b}}$*;*

(ii) $T(X) \subseteq S(X)$ *;*

(iii) $E(\bar{\mathcal{G}})$ *is a transitive set;*

(iv) If $d_{\mathfrak{b}} \not\geq d'_{\mathfrak{b}}$ *assume that* $T : (X, d_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ *is* S*-Cauchy sequence on* X*;*

(v) $T : (X, d'_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ *is* $\bar{\mathcal{G}}_{\mathfrak{b}}$*-continuous and* T, S *are* $d'_{\mathfrak{b}}$*-compatible, then*

$$V(T, S) \neq \; \Leftrightarrow \mathcal{C}(T, S) \neq . \tag{7}$$

Proof. If $\mathcal{C}(T, S) \neq$ *then let* $x \in \mathcal{C}(T, S)$. *This means* $Tx = Sx$. *Then*

$$(Tx, Sx) = (Sx, Sx) \in \Delta \subset E(\bar{\mathcal{G}}).$$

We have, $(Sx, Sx) = (Tx, Sx) \in E(\bar{\mathcal{G}})$ *showing that* $x \in V(T, S)$ *thus,* $V(T, S) \neq$ *. Now, to verify other side, suppose that* $V(T, S) \neq$ *and* $x_0 \in X$ *with* $(Sx_0, Tx_0) \in E(\bar{\mathcal{G}})$. *Now, as (2) holds then we can construct a sequence* $x_n \in X$ *such that* $Sx_n = Tx_{n-1}$ *for all* $n \in \mathbb{N}$. *Note that if* $Sx_n = Sx_{n-1}$ *for some* n, *then* $x_{n-1} \in \mathcal{C}(T, S)$. *This leads to a trivial case so suppose that* $Sx_n \neq Sx_{n-1}$ *for every* $n \in \mathbb{N}$. *Now, as* $(Sx_0, Tx_0) = (Sx_0, Sx_1) \in E(\bar{\mathcal{G}})$, *and inductively we may write* $(Sx_{n-1}, Sx_n) \in E(\bar{\mathcal{G}})$, *for any* n. *Using contraction conditions,*

$$\begin{aligned} \phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})) &= \phi(d_{\mathfrak{b}}(Tx_n, Tx_{n+1})) \\ &\leq \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))\phi(\delta(\mathbb{M}(Sx_n, Sx_{n+1})))), \end{aligned} \tag{8}$$

where

$$\begin{aligned}\mathbb{M}(Sx_n, Sx_{n+1}) = \max\Bigg\{&\frac{d_{\mathfrak{b}}(Sx_n, Tx_n)d_{\mathfrak{b}}(Tx_{n+1}, Sx_{n+1})}{d_{\mathfrak{b}}(Sx_n, Sx_{n+1})}, d_{\mathfrak{b}}(Sx_n, Sx_{n+1}), d_{\mathfrak{b}}(Sx_n, Tx_n),\\ &d_{\mathfrak{b}}(Sx_{n+1}, Tx_{n+1}), \frac{d_{\mathfrak{b}}(Sx_n, Tx_{n+1}) + d_{\mathfrak{b}}(Sx_{n+1}, Tx_n)}{2\mathfrak{b}}\Bigg\}.\\ = \max\Bigg\{&d_{\mathfrak{b}}(Sx_n, Sx_{n+1})d_{\mathfrak{b}}(Sx_n, Sx_{n+1}), d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2}),\\ &\frac{d_{\mathfrak{b}}(Sx_n, Sx_{n+2}) + d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+1})}{2\mathfrak{b}}\Bigg\}\\ = \max\Bigg\{&d_{\mathfrak{b}}(Sx_n, Sx_{n+1}), d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2}), \frac{d_{\mathfrak{b}}(Sx_n, Sx_{n+2})}{2\mathfrak{b}}\Bigg\}\\ \leq \max\{&d_{\mathfrak{b}}(Sx_n, Sx_{n+1}), d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})\}.\end{aligned}$$

We will observe both possibilities separately, if $\mathbb{M}(Sx_n, Sx_{n+1}) = d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})$*. Then, from* (8)*,*

$$\begin{aligned}\phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})) &\leq \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))\phi(\delta(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})))\\ &\leq \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))\phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2}))\\ &\leq \phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})),\end{aligned}$$

for each $n \geq 0$*. As by assumption* $Sx_{n+1} \neq Sx_{n+2}$ *so* $d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2}) > 0$*. Then we have* $\phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})) > 0$*. Showing that* $\lim_{n\to\infty} \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) = 1$*. Thus*

$$d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) = 0.$$

Also, in the same manner we get

$$\begin{aligned}\phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2})) &\leq \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))\phi(\delta(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})))\\ &\leq \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))\phi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})),\end{aligned} \tag{9}$$

Showing that $d_{\mathfrak{b}}(Sx_n, Sx_{n+1})$ *is a non-increasing sequence. Also, by using properties of* ϕ*, we observe that* $\phi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))$ *is a non-increasing sequence and it is also bounded below so that it would be a convergent sequence. Eventually, there exists* $c \geq 0$ *such that* $(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) = c$*. Now, suppose on contrary that* $\lim_{n\to\infty} d_{\mathfrak{b}}(Sx_n, Sx_{n+1}) > 0$*. Also,* $\lim_{n\to\infty} \phi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) > 0$*, then form* (9)*, we have*

$$1 = \lim_{n\to\infty} \frac{\phi(d_{\mathfrak{b}}(Sx_{n+1}, Sx_{n+2}))}{\phi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1}))} \leq \lim_{n\to\infty} \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) \leq 1.$$

Therefore, $\lim_{n\to\infty} \psi(d_{\mathfrak{b}}(Sx_n, Sx_{n+1})) = 1$ *producing* $\lim_{n\to\infty} d_{\mathfrak{b}}(Sx_n, Sx_{n+1}) = 0$*, i.e.,* $c = 0$*, which is a contradiction. So we get*

$$\lim_{n\to\infty} d_{\mathfrak{b}}(Sx_n, Sx_{n+1}) = 0.$$

Lemma (2) shows that $\{Sx_n\}$ is a Cauchy sequence in $(X, d_{\mathfrak{b}})$. Finally, to show that $\{Sx_n\}$ is a Cauchy sequence in $(X, d'_{\mathfrak{b}})$ too. Notice that if $d_{\mathfrak{b}} \geq d'_{\mathfrak{b}}$, proof is trivial. So take $d_{\mathfrak{b}} \ngeq d'_{\mathfrak{b}}$. Now, as $\{Sx_n\}$ is a Cauchy sequence in $(X, d_{\mathfrak{b}})$ and T is S-Cauchy on X so have $\{Tx_n\}$ is a Cauchy sequence in $(X, d'_{\mathfrak{b}})$. This means their exists $n_0 \in \mathbb{N}$ such that

$$d'_{\mathfrak{b}}(Sx_{n+1}, Sx_{m+1}) = d'_{\mathfrak{b}}(Tx_n, Tx_m) < \epsilon \quad \forall m, n \geq n_0.$$

Showing that $\{Sx_n\}$ is a Cauchy sequence in $(X, d'_{\mathfrak{b}})$. Now as $S(X)$ is closed with respect to $d'_{\mathfrak{b}}$, and $(X, d'_{\mathfrak{b}})$ is complete, so there exists $x^* \in S(X)$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x^*. \tag{10}$$

Also, T is $\check{\mathcal{G}}_{\mathfrak{b}}$-continuous together with $d'_{\mathfrak{b}}$-compatibility of T and S. Hence, we conclude

$$\lim_{n\to\infty} d'_{\mathfrak{b}}(TSx_n, STx_n) = 0. \tag{11}$$

Now, consider

$$\begin{aligned} d'_{\mathfrak{b}}(Sx^*, Tx^*) &\leq \mathfrak{b}\{d'_{\mathfrak{b}}(Sx^*, STx_n) + d'_{\mathfrak{b}}(STx_n, Tx^*)\} \\ &\leq \mathfrak{b}d'_{\mathfrak{b}}(Sx^*, STx_n) + \mathfrak{b}^2\{d'_{\mathfrak{b}}(STx_n, TSx_n) + d'_{\mathfrak{b}}(TSx_n, Tx^*)\}. \end{aligned}$$

Taking $n \to \infty$ and by (10), we get

$$d'_{\mathfrak{b}}(Sx^*, Tx^*) = 0.$$

Showing that $Sx^* = Tx^*$, thus $x^* \in \mathcal{C}(T, S) \Rightarrow \mathcal{C}(T, S) \neq$.

