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Hierarchical Uncertainty Modeling via (m, n)-Superhyperuncertain and (h, k)-ary (m, n)-Superhyperuncertain Sets: Unified Extensions of Fuzzy, Neutrosophic, Soft, Rough, and Plithogenic Set Theories

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Abstract. Various set-theoretic frameworks have been widely recognized for their effectiveness in handling uncertainty, including *Fuzzy Sets*, *Neutrosophic Sets*, *Plithogenic Sets*, *Rough Sets*, and *Soft Sets*. These foundational models have been further extended through the use of *hyperstructures*—based on the powerset construction—and *superhyperstructures*—based on the n-th-order powerset, obtained by iteratively applying the powerset operation [1, 2]. These extended constructs are collectively referred to as *HyperUncertain Sets* and *SuperHyperUncertain Sets*.
Research on SuperHyperUncertain Sets is still in its early stages, and investigations into their properties, extended forms, and potential applications are expected to become increasingly significant in the future. In this paper, we propose two new, more general frameworks: the (m, n)-SuperHyperUncertain Set and the (h, k)-ary (m, n)-SuperHyperUncertain Set. These new structures represent a concrete and refined reconsideration of the foundational concepts introduced in [1, 2]. It is anticipated that the concepts developed in this work can be effectively applied to the modeling of more hierarchical forms of uncertainty, as well as to scenarios requiring complex membership functions. Since this paper conducts only theoretical analysis, we also hope that quantitative analysis using computational methods will be carried out in the future.

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Key Words and Phrases: HyperFuzzy Set, HyperNeutrosophic Set, HyperPlithogenic Set, HyperRough Set, HyperSoft Set

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1. Introduction

Set theory is one of the most fundamental areas of mathematics[3]. However, classical sets cannot adequately represent the many forms of uncertainty present in the real world.

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Reasoning with incomplete, vague, or heterogeneous information has therefore motivated the development of a rich family of *uncertain set* models [4]. Some of the most widely studied examples include Fuzzy Sets [5], Intuitionistic Fuzzy Sets [6], Vague Sets [7], Soft Sets [8], Rough Sets [9], Neutrosophic Sets [10], QuadriPartitioned Neutrosophic Sets[11, 12], HeptaPartitioned Neutrosophic Sets[13, 14], and Plithogenic Sets [15]. For the reader's reference, a concise overview of common uncertain–set families is provided in Table 1.

Table 1: Concise overview of common uncertain–set families.

Family	**Core semantics**	**Key ref.**	**Representative extensions**
Fuzzy Set	Membership $\mu(x) \in [0, 1]$.	[5]	Hesitant [16]; Spherical [17].
Intuitionistic / Vague Set	Two degrees (μ, ν) for membership / non–membership.	[7, 18]	—
Soft Set	Parameterized family $E \to \mathcal{P}(U)$.	[8]	—
Rough Set	Lower / upper approximations via indiscernibility.	[9]	Granular [19]; Probabilistic [20].
Neutrosophic Set	Triple (T, I, F) with $0 \leq T+I+F \leq 3$.	[10]	Bipolar [21]; Interval–Valued [22].
Plithogenic Set	Multi–attribute appurtenance with contradiction degree.	[15]	—

These Uncertain Set frameworks have been successfully adapted to diverse mathematical and applied domains—including graph theory, topology, and knowledge representation—offering flexible abstractions for modeling uncertainty at multiple levels of granularity. These Uncertain Sets are studied not only in mathematics but also in a wide range of applications across decision science, engineering, and computer science (e.g.[23–25]).

Next, we explain the notions of *hyperstructures*, *superhyperstructures*, *hyperuncertain sets*, and *superhyperuncertain sets*. Many classical mathematical structures admit two natural and systematic forms of enrichment. The first replaces a base set S with its powerset $\mathcal{P}(S)$ and endows it with *hyperoperations*, producing what are called *hyperstructures* [26, 27]. The second applies the powerset construction iteratively n times, yielding *superhyperstructures* on $\mathcal{P}^n(S)$, which are capable of encoding nested or hierarchical interactions [28, 29]. This perspective is closely related to the theory of hypergraphs [30, 31] and their higher-order analogues, *superhypergraphs* [32, 33], which generalize pairwise connections to multiway and multi-level relationships. Further examples of superhyperstructures include *SuperHyperAlgebra*[34, 35] and *Chemical SuperHyperStructures*[36, 37]. Because of their ability to represent hierarchical, multi-scale relationships, superhyperstructures are expected to find applications in modeling a wide range of complex real-world systems. For reference, a Concise overview of Classical, Hyper, and SuperHyper Structures is provided in Table 2. Unless otherwise specified, values such as n and m in this paper are nonnegative integers. Moreover, throughout we work only with *finite* sets; issues related to infinity are not considered.

Table 2: Concise overview of Classical, Hyper, and SuperHyper Structures.

Notion	**Carrier**	**Essence**	**Typical instances / refs.**
Classical Structure	S (base set)	Operations are defined directly on elements of S (e.g., groups, rings, topologies). Classical structures capture pairwise or elementwise interactions without higher-level nesting.	Groups, Rings, Topological Spaces, Graphs. See [30].
HyperStructure	$\mathcal{P}(S)$ (powerset of S)	Replace S with $\mathcal{P}(S)$ and introduce *hyperoperations* acting on subsets rather than single elements. This enables multi-valued or set-valued outputs, capturing richer interactions.	Hypergroups, Hyperrings, Hypergraphs. See [26, 27, 31].
SuperHyper Structure	$\mathcal{P}^n(S)$ ($n \geq 1$ iterated powerset)	Extend hyperstructures by iterating the powerset n times, thereby modeling *nested and hierarchical* uncertainty and multi-level interactions. Admits multi-ary inputs and multi-layered outputs.	SuperHypergraphs [32, 33]; SuperHyperAlgebra [34, 35]; Chemical Super-HyperStructures [36, 37].

In parallel with these structural enrichments, the notion of *uncertain sets* has itself been 'hyperized" and 'superhyperized." Specifically, HyperFuzzy [38, 39], HyperSoft [40–42], HyperRough [43, 44], HyperNeutrosophic [45], and HyperPlithogenic Sets [46] arise by imposing the corresponding semantics on the powerset $\mathcal{P}(S)$ [1, 2]. Their superhyper counterparts—such as *SuperHyperFuzzy*[47], *SuperHyperNeutrosophic*[2], and related variants—are defined over the n-th powerset $\mathcal{P}^n(S)$ and are designed to capture uncertainty at multiple hierarchical levels [1]. For reference, a concise overview of Uncertain, HyperUncertain, and SuperHyperUncertain Sets is provided in Table 3. Needless to say, these frameworks are expected to find applications in various fields, such as decision–making and beyond. Note that type-n uncertain sets refine uncertainty inside membership values—e.g., type-n fuzzy [48] treats degrees-of-membership as higher-order quantities. SuperHyper-Uncertain sets move uncertainty to the elements themselves via iterated powersets, capturing multi-level groupings and interactions; this enables explicit hierarchical modeling, compositional operations, and scalable reasoning across nested attributes and real-world scenarios.

From the above discussion, it is clear that research on *HyperUncertain Sets* and *Super-HyperUncertain Sets* is of great importance, both mathematically and in applied mathematics. Nevertheless, the systematic study of *HyperUncertain Sets*—including Hyper-Fuzzy, HyperVague, HyperSoft, HyperRough, HyperNeutrosophic, and HyperPlithogenic models—remains comparatively young despite their clear potential for modeling complex data. Existing frameworks such as HyperFuzzy or n–SuperHyperFuzzy cannot adequately address real–world scenarios like multi–modal medical diagnostics, hierarchical IoT sensor fusion, or cross–domain financial risk analysis involving layered uncertainties.

To bridge this gap, in this work we introduce and formalize the (m, n)-*SuperHyperUncertain*

Table 3: Concise overview of Uncertain, HyperUncertain, and SuperHyperUncertain Sets.

Notion	**Carrier**	**Essence**	**Typical instances / refs.**
Uncertain Set	S (or model-dependent $\mathcal{P}(S)$)	Attach uncertainty semantics to elements or subsets, such as graded membership, parameterization, lower/upper approximations, neutrosophic triples, or contradiction degree.	Fuzzy [5]; Soft [8]; Rough [9]; Neutrosophic [10]; Plithogenic [15].
HyperUncertain Set	$\mathcal{P}(S)$	Impose these semantics on the powerset; membership values become set–valued, allowing hesitation and multi–valued appurtenance for richer modeling of uncertainty.	HyperFuzzy [38, 39]; HyperSoft [40, 42]; HyperRough [43, 44]; HyperNeutrosophic [45]; HyperPlithogenic [46].
SuperHyper-Uncertain Set	$\mathcal{P}^n(S)$ $(n \geq 1)$	Lift semantics to the n-th iterated powerset to capture hierarchical and multi–level uncertainty, with the ability to admit multi–ary inputs and outputs simultaneously.	SuperHyperFuzzy, SuperHyperVague, and other variants [1].

Set, which assigns uncertain semantics on the n-fold iterated powerset $\mathcal{P}^n(\cdot)$ and admits m-ary superhyperoperations. We further develop the more general (h, k)*-ary* (m, n)*-SuperHyperUncertain Set*, enabling multiple input and output arities at each hierarchical level. Taken together, these constructions unify and extend existing uncertain–set paradigms within the framework of superhyperstructures, yielding a scalable toolkit for hierarchical and multi–attribute uncertainty while clarifying how classical models arise via restriction, projection, and composition across levels.

For the reader's reference, Table 4 provides a compact overview of n–, (m, n)–, and (h, k)–ary (m, n)–SuperHyperUncertain Sets.

Table 4: Compact overview of n–, (m, n)–, and (h, k)–ary (m, n)–Superhyperuncertain Sets.

Notion	**Carrier / Signature**	**Essence (very brief)**
n–Superhyperuncertain Set	Carrier: $\mathcal{P}^n(S)$ Sig.: $F: \mathcal{P}^n(S) \to \mathcal{D}^{(n)}$	Uncertainty on the n-th powerset (hierarchical levels). Recovers classical for n=0, hyper for n=1.
(m, n)–Superhyperuncertain Set	Carrier: $\mathcal{P}^n(S)$ Sig.: $F: (\mathcal{P}^n(S))^m \to \mathcal{D}^{(n)}$	Adds m-ary interaction at level n; m=1 gives the n–case.
(h, k)–ary (m, n)–Superhyperuncertain Set	Carrier: inputs from $\mathcal{P}^n(S)$ Sig.: $F: (\mathcal{P}^n(S))^h \to (\mathcal{D}^{(n)})^k$	Most general: h inputs, k outputs at level n. Reduces to (m, n)–case when h=1, k=1.

Note. $\mathcal{D}^{(n)}$ denotes an n–level degree family (e.g., $\tilde{\mathcal{P}}_n([0, 1])$ for fuzzy, $\tilde{\mathcal{P}}_n([0, 1]^3)$ for neutrosophic).

Together, these frameworks capture hierarchical uncertainty by assigning set-valued memberships across iterated powersets and permitting variable input–output arities. They unify fuzzy, neutrosophic, soft, rough, and plithogenic semantics, support compositional reasoning, and scale to complex, nested attributes through well-defined operations and cut-based analyses. Since this paper conducts only theoretical analysis, we also hope that quantitative analysis using computational methods will be carried out in the future.

A concise, section-level outline is provided in Table 5.

Table 5: Section-level contents (Introduction omitted).

Section	**Summary**
2. Preliminaries	Notation and core constructions: powerset $\mathcal{P}(S)$ and iterates $\mathcal{P}_n(S)$; hyperoperations and hyperstructures; n-superhyperstructures on $\mathcal{P}_n(S)$; (h, k)-ary SuperHyperstructure; brief, concrete examples.
3. Main Results	Definition of (m, n)–Superhyperuncertain Sets. Subsections: (i) SuperhyperFuzzy—restriction/projection in m, n, closure (union/intersection), nested α-cuts, functoriality; (ii) Superhyper-Neutrosophic—recovery of m=1 case, truth-projection to fuzzy, λ-cuts, closure, functoriality; (iii) SuperhyperPlithogenic—HDAF/DCF axioms and decision-making examples.
4. Additional Result	(h, k)-ary (m, n)–SuperhyperSoft and SuperhyperRough frameworks: formal definitions over $\tilde{\mathcal{P}}_m(S)$ and $\tilde{\mathcal{P}}_n(U)/\tilde{\mathcal{P}}_n(X)$; unary reduction lemmas; fixing/projection of inputs/outputs; closure under pointwise union/intersection; functorial pushforwards (surjections/quotients); illustrative cases (personalized recommendations; medical diagnosis).
5. Conclusion and Future Work	Summary of the unified superhyper framework; prospective applications to hierarchical uncertainty and complex memberships; outlook on algorithms and domain-specific case studies.

