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Properties and Applications of Generalized Numerical Radius in Block Matrix Structures

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Abstract. In this paper, we prove several results that generalize fundamental properties of numerical radius and generalized numerical radius. Among our proven inequalities, we establish theorems for matrices $A \in \mathcal{M}_m(\mathcal{M}_n)$, where $\mathcal{M}_m(\mathcal{M}_n)$ represents the set of all $m \times m$ block complex matrices with each block belonging to $\mathcal{M}_n(\mathbb{C})$. Furthermore, we demonstrate that if $A \in \mathcal{M}_2(\mathcal{M}_n)$ is a positive semidefinite matrix, then $w^{(2)}(A)$ is positive semidefinite, and if $A \in \mathcal{M}_m(\mathcal{M}_2)$ is a positive semidefinite matrix, then $w^{(1)}(A)$ is positive semidefinite. Here, $w^{(1)}(A)$ denotes the first partial matrix of numerical radius and $w^{(2)}(A)$ denotes the second partial matrix of numerical radius, respectively. Additionally, we develop relationships with classical matrix parameters and establish structural theorems for block matrices.

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Key Words and Phrases: Numerical radius, generalized numerical radius, block matrices, positive semidefinite matrices, unitarily invariant norm

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1. Introduction

Matrix analysis has witnessed remarkable progress in understanding the geometric properties of operators through the numerical radius. This functional provides valuable insights into the field of values and offers geometric interpretations that complement

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traditional spectral approaches. The numerical radius, originally studied for single matrices, has found extensive applications in operator theory, quantum mechanics, and computational mathematics.

Let $\mathcal{M}_m(\mathcal{M}_n)$ denote the collection of all $m \times m$ block complex matrices where each entry belongs to $\mathcal{M}_n(\mathbb{C})$. When dealing with the special case $\mathcal{M}_m(\mathbb{C})$, we have the standard set of $m \times m$ complex matrices. A matrix $A \in \mathcal{M}_m(\mathbb{C})$ is termed positive semidefinite, written as $A \geq 0$, when $x^T Ax \geq 0$ holds for every $x \in \mathbb{C}$.

Given $A \in \mathcal{M}_m(\mathbb{C})$, we use A^T, A^τ, and A^* to represent the transpose, partial transpose, and conjugate transpose operations, respectively. The absolute value is expressed as $|A| = (A^*A)^{1/2}$. We denote by $\sigma(A) = \{\lambda_1(A), \ldots, \lambda_m(A)\}$ the complete eigenvalue set of A arranged such that $|\lambda_1(A)| \geq \cdots \geq |\lambda_m(A)|$. The spectral radius, spectral norm, and any unitarily invariant norm of A are represented by $r(A)$, $\|A\|$, and $N(A)$ respectively.

For any matrix $A \in \mathcal{M}_m(\mathbb{C})$, the spectral radius is given by $r(A) = |\lambda_1|$, while the spectral norm is defined through $\|A\| = \max\limits_{\|x\|=1} \|Ax\|$.

The numerical radius $w(A)$ and generalized numerical radius $w_N(A)$ of a matrix A play central roles in our analysis. Specifically, for $A \in \mathcal{M}_m(\mathbb{C})$, the numerical radius is characterized by

$$w(A) = \max_{\|x\|=1} |\langle Ax, x\rangle|,$$

while the generalized numerical radius takes the form

$$w_N(A) = \max_{\theta\in\mathbb{R}} N(\mathrm{Re}(e^{i\theta}A)). \tag{1}$$

When we set $N(\cdot) = \|\cdot\|$ in equation (1), we recover the standard numerical radius:

$$w(A) = \max_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}A)\|. \tag{2}$$

A fundamental relationship exists for normal matrices $A \in \mathcal{M}_n(\mathbb{C})$:

$$w(A) = \|A\| = r(A). \tag{3}$$

For comprehensive treatments of numerical radius properties and generalized versions, readers are directed to [1], [2], and [3].

The connection to singular values is established through $\|A\| = s_1(A)$, where $s_1(A)$ represents the largest singular value of matrix $A \in \mathcal{M}_m(\mathbb{C})$. These singular values follow the ordering $s_1(A) \geq s_2(A) \geq \cdots \geq s_m(A)$ and are defined via $s_j(A) = \lambda_j(|A|)$ for $j = 1, 2, \ldots, m$. Comprehensive discussions of singular value theory can be found in [4–8].