Theorem 2. *Let $(X, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}MS$ equipped with a $\mathcal{DG}$ and a continuous metric $d_{\mathfrak{b}}$, let $d'_{\mathfrak{b}}$ be another continuous metric. Define two functions $T, S : X \to X$ where (T, S) satisfies $\psi - \phi$-contraction with respect to $d_{\mathfrak{b}}$ and we have,*

(1) S is a continuous mapping and $S(X)$ is closed;

(2) $T(X) \subseteq S(X)$;

(3) $E(\bar{\mathcal{G}})$ is a transitive set;

(4) If $d_{\mathfrak{b}} \ngeq d'_{\mathfrak{b}}$ with $T : (X, d_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ is g-Cauchy sequence on X;

(5) $(X, d_{\mathfrak{b}}, \bar{\mathcal{G}})$ have property $\mathcal{A}$ (1).

As a consequence,

$$V(T, S) \neq \ \Leftrightarrow \mathcal{C}(T, S) \neq . \tag{12}$$

Proof. *We can observe that all other conditions of this result coincide with Theorem 1, so it is sufficient to check that even with assumption (5), we will get*

$$\text{if} \ \ V(T, S) \neq \ \ \text{then}\ \mathcal{C}(T, S) \neq .$$

As $\{Sx_n\}$ is a Cauchy sequence and $S(X)$ is closed in X, so for some $x^* \in X$ we have

$$\lim_{n\to\infty} Sx_n = Sx^* = \lim_{n\to\infty} Tx_n. \tag{13}$$

We claim that $x^* \in \mathcal{C}(T,S)$, since if, x^* is not a coincident point of T and S, then $Tx^* \neq Sx^*$. So,

$$d_{\mathfrak{b}}(Tx^*, Sx^*) > 0. \tag{14}$$

Then $(Sx_n, Sx^*) \in E(\bar{\mathcal{G}})$ for all n because $(X, d_{\mathfrak{b}}, \bar{\mathcal{G}})$ attains the property $\mathcal{A}$. Also,

$$d_{\mathfrak{b}}(Sx^*, Tx^*) \leq \mathfrak{b}\{d_{\mathfrak{b}}(Sx^*, Tx_{n(k)}) + d_{\mathfrak{b}}(Tx_{n(k)}, Tx^*)\},$$

or

$$d_{\mathfrak{b}}(Sx^*, Tx^*) - \mathfrak{b}d_{\mathfrak{b}}(Sx^*, Tx_{n(k)}) \leq \mathfrak{b}d_{\mathfrak{b}}(Tx_{n(k)}, Tx^*).$$

We have

$$\begin{aligned} \phi(d_{\mathfrak{b}}(Sx^*, Tx^*) - \mathfrak{b}d_{\mathfrak{b}}(Sx^*, Tx_{n(k)})) &\leq \phi(\mathfrak{b}d_{\mathfrak{b}}(Tx_{n(k)}, Tx^*)) \\ &\leq \psi(d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*))\phi(\delta(\mathfrak{b}\mathbb{M}(Sx_{n(k)}, Sx^*))) \\ &\leq \psi(d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*))\phi(\mathbb{M}(Sx_{n(k)}, Sx^*)), \end{aligned} \tag{15}$$

then

$$\begin{aligned} \mathbb{M}((Sx_{n(k)}, Sx^*) = \max\Bigg\{ &\frac{d_{\mathfrak{b}}(Sx_{n(k)}, Tx_{n(k)})d_{\mathfrak{b}}(Tx^*, Sx^*)}{d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*)}, d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*), d_{\mathfrak{b}}(Sx_{n(k)}, Tx_{n(k)}), d_{\mathfrak{b}}(Sx^*, Tx^*), \\ &\frac{d_{\mathfrak{b}}(Sx_{n(k)}, Tx^*) + d_{\mathfrak{b}}(Sx^*, Tx_{n(k)})}{2\mathfrak{b}} \Bigg\}. \end{aligned}$$

Using (13) and taking limit $n \to \infty$, we will get

$$\lim_{n\to\infty} \mathbb{M}((Sx_{n(k)}, Sx^*) = \lim_{n\to\infty} d_{\mathfrak{b}}(Sx^*, Tx^*) > 0.$$

From the condition (15), we have $\lim_{n\to\infty} \psi(d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*) = 1$, hence $\lim_{n\to\infty} d_{\mathfrak{b}}(Sx_{n(k)}, Sx^*) = d_{\mathfrak{b}}(Tx^*, Sx^*) = 0$. It is a contradiction to (14). Hence, $x^* \in \mathcal{C}(T,S)$ proving that $\mathcal{C}(T,S) \neq$.

Theorem 3. *If all the conditions of Theorem 1 are adopted and for any $x, y \in \mathcal{C}(T,S)$ with $Sx \neq Sy$, and it is true that $(Sx, Sy) \in E(\bar{\mathcal{G}})$ then $V(T,S) \neq\ \Leftrightarrow \mathcal{C}_m(T,S) \neq$.*

Proof. *From Theorem 1, there exists $x \in X$ such that $Sx = Tx$. Suppose that there exists another element $y \in X$ then $Sy = Ty$. We will prove that $Sx = Sy$. On the contrary, suppose it is not true then by assumption $(Sx, Sy) \in E(\bar{\mathcal{G}})$. Now, we may write*

$$\begin{aligned} \phi(d_{\mathfrak{b}}(Ts, Sy)) &\leq \psi(d_{\mathfrak{b}}(Sx, Sy))\phi(\delta(\mathbb{M}(Sx, Sy))) \\ &\leq \psi(d_{\mathfrak{b}}(Sx, Sy))\phi(\mathbb{M}(Sx, Sy)) \\ &\leq \phi(\mathbb{M}(Sx, Sy)) \\ &= \phi d_{\mathfrak{b}}(Tx, Ty)). \end{aligned}$$

Thus, $\psi(Sx, Sy) = 1$, hence $Sy = Sx$. Next, as x is the coincidence point so $x_n = x$. We may construct a sequence such that $Tx_{n-1} = Sx_n = Sx = Ts$ for every $n \in \mathbb{N}$. Now, let $r = Sx$ then $Sr = SSx = STx$. Then, $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Tx$, in $(X, d'_{\mathfrak{b}})$. Also,

$$\lim_{n\to\infty} d'_{\mathfrak{b}}(STx_n, TSx_n) = 0,$$

since S and T are compatible with respect to $d'_{\mathfrak{b}}$. This means $STx = TSx$, so

$$Sr = STx = TSx = Tr,$$

showing that $r \in \mathcal{C}(T, S)$. Also, from the above calculations

$$Sr = Tr = Sx = r$$

this shows $r \in \mathcal{C}_m(S, T)$.