2. Preliminaries

In this section, we summarize the basic definitions and notational conventions used in the paper. Throughout, we work only with *finite* sets; issues related to infinity are not considered.

2.1. SuperHyperstructure

In this context, a *Classical Structure* refers to any mathematical or real–world structure or concept. To capture such structures in a hierarchical manner, *HyperStructures* are introduced. A *Hyperstructure* is a mathematical framework in which operations on elements return sets rather than single outcomes, thereby enabling multivalued algebraic relations on a base set[28, 29]. A *SuperHyperstructure* generalizes hyperstructures by permitting operations on higher–order collections such as iterated powersets, supporting multiple input and output arities, and modeling layered, hierarchical uncertainty [28, 29].

First, we record formal definitions of the powerset, the n-th powerset, and their non-empty variants, followed by an illustrative example. Note that the *n-th powerset* of a set S is obtained by iterating the powerset operator n times, producing $\mathcal{P}_n(S) = \mathcal{P}(\mathcal{P}_{n-1}(S))$, thereby encoding higher-order collections of subsets and supporting layered, hierarchical structures [28, 29].

Definition 1 (Base Set)**.** *A* base set *S is the underlying set from which constructions such as powersets, hyperstructures, and superhyperstructures are built. Formally,*

$$S = \{x \mid x \textit{ is an element in the specified domain}\}.$$

All elements of objects like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are ultimately drawn from S.

Definition 2 (Powerset)**.** *The* powerset *of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S (including $\emptyset$ and S itself):*

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

We also write the non-empty powerset *as*

$$\mathcal{P}^*(S) \;=\; \mathcal{P}(S) \setminus \{\emptyset\}.$$

Definition 3 (n-th powerset and n-th non-empty powerset)**.** *(cf.,[28, 29]) For a set H and an integer $n \geq 1$, define recursively*

$$\mathcal{P}_1(H) = \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) = \mathcal{P}\big(\mathcal{P}_n(H)\big).$$

Similarly, set

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \qquad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*\big(\mathcal{P}_n^*(H)\big).$$

Thus $\mathcal{P}_n(H)$ (resp. $\mathcal{P}_n^*(H)$) is obtained by iterating the (non-empty) powerset operator n times.

Example 1 (n-th powerset: a concrete instance)**.** *Let $H = \{a, b\}$. Then*

$$\mathcal{P}_1(H) = \mathcal{P}(H) = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}\big\}.$$

The second powerset $\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big)$ is the set of all *subsets of $\mathcal{P}_1(H)$. Since $\mathcal{P}_1(H)$ has four elements, $\mathcal{P}_2(H)$ has $2^4 = 16$ elements, which we list by cardinality:*

$$\begin{aligned}
&\textit{size } 0 : \{\ \emptyset\ \},\\
&\textit{size } 1 : \big\{\{\emptyset\}\big\},\ \big\{\{\{a\}\}\big\},\ \big\{\{\{b\}\}\big\},\ \big\{\{\{a,b\}\}\big\},\\
&\textit{size } 2 : \big\{\{\emptyset\},\{\{a\}\}\big\},\ \big\{\{\emptyset\},\{\{b\}\}\big\},\ \big\{\{\emptyset\},\{\{a,b\}\}\big\},\\
&\qquad\quad \big\{\{\{a\}\},\{\{b\}\}\big\},\ \big\{\{\{a\}\},\{\{a,b\}\}\big\},\ \big\{\{\{b\}\},\{\{a,b\}\}\big\},\\
&\textit{size } 3 : \big\{\{\emptyset\},\{\{a\}\},\{\{b\}\}\big\},\ \big\{\{\emptyset\},\{\{a\}\},\{\{a,b\}\}\big\},\\
&\qquad\quad \big\{\{\emptyset\},\{\{b\}\},\{\{a,b\}\}\big\},\ \big\{\{\{a\}\},\{\{b\}\},\{\{a,b\}\}\big\},\\
&\textit{size } 4 : \big\{\{\emptyset\},\{\{a\}\},\{\{b\}\},\{\{a,b\}\}\big\}.
\end{aligned}$$

In particular, $\mathcal{P}_2(H)$ is the Boolean algebra on the four generators $\emptyset, \{a\}, \{b\}, \{a, b\}$. Higher iterations $\mathcal{P}_n(H)$ are obtained by repeating this construction.

To establish a formal foundation for the concepts of Hyperstructures and Superhyperstructures, we present the following definitions and propositions.

Definition 4 (Classical Structure)**.** *(cf.[28, 49]) A* Classical Structure *is a mathematical framework defined on a non-empty set H, equipped with one or more* Classical Operations *that satisfy specified* Classical Axioms*. Specifically:*

A Classical Operation *is a function of the form:*

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 5 (Hyperoperation)**.** *(cf.[50, 51]) A* hyperoperation *is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:*

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 6 (Hyperstructure)**.** *(cf.[26, 28, 52]) A* Hyperstructure *extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:*

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Example 2 (Grocery Substitution as a Hyperstructure)**.** *Let*

$$S = \{\text{Milk}, \text{SoyMilk}, \text{Bread}, \text{GFBread}, \text{Eggs}\}.$$

Define a substitution map $\sigma : S \to \mathcal{P}(S)$ by $\sigma(\text{Milk}) = \{\text{Milk}, \text{SoyMilk}\}$, $\sigma(\text{Bread}) = \{\text{Bread}, \text{GFBread}\}$, and $\sigma(x) = \{x\}$ for $x \in \{\text{SoyMilk}, \text{GFBread}, \text{Eggs}\}$. Set a hyperoperation on items

$$\star : S \times S \to \mathcal{P}(S), \qquad a \star b = \sigma(a) \cup \sigma(b),$$

and induce an operation on baskets (subsets) by

$$\odot : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S), \qquad A \odot B = \bigcup_{a \in A,\, b \in B} (a \star b).$$

Then $\mathcal{H} = (\mathcal{P}(S), \odot)$ is a Hyperstructure modelling shopping lists with permissible substitutions. For instance,

$$\{\text{Milk}, \text{Bread}\} \odot \{\text{Eggs}\} = \{\text{Milk}, \text{SoyMilk}, \text{Bread}, \text{GFBread}, \text{Eggs}\}.$$

Definition 7 (SuperHyperOperations)**.** *(cf.[28]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:*

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A SuperHyperOperation *of order (m, n) is an m-ary operation:*

$$\circ^{(m,n)} : H^m \to \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- *If the codomain is $\mathcal{P}^n_*(H)$ excluding the empty set, it is called a* classical-type (m, n)-SuperHyperOperation*.*
- *If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a* Neutrosophic (m, n)-SuperHyperOperation*.*

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Definition 8 (n-Superhyperstructure)**.** *(cf.[28, 29, 49, 53]) An* n-Superhyperstructure *further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:*

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 3 (Feature-Flag Cohorts as a 2-Superhyperstructure)**.** *In a modern microservice architecture, a "feature flag" system often needs to manage nested groups of users for A/B testing and gradual roll-outs. Let*

$$S = \{\texttt{FeatureA}, \texttt{FeatureB}, \texttt{FeatureC}\}$$

be the base set of all flags. Then

$$\mathcal{P}_1(S) = \mathcal{P}(S) \quad \text{and} \quad \mathcal{P}_2(S) = \mathcal{P}\big(\mathcal{P}(S)\big)$$

are respectively all subsets of flags (cohorts) and all subsets of cohorts (collections of test groups).

Define a binary operation

$$\mathsf{merge} : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \longrightarrow \mathcal{P}_2(S)$$

by

$$\mathsf{merge}(C_1, C_2) = \big\{ g_1 \cup g_2 \mid g_1 \in C_1,\ g_2 \in C_2 \big\}.$$

Then

$$\mathcal{SH}_2 \;=\; \bigl(\mathcal{P}_2(S),\, \mathsf{merge}\bigr)$$

is a concrete 2-Superhyperstructure: each element is a collection of cohorts, and merging two collections produces every possible union of one cohort from each.

The definition of the (h, k)-ary SuperHyperstructure, which is known as a further generalization of the above superhyper-framework, is provided below.

Definition 9 ((h, k)-ary SuperHyperstructure)**.** *Let S be a nonempty base set. For each $i \in \{1, 2, \ldots, h\}$ choose a nonempty subset $A_i \subseteq S$ and fix an integer $m_i \geq 0$. Denote by*

$$\mathcal{P}_{m_i}(A_i)$$

the m_i-th iterated powerset of A_i. Similarly, for each $j \in \{1, 2, \ldots, k\}$ choose a nonempty subset $B_j \subseteq S$ and fix an integer $n_j \geq 0$, and denote by

$$\mathcal{P}_{n_j}(B_j)$$

the n_j-th iterated powerset of B_j.

Define the domain D *and* codomain C *as*

$$D \;=\; \mathcal{P}_{m_1}(A_1) \times \mathcal{P}_{m_2}(A_2) \times \cdots \times \mathcal{P}_{m_h}(A_h),$$

$$C \;=\; \mathcal{P}_{n_1}(B_1) \times \mathcal{P}_{n_2}(B_2) \times \cdots \times \mathcal{P}_{n_k}(B_k).$$

An (h, k)-ary SuperHyperstructure *on S is an algebraic system*

$$\mathcal{SH} = \bigl(D,\; C,\; \{\circ_\alpha\}_{\alpha \in I}\bigr),$$

where $\{\circ_\alpha\}_{\alpha \in I}$ is an indexed family of SuperHyperoperations

$$\circ_\alpha : D \longrightarrow C, \quad \alpha \in I.$$

For each $(X_1, \ldots, X_h) \in D$, one has

$$\circ_\alpha(X_1, \ldots, X_h) \;=\; (Y_1, \ldots, Y_k) \in C,$$

with $Y_j \in \mathcal{P}_{n_j}(B_j)$ for all j. The structural properties imposed on the maps $\circ_\alpha$—such as associativity, commutativity, or distributivity—are specified according to the algebraic framework adopted, thereby extending classical algebraic systems into a higher-order superhyperstructural setting.

Example 4 (Collaborative-Filtering as a $(2, 2)$-ary SuperHyperstructure)**.** *In a recommendation system, one often fuses information about user cohorts and item-sets to produce both new item suggestions and peer cohorts. Let*

$$S = U \;\cup\; P$$

be the union of all users $U = \{\text{Hiroko}, \text{Masahiro}, \text{Shinya}, \ldots\}$ *and all products* $P = \{\text{iPhone}, \text{Galaxy}, \text{Pixel}, \ldots\}$.