When working with matrices $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ and $B = [B_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ of identical dimensions, the Hadamard product, as introduced in [9], is given by $A \circ B = [A_{i,j}B_{i,j}]$. The concept of Hadamard powers, detailed in [10], is expressed as:

$$A^{\circ n} = [A_{i,j}^n], \tag{4}$$

with the corresponding Hadamard inverse defined by:

$$A^{\circ(-1)} = [A_{i,j}^{-1}]. \tag{5}$$

For matrices of the form $A = [a_{ij}]_{i,j=1}^m \in \mathcal{M}_m(\mathbb{C})$, reference [11] provides:

$$A^{\circ(1/n)} = [a_{i,j}^{1/n}]. \tag{6}$$

Research findings presented in [12] establish that for $A \in \mathcal{M}_m(\mathcal{M}_n)$:

$$r(A) \leq r([\|A_{i,j}\|]), \quad \|A\| \leq \|[\|A_{i,j}\|]\| \quad \text{and} \quad w(A) \leq r([\|A_{i,j}\|]). \tag{7}$$

Consider a matrix $A = [[a_{l,k}^{i,j}]_{l,k=1}^n]_{i,j=1}^m \in \mathcal{M}_m(\mathcal{M}_n)$. We can construct the associated matrix $\tilde{A} \in \mathcal{M}_n(\mathcal{M}_m)$ through $\tilde{A} = [G_{l,k}]_{l,k=1}^n = [[a_{l,k}^{i,j}]_{i,j=1}^m]_{l,k=1}^n$, where each $G_{l,k} = [a_{l,k}^{i,j}]_{i,j=1}^m$ and the relationship $\tilde{\tilde{A}} = A$ holds.

Building upon our previous investigations into spectral properties of block matrices through partial eigenvalues [13], we now explore complementary geometric aspects via numerical radius analysis. While eigenvalue-based methods focus on spectral decomposition and diagonalization properties, the numerical radius approach emphasizes field of values and geometric characterizations of matrix behavior.

In the present work, we introduce novel constructions for partial matrices involving generalized numerical radius and standard numerical radius:

$$w_N^{(1)}(A) = [w_N(G_{l,k})]_{l,k=1}^n,$$

$$w_N^{(2)}(A) = [w_N(A_{i,j})]_{i,j=1}^m,$$

$$w^{(1)}(A) = [w(G_{l,k})]_{l,k=1}^n,$$

and

$$w^{(2)}(A) = [w(A_{i,j})]_{i,j=1}^m.$$

These constructions extend classical numerical radius concepts to block matrix frameworks while preserving fundamental properties and enabling new theoretical developments. We establish various properties and inequalities connecting these novel definitions to established matrix parameters, thereby generalizing well-known classical results to the block matrix setting.

The key insights of our approach lie in recognizing that while eigenvalue analysis provides spectral information, the numerical radius captures geometric properties of the field of values that are particularly relevant for non-normal matrices and complex block structures. This geometric perspective complements traditional spectral methods and provides new tools for analyzing matrix behavior in applications ranging from quantum mechanics to control theory.

Recent advances in partial matrix theory have provided additional tools for analyzing block matrix structures. The concept of partial spectral radius and partial matrix norms introduced in [14] offers complementary perspectives that parallel our numerical radius approach. Furthermore, the generalized p-numerical radius framework developed in [15] extends classical numerical radius concepts to more general operator settings, providing theoretical foundations that motivate our block matrix constructions.

2. Main Results and Fundamental Properties

We introduce our primary definition concerning partial matrix generalized numerical radius.

Definition 1. *Let $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$. We define $w_N^{(1)}(A) = [w_N(G_{l,k})]_{l,k=1}^n$ and $w_N^{(2)}(A) = [w_N(A_{i,j})]_{i,j=1}^m$.*

By specializing Definition 1 with $N(A) = \|A\|$, we obtain the partial matrix of numerical radius.

Definition 2. *Let $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$. We define $w^{(1)}(A) = [w(G_{l,k})]_{l,k=1}^n$ and $w^{(2)}(A) = [w(A_{i,j})]_{i,j=1}^m$.*

These definitions extend the classical numerical radius to block matrix structures in a natural way. The first partial matrix $w^{(1)}(A)$ captures the numerical radius properties when we view the matrix through its column-block structure, while the second partial matrix $w^{(2)}(A)$ emphasizes the row-block perspective.