4. Consequences and an Application

If we use $\delta(\mathbb{M}((Sx, Sy))) = d_{\mathfrak{b}}(Sx, Sy))$, then we will get following result:

Corollary 1. *Let $(X, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}MS$ with $\bar{\mathcal{G}}$ as a $\mathcal{DG}$, and $d_{\mathfrak{b}}$ be continuous. Let $d'_{\mathfrak{b}}$ be another continuous function and the functions (T, S) satisfy an $\psi - \phi$-contraction w.r.t $d_{\mathfrak{b}}$ satisfying the axioms given below:*

(1) $S : (X, d'_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ is continuous with $S(X)$ is closed w.r.t $d'_{\mathfrak{b}}$;

(2) $T(X) \subseteq S(X)$;

(3) $E(\bar{\mathcal{G}})$ is a transitive set;

(4) If $d_{\mathfrak{b}} \ngeq d'_{\mathfrak{b}}$, assume that $T : (X, d_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ is S-Cauchy sequence in X;

(5) $T : (X, d'_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ is $\bar{\mathcal{G}}_{\mathfrak{b}}$-continuous with T and S also being $d'_{\mathfrak{b}}$-compatible;

(6) there exist two functions $\phi \in \Phi$ and $\psi \in \Psi$ such that $\phi(d_{\mathfrak{b}}(Ts, Tt)) \leq \psi(Sx, Sy)\phi(d_{\mathfrak{b}}(Sx, Sy)))$, then

$$V(T, S) \neq \ \Rightarrow \mathcal{C}(T, S) \neq . \tag{16}$$

Example 2. *Let $X = [0, \infty) \subseteq \mathbb{R}$ and define $d_{\mathfrak{b}}, d'_{\mathfrak{b}} : X \times X \to X$ such that*

$$d_{\mathfrak{b}}(x, y) = |x - y|^2 \quad and \quad d'_{\mathfrak{b}}(x, y) = \mathcal{R}|x - y|^2,$$

where $\mathcal{R} > 1$ is any constant. It is easy to check that $d_{\mathfrak{b}}$ and $d'_{\mathfrak{b}}$ are $\mathfrak{b}$ metrics on X, also $d_{\mathfrak{b}} < d'_{\mathfrak{b}}$. Now suppose that

$$E(\bar{\mathcal{G}}) = \{(x, y) : x = y \ or \ x, y \in [0, 1] \ with \ x \leq y\}.$$

And, define two continuous self mappings on X as follow

$$Sx = x^2 \quad \text{and} \quad Tx = \ln\left(1+\frac{x^2}{4}\right) \quad \text{for all } x.$$

Now, consider $(Sx, Sy) \in E(\bar{\mathcal{G}})$, note that if $x = y$ then $(Tx, Ty) \in E(\bar{\mathcal{G}})$. Also, if $(Sx, Sy) \in E(\bar{\mathcal{G}})$ and $Sx \leq Sy$ then $Sx = x^2, Sy = y^2 \in [0,1]$ and $x^2 = Sx \leq Sy = y^2$. Therefore, we have

$$Tx = \ln\left(1+\frac{x^2}{4}\right) \leq \ln\left(1+\frac{y^2}{4}\right) = Ty,$$

showing that $(Tx, Ty) \in E(\bar{\mathcal{G}})$. Next, define $\phi(x) = \frac{x}{4}$ and $\psi : X \times X \to [0,1]$ as

$$\psi(x,y) = \begin{cases} \dfrac{\ln\left(2+\frac{\sqrt{|x-y|}}{2}\right)}{\sqrt{|x-y|}}, & \text{if } \quad |x-y| > 1 \\ 0, & \text{if } \ |x-y| \in [0,1). \end{cases}$$

Now, we have

$$\begin{aligned} \phi(d_{\mathfrak{b}}(Tx, Ty) &= \frac{|\ln(1+\frac{x^2}{4}) - \ln(1+\frac{y^2}{4})|^2}{4} \\ &= \frac{\left(\ln(1+\frac{y^2}{4}) - \ln(1+\frac{x^2}{4})\right)^2}{4} \\ &= \frac{\ln\left(\frac{2+\frac{y^2}{2}}{1+\frac{x^2}{4}}\right)}{4} \\ &= \frac{\ln\left(2+\frac{\frac{y^2}{2}-\frac{x^2}{2}}{1+\frac{x^2}{4}}\right)}{4} \\ &\leq \frac{\ln(2+\left|\frac{y^2}{2}-\frac{x^2}{2}\right|)}{4} \\ &= \frac{\ln(2+\frac{1}{2}\left|y^2-x^2\right|)}{4}\frac{\left|y^2-x^2\right|}{\left|y^2-x^2\right|} \\ &\leq \frac{\ln(2+\frac{1}{2}\left|y^2-x^2\right|)}{\left|y^2-x^2\right|}\frac{\left|y^2-x^2\right|^2}{4} \\ &= \psi(Sx, Sy)\phi(d_{\mathfrak{b}}(Sx, Sy), \end{aligned}$$

showing that the contraction condition holds. Now, as $d_{\mathfrak{b}} < d'_{\mathfrak{b}}$ we will show that $T : (X, d_{\mathfrak{b}}) \to (X, d'_{\mathfrak{b}})$ is S-Cauchy. If for $\epsilon > 0$ the sequence $\{x_n\} \in X$ with $\{Sx_n\}$ being Cauchy in $(X, d_{\mathfrak{b}})$ there exists $n_0 \in \mathbb{N}$ such that

$$d_{\mathfrak{b}}(Sx_n, Sx_k) < \frac{\epsilon}{\mathcal{R}} \quad \forall\ n, k \geq n_0.$$

Therefore,

$$
\begin{aligned}
d'_{\mathfrak{b}}(Tx_n, Tx_k) &= \mathcal{R}\left| \ln(1+\frac{(x_n)^2}{4}) - \ln(1+\frac{(x_k)^2}{4})\right|^2 \\
&= \mathcal{R}\left(\ln(1+\frac{(x_k)^2}{4}) - \ln(1+\frac{(x_n)^2}{4})\right)^2 \\
&= \mathcal{R}\ln\left(\frac{2+\frac{(x_k)^2}{2}}{1+\frac{(x_n)^2}{4}}\right) \\
&= \mathcal{R}\ln\left(2+\frac{\frac{(x_k)^2}{2}-\frac{(x_n)^2}{2}}{1+\frac{(x_n)^2}{4}}\right) \\
&\leq \mathcal{R}\ln\left(2+\left|\frac{(x_k)^2}{2}-\frac{(x_n)^2}{2}\right|\right) \\
&= \mathcal{R}\frac{\ln(2+\frac{1}{2}\left|(x_k)^2-(x_n)^2\right|)}{\left|(x_k)^2-(x_n)^2\right|^2}\left|(x_k)^2-(x_n)^2\right|^2 \\
&\leq \mathcal{R}\left|(x_k)^2-(x_n)^2\right|^2 \\
&= \mathcal{R}d_{\mathfrak{b}}(Sx_n, Sx_k) \\
&< \mathcal{R}\frac{\epsilon}{\mathcal{R}} \\
&= \epsilon,
\end{aligned}
$$

showing that T is $S-$Cauchy. Also, by using the fact that T and S are $d_{\mathfrak{b}}$-compatible, we may write

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x,$$

then $\ln(1+\frac{x}{4}) = x$ this implies $x = 0$. And letting $n \to \infty$ we have

$$d'_{\mathfrak{b}}(STx_n, TSx_n) = \mathcal{R}\left|\left(\ln\left(1+\frac{(x_n)^2}{4}\right)\right)^2 - \ln\left(1+\frac{(x_n)^4}{4}\right)\right|^2 \to 0.$$

Finally, we have

$$(S0, T0) = (0, 0)$$

belongs to set of edges, that is, $V(T,S) \neq \ \Rightarrow \mathcal{C}(T,S) \neq$.