- *Choose* $h = 2$ *inputs:*

$$A_1 = U, \quad m_1 = 1, \qquad A_2 = P, \quad m_2 = 1.$$

Then $P_{m_1}(A_1) = \mathcal{P}(U)$ *is the set of all user-cohorts, and* $P_{m_2}(A_2) = \mathcal{P}(P)$ *is the set of all item-sets.*

- *Choose* $k = 2$ *outputs:*

$$B_1 = P, \quad n_1 = 1, \qquad B_2 = U, \quad n_2 = 1.$$

Then $P_{n_1}(B_1) = \mathcal{P}(P)$ *are the recommended items, and* $P_{n_2}(B_2) = \mathcal{P}(U)$ *are the peer cohorts.*

Thus the domain and codomain are

$$D = \mathcal{P}(U) \times \mathcal{P}(P), \qquad C = \mathcal{P}(P) \times \mathcal{P}(U).$$

Define a single SuperHyperOperation

$$\circ : \; D \longrightarrow C$$

by collaborative-filtering logic, for example

$$\circ\bigl(X, Y\bigr) = \bigl(\, \mathrm{RecItems}(X, Y), \; \mathrm{SimUsers}(X, Y)\bigr),$$

where

$$\mathrm{RecItems}(X, Y) = \bigl\{ \, p \in P \mid \exists \, u \in X, \; u \text{ liked } p' \in Y, \text{ and other users in } X \text{ also liked } p\bigr\},$$

$$\mathrm{SimUsers}(X, Y) = \bigl\{ \, u' \in U \mid u' \notin X, \; \exists \, p \in Y \text{ with } u' \text{ liked } p\bigr\}.$$

For instance,

$$\circ\bigl(\{\mathrm{Hiroko}, \mathrm{Masahiro}\}, \, \{\mathrm{iPhone}, \mathrm{Galaxy}\}\bigr) = \bigl(\{\mathrm{Pixel}, \mathrm{OnePlus}\}, \, \{\mathrm{Shinya}, \mathrm{Dan}\}\bigr).$$

Hence

$$\mathcal{SH} = \bigl(D, \; C, \; \{\circ\}\bigr)$$

is a concrete $(2, 2)$*-ary SuperHyperstructure modeling collaborative-filtering in practice.*

Example 5 (Smart Home Automation as a $(2, 2)$-ary SuperHyperstructure)**.** *Consider a smart home system where sensor events and notification preferences determine both device actions and alert recipients. Let the base set be*

$$S = \{\text{MotionSensor, DoorSensor, SmokeSensor,}$$

$$\text{LightSwitch, Alarm, OwnerApp, SecurityService}\}.$$

Choose two input subsets:

$$A_1 = \{\text{MotionSensor, DoorSensor, SmokeSensor}\}, \quad m_1 = 1,$$

$$A_2 = \{\text{OwnerApp, SecurityService}\}, \quad m_2 = 1,$$

so that $P_{m_1}(A_1) = \mathcal{P}(A_1)$ *is the set of all active sensor-subsets, and* $P_{m_2}(A_2) = \mathcal{P}(A_2)$ *is the set of all notification-channel configurations. Similarly, choose two output subsets:*

$$B_1 = \{\text{LightSwitch, Alarm}\}, \quad n_1 = 1,$$

$$B_2 = \{\text{OwnerApp, SecurityService}\}, \quad n_2 = 1,$$

so that $P_{n_1}(B_1) = \mathcal{P}(B_1)$ *are possible device-action sets, and* $P_{n_2}(B_2) = \mathcal{P}(B_2)$ *are possible recipient sets.*

Thus the domain and codomain are

$$D = \mathcal{P}(A_1) \times \mathcal{P}(A_2), \qquad C = \mathcal{P}(B_1) \times \mathcal{P}(B_2).$$

Define a single SuperHyperOperation

$$\circ : D \longrightarrow C$$

by

$$\circ(X, Y) = \big(\text{Actions}(X),\ \text{Recipients}(X, Y)\big),$$

where

$$\text{Actions}(X) = \begin{cases} \{\text{LightSwitch}\}, & \textit{if } \text{MotionSensor} \in X, \\ \{\text{Alarm}\}, & \textit{if } \text{SmokeSensor} \in X, \\ \varnothing, & \textit{otherwise}, \end{cases}$$

$$\text{Recipients}(X, Y) = \{\, r \in Y \mid r \textit{ is configured}\}.$$

For example,

$$\circ\big(\{\text{MotionSensor}, \text{SmokeSensor}\},\ \{\text{OwnerApp}\}\big) = \big(\{\text{LightSwitch}, \text{Alarm}\},\ \{\text{OwnerApp}\}\big).$$

Therefore,

$$\mathcal{SH} = \big(D,\ C,\ \{\circ\}\big)$$

is a concrete $(2, 2)$*-ary SuperHyperstructure modeling a real-world smart home automation workflow.*

2.2. Fuzzy Set

The Fuzzy Set is a well-known concept used to address uncertainty in set theory[5, 54, 55]. These sets can be extended into Hyperfuzzy Sets[56, 57] and SuperHyperfuzzy Sets[58] using hyperstructures and superhyperstructures. The definition is provided below .

Definition 10 (Fuzzy Set). *[5] A fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if*

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 11 (HyperFuzzy Set). *[2, 59–61] Let X be a non-empty set. A hyperfuzzy set over X is defined as a mapping:*

$$\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

where $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ represents the family of all non-empty subsets of the interval $[0,1]$.

For each element $x \in X$, $\tilde{\mu}(x)$ assigns a non-empty subset of $[0,1]$, representing the possible membership degrees of x in the hyperfuzzy set. This definition generalizes classical fuzzy sets by allowing the membership degree of each element to be a range (set of values) instead of a single scalar value.

Example 6 (Restaurant Customer Satisfaction). *Let*

$$X = \{\text{R1}, \text{R2}, \text{R3}\}$$

be a set of restaurants. We define a hyperfuzzy set $\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ modeling customer satisfaction as follows:

$$\tilde{\mu}(\text{R1}) = [0.7,\, 0.9], \quad \tilde{\mu}(\text{R2}) = \{0.5,\, 0.6,\, 0.7\}, \quad \tilde{\mu}(\text{R3}) = [0.2,\, 0.4] \cup \{0.6\}.$$

Here:

- $\tilde{\mu}(\text{R1}) = [0.7, 0.9]$ *reflects that customers' satisfaction scores for R1 range continuously from 0.7 to 0.9.*
- $\tilde{\mu}(\text{R2}) = \{0.5, 0.6, 0.7\}$ *indicates that satisfaction for R2 clusters at three discrete levels.*
- $\tilde{\mu}(\text{R3}) = [0.2, 0.4] \cup \{0.6\}$ *shows a mixed pattern: a continuous range for most customers plus one high outlier at 0.6.*

This hyperfuzzy set captures the variability and uncertainty in customer satisfaction across different restaurants.

Definition 12 (n-SuperHyperFuzzy Set)**.** *[1, 2] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:*

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$*, and for* $k \geq 2$*,*

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ *is similarly defined for the interval* $[0,1]$*.*
- $\tilde{\mu}_n$ *assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.*

Example 7 (2-SuperHyperFuzzy Set for Equipment Reliability)**.** *Let*

$$X = \{\text{Compressor, Pump}\}.$$

Then

$$\tilde{\mathcal{P}}_1(X) = \big\{\{\text{Compressor}\}, \{\text{Pump}\}, \{\text{Compressor}, \text{Pump}\}\big\},$$

and

$$\tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_1(X)\big),$$

the family of all nonempty collections of nonempty subsets of X. We define a 2-SuperHyperFuzzy Set $\tilde{\mu}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0,1])$ by specifying its value on one representative element:

$$A \;=\; \big\{\{\text{Compressor}\}, \{\text{Pump}\}\big\} \;\in\; \tilde{\mathcal{P}}_2(X),$$

$$\tilde{\mu}_2(A) = \Big\{\{0.70,\, 0.75\},\; [0.60,\, 0.65],\; \{0.50,\, 0.55\}\Big\}.$$

Here each inner set (or interval) is a subset of $[0,1]$, representing possible reliability scores from different assessment methods (e.g. expert judgment, sensor analytics, historical data). Thus $\tilde{\mu}_2(A)$ captures the multi-level uncertainty in the joint reliability evaluation of the Compressor and Pump.

Example 8 (3-SuperHyperFuzzy Set for Predictive Maintenance)**.** *Let*

$$X = \{\text{Sensor1, Sensor2}\}.$$

We construct the nested powersets:

$$\tilde{\mathcal{P}}_1(X) = \big\{\{\text{Sensor1}\}, \{\text{Sensor2}\}, \{\text{Sensor1}, \text{Sensor2}\}\big\},$$

$$\tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_1(X)\big), \quad \tilde{\mathcal{P}}_3(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_2(X)\big).$$

Choose the element

$$A \;=\; \big\{\{\{\text{Sensor1}\},\{\text{Sensor2}\}\},\; \{\{\text{Sensor1},\text{Sensor2}\}\}\big\} \;\in\; \tilde{\mathcal{P}}_3(X).$$

A 3-SuperHyperFuzzy Set is a mapping

$$\tilde{\mu}_3 : \tilde{\mathcal{P}}_3(X) \;\longrightarrow\; \tilde{\mathcal{P}}_3([0,1]).$$

We define its value on A by

$$\tilde{\mu}_3(A) = \big\{\, B_1,\; B_2 \big\} \;\subseteq\; \tilde{\mathcal{P}}_2([0,1]),$$

where

$$B_1 = \big\{[0.80, 0.90],\; \{0.85\}\big\}, \qquad B_2 = \big\{[0.60, 0.70],\; \{0.65, 0.68\}\big\}.$$

Each B_i is a non-empty subset of $\tilde{\mathcal{P}}([0,1])$, representing multiple reliability scores from different diagnostic methods (e.g. expert judgment vs. sensor analytics). Thus $\tilde{\mu}_3(A)$ captures the third-level, nested uncertainty in the combined sensor evaluation.

2.3. Neutrosophic Set

A Neutrosophic Set models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F), which satisfy:

$$0 \le T + I + F \le 3.$$

[62, 63]. These sets can be extended into HyperNeutrosophic Sets [64] and SuperHyper-Neutrosophic Sets using hyperstructures and superhyperstructures.

Definition 13 (Neutrosophic Set)**.** *[10, 63] Let X be a non-empty set. A Neutrosophic Set (NS) A on X is characterized by three membership functions:*

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 14 (HyperNeutrosophic Set)**.** *[1, 2] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a* HyperNeutrosophic Set *over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:*

$$0 \le T + I + F \le 3.$$

Example 9 (Medical Diagnosis Application)**.** *Let $X = \{Hiroko, Masahiro\}$ be a set of patients undergoing diagnosis for a particular disease. We model the uncertain evaluation of each patient's condition by a hyperneutrosophic set*

$$\tilde{\mu} : X \longrightarrow \tilde{\mathcal{P}}\big([0,1]^3\big).$$

Here each element of $\tilde{\mu}(x) \subseteq [0,1]^3$ is a triple (T, I, F) giving a possible assessment of Truth (presence), Indeterminacy, and Falsity (absence) of the disease.

For example:

$$\tilde{\mu}(Hiroko) = \big\{(0.85,\, 0.10,\, 0.05),\ (0.80,\, 0.15,\, 0.05)\big\},$$

$$\tilde{\mu}(Masahiro) = \big\{(0.60,\, 0.20,\, 0.20),\ (0.65,\, 0.25,\, 0.10)\big\}.$$

Each triple satisfies $0 \le T + I + F \le 3$. The two points in $\tilde{\mu}(x)$ reflect, for instance, two independent expert opinions or two different diagnostic tests, capturing both the variability among assessments and the underlying neutrosophic uncertainty.

Definition 15 (n-SuperHyperNeutrosophic Set)**.** *[1, 2] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, Hyper-Neutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:*

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$*, and for $k \ge 2$,*

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ *is similarly defined for the unit cube $[0,1]^3$.*
- *The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.*

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \le T + I + F \le 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 10 (Hierarchical Fault Diagnosis in an Industrial System)**.** *Let*

$$X = \{\, TempSensor,\ PressureSensor\}$$

be the set of two critical sensors in an industrial process. We consider $n = 2$, so that

$$\tilde{\mathcal{P}}_1(X) = \big\{\{\textit{TempSensor}\},\ \{\textit{PressureSensor}\},\ \{\textit{TempSensor}, \textit{PressureSensor}\}\big\},$$

and

$$\tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}\big(\tilde{\mathcal{P}}_1(X)\big),$$

whose elements are nonempty collections of nonempty subsets of X. For example, choose

$$A \ = \ \big\{\{\textit{TempSensor}\}, \{\textit{PressureSensor}\}\big\} \ \in \ \tilde{\mathcal{P}}_2(X).$$

A 2-SuperHyperNeutrosophic Set on X is a mapping

$$\tilde{A}_2 : \tilde{\mathcal{P}}_2(X) \ \longrightarrow \ \tilde{\mathcal{P}}_2\big([0,1]^3\big).$$

We may define, for instance,

$$\tilde{A}_2(A) = \Big\{\{(0.85, 0.10, 0.05),\ (0.80, 0.15, 0.05)\},\ \{(0.70, 0.20, 0.10)\}\Big\}.$$

Here:

- *The first inner set $\{(0.85, 0.10, 0.05), (0.80, 0.15, 0.05)\}$ represents two expert judgments about the joint status of the temperature and pressure sensors, each triple (T, I, F) satisfying $0 \leq T + I + F \leq 3$.*
- *The second inner set $\{(0.70, 0.20, 0.10)\}$ could correspond to an automated diagnostic algorithm's assessment.*

Thus $\tilde{A}_2(A)$ captures the hierarchical uncertainty at the second level, aggregating multiple neutrosophic evaluations for the group of sensors.