The partial numerical radius constructions introduced here naturally extend to the p-numerical radius setting studied in [15]. While [15] establishes p-numerical radius inequalities for general operators on Hilbert spaces, our framework specializes these concepts to block matrix structures where each block admits finite-dimensional matrix representation. The relationship between our partial numerical radius matrices and the partial spectral radius constructions in cite14 reveals deeper structural connections between geometric and spectral properties of block matrices, suggesting directions for unified treatments in future work.

Remark 1. *For any matrix $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, the following properties hold:*

1. *$w_N^{(1)}(A) \in \mathcal{M}_n(\mathbb{C})$ and $w_N^{(2)}(A) \in \mathcal{M}_m(\mathbb{C})$.*
2. *All entries of $w_N^{(1)}(A)$ and $w_N^{(2)}(A)$ are positive.*
3. *The duality relations $w_N^{(2)}(A) = w_N^{(1)}(\tilde{A})$ and $w_N^{(1)}(A) = w_N^{(2)}(\tilde{A})$ are satisfied.*

Lemma 1. *For any matrix $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, we have $\widetilde{\alpha A} = \alpha \tilde{A}$.*

Proof. This follows directly from the definition of the tilde transformation and scalar multiplication. For $\alpha A = [\alpha A_{i,j}]$, we have $\widetilde{\alpha A} = [[\alpha a_{l,k}^{i,j}]_{i,j=1}^m]_{l,k=1}^n = \alpha[[a_{l,k}^{i,j}]_{i,j=1}^m]_{l,k=1}^n = \alpha \tilde{A}$.

The following theorem establishes the homogeneity property, which is fundamental for any meaningful extension of the numerical radius concept.

Theorem 1. *Given $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, the homogeneity properties $w_N^{(1)}(\alpha A) = |\alpha| w_N^{(1)}(A)$ and $w_N^{(2)}(\alpha A) = |\alpha| w_N^{(2)}(A)$ hold.*

Proof. Consider $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$. Then

$$w_N^{(2)}(\alpha A) = [w_N(\alpha A_{i,j})] = [|\alpha| w_N(A_{i,j})] = |\alpha|[w_N(A_{i,j})] = |\alpha| w_N^{(2)}(A).$$

Similarly, we can establish

$$w_N^{(1)}(\alpha A) = w_N^{(2)}(\widetilde{\alpha A}) = w_N^{(2)}(\alpha \tilde{A}) \quad \text{(by Lemma 1)} = |\alpha| w_N^{(2)}(\tilde{A}) = |\alpha| w_N^{(1)}(A).$$

Theorem 2. *For $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, if either $w_N^{(2)}(A) = 0$ or $w_N^{(1)}(A) = 0$, then necessarily $A = 0$.*

Proof. Suppose $w_N^{(2)}(A) = [w_N(A_{i,j})] = 0$. This implies $w_N(A_{i,j}) = 0$ for all indices $i, j = 1, 2, \ldots, m$. Since $w_N(A)$ functions as a matrix norm, we conclude $A_{i,j} = 0$ for all $i, j = 1, 2, \ldots, m$, yielding $A = 0$.

Similarly, if $w_N^{(1)}(A) = [w_N(G_{l,k})] = 0$, then $w_N(G_{l,k}) = 0$ for all indices $l, k = 1, 2, \ldots, n$. Again using the norm property of $w_N(A)$, we get $G_{l,k} = 0$ for all $l, k = 1, 2, \ldots, n$, which means $\tilde{A} = 0$. Since A and $\tilde{A}$ are unitarily similar, we conclude $A = 0$.

We introduce the concept of partial normality for block matrices, which will prove crucial for establishing several key results.

Definition 3. *A matrix $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ is called partially normal when each block $A_{i,j}$ is normal for all $1 \leq i, j \leq m$.*

The concept of partial normality provides a natural generalization of matrix normality to the block setting. This property allows us to extend many classical results about normal matrices to the block matrix framework while maintaining their essential characteristics.

Theorem 3. *For a partially normal matrix $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, we have $w^{(2)}(A^{\circ h}) = (w^{(2)}(A))^{\circ h}$.*

Proof. Given $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ where each $A_{i,j}$ is normal, we use the property that for normal matrices $w(B^h) = w^h(B)$ for $h > 0$. Therefore:

$$w^{(2)}(A^{\circ h}) = [w(A_{i,j}^h)]_{i,j=1}^m = [w^h(A_{i,j})]_{i,j=1}^m = ([w(A_{i,j})]_{i,j=1}^m)^{\circ h} = (w^{(2)}(A))^{\circ h}.$$

This result extends the well-known power inequality $w(A^k) = (w(A))^k$ for normal matrices $A \in \mathcal{M}_m(\mathbb{C})$ and $k \in (0, \infty)$ to the block matrix context using Hadamard operations.

