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Stochastic Stability and Controllability Analysis of Impulsive Multi-Term Fractional Differential System

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Abstract. This study investigates the existence, stability, and controllability of multi-term stochastic fractional impulsive differential equations. By employing the contraction mapping principle, sufficient conditions ensuring stochastic stability are rigorously established. Two classes of systems—linear and nonlinear are analyzed in detail. Furthermore, the controllability of these systems is demonstrated using the Gramian operator approach. To validate the theoretical findings, numerical simulations are performed in MATLAB, illustrating the effectiveness and applicability of the proposed results.

2020 Mathematics Subject Classifications: 26A33, 44A10, 60H20, 93E03, 93E15

Key Words and Phrases: Stochastic impulsive systems, multi-term fractional derivative, Laplace transform, stochastic stability, controllability, Rosenblatt process

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1. Introduction

Fractional stochastic differential equations (FSDEs) combine the concepts of fractional calculus and stochastic processes. They extend the classical theory of stochastic differential equations (SDEs) by introducing fractional derivatives, which capture non-local and memory-dependent behaviors in the underlying stochastic processes. FSDEs are employed to model systems in which both randomness and fractional dynamics play significant roles. Recent studies have developed efficient analytical and numerical approaches for solving such equations, including shooting methods, bilinear time-series frameworks, and fractional wave-based models [1–3]. These advances demonstrate the versatility of FSDEs in modeling complex systems that exhibit both stochastic and long-memory characteristics. Their applications span multiple disciplines and contribute to a deeper understanding of processes with intricate dynamical behaviors. As technology and research continue to evolve, FSDEs are expected to find broader and more innovative applications across various scientific and engineering domains (see also [4–7]). The Caputo fractional derivative is commonly used to extend the concept of differentiation to non-integer orders. It provides a practical means of describing fractional-order dynamics, being defined as a combination of the classical derivative and the Riemann–Liouville fractional integral.

A multi-term Caputo fractional derivative extends this concept by considering a sum of multiple terms, each involving a Caputo fractional derivative with different orders and weights. This can lead to more flexible and accurate modeling of complex systems that exhibit a combination of different fractional-order behaviors. In materials science and engineering, viscoelastic materials exhibit time-dependent responses that can be characterized by fractional calculus. Multi-term Caputo fractional derivatives can model viscoelastic materials with different relaxation times and memory effects. Multi-term fractional derivatives can model networks with different types of interactions that have fractional-order characteristics. This is particularly relevant in studies of complex systems and network dynamics (see [8–15]). An impulsive fractional stochastic differential equation (IF-SDE) combines elements of fractional calculus, impulsive differential equations, and stochastic processes. An impulsive differential equation involves sudden changes in the state of a system at discrete moments in time. These changes are modeled as impulses or jumps. IF-SDEs can have various specific forms depending on the precise nature of the fractional derivative, the impulsive term, and the underlying dynamics. These equations find applications in modeling phenomena with memory effects, sudden jumps, and random influences, such as financial processes, biological systems, and anomalous diffusion (see [16]). Stochastic stability refers to the property of a stochastic system to remain bounded and exhibit well-behaved behavior over time, even in the presence of random fluctuations or noise. In other words, a stochastically stable system does not undergo excessive or unpredictable changes due to the inherent randomness in its dynamics. Stochastic stability is particularly relevant when dealing with real-world systems that are subject to inherent randomness, such as biological systems, financial markets, and complex physical systems. Stability analysis provides insights into the long-term behavior of these systems and helps ensure their reliability and predictability despite the presence of uncertain and random factors (see [10, 12, 17, 18]). The controllability analysis of fractional stochastic differential equations presents unique challenges due to the combination of fractional derivatives and stochastic components. Effective strategies require innovative approaches that account for memory effects, stochasticity, and nonlinearity. This field is at the intersection of fractional calculus, stochastic control theory, and applied mathematics, offering opportunities to devise novel control methods for a wide range of real-world systems (see [19, 20]). The Rosenblatt process, also known as the Rosenblatt-Levy process, is a type of stochastic process used in probability theory and statistical analysis. It's named after Murray Rosenblatt, who introduced it in the 1950s. The process is closely related to fractional Brownian motion and has applications in fields like finance, physics, and signal processing. The Rosenblatt process captures this artistic wandering. It's like an intricate dance between memory and randomness. The past informs the artist's decisions, while the randomness of the postcards adds a touch of unpredictability. And just like a piece of art, the Rosenblatt process's path is unique every time you look at it(see [**?**]). We are aware of no study in the literature that deals with the stochastic stability and controllability analysis of a stochastic multi-term fractional impulsive differential system.

Our work makes the following significant contributions: A Fractional stochastic delay differential equation with the stochastic term as a Rosenblatt process is considered for multi-term for the first time. It can reflect the transfer process more effectively compared to existing literature. In this work, we extend the stochastic stability result for multi-term fractional stochastic impulsive differential systems by providing suitable hypothesis. The fixed point theorem was used to introduce controllability for the multi-term equation in the existing system. There are mathematical instances presented.

The following theory is structured as follows: Section 2 introduces basic definitions and lemmas. In Section 3, the multi-term fractional impulsive differential system's state reaction is illustrated. Section 4 deduces the FSDDE solution and the required and sufficient requirements of Controllability, with examples. Section 5 discusses stochastic stability criteria and provides instances. Section 6 provided the conclusion.

2. Preliminaries

Definition 1. *[6] Let $a(r)$ be a function of a real variable $r \in \mathbb{R}^+$. The Laplace transform (LT) of $a(r)$ is defined as*

$$\mathcal{L}[a(r)] = \int_0^\infty e^{-sr} a(r)\, dr = A(s).$$

The Laplace transform of a convolution satisfies

$$\begin{aligned} \mathcal{L}[a * n] &= \mathcal{L}[a(r)] \cdot \mathcal{L}[n(r)] \\ &= A(s)\, N(s), \end{aligned}$$

where

$$(a * n)(r) = \int_0^r a(r-s) n(s)\, ds.$$

Moreover, the inverse Laplace transform of a product can be written as

$$\mathcal{L}^{-1}[A(s)N(s)] = \big(\mathcal{L}^{-1}[A(s)] * \mathcal{L}^{-1}[N(s)]\big)(r).$$

For the Caputo fractional derivative of order $\alpha > 0$, the Laplace transform is given by

$$\mathcal{L}\big[{}^{C}D^{\alpha}_{0^+} f(t)\big] = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0^+),$$

Where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $F(s) = \mathcal{L}[f(t)]$.