2.4. Plithogenic Set

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [65, 66]. The definition is presented below.

Definition 16 (Plithogenic Set)**.** *[15, 65] Let S be a universal set, and $P \subseteq S$. A Plithogenic Set PS is defined as:*

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- *v is an attribute.*
- *Pv is the range of possible values for the attribute v.*

- *pdf* : $P \times Pv \to [0,1]^s$ *is the Degree of Appurtenance Function (DAF).*
- $pCF : Pv \times Pv \to [0,1]^t$ *is the Degree of Contradiction Function (DCF).*

These functions satisfy the following axioms for all $a, b \in Pv$*:*

(i) Reflexivity of Contradiction Function:

$$pCF(a,a) = 0$$

(ii) Symmetry of Contradiction Function:

$$pCF(a,b) = pCF(b,a)$$

Example 11 (Job Candidate English Proficiency)**.** *Let*

$$S = \{\mathit{Hiroko},\ \mathit{Masahiro},\ \mathit{Shinya}\}$$

be the universal set of job candidates, and set $P = S$*. We evaluate each candidate's English proficiency using a plithogenic set.*

$$PS = \big(P,\ v,\ P_v,\ \mathit{pdf},\ pCF\big),$$

where:

- v *is the attribute "English proficiency level."*
- $P_v = \{\mathit{Beginner},\ \mathit{Intermediate},\ \mathit{Advanced}\}$.
- *The* Degree of Appurtenance Function $\mathit{pdf} \colon P \times P_v \to [0,1]^3$ *assigns to each* (x,ℓ) *a triple* (g,w,f) *of degrees for grammar* (g)*, vocabulary* (w)*, and fluency* (f)*. For example:*

$$\mathit{pdf}(\mathit{Hiroko}, \mathit{Beginner}) = (0.10,\ 0.20,\ 0.10),$$

$$\mathit{pdf}(\mathit{Hiroko}, \mathit{Intermediate}) = (0.60,\ 0.70,\ 0.50),$$

$$\mathit{pdf}(\mathit{Hiroko}, \mathit{Advanced}) = (0.80,\ 0.90,\ 0.70).$$

- *The* Degree of Contradiction Function $pCF \colon P_v \times P_v \to [0,1]$ *measures the contradiction between two proficiency levels. It satisfies:*

$$pCF(a,a) = 0, \quad pCF(\mathit{Beginner}, \mathit{Advanced}) = 1, \quad pCF(\mathit{Beginner}, \mathit{Intermediate}) = 0.5,$$

  and by symmetry $pCF(b,a) = pCF(a,b)$.

This plithogenic set captures for each candidate not only a single "level" but a vector of sub-scores (g,w,f)*, while the contradiction function quantifies how mutually incompatible two levels are.*

These definitions establish the foundational framework necessary for exploring the HyperPlithogenic Set and the SuperHyperPlithogenic Set. The definitions of the Hyper-Plithogenic Set and the SuperHyperPlithogenic Set are presented below[46, 67].

Definition 17 (HyperPlithogenic Set)**.** *[1, 46] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A* HyperPlithogenic Set *HPS over X is defined as:*

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- *$P \subseteq X$ is a subset of the universe.*
- *For each attribute v_i, Pv_i is the set of possible values.*
- *For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the Hyper Degree of Appurtenance Function (HDAF), assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.*
- *$pCF : (\bigcup_{i=1}^n Pv_i) \times (\bigcup_{i=1}^n Pv_i) \to [0,1]^t$ is the Degree of Contradiction Function (DCF).*

Example 12 (HyperPlithogenic Set in a Smartphone Purchase Decision)**.** *Let*

$$X = \{\text{A, B, C}\}$$

be a set of smartphone models under consideration, and set $P = X$. We evaluate each model with two attributes:

$$v_1 = \textit{Battery Life}, \quad Pv_1 = \{\textit{Short}, \ \textit{Moderate}, \ \textit{Long}\},$$

$$v_2 = \textit{Camera Quality}, \quad Pv_2 = \{\textit{Low}, \ \textit{Medium}, \ \textit{High}\}.$$

For simplicity, take $s = 1$ so that $\tilde{\mathcal{P}}([0,1]^s) = \tilde{\mathcal{P}}([0,1])$.

The hyper degree of appurtenance functions are defined as follows:

$$\begin{aligned}
&\tilde{pdf}_1(\text{A}, \textit{Short}) = \{0.2, 0.3\}, \quad \tilde{pdf}_1(\text{A}, \textit{Moderate}) = \{0.6, 0.7\}, \quad \tilde{pdf}_1(\text{A}, \textit{Long}) = \{0.4, 0.5\},\\
&\tilde{pdf}_1(\text{B}, \textit{Short}) = \{0.3, 0.4\}, \quad \tilde{pdf}_1(\text{B}, \textit{Moderate}) = \{0.5, 0.6\}, \quad \tilde{pdf}_1(\text{B}, \textit{Long}) = \{0.7, 0.8\},\\
&\tilde{pdf}_1(\text{C}, \textit{Short}) = \{0.1, 0.2\}, \quad \tilde{pdf}_1(\text{C}, \textit{Moderate}) = \{0.4, 0.5\}, \quad \tilde{pdf}_1(\text{C}, \textit{Long}) = \{0.6, 0.7\},
\end{aligned}$$

$$\begin{aligned}
&\tilde{pdf}_2(\text{A}, \textit{Low}) = \{0.1, 0.2\}, \quad \tilde{pdf}_2(\text{A}, \textit{Medium}) = \{0.5, 0.6\}, \quad \tilde{pdf}_2(\text{A}, \textit{High}) = \{0.7, 0.8\},\\
&\tilde{pdf}_2(\text{B}, \textit{Low}) = \{0.2, 0.3\}, \quad \tilde{pdf}_2(\text{B}, \textit{Medium}) = \{0.6, 0.7\}, \quad \tilde{pdf}_2(\text{B}, \textit{High}) = \{0.8, 0.9\},\\
&\tilde{pdf}_2(\text{C}, \textit{Low}) = \{0.3, 0.4\}, \quad \tilde{pdf}_2(\text{C}, \textit{Medium}) = \{0.4, 0.5\}, \quad \tilde{pdf}_2(\text{C}, \textit{High}) = \{0.6, 0.7\}.
\end{aligned}$$

The degree of contradiction function $pCF : (Pv_1 \cup Pv_2) \times (Pv_1 \cup Pv_2) \to [0,1]$ is given by

$$pCF(\textit{Short}, \textit{Long}) = 0.9, \quad pCF(\textit{Low}, \textit{High}) = 0.8,$$

with $pCF(a,a) = 0$ *and symmetry* $pCF(a,b) = pCF(b,a)$*.*

Hence the HyperPlithogenic Set is

$$HPS = \big(P,\ \{v_1, v_2\},\ \{Pv_1, Pv_2\},\ \{\tilde{pdf}_1, \tilde{pdf}_2\},\ pCF\big),$$

which captures for each phone model both the hyper-valued membership degrees and the contradictions among attribute levels.

Definition 18 (n-SuperHyperPlithogenic Set)**.** *[1, 68] Let* X *be a non-empty set, and let* $V = \{v_1, v_2, \ldots, v_n\}$ *be a set of attributes, each associated with a set of possible values* P_{v_i}*. An n-SuperHyperPlithogenic Set (*$SHPS_n$*) is defined recursively as:*

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$*, and for* $k \geq 2$*,*
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k*-th nested family of non-empty subsets of* P_1*.*
- *For each attribute* $v_i \in V$*,* P_{v_i} *is the set of possible values of the attribute* v_i*.*
- *For each* k*-th level subset* P_k*,* $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ *is the Hyper Degree of Appurtenance Function (HDAF), assigning to each element* $x \in P_n$ *and attribute value* $a_i \in P_{v_i}$ *a subset of* $[0,1]^s$*.*
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ *is the Degree of Contradiction Function (DCF), satisfying:*
  - *(i) Reflexivity:* $pCF^{(n)}(a,a) = 0$ *for all* $a \in \bigcup_{i=1}^n P_{v_i}$*,*
  - *(ii) Symmetry:* $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ *for all* $a, b \in \bigcup_{i=1}^n P_{v_i}$*.*
- s *and* t *are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.*

Example 13 (Nested Project Evaluation Under Uncertainty)**.** *Let*

$$X = \{\text{Proj1}, \text{Proj2}\},$$

and take $n = 2$*. Then*

$$P_1 = X, \quad P_2 = \tilde{\mathcal{P}}(P_1) = \big\{\{\text{Proj1}\}, \{\text{Proj2}\}, \{\text{Proj1}, \text{Proj2}\}\big\}.$$

Choose two attributes:

$$v_1 = \textit{Risk Level}, \quad P_{v_1} = \{\textit{Low}, \textit{High}\},$$

$$v_2 = Cost, \quad P_{v_2} = \{Cheap,\ Expensive\}.$$

Let $s = 1$, so that $\tilde{\mathcal{P}}([0,1]^s) = \tilde{\mathcal{P}}([0,1])$.

Define the hyper degree of appurtenance functions $\tilde{pdf}_i^{(2)} : P_2 \times P_{v_i} \to \tilde{\mathcal{P}}([0,1])$ by:

$$\begin{aligned}
\tilde{pdf}_1^{(2)}\big(\{\text{Proj1}\}, Low\big) &= \{0.2,\ 0.3\}, & \tilde{pdf}_1^{(2)}\big(\{\text{Proj1}\}, High\big) &= \{0.7,\ 0.8\},\\
\tilde{pdf}_1^{(2)}\big(\{\text{Proj2}\}, Low\big) &= \{0.3,\ 0.4\}, & \tilde{pdf}_1^{(2)}\big(\{\text{Proj2}\}, High\big) &= \{0.6,\ 0.7\},\\
\tilde{pdf}_1^{(2)}\big(\{\text{Proj1}, \text{Proj2}\}, Low\big) &= \{0.4,\ 0.5\}, & \tilde{pdf}_1^{(2)}\big(\{\text{Proj1}, \text{Proj2}\}, High\big) &= \{0.8,\ 0.9\},
\end{aligned}$$

$$\begin{aligned}
\tilde{pdf}_2^{(2)}\big(\{\text{Proj1}\}, Cheap\big) &= \{0.5,\ 0.6\}, & \tilde{pdf}_2^{(2)}\big(\{\text{Proj1}\}, Expensive\big) &= \{0.7,\ 0.8\},\\
\tilde{pdf}_2^{(2)}\big(\{\text{Proj2}\}, Cheap\big) &= \{0.4,\ 0.5\}, & \tilde{pdf}_2^{(2)}\big(\{\text{Proj2}\}, Expensive\big) &= \{0.6,\ 0.7\},\\
\tilde{pdf}_2^{(2)}\big(\{\text{Proj1}, \text{Proj2}\}, Cheap\big) &= \{0.6,\ 0.7\}, & \tilde{pdf}_2^{(2)}\big(\{\text{Proj1}, \text{Proj2}\}, Expensive\big) &= \{0.8,\ 0.9\}.
\end{aligned}$$

Define the degree of contradiction $pCF^{(2)} : (P_{v_1} \cup P_{v_2})^2 \to [0,1]$ by:

$$pCF^{(2)}(Low, High) = 0.9, \quad pCF^{(2)}(Cheap, Expensive) = 0.8,$$

with reflexivity $pCF^{(2)}(a,a) = 0$ and symmetry $pCF^{(2)}(a,b) = pCF^{(2)}(b,a)$.

Thus the 2-SuperHyperPlithogenic Set is

$$SHPS_2 \ = \ \big(P_2,\ \{v_1, v_2\},\ \{P_{v_1}, P_{v_2}\},\ \{\tilde{pdf}_1^{(2)}, \tilde{pdf}_2^{(2)}\},\ pCF^{(2)}\big),$$

which captures nested, hyper-valued appurtenance degrees and contradictions for multi-criteria project evaluation.