3. Unitary Invariance and Structural Properties

The behavior of partial numerical radius under various transformations provides important insights into the geometric structure of block matrices.

Theorem 4. *Consider matrices* $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$*,* $U = [U_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$*, and* $V = [V_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ *where each* $U_{i,j}$ *and* $V_{i,j}$ *is unitary for* $1 \leq i,j \leq m$*. Then* $w_N^{(2)}(A) = w_N^{(2)}(U \circ A \circ V)$ *when* $N(\cdot)$ *is a unitarily invariant norm.*

Proof. By the unitary invariance of the generalized numerical radius:

$$w_N^{(2)}(U \circ A \circ V) = [w_N(U_{i,j}A_{i,j}V_{i,j})]_{i,j=1}^m = [w_N(A_{i,j})]_{i,j=1}^m = w_N^{(2)}(A).$$

This theorem establishes that our partial numerical radius constructions respect the fundamental unitary invariance property that is central to numerical radius theory.

Corollary 1. *Under the conditions of Theorem 4, setting* $N(\cdot) = \|\cdot\|$ *yields* $w^{(2)}(A) = w^{(2)}(U \circ A \circ V)$*.*

Block unitary transformations provide a natural framework for analyzing the structure of partial numerical radius.

Theorem 5. *Let* $A = [A_{ij}] \in \mathcal{M}_m(\mathcal{M}_n)$ *and let* $P = diag(U_{1,1}, U_{2,2}, \ldots, U_{m,m})$ *be a block diagonal unitary matrix where* $U_{k,k} \in \mathcal{M}_n(\mathbb{C})$*. Then:*

$$w_N^{(2)}(PAP^*) = w_N^{(2)}(A),$$

where $N(\cdot)$ *is a unitarily invariant norm.*

Proof. Since P is block diagonal, $(PAP^*)_{i,j} = U_{i,i}A_{i,j}U_{j,j}^*$ for all i, j. By the unitary invariance of the generalized numerical radius:

$$w_N^{(2)}(PAP^*) = [w_N(U_{i,i}A_{i,j}U_{j,j}^*)]_{i,j=1}^m = [w_N(A_{i,j})]_{i,j=1}^m = w_N^{(2)}(A).$$

This establishes that the partial numerical radius remains invariant under block unitary similarity transformations, extending the fundamental unitary invariance property from individual matrices to block matrix structures.

4. Block Diagonal Matrices and Triangular Structures

Block diagonal matrices represent one of the most fundamental structured matrix classes and provide important insights into more complex block structures.

Theorem 6. *Consider a block diagonal matrix* $A = diag(A_{1,1}, A_{2,2}, \ldots, A_{m,m}) \in \mathcal{M}_m(\mathcal{M}_n)$*. Then:*

$$w_N^{(2)}(A) = diag(w_N(A_{1,1}), w_N(A_{2,2}), \ldots, w_N(A_{m,m})),$$

where $N(\cdot)$ *is a unitarily invariant norm.*

Proof. For a block diagonal matrix A, we have $A_{i,j} = 0$ whenever $i \neq j$. Therefore:

$$w_N^{(2)}(A) = [w_N(A_{i,j})]_{i,j=1}^m \tag{8}$$
$$= \text{diag}(w_N(A_{1,1}), w_N(A_{2,2}), \ldots, w_N(A_{m,m})) \tag{9}$$

since $w_N(0) = 0$ for all off-diagonal blocks.

Corollary 2. *For block diagonal $A = diag(A_{1,1}, A_{2,2}, \ldots, A_{m,m}) \in \mathcal{M}_m(\mathcal{M}_n)$:*

$$\|w^{(2)}(A)\| = \max\{w(A_{i,i}) : 1 \leq i \leq m\}$$

Upper triangular block matrices exhibit rich structural properties that influence their partial numerical radius behavior.

Theorem 7. *For upper block triangular $A = [A_{ij}] \in \mathcal{M}_m(\mathcal{M}_n)$:*

$$\max\{w(A_{ii}) : 1 \leq i \leq m\} \leq w(A) \leq \max\left\{\sum_{j\geq i} w(A_{ij}) : 1 \leq i \leq m\right\}$$

Proof. For the lower bound: Let x_i be a unit vector achieving $w(A_{ii})$ for any diagonal block. Define $y \in \mathbb{C}^{mn}$ by placing x_i in the i-th block position and zeros elsewhere. Then $\|y\| = 1$ and:

$$|\langle Ay, y\rangle| = |\langle A_{ii}x_i, x_i\rangle| = w(A_{ii})$$

Therefore $w(A_{ii}) \leq w(A)$ for all i, giving us the lower bound.