Now, we will prove that our results can be used to analyze the existence of solutions for fractional boundary value problems of the Caputo variety with fractional order γ. Here, we have $\gamma \in (n-1, n]$ where the integer n is such that $n \geq 2$.
Let $\mathcal{Q}(x)$ be a continuous function and γ be a real number. The γ order Caputo derivative is as follows

$$^{c}D^{\gamma}\mathcal{Q} = I^{\lceil\gamma\rceil-\gamma}D^{\lceil\gamma\rceil}\mathcal{Q}.$$

Here, I^{γ} is the Riemann-Liouville integral operator defined below

$$I^{\gamma}\mathcal{Q}(x)=\frac{1}{\Gamma(\gamma)}\int_0^x (x-\xi)^{\gamma-1}\ \mathcal{Q}(\xi)\ d\xi,\ \Gamma(\gamma)=\int_0^{\infty} x^{\gamma-1}e^{-x}\ dx.$$

I^0 is the identity operator. We will consider a nonlinear $\mathcal{CFDE}$ of the form

$$({}^{c}D^{\gamma}\mathfrak{y})(x)=\upsilon\big(x,\mathfrak{y}(x)\big),\ \ x\in[0,1] \text{ and } \gamma\in(n-1,n] \tag{17}$$

along with

$$\mathfrak{y}=\mathfrak{y}'=\cdots=\mathfrak{y}^{(n-2)}=0 \text{ at } x=0, \text{ and } \mathfrak{y}=\int_0^{\eta}\mathfrak{y}(\xi)\ d\xi \text{ at } x=1, \tag{18}$$

where $\upsilon:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $\eta\in[0,1]$.
This is a specific Volterra integral equation and the solution $\mathfrak{y}\in C[0,1]$ is of the form

$$\mathfrak{y}(x)=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+\frac{1}{\Gamma(\gamma)}\int_0^x (x-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi,$$

where a_i are real numbers.
The boundary conditions given in (4.3) mean

$$\begin{aligned}&a_0=a_1=\cdots=a_{n-2}=0\\&\mathfrak{y}(x)=a_{n-1}x^{n-1}+I^{\gamma}\upsilon\big(x,\mathfrak{y}(x)\big)\end{aligned}$$

and

$$\mathfrak{y}(1)=\int_0^{\eta}\mathfrak{y}(\xi)\ d\xi=a_{n-1}+\frac{1}{\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi,$$

$$\begin{aligned}\Rightarrow\ a_{n-1}&=\int_0^{\eta}\mathfrak{y}(\xi)\ d\xi-\frac{1}{\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi\\&=\int_0^{\eta}\Big(a_{n-1}\xi^{n-1}+\frac{1}{\Gamma(\gamma)}\int_0^{\xi}(\xi-\vartheta)^{\gamma-1}\ \upsilon\big(\vartheta,\mathfrak{y}(\vartheta)\big)\ d\vartheta\Big)d\xi-\frac{1}{\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi\\&=\frac{\eta^n}{n}a_{n-1}+\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\int_0^{\xi}(\xi-\vartheta)^{\gamma-1}\ \upsilon\big(\vartheta,\mathfrak{y}(\vartheta)\big)\ d\vartheta d\xi-\frac{1}{\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi\\&=\frac{n}{(n-\eta^n)\Gamma(\gamma)}\int_0^{\eta}\int_0^{\xi}(\xi-\vartheta)^{\gamma-1}\ \upsilon\big(\vartheta,\mathfrak{y}(\vartheta)\big)\ d\vartheta d\xi-\frac{n}{(n-\eta^n)\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi.\end{aligned}$$

Substituting this value in $\mathfrak{y}(x)$ gives the solution of the boundary value problem given above to be the same as that of the Volterra integral equation, i.e.

$$\begin{aligned}\mathfrak{y}(x)=&\frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)}\int_0^{\eta}\int_0^{\xi}(\xi-\vartheta)^{\gamma-1}\ \upsilon\big(\vartheta,\mathfrak{y}(\vartheta)\big)\ d\vartheta d\xi\\&-\frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)}\int_0^1(1-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi+\frac{1}{\Gamma(\gamma)}\int_0^x(x-\xi)^{\gamma-1}\ \upsilon\big(\xi,\mathfrak{y}(\xi)\big)\ d\xi.\end{aligned}$$

To transform this into a fixed point problem, construct $\mathbb{T} : C[0,1] \to C[0,1]$ as below

$$\mathbb{T}\big(\mathfrak{y}(x)\big) = \frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)} \int_0^\eta \int_0^\xi (\xi-\vartheta)^{\gamma-1}\, \upsilon\big(\vartheta, \mathfrak{y}(\vartheta)\big)\, d\vartheta d\xi - \frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)} \int_0^1 (1-\xi)^{\gamma-1}\, \upsilon\big(\xi, \mathfrak{y}(\xi)\big)\, d\xi + \frac{1}{\Gamma(\gamma)} \int_0^x (x-\xi)^{\gamma-1}\, \upsilon\big(\xi, \mathfrak{y}(\xi)\big)\, d\xi.$$

The solution of the boundary value problem described in equations (4.2)-(4.3) is given by $\mathbb{T}\mathfrak{y} = \mathfrak{y}$. We will now show that a solution for this fixed point problem exists by considering $\big(C[0,1], ||\cdot||_\infty^*\big)$ as a base space with a directed graph $\bar{\mathcal{G}}$. We define the $\mathfrak{b}$-metric $||\cdot||_\infty^*$ as

$$||\mathcal{Q}-\mathcal{R}||_\infty^* = \sup_{x\in[0,1]} \big|\mathcal{Q}(x)-\mathcal{R}(x)\big|^2, \text{ with } \mathfrak{b}=2.$$

The following conditions are also important for our discussion.

I. There exists $\mathcal{Q}_0 \in C[0,1]$ such that $(\mathcal{Q}_0, \mathbb{T}\mathcal{Q}_0) \in E(\bar{\mathcal{G}})$.

II. $\mathbb{T}$ is edge preserving on $C[0,1]$,

III. For a sequence $\{\mathcal{Q}_n\} \in C[0,1]$ converging to $\mathcal{Q}$ with $(\mathcal{Q}_n, \mathcal{Q}_{n+1}) \in E(\bar{\mathcal{G}})\ \forall\ n \in \mathbb{N}$, we must have $(\mathcal{Q}_n, \mathcal{Q})$ is an edge for each $n \in \mathbb{N}$,

IV. Set of edges is transitive,

V. $\exists\ \phi \in \Phi$ with $\phi(r) < r,\ \forall\ r \in (0,1]$ such that

$$\Big|\upsilon\big(x, \mathcal{Q}(x)\big) - \upsilon\big(x, \mathcal{R}(x)\big)\Big| \leq K_1\phi\Big(\big|\mathcal{Q}(x)-\mathcal{R}(x)\big|\Big), \tag{19}$$

and

$$\Big|\upsilon\big(x, \mathcal{Q}(x)\big) - \upsilon\big(x, \mathcal{R}(x)\big)\Big|^2 \leq K_2\phi\Big(\big|\mathcal{Q}(x)-\mathcal{R}(x)\big|^2\Big), \tag{20}$$

where

$$K_1 \leq \frac{(n-\eta^n)\Gamma(\gamma+2)}{n\eta^{\gamma+1} + (\gamma+1)(2n-\eta^n)},$$

$$K_2 \leq \frac{(2\gamma-1)(n-\eta^n)^2\Gamma(\gamma+1)\Gamma(\gamma+2)}{\gamma(\gamma+1)\Big[n^2\eta^{2\gamma} + 2\gamma\big[n^2+(n-\eta^n)^2\big]\Big] + 2(2\gamma-1)n\big[n^{\gamma+2} + \eta^{\gamma+1}(n-\eta^n) + (\gamma+1)(n-\eta^n)\big]},$$

and

$$K_1^2\big(\phi(||\ \mathcal{Q}-\mathcal{R}\ ||_\infty)\big)^2 \leq K_2\big(\phi(||\ \mathcal{Q}-\mathcal{R}\ ||_\infty^*)\big).$$