3. Main Results of this paper

This section presents the main results of this paper.

3.1. (m,n)-Superhyperuncertain Set

We provide the definition of the (m,n)-Superhyperuncertain Set.

3.1.1. (m,n)-SuperhyperFuzzy Set

A (m,n)-Superhyperfuzzy Set maps m-level nested subsets of X to n-level nested fuzzy membership-value subsets in $[0,1]$, capturing distinct hierarchical fuzzy uncertainty patterns. The definition of the (m,n)-SuperhyperFuzzy Set is presented below.

Definition 19 ((m,n)-SuperhyperFuzzy Set)**.** *Let U be a universe of discourse and let $A \subseteq U$ be nonempty. Fix integers $m,n \geq 0$. Define the m-th nested power-set of A by*

$$\mathcal{P}^0(A) = A, \qquad \mathcal{P}^k(A) = \mathcal{P}\big(\mathcal{P}^{k-1}(A)\big) \quad (k \geq 1),$$

and similarly define $\mathcal{P}^n([0,1])$ *for the unit interval. An* (m,n)-SuperhyperFuzzy Set *on* A *is a function*

$$\tau : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n([0,1])$$

such that for each $X \in \mathcal{P}^m(A)$*, the image* $\tau(X) \subseteq [0,1]$ *is nonempty. In other words, the "membership grade" of the (possibly nested) element* X *is not a single number but a set of values—intervals and/or discrete points—drawn from* $[0,1]$*.*

Example 14 (Student Group Assignment)**.** *Let* $U = \{\text{Hiroko}, \text{Masahiro}, \text{Shinya}\}$ *be a class roster, and set* $A = U$*. Choose* $m = 1$ *and* $n = 1$*. Then*

$$\mathcal{P}^1(A) = \mathcal{P}(A) = \big\{\emptyset, \{\text{Hiroko}\}, \ldots, \{\text{Hiroko}, \text{Masahiro}, \text{Shinya}\}\big\}, \quad \mathcal{P}^1([0,1]) = \mathcal{P}([0,1]).$$

Define

$$\tau(\{\text{Hiroko}, \text{Masahiro}\}) = [0.75, 0.85], \quad \tau(\{\text{Shinya}\}) = \{0.60, 0.65\}, \quad \tau(\emptyset) = \{0\},$$

and similarly for the remaining subsets. Here τ *assigns each student group a range or set of possible average grades (e.g. from different project evaluations), rather than a single score.*

Example 15 (Grocery–Delivery Freshness (simple $(1,1)$-SuperhyperFuzzy Set))**.** *Let the universe be the set of items in a small grocery order,*

$$U = \{\, \textit{Milk}, \ \textit{Lettuce}, \ \textit{Fish} \,\},$$

and take $A = U$*. Fix* $m = n = 1$*. We interpret*

$$\tau : \ \mathcal{P}^1(A) = \mathcal{P}(A) \ \longrightarrow \ \mathcal{P}^1([0,1]) = \mathcal{P}([0,1])$$

so that, for each subset $X \subseteq A$*, the value* $\tau(X) \subseteq [0,1]$ *is a* set *of plausible freshness degrees for the delivery (aggregating uncertainty from traffic, temperature, and handling). Concretely, define*

$$\begin{aligned}
\tau(\varnothing) &= \{\, 0 \,\}, \\
\tau(\{\textit{Milk}\}) &= [0.60,\ 0.80], \qquad \tau(\{\textit{Lettuce}\}) = [0.50,\ 0.70], \qquad \tau(\{\textit{Fish}\}) = [0.40,\ 0.60], \\
\tau(\{\textit{Milk}, \textit{Lettuce}\}) &= [0.55,\ 0.75], \qquad \tau(\{\textit{Milk}, \textit{Fish}\}) = [0.45,\ 0.65], \\
\tau(\{\textit{Lettuce}, \textit{Fish}\}) &= [0.40,\ 0.60], \qquad \tau(\{\textit{Milk}, \textit{Lettuce}, \textit{Fish}\}) = [0.35,\ 0.55].
\end{aligned}$$

Each image is a nonempty subset of $[0,1]$*, so* τ *is an* $(1,1)$*-SuperhyperFuzzy Set. Intervals narrow or widen to encode uncertainty (e.g., fish is more temperature–sensitive, hence a lower range; mixing sensitive items widens risk and lowers the aggregated plausibility of "fresh on arrival").*

Notation 1 (Iterated embeddings and flattenings)**.** *For a set* S *and integers* $r \le s$ *with* $r \ge 1$*, define the* canonical embedding

$$\iota_{r \to s} : \ \mathcal{P}^r(S) \longrightarrow \mathcal{P}^s(S)$$

recursively by $\iota_{r\to r} = \mathrm{id}$ *and*

$$\iota_{r\to(t+1)}(X) \;=\; \{\ \iota_{r\to t}(X)\ \} \qquad (t \geq r).$$

Thus $\iota_{r\to s}$ *nests* X *inside* $s - r$ *successive singletons.*

For a set S *and integers* $s > r \geq 0$, *define the* level-flattening *(union) map*

$$\mathrm{U}_{s\to r} : \ \mathcal{P}^s(S) \longrightarrow \mathcal{P}^r(S)$$

by $\mathrm{U}_{s\to(s-1)}(Y) = \bigcup Y$ *and, for* $r < s-1$, *by* $\mathrm{U}_{s\to r} = \mathrm{U}_{(r+1)\to r} \circ \cdots \circ \mathrm{U}_{s\to(s-1)}$. *When* $S = [0,1]$, *nonemptiness is preserved because the families we consider consist of nonempty members at every level (see the definitions of superhyperfuzzy codomains used in this paper).*

Theorem 1. *If* $\tilde{\mu}_n : \mathcal{P}^n(X) \to \mathcal{P}^n([0,1])$ *is an n-SuperhyperFuzzy Set, then*

$$\tau \ := \ \tilde{\mu}_n \circ \iota_{1\to n} \ : \ \mathcal{P}^1(X) \longrightarrow \mathcal{P}^n([0,1])$$

is an $(1,n)$*-SuperhyperFuzzy Set. Moreover, for every* Z *in the image of* $\iota_{1\to n}$, *writing* $Z = \iota_{1\to n}(A)$ *with* $A \in \mathcal{P}(X)$, *one has*

$$\tau(A) \ = \ \tilde{\mu}_n(Z),$$

i.e. τ *agrees with* $\tilde{\mu}_n$ *on all level-n inputs obtained by the canonical embedding from level 1.*

Proof. By hypothesis $\tilde{\mu}_n$ has codomain $\mathcal{P}^n([0,1])$. The canonical embedding $\iota_{1\to n} : \mathcal{P}(X) \to \mathcal{P}^n(X)$ is well-defined by the recursion in the notation: $\iota_{1\to 1} = \mathrm{id}$ and $\iota_{1\to(t+1)}(A) = \{\iota_{1\to t}(A)\}$ for $t \geq 1$. Therefore the composition

$$\tau \ = \ \tilde{\mu}_n \circ \iota_{1\to n}$$

is a map $\mathcal{P}(X) \to \mathcal{P}^n([0,1])$, which is precisely an $(1,n)$-SuperhyperFuzzy Set by definition.

For the agreement statement, take any Z in the image of $\iota_{1\to n}$, so $Z = \iota_{1\to n}(A)$ for some $A \in \mathcal{P}(X)$. Then by construction

$$\tau(A) \ = \ \tilde{\mu}_n\big(\iota_{1\to n}(A)\big) \ = \ \tilde{\mu}_n(Z),$$

which establishes that τ reproduces $\tilde{\mu}_n$ on those level-n elements canonically obtained from level 1.

Theorem 2 (Restriction to Lower m)**.** *Let* $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ *be an* (m,n)*-SuperhyperFuzzy Set and let* m' *satisfy* $0 \leq m' < m$. *Define the inclusion*

$$\iota_{m'\to m} : \ \mathcal{P}^{m'}(A) \hookrightarrow \mathcal{P}^m(A), \qquad \iota_{m'\to m} = \iota_{(m-1)\to m} \circ \cdots \circ \iota_{m'\to(m'+1)},$$

where each step $\iota_{t\to(t+1)}(X) = \{X\}$. *Then*

$$\tau' = \tau \circ \iota_{m'\to m} : \ \mathcal{P}^{m'}(A) \longrightarrow \mathcal{P}^n([0,1])$$

is an (m',n)*-SuperhyperFuzzy Set.*

Proof. Fix $X \in \mathcal{P}^{m'}(A)$. By construction $\iota_{m'\to m}(X) \in \mathcal{P}^m(A)$, so $\tau\big(\iota_{m'\to m}(X)\big) \in \mathcal{P}^n([0,1])$. Hence τ' is well-defined with the required codomain. No additional axioms are needed, so τ' is an (m', n)-SuperhyperFuzzy Set.

Theorem 3 (Projection to Lower n)**.** *Let* $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ *be an* (m,n)*-SuperhyperFuzzy Set and let* n' *satisfy* $0 \le n' < n$*. Define the level-flattening map*

$$\pi_{n\to n'} \;:=\; \mathrm{U}_{n\to n'} :\; \mathcal{P}^n([0,1]) \longrightarrow \mathcal{P}^{n'}([0,1]),$$

i.e. iterated unions taken $(n-n')$ *times. Then*

$$\tau'' = \pi_{n\to n'} \circ \tau :\; \mathcal{P}^m(A) \longrightarrow \mathcal{P}^{n'}([0,1])$$

is an (m,n')*-SuperhyperFuzzy Set.*

Proof. Fix $X \in \mathcal{P}^m(A)$. Since $\tau(X) \in \mathcal{P}^n([0,1])$, applying $\mathrm{U}_{n\to(n-1)}$ produces $\bigcup \tau(X) \in \mathcal{P}^{n-1}([0,1])$. By the convention of this paper, members at each level are nonempty, so the union is nonempty. Iterating this $(n-n')$ times yields

$$\pi_{n\to n'}\big(\tau(X)\big) \in \mathcal{P}^{n'}([0,1]).$$

Therefore τ'' is well-defined with codomain $\mathcal{P}^{n'}([0,1])$, and hence is an (m,n')-SuperhyperFuzzy Set.

Theorem 4 (Closure under Pointwise Union)**.** *If* $\tau_1, \tau_2 : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ *are* (m,n)*-SuperhyperFuzzy Sets, then the map*

$$(\tau_1 \cup \tau_2)(X) \;:=\; \tau_1(X) \,\cup\, \tau_2(X) \qquad (X \in \mathcal{P}^m(A))$$

is also an (m,n)*-SuperhyperFuzzy Set.*

Proof. For each X, both $\tau_1(X)$ and $\tau_2(X)$ are elements of $\mathcal{P}^n([0,1])$, i.e. subsets of $\mathcal{P}^{n-1}([0,1])$. Their union is again a subset of $\mathcal{P}^{n-1}([0,1])$ and is nonempty because each term is nonempty. Thus $(\tau_1 \cup \tau_2)(X) \in \mathcal{P}^n([0,1])$ for all X, which proves the claim.

Theorem 5 (Closure under Pointwise Intersection)**.** *Let* $\tau_1, \tau_2 : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ *be* (m,n)*-SuperhyperFuzzy Sets. Define*

$$(\tau_1 \cap \tau_2)(X) \;:=\; \tau_1(X) \,\cap\, \tau_2(X) \qquad (X \in \mathcal{P}^m(A)).$$

Then $(\tau_1 \cap \tau_2)$ *is an* (m,n)*-SuperhyperFuzzy Set provided each intersection is nonempty. If some intersections are empty, one may either restrict the effective domain to those* X *with nonempty intersection, or define the modified map*

$$(\tau_1 \sqcap \tau_2)(X) \;:=\; \begin{cases} \tau_1(X) \cap \tau_2(X), & \text{if } \tau_1(X) \cap \tau_2(X) \neq \varnothing, \\ \{\,0\,\}, & \text{otherwise,} \end{cases}$$

which is again (m,n)*-SuperhyperFuzzy.*

Proof. Fix X. Since $\tau_i(X) \in \mathcal{P}^n([0,1])$ $(i = 1, 2)$, the intersection $\tau_1(X) \cap \tau_2(X)$ is a subset of $\mathcal{P}^{n-1}([0,1])$. If it is nonempty, then it lies in $\mathcal{P}^n([0,1])$ and we are done. If it happens to be empty for some X, the two remedies stated in the theorem ensure the resulting value remains a (nonempty) element of $\mathcal{P}^n([0,1])$.