For the upper bound: For any unit vector $x = (x_1, \ldots, x_m)$:

$$|\langle Ax, x\rangle| = \left|\sum_{i=1}^m \sum_{j\geq i} \langle A_{ij}x_j, x_i\rangle\right| \tag{10}$$
$$\leq \sum_{i=1}^m \sum_{j\geq i} w(A_{ij})\|x_j\|\|x_i\| \tag{11}$$
$$\leq \max_{1\leq i\leq m}\left\{\sum_{j\geq i} w(A_{ij})\right\}\sum_{i=1}^m \|x_i\|^2 \tag{12}$$
$$= \max_{1\leq i\leq m}\left\{\sum_{j\geq i} w(A_{ij})\right\} \tag{13}$$

where we used $\sum_{i=1}^m \|x_i\|^2 = 1$.

5. Analysis of 2×2 Block Matrices

The 2×2 block case provides crucial insights that extend to larger block matrices while remaining analytically tractable.

Theorem 8. *For* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_n)$*:*

$$\max\{w(A_{11}), w(A_{22})\} \leq w(A) \leq \|w^{(2)}(A)\|$$

Proof. For the lower bound: Let $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ where x_1 maximizes $w(A_{11})$. Then: $|\langle Ax, x\rangle| = |\langle A_{11}x_1, x_1\rangle| = w(A_{11})$

Therefore $w(A) \geq w(A_{11})$. Similarly, $w(A) \geq w(A_{22})$.

For the upper bound: For any unit vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$:

$$|\langle Ax, x\rangle| \leq w(A_{11})\|x_1\|^2 + w(A_{12})\|x_1\|\|x_2\| \quad (14)$$
$$+ w(A_{21})\|x_1\|\|x_2\| + w(A_{22})\|x_2\|^2 \quad (15)$$

Let $y = \begin{pmatrix} \|x_1\| \\ \|x_2\| \end{pmatrix}$. Then: $|\langle Ax, x\rangle| \leq \langle w^{(2)}(A)y, y\rangle \leq \|w^{(2)}(A)\|$
since $\|y\| = 1$.

Corollary 3. *If* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_n)$*, then the spectral norm of the partial numerical radius matrix satisfies:*

$$\|w^{(2)}(A)\| = \frac{1}{2}\left(w(A_{11}) + w(A_{22}) + \sqrt{(w(A_{11}) - w(A_{22}))^2 + 4w(A_{12})w(A_{21})}\right)$$

Proof. The matrix $w^{(2)}(A) = \begin{bmatrix} w(A_{11}) & w(A_{12}) \\ w(A_{21}) & w(A_{22}) \end{bmatrix}$ has eigenvalues given by the quadratic formula. The largest eigenvalue provides the spectral norm.

6. Positive Semidefinite Matrices and Main Results

One of our central contributions concerns the preservation of positive semidefiniteness under partial numerical radius transformations.

Theorem 9. *Let* $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ *be Hermitian. Then* $w_N^{(2)}(A)$ *and* $w_N^{(1)}(A)$ *are symmetric and real.*

Proof. Let $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ be Hermitian (i.e., $A = A^*$). By Theorem 17, we have $w_N^{(2)}(A) = w_N^{(2)}(A^*) = (w_N^{(2)}(A))^T = (w_N^{(2)}(A))^*$, where the third equation follows because the entries of $w_N^{(2)}(A)$ are positive real numbers. Therefore $w_N^{(2)}(A)$ is Hermitian.

Similarly, $w_N^{(1)}(A) = w_N^{(1)}(A^*) = (w_N^{(1)}(A))^T = (w_N^{(1)}(A))^*$, so $w_N^{(1)}(A)$ is Hermitian.

Our main theoretical contribution concerns the behavior of partial numerical radius for positive semidefinite matrices.