Definition 2. *[6] The fractional Caputo derivative(CD) can be defined by*

$$({}^{C}D^{x}a)(r) = \frac{1}{\Gamma(n-x)} \int_0^r (r-s)^{n-x-1} a^(n)(s)ds, \; n-1 < x \leq n, \; n \in \mathbb{N}$$

If $0 < x \leq 1$,

$$({}^{C}D^{x}a)(r) = \frac{1}{\Gamma(n-x)} \int_0^r \frac{a^1(s)}{(r-s)^x} ds$$

The LT of CDis

$$L[{}^{C}D^{x}a(r)](s) = s^x A(s) - \sum_{l=0}^{n-1} a^l(0^+) s^{x-1-l}$$

If $0 < x \leq 1$,

$$L[{}^{C}D^{x}a(r)](s) = s^x A(s) - s^{x-1}a(0^+)$$

For $1 < x \leq 2$,

$$L[{}^{C}D^{x}a(r)](s) = s^x A(s) - s^{x-1}a(0^+) - s^{x-2}a'(0^+)$$

Lemma 1. *[21]* ***Banach Contraction Principle***
$\ominus$ has a singular fixed point if X is a Banach space and $\ominus : X \to X$ is a contraction mapping.

2.1. Rosenblatt process (RP)

In a filtered probability space$(\Omega, \mathcal{F}, \{\mathcal{F}_q\}_{q\geq 0}, P)$ where $\{\mathcal{F}_q\}_{q\geq 0}$ is the filtration and it is right continuous. The Rosenblatt process, which is defined on this filtered probability space is denoted by(see [**?**])

$$R_{\mathbf{H}}(q) = d(\mathbf{H}) \int_0^q \int_0^q \{ \int_{t_1 \vee t_2}^q \frac{\partial K^{\mathbf{H}'}}{\partial u}(u,t_1) \frac{\partial K^{\mathbf{H}'}}{\partial u}(u,t_2) du \} dB(t_1) dB(t_2) \tag{1}$$

Where $q, r \in [0,a], a > 0$, $q \leq r$, $\{B(q)\}$ is a Brownian motion, Hurst parameter $\mathbf{H} > \frac{1}{2}$, $d(\mathbf{H}) = \frac{1}{\mathbf{H}+1}(\frac{\mathbf{H}}{2(2\mathbf{H}-1)})^{\frac{-1}{2}}$ and $\mathbf{H}' = \frac{\mathbf{H}+1}{2}$.
Here $K_{\mathbf{H}}(q,r)$ is the kernel,

$$K_{\mathbf{H}}(q,r) = C_{\mathbf{H}} r^{\frac{1}{2}-\mathbf{H}} \int_r^q (u-r)^{\mathbf{H}-\frac{3}{2}} u^{\mathbf{H}-\frac{1}{2}} du$$

for $q > r$, where $C_{\mathbf{H}} = \sqrt{\frac{\mathbf{H}(2\mathbf{H}-1)}{\beta(2-2\mathbf{H},\mathbf{H}-\frac{1}{2})}}$ and $K_{\mathbf{H}}(q,r) = 0$ if $q \leq r$.

- Let A be a Separable Hilbert Space with $< .,. >_A$ and $|.|_A$
- Υ is a Hilbert space of all square integrable and $\mathcal{F}_r$- measurable processes.

3. System and Solution Representation

Evaluate the following linear system

$$\begin{aligned} {}^C D^M a(r) + \sum_{g=1}^{x} u_g \, {}^C D^{N_g} a(r) &= Ka(r) + H(r)d\xi(r), \quad r \in i = (0,z] \; r \neq r_t, \\ \Delta a(r_t) &= J_t(a(r_t^-)), \quad t = 1,2,...,s, \\ a'(0) &= \delta \end{aligned} \tag{2}$$

- Where, ${}^C D^M$ is M order Caputo derivative and $1 < M < 2$, ${}^C D^{N_g}$ is N_g order Caputo
derivative and $0 < N_g < M$.
- $K < 0$ and $u_g \in \Re$,
- $a(r) \in \Re$,
- $H : i \to L_2^0$ is a Hilbert-Schmidt operator,
- $d\xi(r)$ is a Rosenblatt process,
- $\Delta a(r_t) = a(r_t^+) - a(r_t^-)$ is the impulse term, $a(r_t^+) = \lim_{h\to 0} a(r_t + h)$ is the right limit and $a(r_t^-) = \lim_{h\to 0} a(r_t - h)$ is the left limit of $a(r)$ on $r = r_t$.
- δ is initial function,

Consider

$$\begin{aligned} {}^C D^M a(r) + \sum_{g=1}^{x} u_g \, {}^C D^{N_g} a(r) &= Ka(r) + H(r)d\xi(r), \quad r \in i = (0,z], \\ a'(0) &= \delta \end{aligned} \tag{3}$$

By taking LT of (3),

$$a(r) = \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \tag{4}$$

$$+\int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q)$$

Here,

$$\begin{aligned} p_m(r) &= L^{-1}\Big[\frac{q^{m-1}}{q^m+\sum_{g=1}^{x} u_g q^{N_g}-K}\Big] \\ r^{m-1} p_{m,1+m-N_g}(r) &= L^{-1}\Big[\frac{q^{N_g-2}}{q^m+\sum_{g=1}^{x} u_g q^{N_g}-K}\Big] \\ r^{m-1} p_{m,m}(r) &= L^{-1}\Big[\frac{1}{q^m+\sum_{g=1}^{x} u_g q^{N_g}-K}\Big] \end{aligned}$$