Theorem 6 (Nested α-Cuts). *Let $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ be an (m,n)-SuperhyperFuzzy Set with $n \geq 1$. For $\alpha \in [0,1]$, define*

$$C_\alpha \;=\; \big\{\, X \in \mathcal{P}^m(A) \;\big|\; \exists T \in \mathrm{U}_{n\to 1}\big(\tau(X)\big) \text{ with } T \geq \alpha \,\big\}.$$

Then for $\alpha_1 < \alpha_2$ one has $C_{\alpha_2} \subseteq C_{\alpha_1}$.

Proof. Suppose $X \in C_{\alpha_2}$. By definition there exists $T \in \mathrm{U}_{n\to 1}(\tau(X)) \subseteq [0,1]$ with $T \geq \alpha_2$. Since $\alpha_2 > \alpha_1$, the same T satisfies $T \geq \alpha_1$, hence $X \in C_{\alpha_1}$. Therefore $C_{\alpha_2} \subseteq C_{\alpha_1}$.

Theorem 7 (Functoriality under Surjections). *Let $f : U \to V$ be a surjection and let $\tau : \mathcal{P}^m(U) \to \mathcal{P}^n([0,1])$ be an (m,n)-SuperhyperFuzzy Set. Define the* lifted preimage *(by recursion on m)*

$$f^{-1}_{(1)} :\; \mathcal{P}(V) \to \mathcal{P}(U), \qquad f^{-1}_{(1)}(Y) = \{\, u \in U \mid f(u) \in Y \,\},$$

and for $t \geq 1$,

$$f^{-1}_{(t+1)} :\; \mathcal{P}^{t+1}(V) \to \mathcal{P}^{t+1}(U), \qquad f^{-1}_{(t+1)}(\mathcal{Y}) = \{\, f^{-1}_{(t)}(Y) \mid Y \in \mathcal{Y} \,\}.$$

Then the pushforward

$$f_*(\tau) :\; \mathcal{P}^m(V) \longrightarrow \mathcal{P}^n([0,1]), \qquad f_*(\tau)(\mathcal{Y}) \;=\; \tau\big(f^{-1}_{(m)}(\mathcal{Y})\big)$$

is an (m,n)-SuperhyperFuzzy Set on V.

Proof. Let $\mathcal{Y} \in \mathcal{P}^m(V)$. By the recursive definition of $f^{-1}_{(m)}$, we have $f^{-1}_{(m)}(\mathcal{Y}) \in \mathcal{P}^m(U)$. Therefore $\tau\big(f^{-1}_{(m)}(\mathcal{Y})\big) \in \mathcal{P}^n([0,1])$, showing $f_*(\tau)$ is well-defined with the required codomain. No additional properties are needed to conclude that $f_*(\tau)$ is an (m,n)-SuperhyperFuzzy Set.

3.1.2. (m,n)-SuperhyperNeutrosophic Set

An (m,n)-SuperhyperNeutrosophic Set maps m-level nested subsets to nonempty families of n-level neutrosophic triples $(T, I, F) \in [0,1]^3$, thereby modeling hierarchical uncertainty across levels.

Definition 20 ((m,n)-SuperhyperNeutrosophic Set)**.** *Let U be a universe of discourse and let $A \subseteq U$ be nonempty. Fix $m, n \in \mathbb{N} \cup \{0\}$. Define recursively*

$$\mathcal{P}^0(A) = A, \qquad \mathcal{P}^k(A) = \mathcal{P}\big(\mathcal{P}^{k-1}(A)\big) \quad (k \geq 1),$$

*and analogously for $\mathcal{P}^n([0,1]^3)$. An (m,n)-*SuperhyperNeutrosophic Set *on A is a mapping*

$$\mu : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n\big([0,1]^3\big)$$

such that for each $X \in \mathcal{P}^m(A)$ the value $\mu(X)$ is nonempty and every triple (T, I, F) occurring at level 1 inside $\mu(X)$ satisfies

$$0 \leq T + I + F \leq 3.$$

Equivalently, $\mathrm{U}_{n\to 1}\big(\mu(X)\big) \subseteq [0,1]^3$ is nonempty and consists of triples obeying $T+I+F \leq 3$, where $\mathrm{U}_{n\to 1}$ denotes iterated union (flattening) from level n to level 1.

Notation 2 (Canonical embedding and flattening)**.** *For $r \leq s$ with $r \geq 1$, the* canonical embedding

$$\iota_{r\to s} : \mathcal{P}^r(S) \to \mathcal{P}^s(S)$$

is defined by $\iota_{r\to r} = \mathrm{id}$ and $\iota_{r\to(t+1)}(X) = \{\iota_{r\to t}(X)\}$ for $t \geq r$; hence it nests X by $s-r$ singletons. For $s > r \geq 0$, the level-flattening *map*

$$\mathrm{U}_{s\to r} : \mathcal{P}^s(S) \to \mathcal{P}^r(S)$$

is given by $\mathrm{U}_{s\to(s-1)}(Y) = \bigcup Y$ and $\mathrm{U}_{s\to r} = \mathrm{U}_{(r+1)\to r} \circ \cdots \circ \mathrm{U}_{s\to(s-1)}$.

Example 16 (Multi-Expert Risk Assessment)**.** *Let $A = \{a_1, a_2\}$ and $m = n = 1$, so $\mathcal{P}^1(A) = \mathcal{P}(A)$ and $\mathcal{P}^1([0,1]^3) = \mathcal{P}([0,1]^3)$. Suppose two experts assess $\{a_1, a_2\}$. Define*

$$\mu(\{a_1, a_2\}) = \{(0.80, 0.15, 0.05),\ (0.75, 0.20, 0.05)\}, \quad \mu(\{a_1\}) = \{(0.60, 0.25, 0.15)\},$$

$$\mu(\{a_2\}) = \{(0.50, 0.30, 0.20)\}, \qquad \mu(\emptyset) = \{(0, 0, 0)\}.$$

Each triple satisfies $0 \leq T + I + F \leq 3$, so μ is $(1,1)$-SuperhyperNeutrosophic.

Example 17 (Hierarchical Fault Diagnosis in an Industrial IoT Network)**.** *Let $A = \{\mathrm{TempSensor}, \mathrm{VibSensor}\}$, $m = 1$, $n = 2$. Define*

$$\mu : \mathcal{P}(A) \longrightarrow \mathcal{P}\big(\mathcal{P}([0,1]^3)\big)$$

by, for $X = \{\mathrm{TempSensor}, \mathrm{VibSensor}\}$,

$$\mu(X) = \big\{\{(0.85, 0.10, 0.05), (0.80, 0.15, 0.05)\},\ \{(0.60, 0.25, 0.15)\}\big\}.$$

Flattening gives $\mathrm{U}_{2\to 1}(\mu(X)) = \{(0.85, 0.10, 0.05), (0.80, 0.15, 0.05), (0.60, 0.25, 0.15)\}$, all obeying $T + I + F \leq 3$.

Theorem 8. *Every n-SuperhyperNeutrosophic Set $\tilde{A}_n : \mathcal{P}^n(X) \to \mathcal{P}^n([0,1]^3)$ is obtained as the special case $m = 1$ of an (m,n)-SuperhyperNeutrosophic Set on X.*

Proof. Define $\mu := \tilde{A}_n \circ \iota_{1\to n} : \mathcal{P}^1(X) \to \mathcal{P}^n([0,1]^3)$. This composition is well-defined since $\iota_{1\to n}$ maps $\mathcal{P}(X)$ into $\mathcal{P}^n(X)$. For any $A \in \mathcal{P}(X)$ we have

$$\mu(A) = \tilde{A}_n\big(\iota_{1\to n}(A)\big),$$

hence μ agrees with $\tilde{A}_n$ on the level-n images of level-1 inputs. Because $\tilde{A}_n(A')$ is nonempty for each $A' \in \mathcal{P}^n(X)$ and its level-1 elements satisfy $T + I + F \leq 3$, the same properties hold for $\mu(A)$. Thus μ is $(1,n)$-SuperhyperNeutrosophic, realizing $\tilde{A}_n$ as the $m = 1$ case.

Theorem 9 (Restriction to Lower m)**.** *Let*

$$\mu : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^3)$$

be (m,n)-SuperhyperNeutrosophic and let $0 \leq m' < m$. Let $\iota_{m'\to m}$ be the canonical embedding. Then

$$\mu' = \mu \circ \iota_{m'\to m} : \ \mathcal{P}^{m'}(A) \longrightarrow \mathcal{P}^n([0,1]^3)$$

is (m',n)-SuperhyperNeutrosophic.

Proof. Fix $X \in \mathcal{P}^{m'}(A)$. Then $\iota_{m'\to m}(X) \in \mathcal{P}^m(A)$ and $\mu\big(\iota_{m'\to m}(X)\big) \in \mathcal{P}^n([0,1]^3)$ is nonempty with all (T,I,F) at level 1 obeying $T + I + F \leq 3$. Hence $\mu'(X)$ has the required codomain and constraints.

Theorem 10 (Projection onto Truth Component)**.** *Let $\mu : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^3)$ be (m,n)-SuperhyperNeutrosophic. Define $\pi_T : [0,1]^3 \to [0,1]$, $\pi_T(T,I,F) = T$, and lift it levelwise to*

$$\Pi_T : \ \mathcal{P}^n([0,1]^3) \longrightarrow \mathcal{P}^n([0,1]), \qquad \Pi_T = \mathcal{P}^n(\pi_T).$$

Then $\tau := \Pi_T \circ \mu : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1])$ is an (m,n)-SuperhyperFuzzy Set.

Proof. For any $X \in \mathcal{P}^m(A)$, the inner-most (level 1) set $\mathrm{U}_{n\to 1}(\mu(X)) \subseteq [0,1]^3$ is nonempty. Applying π_T pointwise yields the nonempty set $\pi_T\big(\mathrm{U}_{n\to 1}(\mu(X))\big) \subseteq [0,1]$. Lifting this operation consistently to higher levels defines Π_T, so $\tau(X) = \Pi_T(\mu(X)) \in \mathcal{P}^n([0,1])$. Thus τ is (m,n)-SuperhyperFuzzy.

Theorem 11 (Pointwise Union)**.** *If*

$$\mu_1, \mu_2 : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^3)$$

are (m,n)-SuperhyperNeutrosophic, then

$$(\mu_1 \cup \mu_2)(X) := \mu_1(X) \cup \mu_2(X)$$

defines another (m,n)-SuperhyperNeutrosophic Set.

Proof. For each X, both $\mu_1(X)$ and $\mu_2(X)$ lie in $\mathcal{P}^n([0,1]^3)$ and are nonempty. Their union is again a nonempty element of $\mathcal{P}^n([0,1]^3)$. Every triple in the union comes from either $\mu_1(X)$ or $\mu_2(X)$ and thus satisfies $T+I+F \leq 3$.

Theorem 12 (Pointwise Intersection)**.** *Under the same hypotheses as Theorem 11, let*

$$(\mu_1 \cap \mu_2)(X) := \mu_1(X) \cap \mu_2(X).$$

If $\mu_1(X) \cap \mu_2(X) \neq \varnothing$ *for all* X*, then* $(\mu_1 \cap \mu_2)$ *is* (m,n)*-SuperhyperNeutrosophic.*

Proof. For each X, the intersection is a (by assumption nonempty) subset of $\mathcal{P}^{n-1}([0,1]^3)$, hence an element of $\mathcal{P}^n([0,1]^3)$. Any triple in the intersection already satisfies $T+I+F \leq 3$ because it belongs to both $\mu_1(X)$ and $\mu_2(X)$.