VI.

$$\sqrt{\int_0^\eta \int_0^\xi |\ \xi-\vartheta\ |^{2(\gamma-1)}\ d\vartheta d\xi}\ \sqrt{\int_0^\eta \int_0^\xi \Big|\upsilon\big(\vartheta, \mathcal{Q}(\vartheta)\big) - \upsilon\big(\vartheta, \mathcal{R}(\vartheta)\big)\Big|^2 d\vartheta d\xi} \geq 1,$$

$$\sqrt{\int_0^1 \mid 1-\xi \mid^{2(\gamma-1)} \ d\xi} \sqrt{\int_0^1 \Big|\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big|^2 \ d\xi} \geq 1$$

and

$$\sqrt{\int_0^x \mid x-\xi \mid^{2(\gamma-1)} \ d\xi} \sqrt{\int_0^x \Big|\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big|^2 \ d\xi} \geq 1.$$

Using these conditions, we will now consider the following result.

Theorem 4. *Consider a complete* $\mathfrak{b}MS$ $(X, d_{\mathfrak{b}})$ *accompanied with a directed graph* $\bar{\mathcal{G}} = \big(V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}})\big)$ *and conditions **I** to **VI** are satisfied. Then the integral operator* $\mathbb{T}$ *defined above has a fixed point* $\mathcal{Q}^* \in C[0,1]$*. Further, this fixed point is a solution for the boundary value problem given in (4.2) - (4.3).*

Proof. Let S be the identity map defined on $C[0,1]$. From **III**, we see that $\mathbb{T}$ is S-edge preserving with respect to $\bar{\mathcal{G}}$. From **IV**, we have $(C[0,1], ||\cdot||_\infty, \bar{\mathcal{G}})$ has the property I. Finally, **V** shows that $E(\bar{\mathcal{G}})$ has the transitivity property.
Using **I**, $\forall\ (\mathcal{Q}, \mathcal{R}) \in E(\bar{\mathcal{G}})$, consider

$$\begin{aligned}\Big|\mathbb{T}\big(\mathcal{Q}(x)\big) - \mathbb{T}\big(\mathcal{R}(x)\big)\Big|^2 = \Bigg| & \frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)} \int_0^\eta \int_0^\xi (\xi-\vartheta)^{\gamma-1} \ \Big(\upsilon\big(\vartheta, \mathcal{Q}(\vartheta)\big) - \upsilon\big(\vartheta, \mathcal{R}(\vartheta)\big)\Big) \ d\vartheta d\xi \\ & - \frac{nx^{n-1}}{(n-\eta^n)\Gamma(\gamma)} \int_0^1 (1-\xi)^{\gamma-1} \Big(\upsilon\big(\vartheta, \mathcal{Q}(\xi)\big) - \upsilon\big(\vartheta, \mathcal{R}(\xi)\big)\Big) \ d\xi \\ & + \frac{1}{\Gamma(\gamma)} \int_0^x (x-\xi)^{\gamma-1} \ \Big(\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big) \ d\xi \Bigg|^2.\end{aligned}$$

Expanding the square on the right hand side and further manipulation gives

$$\begin{aligned}\Big|\mathbb{T}\big(\mathcal{Q}(x)\big) - \mathbb{T}\big(\mathcal{R}(x)\big)\Big|^2 \leq & \frac{n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi \mid \xi-\vartheta \mid^{\gamma-1} \ \Big|\upsilon\big(\vartheta, \mathcal{Q}(\vartheta)\big) - \upsilon\big(\vartheta, \mathcal{R}(\vartheta)\big)\Big| \ d\vartheta d\xi\right]^2 \\ & + \frac{n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^1 \mid 1-\xi \mid^{\gamma-1} \ \Big|\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big| \ d\xi\right]^2 \\ & + \frac{1}{\Gamma^2(\gamma)} \left[\int_0^x \mid x-\xi \mid^{\gamma-1} \ \Big|\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big| \ d\xi\right]^2 \\ & + \frac{2n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi \mid \xi-\vartheta \mid^{\gamma-1} \ \Big|\upsilon\big(\vartheta, \mathcal{Q}(\vartheta)\big) - \upsilon\big(\vartheta, \mathcal{R}(\vartheta)\big)\Big| \ d\vartheta d\xi\right] \\ & \left[\int_0^1 \mid 1-\xi \mid^{\gamma-1} \ \Big|\upsilon\big(\xi, \mathcal{Q}(\xi)\big) - \upsilon\big(\xi, \mathcal{R}(\xi)\big)\Big| \ d\xi\right]\end{aligned}$$

$$+ \frac{2nx^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)}\left[\int_0^\eta\int_0^\xi \mid \xi-\vartheta \mid^{\gamma-1} \left|\upsilon\big(\vartheta,\mathcal{Q}(\vartheta)\big)-\upsilon\big(\vartheta,\mathcal{R}(\vartheta)\big)\right| d\vartheta d\xi\right]$$
$$\left[\int_0^x \mid x-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right]$$
$$+ \frac{2nx^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)}\left[\int_0^1 \mid 1-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right]$$
$$\left[\int_0^x \mid x-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right].$$

The Cauchy-Schwarz inequality states

$$\int_a^b T(x)S(x)\ dx \le \sqrt{\int_a^b T^2(x)\ dx}\ \sqrt{\int_a^b x^2(x)\ dx}\,.$$

Assuming that the value on the right hand side is not less than 1 and squaring both sides, we get

$$\left[\int_a^b T(x)S(x)\ dx\right]^2 \le \int_a^b T^2(x)\ dx \int_a^b x^2(x)\ dx.$$

Applying this to the first 3 terms of the right-hand side in the contraction condition above, we get

$$\left|\mathbb{T}\big(\mathcal{Q}(x)\big)-\mathbb{T}\big(\mathcal{R}(x)\big)\right|^2 \le \frac{n^2x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)}\int_0^\eta\int_0^\xi \mid \xi-\vartheta \mid^{2(\gamma-1)} d\vartheta d\xi \int_0^\eta\int_0^\xi \left|\upsilon\big(\vartheta,\mathcal{Q}(\vartheta)\big)-\upsilon\big(\vartheta,\mathcal{R}(\vartheta)\big)\right|^2 d\vartheta d\xi$$
$$+ \frac{n^2x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)}\int_0^1 \mid 1-\xi \mid^{2(\gamma-1)}\ d\xi \int_0^1 \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right|^2 d\xi$$
$$+ \frac{1}{\Gamma^2(\gamma)}\int_0^x \mid x-\xi \mid^{2(\gamma-1)}\ d\xi \int_0^x \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right|^2 d\xi$$
$$+ \frac{2n^2x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)}\left[\int_0^\eta\int_0^\xi \mid \xi-\vartheta \mid^{\gamma-1} \left|\upsilon\big(\vartheta,\mathcal{Q}(\vartheta)\big)-\upsilon\big(\vartheta,\mathcal{R}(\vartheta)\big)\right| d\vartheta d\xi\right]$$
$$\left[\int_0^1 \mid 1-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right]$$
$$+ \frac{2nx^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)}\left[\int_0^\eta\int_0^\xi \mid \xi-\vartheta \mid^{\gamma-1} \left|\upsilon\big(\vartheta,\mathcal{Q}(\vartheta)\big)-\upsilon\big(\vartheta,\mathcal{R}(\vartheta)\big)\right| d\vartheta d\xi\right]$$
$$\left[\int_0^x \mid x-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right]$$
$$+ \frac{2nx^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)}\left[\int_0^1 \mid 1-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right]$$
$$\left[\int_0^x \mid x-\xi \mid^{\gamma-1} \left|\upsilon\big(\xi,\mathcal{Q}(\xi)\big)-\upsilon\big(\xi,\mathcal{R}(\xi)\big)\right| d\xi\right].$$