Lemma 2. *For $1 < M < 2$, the solution of the system (2) can be represented as*

$$\begin{aligned} a(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ &\quad + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \;\; for \; r \in (o, r_1] \\ a(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta + J_1(a(r_1^-)) \\ &\quad + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \;\; for \; r \in (r_1, r_2] \\ &. \\ &. \\ &. \\ a(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) \\ &\quad + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \\ &\quad for \; r \in (r_t, r_t+1], \; t = 1, 2, 3, ..., s \end{aligned}$$

Proof: If $r \in (0, r_1]$, then

$$\begin{aligned} a(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ &\quad + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \\ a(r_1) &= \int_0^{r_1} p_m(r_1-q)\delta dq + \sum_{g=1}^{x} u_g \, r_1^{m-1} p_{m,1+m-N_g}(r_1)\delta \end{aligned}$$

$$+\int_0^{r_1}(r_1-q)^{m-1}p_{m,m}(r_1-q)H(q)d\xi(q)$$

If $r \in (r_1, r_2]$,

$$\begin{aligned}
a(r) &= a(r_1^+) + J_1(a(r_1^-)) - \int_0^{r_1} p_m(r_1-q)\delta dq + \sum_{g=1}^{x} u_g\, r_1^{m-1} p_{m,1+m-N_g}(r_1)\delta \\
&\quad + \int_0^{r_1}(r_1-q)^{m-1}p_{m,m}(r_1-q)H(q)d\xi(q) + \int_0^{r} p_m(r-q)\delta dq \\
&\quad + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta \\
&\quad + \int_0^{r}(r-q)^{m-1}p_{m,m}(r-q)H(q)d\xi(q) \\
a(r) &= \int_0^{r} p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta + J_1(a(r_1^-)) \\
&\quad + \int_0^{r}(r-q)^{m-1}p_{m,m}(r-q)H(q)d\xi(q)
\end{aligned}$$

And so on, If $(r_t, r_{t+1}], t = 1, 2, 3, ..., s$, then the same argument implies the following expression as

$$\begin{aligned}
a(r) &= \int_0^{r} p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) \\
&\quad + \int_0^{r}(r-q)^{m-1}p_{m,m}(r-q)H(q)d\xi(q)
\end{aligned}$$

4. Controllability Analysis

$$\begin{aligned}
{}^{C}D^{M}a(r) + \sum_{g=1}^{x} u_g\, {}^{C}D^{N_g}a(r) &= Ka(r) + Ey(r) + H(r)d\xi(r), \quad r \in i = (0, z]\ r \neq r_t, \\
\Delta a(r_t) &= J_t(a(r_t^-)), \quad t = 1, 2, ..., s, \\
a'(0) &= \delta
\end{aligned} \tag{5}$$

Where $E : \Upsilon \to A$ is a linear bounded operator and $y(.) \in \Upsilon$.
The solution of the system (5) as:

$$a(r) = \begin{cases} \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) E y(q) dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \;\; for \; r \in (o, r_1] \\ \\ \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) E y(q) dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q) d\xi(q) \\ for \; r \in (r_t, r_t+1], \; t = 1, 2, 3, ..., s \end{cases}$$

Now, Consider

$$\begin{aligned} {}^C D^M a(r) + \sum_{g=1}^{x} u_g \, {}^C D^{N_g} a(r) &= K a(r) + E y(r) + H(r) d\xi(r), \quad r \in i = (0, z], \\ a'(0) &= \delta \end{aligned} \tag{6}$$

Theorem 1. *The system* (6) *is controllable on [0,z] iff the Grammian operator*

$$B^K_{M,N_g}[0, z] = \int_0^z (z-q)^{2M-2} [p_{m,m}(z-q)] E E^* [p_{m,m}(z-q)]^* dq \tag{7}$$

*is positive definite, here * represents adjoint operator.*

Proof: Let system (6) is controllable on [0,z]. Now, our claim is $B^K_{M,N_g}[0, z] \neq 0$
Suppose take the operator $B^K_{M,N_g}[0, z] = 0$, $\exists$ a vector $d \in \mathfrak{R} \ni$:

$$d^* B^K_{M,N_g}[0, z] d = 0$$

ie)

$$d^* \int_0^z (z-q)^{2M-2} [p_{m,m}(z-q)] E E^* [p_{m,m}(z-q)]^* dq d = 0$$

Then,

$$d^* (z-q)^{M-1} [p_{m,m}(z-q)] E = 0 \tag{8}$$

on [0,z]. $\exists$ a control function $y(r)$ steers the response from initial 0 to final $a(z) = d$ at $r = z$. Then,

$$d = \int_0^z (z-q)^{2M-2} [p_{m,m}(z-q)] E y(q) dq$$

By apply d^* on both side,

$$d^* d = \int_0^z (z-q)^{2M-2} d^* [p_{m,m}(z-q)] E y(q) dq$$

Which implies $d^*d = 0$ by equation (8) $\Rightarrow d = 0$. which is a $\Rightarrow\Leftarrow$ to assumption $d \neq 0$. Hence $B^K_{M,N_g}[0, z] \neq 0$.
Conversely, $B^K_{M,N_g}[0, z] \neq 0$, so the linear operator $B^K_{M,N_g}[0, z]$ is invertible. If $r = 0$ and $r = z$ on the solution of the equation (6), we get

$$a(0) = 0; \;\; a(z) = \delta$$

Evaluate the following Non-linear control system

$$\begin{aligned} {}^CD^M a(r) + \sum_{g=1}^{x} u_g \, {}^CD^{N_g} a(r) &= Ka(r) + Ey(r) + V(r, a(r)) + H(r, a(r))d\xi(r), \quad (9) \\ & \qquad r \in i = (0, z] \; r \neq r_t, \\ \Delta a(r_t) &= J_t(a(r_t^-)), \quad t = 1, 2, ..., s, \\ a'(0) &= \delta \end{aligned}$$