Theorem 13 (Nested $\boldsymbol{\lambda}$-Cuts)**.** *Let* $\mu : \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^3)$ *be* (m,n)*-SuperhyperNeutrosophic. For* $\lambda = (\alpha, \beta, \gamma) \in [0,1]^3$ *define*

$$C_\lambda = \Big\{ X \in \mathcal{P}^m(A) \;\Big|\; \exists (T,I,F) \in \mathrm{U}_{n\to 1}(\mu(X)) \text{ with } T \geq \alpha,\ I \leq \beta,\ F \leq \gamma \Big\}.$$

If $\lambda' = (\alpha', \beta', \gamma')$ *satisfies* $\alpha' \geq \alpha$*,* $\beta' \leq \beta$*,* $\gamma' \leq \gamma$*, then* $C_{\lambda'} \subseteq C_\lambda$*.*

Proof. Take $X \in C_{\lambda'}$. Then some $(T,I,F) \in \mathrm{U}_{n\to 1}(\mu(X))$ obeys $T \geq \alpha'$, $I \leq \beta'$, $F \leq \gamma'$. Since $\alpha' \geq \alpha$, $\beta' \leq \beta$, and $\gamma' \leq \gamma$, the same triple witnesses $X \in C_\lambda$. Hence $C_{\lambda'} \subseteq C_\lambda$.

Theorem 14 (Functoriality under Surjections)**.** *Let* $f : A \to B$ *be surjective and let*

$$\mu : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n\big([0,1]^3\big)$$

be an (m,n)*-SuperhyperNeutrosophic map. Define the lifted preimage at level* 1 *by*

$$f^{-1}_{(1)} : \mathcal{P}(B) \longrightarrow \mathcal{P}(A), \qquad f^{-1}_{(1)}(Y) := \{\, a \in A : f(a) \in Y \,\}.$$

Inductively

$$f^{-1}_{(t+1)}(\mathcal{Y}) = \big\{\, f^{-1}_{(t)}(Y) \;:\; Y \in \mathcal{Y} \,\big\} \quad (t \geq 1).$$

Then

$$f_*\mu : \mathcal{P}^m(B) \longrightarrow \mathcal{P}^n([0,1]^3), \qquad f_*\mu(\mathcal{Y}) = \mu\big(f^{-1}_{(m)}(\mathcal{Y})\big)$$

is (m,n)*-SuperhyperNeutrosophic on* B*.*

Proof. Let $\mathcal{Y} \in \mathcal{P}^m(B)$. By construction $f^{-1}_{(m)}(\mathcal{Y}) \in \mathcal{P}^m(A)$; surjectivity of f ensures this recursion does not collapse to the empty family when $\mathcal{Y} \neq \varnothing$. Hence $\mu\big(f^{-1}_{(m)}(\mathcal{Y})\big) \in \mathcal{P}^n([0,1]^3)$ is nonempty, and all its level-1 elements satisfy $T+I+F \leq 3$. Therefore $f_*\mu$ has the required codomain and constraints, completing the proof.

3.1.3. (m, n)-SuperhyperPlithogenic Set

A (m, n)-SuperhyperPlithogenic Set assigns m-level parameter-subsets and their attribute values to n-level fuzzy-contradiction degree-sets, capturing multi-faceted membership and inter-attribute conflicts and uncertainty patterns. The definition of the (m, n)-SuperhyperPlithogenic Set is presented below.

Definition 21 ((m, n)-SuperhyperPlithogenic Set)**.** *Let X be a nonempty set and let $V = \{v_1, \ldots, v_k\}$ be a finite set of attributes. For each $v \in V$, let P_v be the set of its possible values. Fix positive integers m, n and positive dimensions s, t. Define the m-th nested powerset of X by*

$$\mathcal{P}^0(X) = X, \quad \mathcal{P}^r(X) = \mathcal{P}\big(\mathcal{P}^{r-1}(X)\big) \quad (r \geq 1),$$

*and similarly $\mathcal{P}^n\big([0,1]^s\big)$ for the s-dimensional unit cube. An (m, n)-*SuperhyperPlithogenic Set *over X is the quintuple*

$$SHP^{(m,n)} = \Big(\mathcal{P}^m(X),\ V,\ \{P_v\}_{v\in V},\ \{\tilde{pdf}_v^{(m,n)}\}_{v\in V},\ pCF^{(m,n)}\Big),$$

where

- *(i)* *$\mathcal{P}^m(X)$ is the domain of "super-elements" of level m.*
- *(ii)* *For each $v \in V$, P_v is the finite set of its values.*
- *(iii)* *The* Hyper Degree of Appurtenance Function

$$\tilde{pdf}_v^{(m,n)} : \mathcal{P}^m(X) \times P_v \longrightarrow \mathcal{P}^n\big([0,1]^s\big)$$

assigns to each (A, a) with $A \in \mathcal{P}^m(X)$ and $a \in P_v$ a nonempty subset $\tilde{pdf}_v^{(m,n)}(A, a) \subseteq [0,1]^s$ representing all possible membership-degree vectors of dimension s.

- *(iv)* *The* Degree of Contradiction Function

$$pCF^{(m,n)} : \Big(\bigcup_{v\in V} P_v\Big) \times \Big(\bigcup_{v\in V} P_v\Big) \longrightarrow [0,1]^t$$

satisfies for all a, b:

  - *(a)* *$pCF^{(m,n)}(a, a) = 0$ (reflexivity),*
  - *(b)* *$pCF^{(m,n)}(a, b) = pCF^{(m,n)}(b, a)$ (symmetry).*

Example 18 (Simple $(1, 1)$-SuperhyperPlithogenic Set)**.** *Let $X = \{a_1, a_2\}$ and $V = \{v\}$ with $P_v = \{p_1, p_2\}$. Take $m = n = 1$ and $s = t = 1$. Then*

$$\mathcal{P}^1(X) = \mathcal{P}(X), \quad \mathcal{P}^1([0,1]) = \mathcal{P}([0,1]).$$

Define

$$\tilde{pdf}_v^{(1,1)} : \mathcal{P}(X) \times P_v \to \mathcal{P}([0,1])$$

by

$$\tilde{pdf}_v^{(1,1)}(\{a_1, a_2\}, p_1) = [0.7, 0.8], \quad \tilde{pdf}_v^{(1,1)}(\{a_1\}, p_1) = \{0.5, 0.6\},$$
$$\tilde{pdf}_v^{(1,1)}(\{a_2\}, p_1) = [0.3, 0.4], \qquad \tilde{pdf}_v^{(1,1)}(\emptyset, p_1) = \{0\},$$

and similarly for p_2*. For the contradiction,*

$$pCF^{(1,1)} : P_v \times P_v \to [0,1],$$

set

$$pCF^{(1,1)}(p_1, p_1) = 0, \quad pCF^{(1,1)}(p_1, p_2) = 0.2, \quad pCF^{(1,1)}(p_2, p_2) = 0,$$

with symmetry $pCF(p_2, p_1) = 0.2$*. Then*

$$SHP^{(1,1)} = \Big(\mathcal{P}(X),\, \{v\},\, \{P_v\},\, \{\tilde{pdf}_v^{(1,1)}\},\, pCF^{(1,1)}\Big)$$

is a valid $(1,1)$*-SuperhyperPlithogenic Set.*

Example 19 (Hackathon Team Evaluation)**.** *Let*

$$X = \{\text{Hiroko, Masahiro, Shinya, Dave}\},$$

and take $m = n = 1$*, so that* $\mathcal{P}^1(X) = \mathcal{P}(X)$*. We consider two evaluation attributes:*

$$v_1 = \textit{Innovation}, \quad P_{v_1} = \{\textit{Low, Medium, High}\},$$

$$v_2 = \textit{Feasibility}, \quad P_{v_2} = \{\textit{Low, Medium, High}\}.$$

For richness, let the membership–degree vectors be two–dimensional (peer score, judge score), so $s = 2$*, and let the contradiction measure be scalar, so* $t = 1$*. Then*

$$\tilde{pdf}_{v_i}^{(1,1)} : \mathcal{P}(X) \times P_{v_i} \longrightarrow \mathcal{P}\big([0,1]^2\big)$$

assigns to each team $A \subseteq X$ *and level* $a \in P_{v_i}$ *a nonempty set of vectors in* $[0,1]^2$*.*

For example, for the team $A = \{\text{Hiroko}, \text{Masahiro}\}$*:*

$$\tilde{pdf}_{\textit{Innovation}}^{(1,1)}(A, \textit{High}) = \big\{\, (0.90, 0.85),\ (0.88, 0.82)\big\},$$

$$\tilde{pdf}_{\textit{Innovation}}^{(1,1)}(A, \textit{Medium}) = \big\{\, (0.70, 0.75)\big\},$$

$$\tilde{pdf}_{\textit{Feasibility}}^{(1,1)}(A, \textit{High}) = \big\{\, (0.80, 0.78),\ (0.82, 0.80)\big\},$$

$$\tilde{pdf}_{\textit{Feasibility}}^{(1,1)}(A, \textit{Low}) = \big\{\, (0.50, 0.55)\big\}.$$

The degree of contradiction function

$$pCF^{(1,1)} : \big(P_{v_1} \cup P_{v_2}\big) \times \big(P_{v_1} \cup P_{v_2}\big) \;\longrightarrow\; [0,1]$$

is defined by

$$pCF^{(1,1)}(High, Low) = 1.0, \quad pCF^{(1,1)}(High, Medium) = 0.5, \quad pCF^{(1,1)}(a,a) = 0,$$

with symmetry $pCF(b,a) = pCF(a,b)$.

Hence

$$SHP^{(1,1)} = \Big(\mathcal{P}(X),\ \{v_1, v_2\},\ \{P_{v_1}, P_{v_2}\},\ \{\tilde{pdf}^{(1,1)}_{v_1}, \tilde{pdf}^{(1,1)}_{v_2}\},\ pCF^{(1,1)}\Big)$$

is a concrete $(1,1)$*-SuperhyperPlithogenic Set modeling multi-dimensional team evaluations in a hackathon.*

Example 20 (Supply Chain Resilience Evaluation with $(2,2)$-SuperhyperPlithogenic Set)**.** *Let*

$$X = \{\text{FactoryA, FactoryB}\},$$

and take $m = n = 2$. *Then*

$$\mathcal{P}^1(X) = \mathcal{P}(X) = \{\emptyset, \{\text{FactoryA}\}, \{\text{FactoryB}\}, \{\text{FactoryA}, \text{FactoryB}\}\},$$

$$\mathcal{P}^2(X) = \mathcal{P}\big(\mathcal{P}^1(X)\big),$$

the set of all subsets of $\mathcal{P}^1(X)$. *For concreteness, choose the level-2 element*

$$A \;=\; \big\{\{\text{FactoryA}\},\, \{\text{FactoryA}, \text{FactoryB}\}\big\} \;\in\; \mathcal{P}^2(X).$$

We evaluate two attributes:

$$v_1 = \textit{Cost Efficiency}, \quad P_{v_1} = \{Low, Medium, High\},$$

$$v_2 = \textit{Delivery Reliability}, \quad P_{v_2} = \{Slow, Moderate, Fast\}.$$

Let the membership–degree vectors be one-dimensional ($s = 1$) *and the contradiction measure scalar* ($t = 1$).

Define the hyper–degree functions

$$\tilde{pdf}^{(2,2)}_{v_i} : \mathcal{P}^2(X) \times P_{v_i} \longrightarrow \mathcal{P}^2([0,1])$$

by giving their values on A*:*

$$\tilde{pdf}^{(2,2)}_{v_1}(A, Low) = \big\{\{0.40, 0.50\},\, \{0.45, 0.55\}\big\},$$

$$\tilde{pdf}^{(2,2)}_{v_1}(A, Medium) = \big\{\, [0.55, 0.65]\, \big\},$$

$$\tilde{pdf}^{(2,2)}_{v_1}(A, High) = \big\{\, [0.70, 0.80],\, \{0.75\}\big\},$$

$$\tilde{pdf}_{v_2}^{(2,2)}(A, Slow) = \{\{0.30, 0.40\}\},$$

$$\tilde{pdf}_{v_2}^{(2,2)}(A, Moderate) = \{\{0.60\},\ [0.60, 0.70]\},$$

$$\tilde{pdf}_{v_2}^{(2,2)}(A, Fast) = \{\ [0.80, 0.90],\ \{0.85\}\}.$$

The degree of contradiction function

$$pCF^{(2,2)} : (P_{v_1} \cup P_{v_2}) \times (P_{v_1} \cup P_{v_2}) \longrightarrow [0, 1]$$

is specified by:

$$pCF^{(2,2)}(Low, High) = 1.0, \quad pCF^{(2,2)}(Low, Medium) = 0.3, \quad pCF^{(2,2)}(Medium, High) = 0.5,$$

$$pCF^{(2,2)}(Slow, Fast) = 0.8, \quad pCF^{(2,2)}(a, a) = 0, \quad pCF^{(2,2)}(a, b) = pCF^{(2,2)}(b, a).$$

Thus the structure

$$SHP^{(2,2)} = \left(\mathcal{P}^2(X),\ \{v_1, v_2\},\ \{P_{v_1}, P_{v_2}\},\ \{\tilde{pdf}_{v_1}^{(2,2)}, \tilde{pdf}_{v_2}^{(2,2)}\},\ pCF^{(2,2)}\right)$$

models, at two hierarchical levels, the cost-efficiency and delivery-reliability uncertainties of the supply chain with hyper-valued degrees and quantified contradictions.