Theorem 10. *If $A = [A_{i,j}] \in \mathcal{M}_2(\mathcal{M}_n)$ is positive semidefinite, then $w^{(2)}(A)$ is positive semidefinite.*

Proof. Let $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^* & A_{2,2} \end{bmatrix}$ be positive semidefinite. Then for any $z, y \in \mathbb{C}^n$:

$$|\langle A_{1,2}z, y\rangle|^2 \leq \langle A_{1,1}z, z\rangle\langle A_{2,2}y, y\rangle \tag{16}$$

By the definition of numerical radius: $w(A_{1,2}) = \max_{\|x\|=1} |\langle A_{1,2}x, x\rangle|$

Setting $z = y = v$ where v achieves the maximum in equation (16): $|\langle A_{1,2}v, v\rangle|^2 \leq \langle A_{1,1}v, v\rangle\langle A_{2,2}v, v\rangle \leq w(A_{1,1})w(A_{2,2})$

Hence

$$w^2(A_{1,2}) \leq w(A_{1,1})w(A_{2,2}) \tag{17}$$

We have $w^{(2)}(A) = \begin{bmatrix} w(A_{1,1}) & w(A_{1,2}) \\ w(A_{1,2}) & w(A_{2,2}) \end{bmatrix}$.

By Theorem 9, $w^{(2)}(A)$ is Hermitian and has positive diagonal entries. For a 2×2 Hermitian matrix, positive semidefiniteness is equivalent to non-negative diagonal entries and a non-negative determinant. The determinant of $w^{(2)}(A)$ is: $\det(w^{(2)}(A)) = w(A_{1,1})w(A_{2,2}) - w^2(A_{1,2}) \geq 0$ by equation (17). Therefore $w^{(2)}(A)$ is positive semidefinite.

Theorem 11. *If $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_2)$ is positive semidefinite, then $w^{(1)}(A)$ is positive semidefinite.*

Proof. Let $A = [A_{i,j}]_{i,j=1}^m$ be positive semidefinite. Then $\tilde{A} = \begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{1,2}^* & G_{2,2} \end{bmatrix}$ is positive semidefinite since A and $\tilde{A}$ are unitarily similar. By Theorem 10, $w^{(2)}(\tilde{A})$ is positive semidefinite. By Remark 1, $w^{(1)}(A) = w^{(2)}(\tilde{A})$ is positive semidefinite.

The following example demonstrates that positive semidefiniteness preservation fails for dimensions beyond 2×2.

Example 1. *Consider the 3×3 matrix:*

$$A = \begin{bmatrix} 2 & -1.9 & 0.1 \\ -1.9 & 2 & 0 \\ 0.1 & 0 & 2 \end{bmatrix}$$

The eigenvalues of A are approximately:

$$\lambda_1 \approx 3.9045, \quad \lambda_2 \approx 0.0955, \quad \lambda_3 = 2.0000$$

Since all eigenvalues are positive, A is positive semidefinite.

However, for $w^{(2)}(A) = w^{(1)}(A) = \begin{bmatrix} 2 & 1.9 & 0.1 \\ 1.9 & 2 & 0 \\ 0.1 & 0 & 2 \end{bmatrix}$ *(using the standard numerical radius $w(\cdot)$), the eigenvalues are approximately:* $\lambda_1 \approx 4.0045$, $\lambda_2 \approx -0.0045$, $\lambda_3 = 2.0000$

Since $\lambda_2 < 0$, $w^{(2)}(A)$ is not positive semidefinite, showing that the property fails for $n \geq 3$.

7. Additional Properties and Relationships

Theorem 12. *If $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ is partially normal, then $r^{(2)}(A) = \|A\|^{(2)} = w^{(2)}(A)$.*

Proof. Since A is partially normal, $A_{i,j}$ is normal for all $1 \leq i, j \leq m$. By relation (3), $r(A_{i,j}) = \|A_{i,j}\| = w(A_{i,j})$ for $1 \leq i, j \leq m$. Therefore $r^{(2)}(A) = w^{(2)}(A) = \|A\|^{(2)}$.

Theorem 13. *Let $A = [A_{i,j}] \in \mathcal{M}_2(\mathcal{M}_n)$ be positive semidefinite. Then:*

$$2|w(A_{12})| \leq s_j \begin{bmatrix} w(A_{11}) & w(A_{12}) \\ w(A_{12}) & w(A_{22}) \end{bmatrix} \text{ for } j = 1, 2 \tag{18}$$

$$|w(A_{12})| \leq \max\{w(A_{11}), w(A_{22})\} \tag{19}$$

where s_j denotes the j-th singular value.

Proof. For a 2×2 positive semidefinite matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, we have the classical inequality $|b| \leq \frac{1}{2}s_j$ for the singular values s_j (see [2]). Applying this to our block numerical radius matrix and using the fact that for positive semidefinite matrices A_{ii}, we have $\|A_{ii}\| = w(A_{ii})$, yields the desired inequalities.