Where $H : i \times \mathfrak{R} \to L^0_2$ is a Hilbert Schmidt operator,and $V : i \times \mathfrak{R} \to \mathfrak{R}$ is a real continuous function in $\mathfrak{R}$.
The solution of the system (9) as:

$$a(r) = \begin{cases} \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q, a(q)) + Ey(q)]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q, a(q))d\xi(q) \;\; for \; r \in (o, r_1] \\ \\ \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g \, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q, a(q)) + Ey(q)]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q, a(q))d\xi(q) \\ for \; r \in (r_t, r_t + 1], \; t = 1, 2, 3, ..., s \end{cases}$$

Let us consider the assumptions
($\mathcal{F}_{11}$) : For any $a, \gamma \in C'(i, \mathfrak{R})$ and $r \in [0, i]$ $\exists$ a constant $\beta_0 > 0 \ni$:

$$\|V(r, a) - V(r, \gamma)\|^2 \leq \beta_0 \|a - \gamma\|^2 \; and \; V(r, 0) = 0$$

($\mathcal{F}_{12}$) : For any $a, \gamma \in C'(i, \mathfrak{R})$ and $r \in [0, i]$ $\exists$ a constant $\beta_1 > 0 \ni$:

$$\|H(r, a) - H(r, \gamma)\|^2 \leq \beta_1 \|a - \gamma\|^2 \; and \; H(r, 0) = 0$$

($\mathcal{F}_{13}$) : $[\beta_0 + \beta_1]Q_0 < 1$, Where $Q_0 = \int_0^r (r-q)^{m-1} p_{m,m}(r-q)dqs$

Theorem 2. *The system* (9) *is completely controllable on [0,i] with* $m \in (1, 2)$ *and* $N_g \in (0, m)$*, g= 1,2,..,x provided that*

$$[\beta_0 + \beta_1]Q_0 < 1$$

Proof: Define $\Lambda : c'(i,\mathfrak{R}) \to c'(i,\mathfrak{R})$, for each $r \in (0, r_1]$ we obtain

$$\begin{aligned} (\Lambda a)(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ &+ \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q)) + Ey(q)]dq \\ &+ \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q) \end{aligned} \tag{10}$$

For each $r \in (r_t, r_{t+1}]$, $t = 1, 2, 3..., s$, we obtain

$$\begin{aligned} (\Lambda a)(r) &= \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) \\ &+ \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q)) + Ey(q)]dq \\ &+ \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q) \end{aligned} \tag{11}$$

The control of (9), for each $r \in (0, r_1]$, for $a_0 \in \mathbf{R}$

$$\begin{aligned} y(r) &= (r-q)^{m-1} d^* p_{m,m}(r-q) E\{a_0 - \int_0^r p_m(r-q)\delta dq \\ &- \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ &- \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q))]dq \\ &- \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q)/\mathcal{G}_s\} \end{aligned} \tag{12}$$

The control of (9), for each $r \in (r_t, r_{t+1}]$, $t = 1, 2, 3, ..., s$, for $a_0 \in \mathbf{R}$

$$\begin{aligned} y(r) &= (r-q)^{m-1} d^* p_{m,m}(r-q) E\{a_0 - \int_0^r p_m(r-q)\delta dq \\ &- \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta - \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) \\ &- \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q))]dq \\ &- \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q)/\mathcal{G}_s\} \end{aligned} \tag{13}$$

By taking the norm

$$\sup_{q\in i} E\|y(r)\|^2 \leq E\|(r-q)^{m-1} d^* p_{m,m}(r-q)\|^2 * E\|a_0\|^2 + E\|\int_0^r p_m(r-q)\delta dq\|^2$$

$$
\begin{aligned}
&+E\|\sum_{g=1}^{x} u_g\ r^{m-1} p_{m,1+m-N_g}(r)\delta\|^2 + E\|\sum_{\theta=1}^{s} J_\theta(a(r_\theta^-))\|^2 \\
&+E\|\int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q))]dq\|^2 \\
&+E\|\int_0^r (r-q)^{m-1} p_{m,m}(r-q)H(q,a(q))d\xi(q)\|^2 \\
\leq\ & \|d^*\|^2\|B^{-1}\|^2\{\|a_0\|^2 + \alpha + \psi + [\beta_2+\beta_3]Q_0\} := \phi
\end{aligned}
$$

Where $\alpha := E\|\sum_{\theta=1}^{s} J_\theta(a(r_\theta^-))\|^2$, $\psi := E\|\int_0^r p_m(r-q)\delta dq\|^2 + E\|\sum_{g=1}^{x} u_g\ r^{m-1} p_{m,1+m-N_g}(r)\delta\|^2$, $\beta_2 := \|V(q,a(q))\|^2$ and $\beta_3 := \|H(q,a(q))\|^2$

$$
\begin{aligned}
\|\Lambda a(r) - \Lambda\gamma(r)\|^2 \ \leq\ & \|\int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q)) - V(q,\gamma(q))]dq \\
& + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[H(q,a(q)) - H(q,\gamma(q))]d\xi(q)\|^2 \\
\leq\ & [\beta_0+\beta_1]Q_0\|a-\gamma\|^2
\end{aligned}
$$

From $(\boldsymbol{\mathcal{F}_{13}})$, $[\beta_0+\beta_1]Q_0 < 1$. This concludes that Λ is a contraction map, from Lemma (1) it has a unique fixed point on the interval [0,z].

$$
\begin{aligned}
\sup_{q\in i} E\|\Lambda a(r)\|^2 \ \leq\ & \|\int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\ r^{m-1} p_{m,1+m-N_g}(r)\delta + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) \\
& + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q)) + Ey(q)]dq \\
& + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)H(q,a(q))d\xi(q)\|^2 \\
\leq\ & \|\psi + \alpha + [\beta_2\phi + \beta_3]Q_0\|^2 := \mu
\end{aligned}
$$

Hence, $\sup_{q\in i} E\|\Lambda a(r)\|^2 \leq \mu$, we conclude that the equation (9) is controllable on i.