Using (4.4) and (4.5), we get

$$\begin{aligned}
\Big|\mathbb{T}(\mathcal{Q}(x)) - \mathbb{T}(\mathcal{R}(x))\Big|^2 \leq{}& \frac{n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{2(\gamma-1)} \; d\vartheta d\xi \; \int_0^\eta \int_0^\xi K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\vartheta d\xi \\
&+ \frac{n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \int_0^1 |\, 1 - \xi \,|^{2(\gamma-1)} \; d\xi \int_0^1 K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\xi \\
&+ \frac{1}{\Gamma^2(\gamma)} \int_0^x |\, x - \xi \,|^{2(\gamma-1)} \; d\xi \; \int_0^x K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\xi \\
&+ \frac{2n^2 x^{2(n-1)}}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\vartheta d\xi \right] \\
&\left[\int_0^1 |\, 1 - \xi \,|^{\gamma-1} \, K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&+ \frac{2n x^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\vartheta d\xi \right] \\
&\left[\int_0^x |\, x - \xi \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&+ \frac{2n x^{(n-1)}}{(n-\eta^n)\Gamma^2(\gamma)} \left[\int_0^1 |\, 1 - \xi \,|^{\gamma-1} \, K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&\left[\int_0^x |\, x - \xi \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right].
\end{aligned}$$

Since $x \in [0,1]$, we have

$$\begin{aligned}
\Big|\mathbb{T}(\mathcal{Q}(x)) - \mathbb{T}(\mathcal{R}(x))\Big|^2 \leq{}& \int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{2(\gamma-1)} \; d\vartheta d\xi \; \int_0^\eta \int_0^\xi K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\vartheta d\xi \\
&+ \frac{n^2}{(n-\eta^n)^2\Gamma^2(\gamma)} \int_0^1 |\, 1 - \xi \,|^{2(\gamma-1)} \; d\xi \int_0^1 K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\xi \\
&+ \frac{1}{\Gamma^2(\gamma)} \int_0^x |\, x - \xi \,|^{2(\gamma-1)} \; d\xi \; \int_0^x K_2 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,|^2 \,\big) \; d\xi \\
&+ \frac{2n^2}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\vartheta d\xi \right] \\
&\left[\int_0^1 |\, 1 - \xi \,|^{\gamma-1} \, K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&+ \frac{2n}{(n-\eta^n)\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\vartheta d\xi \right] \\
&\left[\int_0^x |\, x - \xi \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&+ \frac{2n}{(n-\eta^n)\Gamma^2(\gamma)} \left[\int_0^1 |\, 1 - \xi \,|^{\gamma-1} \, K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
&\left[\int_0^x |\, x - \xi \,|^{\gamma-1} \; K_1 \phi\big(\, |\, \mathcal{Q} - \mathcal{R} \,| \,\big) \; d\xi \right] \\
\leq{}& \frac{n^2}{(n-\eta^n)^2\Gamma^2(\gamma)} \left[\int_0^\eta \int_0^\xi |\, \xi - \vartheta \,|^{2(\gamma-1)} \; d\vartheta d\xi + \int_0^1 |\, 1 - \xi \,|^{2(\gamma-1)} \; d\xi \right.
\end{aligned}$$

$$+\frac{(n-\eta^n)^2}{n^2}\int_0^x |\, x-\xi \,|^{2(\gamma-1)}\ d\xi\Bigg]\Bigg[K_2\phi\big(\,||\,\mathcal{Q}-\mathcal{R}\,||_\infty^*\,\big)\Bigg]$$
$$+\frac{2n^2}{(n-\eta^n)^2\Gamma^2(\gamma)}\Bigg[\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\ d\vartheta d\xi\int_0^1 |\,1-\xi\,|^{\gamma-1}\ d\xi$$
$$+\frac{(n-\eta^n)}{n}\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\ d\vartheta d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\ d\xi$$
$$+\frac{(n-\eta^n)}{n}\int_0^1 |\,1-\xi\,|^{\gamma-1}\ d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\ d\xi\Bigg]\Bigg[K_1^2\Big(\phi\big(\,||\,\mathcal{Q}-\mathcal{R}\,||_\infty\,\big)\Big)^2\Bigg]$$
$$\leq\frac{2n^2}{(n-\eta^n)^2\Gamma^2(\gamma)}\Bigg[\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{2(\gamma-1)}\ d\vartheta d\xi+\int_0^1 |\,1-\xi\,|^{2(\gamma-1)}\ d\xi$$
$$+\frac{(n-\eta^n)^2}{n^2}\int_0^x |\,x-\xi\,|^{2(\gamma-1)}\ d\xi$$
$$+\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\ d\vartheta d\xi\int_0^1 |\,1-\xi\,|^{\gamma-1}\ d\xi$$
$$+\frac{(n-\eta^n)}{n}\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\ d\vartheta d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\ d\xi$$
$$+\frac{(n-\eta^n)}{n}\int_0^1 |\,1-\xi\,|^{\gamma-1}\ d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\ d\xi\Bigg]\Bigg[K_2\phi\big(\,||\,\mathcal{Q}-\mathcal{R}\,||_\infty^*\,\big)\Bigg].$$

Let

$$c=\frac{2n^2}{(n-\eta^n)^2\Gamma^2(\gamma)}\sup_{x\in(0,1)}\Bigg[\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{2(\gamma-1)}\,d\vartheta d\xi\ +\ \int_0^1 |\,1-\xi\,|^{2(\gamma-1)}\,d\xi$$
$$+\ \frac{(n-\eta^n)^2}{n^2}\int_0^x |\,x-\xi\,|^{2(\gamma-1)}\,d\xi\ +\ \int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\,d\vartheta d\xi\ \int_0^1 |\,1-\xi\,|^{\gamma-1}\,d\xi$$
$$+\ \frac{(n-\eta^n)}{n}\int_0^\eta\int_0^\xi |\,\xi-\vartheta\,|^{\gamma-1}\,d\vartheta d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\,d\xi$$
$$+\ \frac{(n-\eta^n)}{n}\int_0^1 |\,1-\xi\,|^{\gamma-1}\,d\xi\int_0^x |\,x-\xi\,|^{\gamma-1}\,d\xi\Bigg].$$

Further calculations give

$$c=\frac{2n^2}{(n-\eta^n)^2\Gamma^2(\gamma)}\Bigg[\frac{\eta^{2\gamma}}{2\gamma(2\gamma-1)}+\frac{1}{2\gamma-1}+\frac{(n-\eta^n)^2}{n^2(2\gamma-1)}$$
$$+\frac{n^{\gamma+1}}{\gamma^2(\gamma+1)}+\frac{\eta^{\gamma+1}(n-\eta^n)}{n\gamma^2(\gamma+1)}+\frac{n-\eta^n}{n\gamma^2}\Bigg].$$