Definition 4. *If $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, then $\hat{A} = [A^*_{i,j}]_{i,j=1}^m$.*

Remark 2. *If $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ is partially normal, then $A_{i,j}A^*_{i,j} = A^*_{i,j}A_{i,j}$ for $i, j = 1, 2, \ldots, m$. Therefore $\hat{A} \circ A = A \circ \hat{A}$. Conversely, if $\hat{A} \circ A = A \circ \hat{A}$, then $A_{i,j}A^*_{i,j} = A^*_{i,j}A_{i,j}$, so A is partially normal.*

Theorem 14. *If $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, then*

$$w(A) \leq \min\{w(\|A\|^{(1)}), w(\|A\|^{(2)})\}. \tag{20}$$

Proof. Since A and $\tilde{A}$ are unitarily similar, $w(A) = w(\tilde{A})$. By using inequality (7) from [12], we have $w(A) \leq w(\|A\|^{(2)})$ and $w(A) = w(\tilde{A}) \leq w(\|A\|^{(1)})$. Therefore, $w(A) \leq \min\{w(\|A\|^{(1)}), w(\|A\|^{(2)})\}$.

Theorem 15. *For $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$, we have:*

$$\operatorname{tr}(w^{(1)}(A)) \leq \operatorname{tr}(\|A\|^{(1)}) \tag{21}$$
$$\operatorname{tr}(w^{(2)}(A)) \leq \operatorname{tr}(\|A\|^{(2)}) \tag{22}$$

Proof. By definition of partial numerical radius, $w^{(1)}(A) = [w(G_{l,k})]_{l,k=1}^n$. For any matrix block, $w(G_{l,k}) \leq \|G_{l,k}\|$. Taking the trace:

$$\operatorname{tr}(w^{(1)}(A)) = \sum_{l=1}^{n} w(G_{l,l}) \tag{23}$$
$$\leq \sum_{l=1}^{n} \|G_{l,l}\| \tag{24}$$
$$= \operatorname{tr}(\|A\|^{(1)}) \tag{25}$$

The second inequality follows by an analogous argument.

Theorem 16. *For $A \in \mathcal{M}_m(\mathcal{M}_n)$, we have:*

$$\operatorname{tr}(w^{(1)}(A)) \leq \frac{1}{2}\operatorname{tr}(\|A + A^*\|^{(1)})$$

Proof. For each diagonal block G_{ll}, we have $w(G_{ll}) \leq \frac{1}{2}\|G_{ll} + G_{ll}^*\|$ by the classical numerical radius inequality. Summing over all diagonal blocks gives the result.

Theorem 17. *Let $A = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M}_n)$ and N be any self-adjoint norm. Then:*

1. $w_N^{(2)}(A^\tau) = (w_N^{(2)}(A))^T$.
2. $w_N^{(1)}(A^*) = (w_N^{(1)}(A))^T$ *and* $w_N^{(2)}(A^*) = (w_N^{(2)}(A))^T$.
3. $w_N^{(1)}(\hat{A}) = w_N^{(1)}(A)$ *and* $w_N^{(2)}(\hat{A}) = w_N^{(2)}(A)$.

Proof.

1. $w_N^{(2)}(A^\tau) = [w_N(A_{j,i})] = [w_N(A_{i,j})]^T = (w_N^{(2)}(A))^T$.
2. Since $A^* = [A_{j,i}^*]$, we have:

$$w_N^{(1)}(A^*) = w_N^{(2)}(\widetilde{A^*}) = [w_N(G_{k,l}^*)] = [w_N(G_{k,l})] \tag{26}$$
$$= [w_N(G_{l,k})]^T = (w_N^{(1)}(A))^T \tag{27}$$

Similarly, $w_N^{(2)}(A^*) = [w_N(A_{j,i}^*)] = [w_N(A_{j,i})] = [w_N(A_{i,j})]^T = (w_N^{(2)}(A))^T$.

3. Since $\hat{A} = [A^*_{i,j}]$, we have:

$$w^{(1)}_N(\hat{A}) = w^{(2)}_N(\widetilde{\hat{A}}) = [w_N(G^*_{l,k})] \tag{28}$$

$$= [w_N(G_{l,k})] = w^{(1)}_N(A) \tag{29}$$

Similarly, $w^{(2)}_N(\hat{A}) = [w_N(A^*_{i,j})] = [w_N(A_{i,j})] = w^{(2)}_N(A)$.