4.1. Examples

Example 1. *Evaluate the following linear control system*

$$
\begin{aligned}
{}^C D^M a(r) + \sum_{g=1}^{3} u_g\ {}^C D^{N_g} a(r) &= Ka(r) + Ey(r) + H(r)d\xi(r), \quad r \in i = (0,5]\ r \neq r_t, \\
\Delta a(r_t) &= \frac{1.5 + a(t_i^-)}{4},\ t_i = i,\ i = 1,2,3. \\
a'(0) &= \delta
\end{aligned} \tag{14}
$$

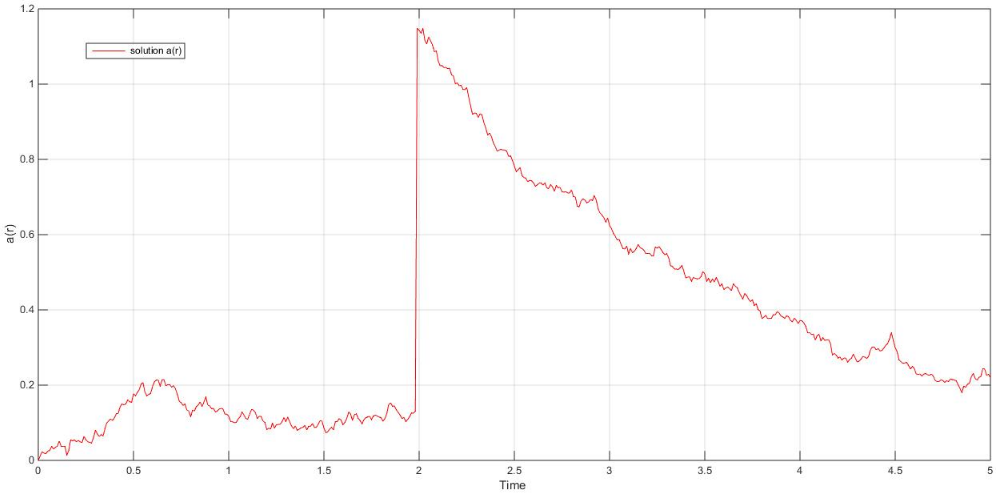

Figure 1: Solution of the system (14) on $[0,5]$ for $M = 1.5,\ K = -0.0834$.

Where $M = 1.5, K = -0.0834$, $H = 10$, $E = 0.25$, $R = 2.5$, $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, *and* $a'(0) = 1$ *The solution of the system* (14) *as:*

$$a(r) = \begin{cases} \int_0^r p_{1.5}(r-q)\delta dq + \sum_{g=1}^{3} u_g\, r^{1.5-1} p_{1.5,1+1.5-N_g}(r)\delta \\ + \int_0^r (r-q)^{1.5-1} p_{1.5,1.5}(r-q) E y(q) dq \\ + \int_0^r (r-q)^{1.5-1} p_{1.5,1.5}(r-q) H(q) d\xi(q) \quad for\ r \in (o,5] \\ \\ \int_0^r p_{1.5}(r-q)\delta dq + \sum_{g=1}^{3} u_g\, r^{1.5-1} p_{1.5,1+1.5-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{1.5-1} p_{1.5,1.5}(r-q) E y(q) dq \\ + \int_0^r (r-q)^{1.5-1} p_{1.5,1.5}(r-q) H(q) d\xi(q) \\ for\ r \in (r_t, r_t+1],\ t = 1,2,3. \end{cases}$$

The controllability Grammian operator,

$$B_{M,N_g}^{K}[0,z] \quad = \quad 3.9338 > 0$$

Which is positive definite, hence by theorem (1) *system* (14) *is completely controllable on [0,5]*

Example 2. *Evaluate the following non-linear control system*

$$^{C}D^{M}a(r) + \sum_{g=1}^{2} u_g\, {}^{C}D^{N_g}a(r) \quad = \quad Ka(r) + Ey(r) + V(r, a(r)) + H(r, a(r))d\xi(r), \quad (15)$$

$$r \in i = (0, 8]\ r \neq r_t,$$

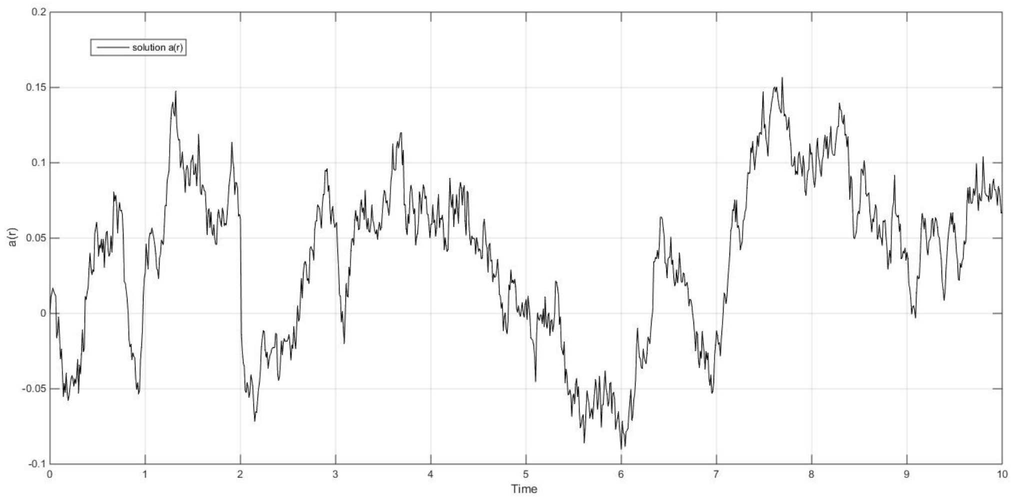

Figure 2: Solution of the system (14) on $[0, 10]$ for $M = 1.9,\ K = -0.0632$.

$$\begin{aligned} \Delta a(r_t) &= \frac{1.8 + a(t_i^-)}{3}\ t = \frac{i}{2}, i = 1, 2, 3, 4 \\ a'(0) &= \delta \end{aligned}$$