We see that $K_2\leq\dfrac{1}{c}$. This gives us

$$\Big|\mathbb{T}\big(\mathcal{Q}(x)\big)-\mathbb{T}\big(\mathcal{R}(x)\big)\Big|^2\leq\phi\big(||\mathcal{Q}-\mathcal{R}||_\infty^*\big)$$

$$= \phi\big(d_{\mathfrak{b}}(\mathcal{Q},\mathcal{R})\big).$$

Define an auxiliary function ψ with domain in $C[0,1]\times C[0,1]$ and range in $[0,1]$ by

$$\psi(\mathcal{Q},\mathcal{R}) = \begin{cases} \dfrac{\phi\big(||\mathcal{Q}-\mathcal{R}||^*_\infty\big)}{||\mathcal{Q}-\mathcal{R}||^*_\infty}, & \text{if } \mathcal{Q}\neq\mathcal{R},\\ 0, & \text{if } \mathcal{Q}=\mathcal{R}. \end{cases}$$

We see that all conditions of Theorem 3.5 are satisfied, and so there must be a fixed point $\mathcal{Q}^*$ of $\mathbb{T}$ in $C[0,1]$.

We now consider the case where $n = 2$. This gives

$$K_1 \leq \frac{(2-\eta^2)\Gamma(\gamma+2)}{2\eta^{\gamma+1}+(\gamma+1)(4-\eta^2)}.$$

Since $\eta \in [0,1]$, putting $\eta = 1$ gives

$$K_1 \leq \frac{\Gamma(\gamma+2)}{5+3\gamma}.$$

Similarly, we get

$$K_2 \leq \frac{(2\gamma-1)\Gamma(\gamma+1)\Gamma(\gamma+2)}{2\big[\gamma(\gamma+1)(5\gamma+2)+2(2\gamma-1)(\gamma+2+2^{\gamma+2})\big]}.$$

This gives the following corollary.

Corollary 2. *Let $(C[0,1], d_{\mathfrak{b}})$ be a complete $\mathfrak{b}MS$ with a $\mathcal{DG}$. We define the transitive set of edges of the graph $\bar{\mathcal{G}}$ by*

$$E(\bar{\mathcal{G}}) = \{(\mathcal{Q},\mathcal{R}) : \epsilon\big(\mathcal{Q}(x),\mathcal{R}(x)\big)\geq 0\}.$$

Suppose the following conditions are satisfied for all $x\in[0,1]$.

i. *There exist $\epsilon : \mathbb{R}^2 \to \mathbb{R}$ and $\phi\in\Phi$ with $\phi(r) < r\ \forall\ r\in(0,1]\ \ni\ \forall\ \mathcal{Q},\mathcal{R}\in C[0,1]$ with $\epsilon\big(\mathcal{Q}(\xi),\mathcal{R}(\xi)\big)\geq 0\ \forall\ \xi\in[0,1]$,*

$$\big|\upsilon\big(x,\mathcal{Q}(x)\big)-\upsilon\big(x,\mathcal{R}(x)\big)\big| \leq K_1\phi\big(\ |\ \mathcal{Q}(x)-\mathcal{R}(x)\ |\ \big) \leq \frac{\Gamma(\gamma+2)}{5+3\gamma}\phi\big(\ |\ \mathcal{Q}(x)-\mathcal{R}(x)\ |\ \big)$$

and

$$\begin{aligned}\big|\upsilon\big(x,\mathcal{Q}(x)\big)-\upsilon\big(x,\mathcal{R}(x)\big)\big|^2 &\leq K_2\phi\big(\ |\ \mathcal{Q}(x)-\mathcal{R}(x)\ |^2\ \big)\\ &\leq \frac{(2\gamma-1)\Gamma(\gamma+1)\Gamma(\gamma+2)}{2\big[\gamma(\gamma+1)(5\gamma+2)+2(2\gamma-1)(\gamma+2+2^{\gamma+2})\big]}\phi\big(\ |\ \mathcal{Q}(x)-\mathcal{R}(x)\ |^2\ \big).\end{aligned}$$

ii. *There exists $\mathcal{Q}_0\in C[0,1]$ such that $\epsilon\big(\mathcal{Q}_0(x),\mathbb{T}\mathcal{Q}_0(x)\big)\geq 0$.*

iii. *For all* $\mathcal{Q}, \mathcal{R} \in C[0,1]$, *we have*

$$\epsilon\big(\mathcal{Q}(x), \mathcal{R}(x)\big) \geq 0 \epsilon\big(\mathbb{T}\mathcal{Q}(x), \mathbb{T}\mathcal{R}(x)\big) \geq 0.$$

iv. *Let the sequence* $\{\mathcal{Q}_n\} \in C[0,1]$ *converge to* $\mathcal{Q} \in C[0,1]$ *with*

$$\epsilon\big(\mathcal{Q}_n(x), \mathcal{Q}_{n+1}(x)\big) \geq 0 \;\forall\; n \in \mathbb{N} \;\Rightarrow\; \epsilon\big(\mathcal{Q}_n(x), \mathcal{Q}(x)\big) \geq 0 \;\forall\; n \in \mathbb{N}.$$

v. *For all* $\mathcal{Q}, \mathcal{R}, \mathcal{W} \in C[0,1]$, *we have*

$$\epsilon\big(\mathcal{Q}(x), \mathcal{R}(x)\big) \geq 0 \text{ and } \epsilon\big(\mathcal{R}(x), \mathcal{W}(x)\big) \geq 0 \; \epsilon\big(\mathcal{Q}(x), \mathcal{W}(x)\big) \geq 0 \;.$$

Then the boundary value problem

$$({}^{c}D^{\gamma}\mathfrak{y})(x) = \upsilon\big(x, \mathfrak{y}(x)\big), \;\; x \in [0,1] \text{ and } \gamma \in (1,2]$$

along with

$$\mathfrak{y}(0) = 0 \text{ and } \mathfrak{y}(1) = \int_0^{\eta} \mathfrak{y}(\xi)d\xi, \;\; \eta \in [0,1]$$

has a solution $\mathcal{Q}^* \in C[0,1]$.

Next, we take the case when $E(\bar{\mathcal{G}}) = C[0,1] \times C[0,1]$. Here, the conditions **i** - **v** can be given by the following single condition satisfied for all $x \in [0,1]$.

$\forall\; \mathcal{Q}, \mathcal{R} \in C[0,1], \;\exists\; \phi \in \Phi$ with $\phi(r) < r \;\forall\; r \in (0,1], \;\ni$

$$\big|\upsilon\big(x, \mathcal{Q}(x)\big) - \upsilon\big(x, \mathcal{R}(x)\big)\big| \leq K_1 \phi\big(\mid \mathcal{Q}(x) - \mathcal{R}(x) \mid\big),$$

$$\big|\upsilon\big(x, \mathcal{Q}(x)\big) - \upsilon\big(x, \mathcal{R}(x)\big)\big|^2 \leq K_2 \phi\big(\mid \mathcal{Q}(x) - \mathcal{R}(x) \mid^2\big),$$

where $K_1 \leq \dfrac{(n-\eta^n)\Gamma(\gamma+2)}{n\eta^{\gamma+1} + (\gamma+1)(2n-\eta^n)}$,

and

$$K_2 \leq \frac{(2\gamma-1)(n-\eta^n)^2\Gamma(\gamma+1)\Gamma(\gamma+2)}{\gamma(\gamma+1)\Big[n^2\eta^{2\gamma} + 2\gamma\big[n^2 + (n-\eta^n)^2\big]\Big] + 2(2\gamma-1)n\big[n^{\gamma+2} + \eta^{\gamma+1}(n-\eta^n) + (\gamma+1)(n-\eta^n)\big]}.$$

This gives the following corollary.