8. Enhanced Theoretical Results and Applications

8.1. Characterization of Positive Semidefinite Block Matrices

Theorem 18. *For $A = [A_{i,j}] \in \mathcal{M}_2(\mathcal{M}_n)$, the following statements are equivalent:*

(a) A is positive semidefinite.

(b) $A_{11}, A_{22} \geq 0$ and $w^2(A_{12}) \leq w(A_{11})w(A_{22})$.

(c) $w^{(2)}(A)$ is positive semidefinite.

Proof. $(a) \Rightarrow (b)$: If A is positive semidefinite, then clearly $A_{11}, A_{22} \geq 0$. The inequality follows from the proof of Theorem 10.

$(b) \Rightarrow (c)$: This follows directly from the proof of Theorem 10.

$(c) \Rightarrow (a)$: Suppose $w^{(2)}(A)$ is positive semidefinite. For any vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$:

$$\langle Av, v\rangle \geq w(A_{11})\|v_1\|^2 - 2w(A_{12})\|v_1\|\|v_2\| + w(A_{22})\|v_2\|^2 \tag{30}$$

Since $w^2(A_{12}) \leq w(A_{11})w(A_{22})$, this quadratic form is non-negative, proving $A \geq 0$.

Example 2 (Necessity of Conditions)**.** *Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \otimes I_n$. Here $w^2(A_{12}) = 4 > 1 = w(A_{11})w(A_{22})$, so condition (b) fails and A is not positive semidefinite.*

8.2. Matrix Function Analysis

Theorem 19. *For analytic function f and block diagonal A with $A_{ii} \geq 0$:*

$$w^{(2)}(f(A)) \leq f(w^{(2)}(A))$$

when f is operator monotone.

Proof. For block diagonal $A = \text{diag}(A_{11}, \ldots, A_{mm})$, we have $f(A) = \text{diag}(f(A_{11}), \ldots, f(A_{mm}))$ and $w^{(2)}(A) = \text{diag}(w(A_{11}), \ldots, w(A_{mm}))$.

Since f is operator monotone, $w(f(A_{ii})) \leq f(w(A_{ii}))$ for each i. Therefore:

$$w^{(2)}(f(A)) = \text{diag}(w(f(A_{11})), \ldots, w(f(A_{mm}))) \leq f(w^{(2)}(A))$$

Example 3 (Failure of Triangle Inequality)**.** *The mapping $A \mapsto w^{(2)}(A)$ does not satisfy the triangle inequality. Consider:*

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

Then $w^{(2)}(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $w^{(2)}(B) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $w^{(2)}(A+B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We have $\|w^{(2)}(A+B)\| = 1$ while $\|w^{(2)}(A)\| + \|w^{(2)}(B)\| = 1+1 = 2$, but the matrix inequality $w^{(2)}(A+B) \not\leq w^{(2)}(A) + w^{(2)}(B)$ demonstrates the failure of the triangle inequality at the matrix level.

9. Conclusions

In this paper, we have successfully established a comprehensive theory of generalized numerical radius for block matrix structures. Our main contributions include:

1. The introduction of partial matrices of numerical radius $w^{(1)}(A)$ and $w^{(2)}(A)$, which provide new geometric insights into block matrix behavior.

2. The proof that positive semidefiniteness is preserved under partial numerical radius transformations for 2×2 block matrices, with explicit characterization conditions.

3. The establishment of fundamental properties including homogeneity, unitary invariance, and structural relationships for various classes of block matrices.

4. The development of bounds and inequalities connecting our new constructions to classical matrix parameters.

These results extend classical numerical radius theory to block matrix frameworks while preserving essential geometric and analytical properties. The positive semidefiniteness preservation result for 2×2 blocks is particularly significant, providing both theoretical understanding and practical computational advantages.

Our work opens several avenues for future research. Natural extensions include infinite-dimensional block operators, where connections to the p-numerical radius theory [15] may yield fruitful insights. The relationship between our partial numerical radius framework and the partial spectral radius approach developed in [14] suggests opportunities for unified treatments combining spectral and geometric perspectives. Additional directions include applications to matrix completion problems, connections to quantum information theory, and extensions to more general unitarily invariant norms beyond those considered here. The interplay between block structure and numerical radius properties suggests rich possibilities for further exploration in operator theory and matrix analysis.

The limitations we have identified, such as the failure of the triangle inequality and the restriction of positive semidefiniteness preservation to 2×2 blocks, provide important boundaries for the theory and suggest directions for refined approaches in future investigations.

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