Where $M = 1.8$,$K - 0.1429$, $H = 5$, $E = 2$, $u_1 = 0.5$, $u_2 = 1$ $V(r, a(r)) = \frac{a(r)+0.074}{r+0.34}$ *and* $a'(0) = 0.2$ *The solution of the system* (15) *as:*

$$a(r) = \begin{cases} \int_0^r p_{1.8}(r-q)\delta dq + \sum_{g=1}^{2} u_g\ r^{1.8-1} p_{1.8,1+1.8-N_g}(r)\delta \\ + \int_0^r (r-q)^{1.8-1} p_{1.8,1.8}(r-q)[V(q, a(q)) + Ey(q)]dq \\ + \int_0^r (r-q)^{1.8-1} p_{1.8,1.8}(r-q)H(q, a(q))d\xi(q) \ \ for\ r \in (o, 8] \\ \\ \int_0^r p_{1.8}(r-q)\delta dq + \sum_{g=1}^{2} u_g\ r^{1.8-1} p_{1.8,1+1.8-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{1.8-1} p_{1.8,1.8}(r-q)[V(q, a(q)) + Ey(q)]dq \\ + \int_0^r (r-q)^{1.8-1} p_{1.8,1.8}(r-q)H(q, a(q))d\xi(q) \\ for\ r \in (r_t, r_t + 1],\ t = 1, 2, 3, 4. \end{cases}$$

The controllability Grammian operator,

$$B^K_{M,N_g}[0, z] = 2.7391 > 0$$

Which is positive definite, hence by theorem (2) *system* (15) *is completely controllable on [0,8]*

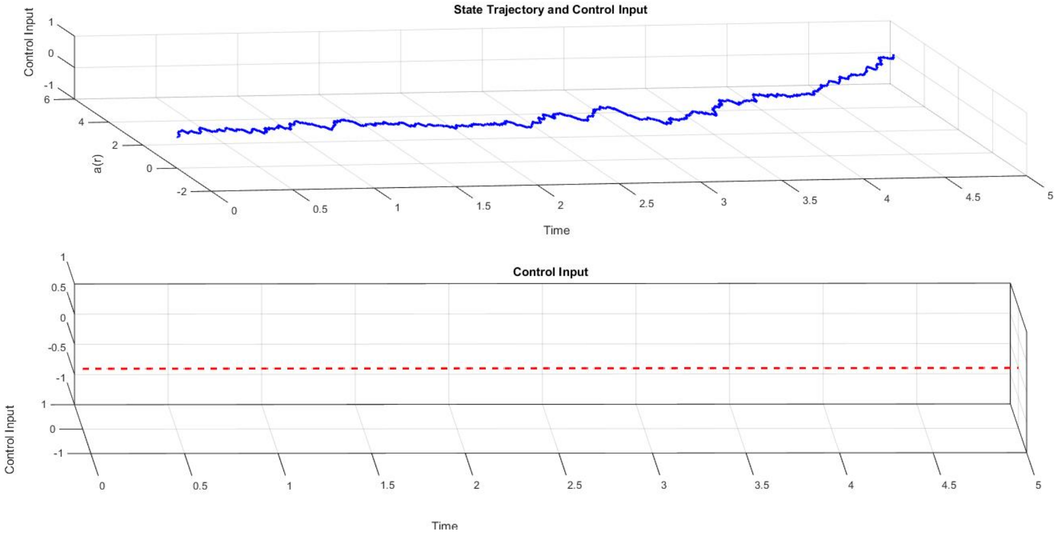

Figure 3: Solution and Control of the system (14) on $[0,5]$ for $M = 1.2,\ \ K = -0.0543$.

5. Stochastic Stability

Evaluate the following system

$$\begin{aligned} {}^{C}D^{M}a(r) + \sum_{g=1}^{x} u_g\, {}^{C}D^{N_g}a(r) &= Ka(r) + V(r,a(r)) + H(r,a(r))d\xi(r), \qquad (16) \\ & r \in i = (0,z]\ r \neq r_t, \\ \Delta a(r_t) &= J_t(a(r_t^-)), \quad t = 1,2,...,s, \\ a'(0) &= \delta \end{aligned}$$

The solution of (16) is

$$a(r) = \begin{cases} \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\ r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q))]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q) \quad for\ r \in (o, r_1] \\ \\ \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\ r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q,a(q))]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q,a(q))d\xi(q) \\ for\ r \in (r_t, r_t+1],\ t = 1,2,3,...,s \end{cases}$$

Definition 3. *([17]) The solution $a(r) = 0$ of equation* (16) *is stochastically stable in the norm. If, given $\epsilon > 0$, $\exists\ \delta(\epsilon) > 0 \ni$: for any initial condition whose norm satisfies $\|a_1\|^2 < \delta$ the norm of the solution process satisfies $E\|a(r)\|^2 < \epsilon$, $\forall g \geq g_0$*

Theorem 3. *Under the assumptions* $(\boldsymbol{\mathcal{F}_{11}}) - (\boldsymbol{\mathcal{F}_{13}})$, *the system* (16) *is stochastic stable on* $[0, j]$.

Proof: Define $\Omega : c'(i, \mathfrak{R}) \to c'(i, \mathfrak{R})$

$$(\Omega a)(r) = \begin{cases} \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q, a(q))]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q, a(q)) d\xi(q) \quad for\ r \in (o, r_1] \\ \\ \int_0^r p_m(r-q)\delta dq + \sum_{g=1}^{x} u_g\, r^{m-1} p_{m,1+m-N_g}(r)\delta \\ + \sum_{\theta=1}^{s} J_\theta(a(r_\theta^-)) + \int_0^r (r-q)^{m-1} p_{m,m}(r-q)[V(q, a(q))]dq \\ + \int_0^r (r-q)^{m-1} p_{m,m}(r-q) H(q, a(q)) d\xi(q) \\ for\ r \in (r_t, r_t+1],\ t = 1, 2, 3, ..., s \end{cases}$$

We omit the proof. Since we have enough to prove Ω is a contraction mapping, and this proof is similar to the proof in the theorem (2).