Corollary 3. *Let* $(X, d_{\mathfrak{b}})$ *be a complete* $\mathfrak{b}MS$ *accompanied with a* $\mathcal{DG}$, $\bar{\mathcal{G}} = \big(V(\bar{\mathcal{G}}), E(\bar{\mathcal{G}})\big)$ *with* $E(\bar{\mathcal{G}})$ *as defined above. Then the boundary value problem given by (4.2)-(4.3) has a solution* $\mathcal{Q}^* \in C[0,1]$.

5. Conclusion

This manuscript presents the following contributions:

- We introduce $\psi - \phi$ contraction mappings in the setting of $\mathfrak{b}$MSs with a directed graph. For these mappings, we derive conditions under which common fixed points exist and we present supporting examples. We also investigate generalized rational contractions for spaces with two metrics.
- We validate our theoretical results by applying them to prove the existence of a solution for a Caputo-type fractional boundary value problem.
- Our planned research includes developing iterative solution methods. We also suggest that these results can be extended to other fractional problems and to the more general framework of extended$\mathfrak{b}$MSs.

Acknowledgements

We acknowledge the support of this research from Al-Zaytoonah University.

Authors' Contributions

All authors contribute equally in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] Haitham Qawaqneh, Hasanen A Hammad, and Hassen Aydi. Exploring new geometric contraction mappings and their applications in fractional metric spaces. *AIMS Mathematics*, 9(1):521–541, 2024.
[2] Muhammad Nazam, Hassen Aydi, Mohd Salmi Noorani, and Haitham Qawaqneh. Existence of fixed points of four maps for a new generalized f-contraction and an application. *Journal of Function Spaces*, 2019(1):5980312, 2019.
[3] Stefan Banach. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fundamenta mathematicae*, 3(1):133–181, 1922.
[4] Stefan Czerwik. Contraction mappings in b-metric spaces. *Acta mathematica et informatica universitatis ostraviensis*, 1(1):5–11, 1993.
[5] I.A. Bakhtin. The contraction mapping principle in quasimetric spaces. *Functional analysis*, 30:26–37, 1989.
[6] Hojjat Afshari, Hassen Aydi, and Erdal Karapinar. Existence of fixed points of set-valued mappings in b-metric spaces. *East Asian Math. J*, 32(3):319–332, 2016.

[7] Hassen Aydi, Monica-Felicia Bota, Erdal Karapinar, and Slobodanka Mitrović. A fixed point theorem for set-valued quasi-contractions in b-metric spaces. *Fixed Point Theory and Applications*, 2012(1):88, 2012.

[8] Thounaojam Stephen, Yumnam Rohen, M Kuber Singh, and Konthoujam Sangita Devi. Some rational F-contractions in b-metric spaces and fixed points. *Nonlinear Functional Analysis and Applications*, pages 309–322, 2022.

[9] Mudasir Younis, Deepak Singh, Ishak Altun, and Varsha Chauhan. Graphical structure of extended b-metric spaces: an application to the transverse oscillations of a homogeneous bar. *International Journal of Nonlinear Sciences and Numerical Simulation*, 23(7-8):1239–1252, 2022.

[10] Haitham Qawaqneh, Mohd Salmi Md Noorani, Hassen Aydi, Amjed Zraiqat, and Arslan Hojat Ansari. On fixed point results in partial b-metric spaces. *Journal of Function Spaces*, 2021(1):8769190, 2021.

[11] Haitham Qawaqneh, Mohd Salmi Md Noorani, and Hassen Aydi. Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications. *AIMS Mathematics*, 8(3):6682–6696, 2023.

[12] Samina Batul, Faisar Mehmood, Azhar Hussain, D Sagheer, Hassen Aydi, and Aiman Mukheimer. Multivalued contraction maps on fuzzy b-metric spaces and an application. *AIMS Math*, 7(4):5925–5942, 2022.

[13] Muhammad Anwar, Dur-e-Shehwar Sagheer, and Rashid Ali. Fixed point theorems of wardowski type mappings in s_b-metric spaces. *Thai Journal of Mathematics*, 20(2):945–956, 2022.

[14] Erdal Karapinar and Cristian Chifu. Results in wt-distance over b-metric spaces. *Mathematics*, 8(2):220, 2020.

[15] M. Elbes, Kanan T., M. Alia, and Ziad M. COVD-19 Detection Platform from X-ray Images using Deep Learning. *International Journal of Advances in Soft Computing and its Applications*, 14(1), 2022.

[16] Haitham Qawaqneh. New Functions For Fixed Point Results In Metric Spaces With Some Applications. *Indian Journal of Mathematics*, 66(1):55–84, 2024.

[17] Monther Rashed Alfuraidan. The contraction principle for multivalued mappings on a modular metric space with a graph. *Canadian Mathematical Bulletin*, 59(1):3–12, 2016.

[18] Ismat Beg, Asma Rashid Butt, and Slobodan Radojevic. The contraction principle for set valued mappings on a metric space with a graph. *Computers & Mathematics with Applications*, 60(5):1214–1219, 2010.

[19] Florin Bojor. Fixed point theorems for reich type contractions on metric spaces with a graph. *Nonlinear Analysis: Theory, Methods & Applications*, 75(9):3895–3901, 2012.

[20] Mudasir Younis and Dhirendra Bahuguna. A unique approach to graph-based metric spaces with an application to rocket ascension. *Computational and Applied Mathematics*, 42(1):44, 2023.

[21] Haitham Qawaqneh, Mohd Salmi Noorani, Hassen Aydi, and Wasfi Shatanawi. On common fixed point results for new contractions with applications to graph and integral equations. *Mathematics*, 7(11):1082, 2019.

[22] Erdal Karapinar, Thabet Abdeljawad, and Fahd Jarad. Applying new fixed point theorems on fractional and ordinary differential equations. *Advances in Difference Equations*, 2019(1):1–25, 2019.

[23] Ben Wongsaijai, Phakdi Charoensawan, Teeranush Suebcharoen, and Watchareepan Atiponrat. Common fixed point theorems for auxiliary functions with applications in fractional differential equation. *Advances in Difference Equations*, 2021(1):503, 2021.

[24] Rezan Sevinik Adiguzel, Umit Aksoy, Erdal Karapinar, and Inci M Erhan. On the solution of a boundary value problem associated with a fractional differential equation. *Mathematical Methods in the Applied Sciences*, 47(13):10928–10939, 2024.

[25] Rezan Sevinik Adiguzel, Umit Aksoy, Erdal Karapinar, and Inci M Erhan. On the solutions of fractional differential equations via geraghty type hybrid contractions. 2021.

[26] Duha Abu Judeh and M Abu Hammad. Applications of conformable fractional pareto probability distribution. *Int. J. Advance Soft Compu. Appl*, 14(2):115–124, 2022.

[27] William Kirk and Naseer Shahzad. *Fixed point theory in distance spaces*, volume 1. Springer, 2014.

[28] P Charoensawan and W Atiponrat. Common fixed point and coupled coincidence point theorems for geraghty's type contraction mapping with two metrics endowed with a directed graph. *Journal of Mathematics*, 2017(1):5746704, 2017.

[29] Ravi P Agarwal and Donal O′Regan. Fixed point theory for generalized contractions on spaces with two metrics. *Journal of Mathematical Analysis and Applications*, 248(2):402–414, 2000.

[30] Asadollah Aghajani, Mujahid Abbas, and Jamal Roshan. Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. *Mathematica Slovaca*, 64(4):941–960, 2014.