5.1. Examples

Example 3. *Stochastic stability in impulsive fractional stochastic differential equations can be illustrated with a real-life example involving a financial model. Consider a scenario where an investor is managing a portfolio of stocks, and they are employing a trading strategy that involves making periodic decisions based on market conditions. The investor's goal is to achieve stability in their portfolio returns despite the inherent uncertainty and randomness in the financial markets.*

Let's model this scenario using an impulsive fractional stochastic differential equation (SFDE) to represent the investor's portfolio value. Evaluate the following system

$$\begin{aligned} {}^{C}D^{M} a(r) + \sum_{g=1}^{x} u_g\, {}^{C}D^{N_g} a(r) &= Ka(r) + V(r, a(r)) + H(r, a(r)) d\xi(r), \qquad (17) \\ & \quad r \in i = (0, 5]\ r \neq r_t, \\ \Delta a(r_t) &= J_t(a(r_t^-)), \quad t = 1, 2, ..., 7, \\ a'(0) &= \delta \end{aligned}$$

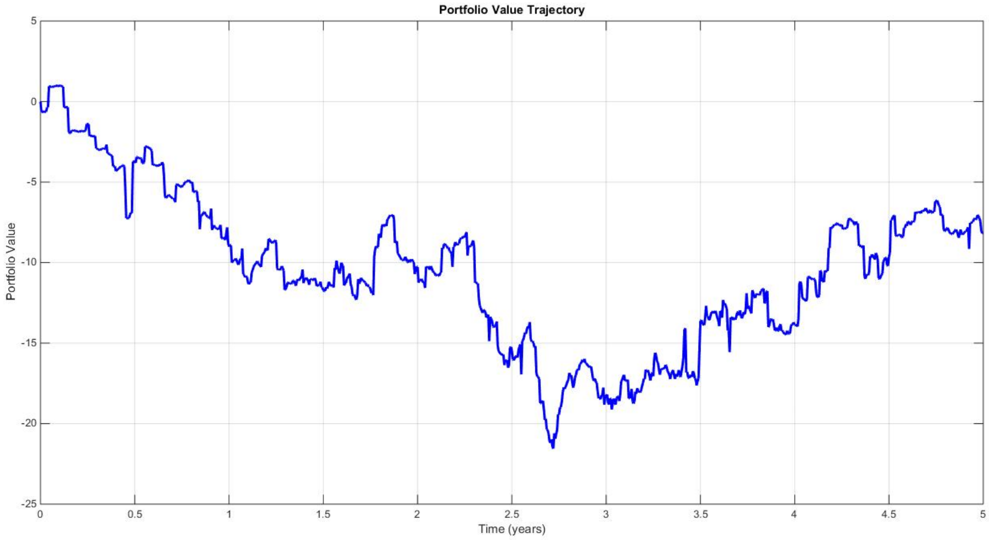

Figure 4: Solution of the system (17) on $[0,5]$ for $M = 1.5,\ \ K = -0.0834$.

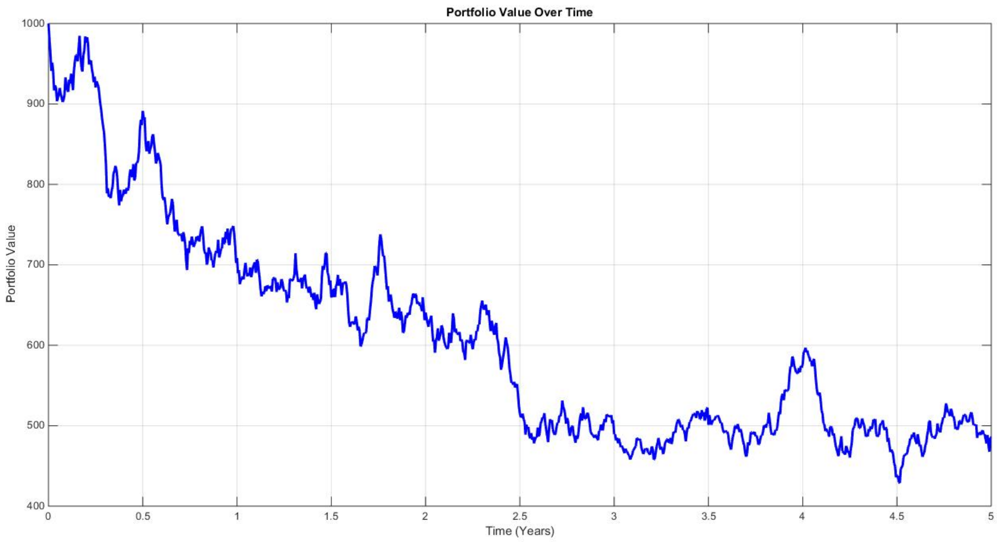

Figure 5: Solution of the system (17) on $[0,5]$ for $M = 1.9,\ \ K = -0.0632$.

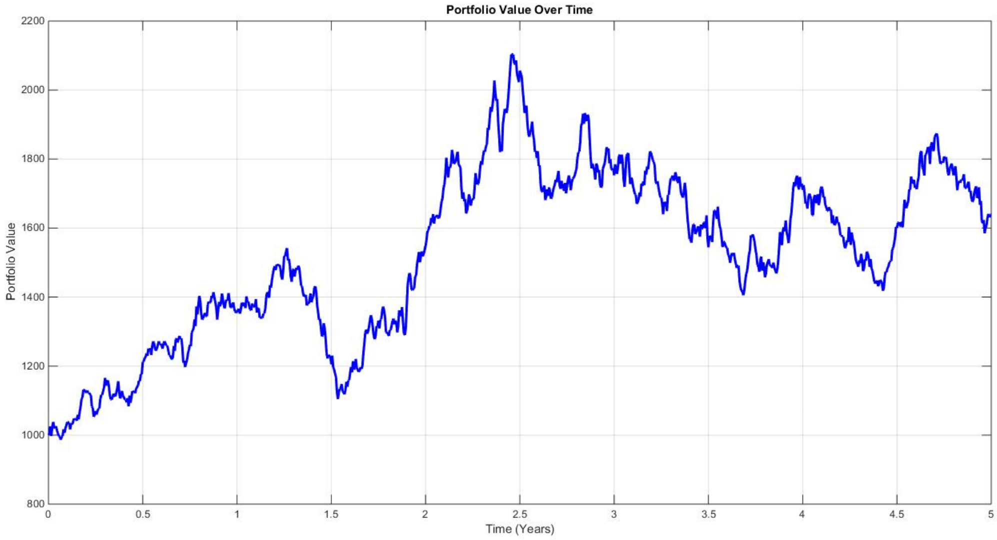

Figure 6: Solution of the system (17) on $[0, 5]$ for $M = 1.2,\ \ K = -0.0543$.

In this financial model:

1. Stochastic Component: The continuous stochastic processes involved in the portfolio's growth. The investor's portfolio value is subject to random market fluctuations (modeled by $d\xi(r)$).

2. Impulsive Component: The impulsive trading decisions made by the investor.

Stochastic stability in this context means that, over time, despite the investor's trading decisions and the inherent randomness in the financial markets, the portfolio value tends to remain within a certain bounded region. This stability is crucial for the investor's financial well-being and long-term success. The goal is to avoid excessive volatility and losses that could jeopardize the portfolio. Achieving stochastic stability would mean that the investor's trading decisions are well-balanced, and the impact of these decisions, combined with market fluctuations, results in a portfolio that grows over time or at least remains within acceptable risk parameters. This stability is a key objective in financial modeling and decision-making, where the balance between proactive trading actions and risk management is critical.

Example 4. *Stochastic stability in impulsive fractional stochastic differential equations (IF-SDEs) can be illustrated with the example of a bouncing ball, which incorporates both impulsive events and stochastic dynamics. This example demonstrates how stochastic stability analysis can be applied to real-world physics problems.*

Consider a ball dropped from a certain height above the ground. The ball experiences random air resistance, which is modeled as a stochastic process due to the turbulent nature of the surrounding air. Additionally, the ball undergoes impulsive collisions with the ground when it bounces back up. The motion of the ball can be described by an IF-SDE.

The IF-SDE for the vertical position of the ball could be represented as follows:

$$\begin{aligned} {}^{C}D^{M}a(r)+\sum_{g=1}^{x}u_g\,{}^{C}D^{N_g}a(r) &= Ka(r)+V(r,a(r))+H(r,a(r))d\xi(r), \qquad (18)\\ & \quad r\in i=(0,10]\ r\neq r_t,\\ \Delta a(r_t) &= J_t(a(r_t^-)),\quad t=1,2,...,9,\\ a'(0) &= \delta \end{aligned}$$

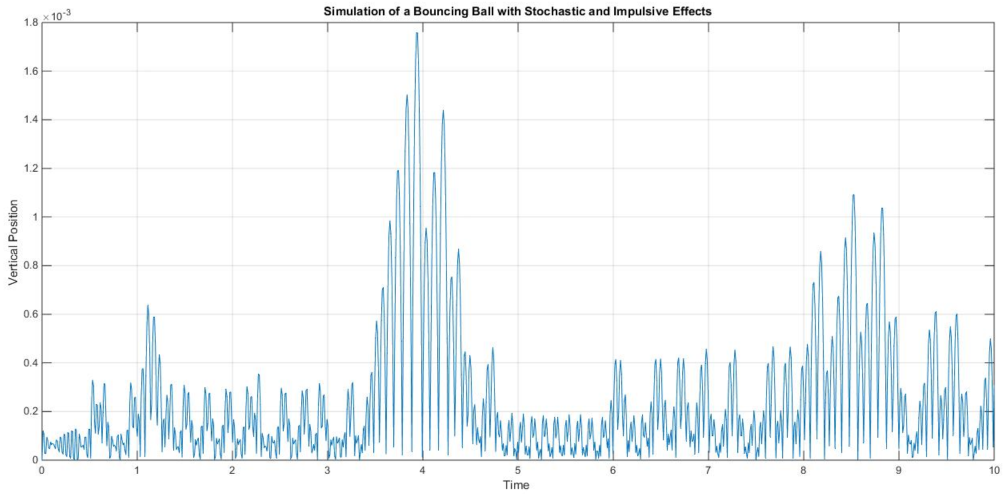

Figure 7: Solution of the system (18) on $[0, 10]$ for $M = 1.2$, $K = -0.0543$.

In this example, stochastic stability analysis would involve studying how the behavior of the ball evolves over time in the presence of both stochastic fluctuations (due to air resistance) and impulsive events (bouncing off the ground). The goal would be to determine whether, on average, the ball's position remains bounded (i.e., it doesn't bounce off into infinity or hit the ground with infinite force), despite the stochastic and impulsive nature of the system. By analyzing the properties of this IF-SDE and assessing the conditions under which the ball's position remains bounded, physicists can gain insights into the stochastic stability of this real-life physical system. The results of such an analysis could be used to design more robust systems or optimize the behavior of bouncing objects subjected to random forces and collisions.

6. Conclusions

This work uses the Rosenblatt process along with a multi-term Caputo fractional derivative to evaluate the stability and controllability of stochastic fractional impulsive differential equations. Using the fixed point theory of the Banach contraction mapping principle, an attempt was made to construct acceptable criteria for stochastic stability and controllability analysis. The full study has been provided with examples. Last but not least, it is possible to investigate the dynamical behaviour of numerous real-world objects by combining the multi-term Caputo fractional derivative with the Rosenblatt process stochastic differential equation. As a direction for future research, the present analysis can be extended to multi-term stochastic fractional differential inclusions, where the control functions belong to a set-valued map. This would allow us to investigate more general systems with uncertainties and constraints. Potential applications of such models can be explored in engineering systems with random perturbations, financial mathematics for modeling market volatility, and biological systems where impulsive effects and memory play a significant role. Further studies may also focus on developing numerical algorithms to approximate the controllability and stability conditions for such inclusions.

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