Theorem 15. *Every n-SuperhyperPlithogenic Set* $(\mathcal{P}^1(X),\ V, \{P_v\}, \{\tilde{pdf}_v^{(1,n)}\}, pCF^{(1,n)})$ *arises as the special case* $m = 1$ *of an* (m, n)*-SuperhyperPlithogenic Set.*

Proof. By definition, an n-SuperhyperPlithogenic Set has domain $\mathcal{P}^1(X) = \mathcal{P}(X)$. In the general (m, n) framework, setting $m = 1$ replaces $\mathcal{P}^m(X)$ by $\mathcal{P}^1(X)$ and leaves all other components identical. Hence the two structures coincide exactly when $m = 1$.

Theorem 16 (Restriction to Lower m)**.** *Let*

$$SHP^{(m,n)} = (\mathcal{P}^m(X), V, \{P_v\}, \{\tilde{pdf}_v^{(m,n)}\}, pCF^{(m,n)})$$

be an (m, n)*-SuperhyperPlithogenic Set and let* m' *satisfy* $0 \le m' < m$*. Define, for each* $v \in V$ *and* $a \in P_v$*,*

$$\tilde{pdf}_{v,\downarrow m}^{(m',n)}(A, a) \ := \ \tilde{pdf}_v^{(m,n)}(\iota_{m' \to m}(A),\ a) \quad \text{for } A \in \mathcal{P}^{m'}(X),$$

and keep $pCF^{(m,n)}$ *unchanged. Then*

$$SHP_{\downarrow}^{(m',n)} = (\mathcal{P}^{m'}(X), V, \{P_v\}, \{\tilde{pdf}_{v,\downarrow m}^{(m',n)}\}, pCF^{(m,n)})$$

is an (m', n)*-SuperhyperPlithogenic Set.*

Proof. Fix $A \in \mathcal{P}^{m'}(X)$, $v \in V$, $a \in P_v$. Since $\iota_{m'\to m}(A) \in \mathcal{P}^m(X)$, the value $\tilde{pdf}_v^{(m,n)}\big(\iota_{m'\to m}(A), a\big) \in \mathcal{P}^n([0,1]^s)$ is well-defined and nonempty by the (m,n)-definition. Hence $\tilde{pdf}_{v,\downarrow m}^{(m',n)}(A,a)$ has the required codomain. The contradiction map $pCF^{(m,n)}$ is unchanged, so all axioms are preserved.

Theorem 17 (Projection to Lower n)**.** *Let $SHP^{(m,n)}$ be as above and let n' satisfy $0 \leq n' < n$. For each $v \in V$, $a \in P_v$, define*

$$\tilde{pdf}_{v,\downarrow n}^{(m,n')}(A,a) \ := \ \mathrm{U}_{n\to n'}\Big(\tilde{pdf}_v^{(m,n)}(A,a)\Big) \in \mathcal{P}^{n'}([0,1]^s),$$

and keep $pCF^{(m,n)}$ unchanged. Then

$$SHP_{\downarrow}^{(m,n')} = \big(\mathcal{P}^m(X), V, \{P_v\}, \{\tilde{pdf}_{v,\downarrow n}^{(m,n')}\}, pCF^{(m,n)}\big)$$

is an (m,n')-SuperhyperPlithogenic Set.

Proof. Fix $A \in \mathcal{P}^m(X)$, v, a. Since $\tilde{pdf}_v^{(m,n)}(A,a) \in \mathcal{P}^n([0,1]^s)$ is nonempty and every level is nonempty by definition, applying $\mathrm{U}_{n\to(n-1)}$ gives a nonempty element of $\mathcal{P}^{n-1}([0,1]^s)$. Iterating $(n-n')$ times yields a nonempty element of $\mathcal{P}^{n'}([0,1]^s)$. The contradiction map is unaffected, so the structure satisfies the definition at level n'.

Theorem 18 (Closure under Pointwise Union)**.** *Let $SHP_1^{(m,n)}$ and $SHP_2^{(m,n)}$ be (m,n)-SuperhyperPlithogenic Sets on the* same $(X, V, \{P_v\})$ *and with the* same $pCF^{(m,n)}$. *Define, for each $v \in V$, $a \in P_v$,*

$$\big(\tilde{pdf}_v^{(m,n)}\big)^{\cup}(A,a) \ := \ \tilde{pdf}_{v,1}^{(m,n)}(A,a) \ \cup \ \tilde{pdf}_{v,2}^{(m,n)}(A,a) \quad \text{for } A \in \mathcal{P}^m(X).$$

Then

$$SHP_{\cup}^{(m,n)} = \big(\mathcal{P}^m(X), V, \{P_v\}, \{\big(\tilde{pdf}_v^{(m,n)}\big)^{\cup}\}, pCF^{(m,n)}\big)$$

is an (m,n)-SuperhyperPlithogenic Set.

Proof. For each (A,a), both $\tilde{pdf}_{v,1}^{(m,n)}(A,a)$ and $\tilde{pdf}_{v,2}^{(m,n)}(A,a)$ lie in $\mathcal{P}^n([0,1]^s)$ and are nonempty. Their union is a nonempty subset of $\mathcal{P}^{n-1}([0,1]^s)$, hence an element of $\mathcal{P}^n([0,1]^s)$. The shared $pCF^{(m,n)}$ remains valid.

Theorem 19 (Attribute–Value $\boldsymbol{\alpha}$-Cuts are Nested)**.** *Fix $v \in V$ and $a \in P_v$. For a threshold vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s) \in [0,1]^s$, define*

$$C_{v,a}(\boldsymbol{\alpha}) = \Big\{ \ A \in \mathcal{P}^m(X) \ \Big| \ \exists\, \mathbf{d} \in \mathrm{U}_{n\to 1}\big(\tilde{pdf}_v^{(m,n)}(A,a)\big) \text{ with } \mathbf{d} \geq \boldsymbol{\alpha} \text{ (componentwise) } \Big\}.$$

If $\boldsymbol{\alpha}', \boldsymbol{\alpha} \in [0,1]^s$ satisfy $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$ componentwise, then

$$C_{v,a}(\boldsymbol{\alpha}') \ \subseteq \ C_{v,a}(\boldsymbol{\alpha}).$$

Proof. Let $A \in C_{v,a}(\boldsymbol{\alpha}')$. Then there exists $\mathbf{d} = (d_1, \ldots, d_s) \in \mathrm{U}_{n\to 1}(\tilde{pdf}_v^{(m,n)}(A,a))$ with $d_i \geq \alpha'_i$ for all i. Since $\alpha'_i \geq \alpha_i$ for each i, we also have $d_i \geq \alpha_i$; hence the same $\mathbf{d}$ witnesses $A \in C_{v,a}(\boldsymbol{\alpha})$.

Theorem 20 (Functoriality under Surjections on the Base). *Let $f : X \to Y$ be surjective and let*

$$SHP_X^{(m,n)} = \big(\mathcal{P}^m(X), V, \{P_v\}, \{\tilde{pdf}_v^{(m,n)}\}, pCF^{(m,n)}\big)$$

be an (m,n)-SuperhyperPlithogenic Set on X. Define lifted preimages recursively:

$$f_{(1)}^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \quad f_{(1)}^{-1}(B) = \{x \in X : \ f(x) \in B\},$$

$$f_{(t+1)}^{-1}(\mathcal{B}) = \{\, f_{(t)}^{-1}(B) \mid B \in \mathcal{B}\} \quad (t \geq 1).$$

Define, for $A' \in \mathcal{P}^m(Y)$, $v \in V$, $a \in P_v$,

$$\tilde{pdf}_{v,f}^{(m,n)}(A', a) := \tilde{pdf}_v^{(m,n)}\big(f_{(m)}^{-1}(A'), a\big),$$

and keep $pCF^{(m,n)}$ unchanged. Then

$$SHP_Y^{(m,n)} = \big(\mathcal{P}^m(Y), V, \{P_v\}, \{\tilde{pdf}_{v,f}^{(m,n)}\}, pCF^{(m,n)}\big)$$

is an (m,n)-SuperhyperPlithogenic Set on Y.

Proof. For each $A' \in \mathcal{P}^m(Y)$, the recursive construction yields $f_{(m)}^{-1}(A') \in \mathcal{P}^m(X)$. Hence $\tilde{pdf}_v^{(m,n)}\big(f_{(m)}^{-1}(A'), a\big) \in \mathcal{P}^n([0,1]^s)$ is nonempty, so $\tilde{pdf}_{v,f}^{(m,n)}(A', a)$ is well-defined with the required codomain. The contradiction map is copied verbatim, preserving its axioms.

Theorem 21 (Monotone Scalarization to SuperhyperFuzzy). *Let $\phi : [0,1]^s \to [0,1]$ be componentwise nondecreasing. Lift ϕ levelwise to*

$$\Phi : \ \mathcal{P}^n([0,1]^s) \longrightarrow \mathcal{P}^n([0,1]), \qquad \Phi = \mathcal{P}^n(\phi).$$

For fixed $v \in V$, $a \in P_v$, define

$$\tau_{v,a}(A) \ := \ \Phi\big(\tilde{pdf}_v^{(m,n)}(A,a)\big) \quad \text{for } A \in \mathcal{P}^m(X).$$

Then $\tau_{v,a} : \mathcal{P}^m(X) \to \mathcal{P}^n([0,1])$ is an (m,n)-SuperhyperFuzzy Set (parametrized by (v,a)).

Proof. Fix A. Since $\tilde{pdf}_v^{(m,n)}(A,a) \in \mathcal{P}^n([0,1]^s)$ is nonempty, applying ϕ pointwise at level 1 yields a nonempty subset of $[0,1]$. Lifting to higher levels preserves nonemptiness and nesting, so $\tau_{v,a}(A) \in \mathcal{P}^n([0,1])$. No further axioms are needed for the fuzzy case; hence $\tau_{v,a}$ is (m,n)-SuperhyperFuzzy.

Theorem 22 (Contradiction-Bounded Selections are Stable)**.** *Fix $v \in V$ and a bound $\boldsymbol{\delta} \in [0,1]^t$. For $A \in \mathcal{P}^m(X)$ define the $\boldsymbol{\delta}$*-admissible value set

$$\mathcal{A}_{v,\boldsymbol{\delta}}(A) = \Big\{ a \in P_v \;\Big|\; \forall b \in P_v, \quad \|pCF^{(m,n)}(a,b)\|_\infty \le \|\boldsymbol{\delta}\|_\infty \Big\}.$$

Then for any $\boldsymbol{\delta}' \le \boldsymbol{\delta}$ (componentwise) one has

$$\mathcal{A}_{v,\boldsymbol{\delta}'}(A) \;\subseteq\; \mathcal{A}_{v,\boldsymbol{\delta}}(A).$$

Proof. If $a \in \mathcal{A}_{v,\boldsymbol{\delta}'}(A)$, then $\|pCF^{(m,n)}(a,b)\|_\infty \le \|\boldsymbol{\delta}'\|_\infty$ for all $b \in P_v$. Since $\|\boldsymbol{\delta}'\|_\infty \le \|\boldsymbol{\delta}\|_\infty$, the same a satisfies the bound for $\boldsymbol{\delta}$, hence $a \in \mathcal{A}_{v,\boldsymbol{\delta}}(A)$.

3.2. (h,k)-ary (m,n)-Superhyperuncertain Set

We provide an explanation of the (h,k)-ary (m,n)-Superhyperuncertain Set below.

3.2.1. (h,k)-ary (m,n)-SuperhyperFuzzy Set

A (h,k)-ary (m,n)-SuperhyperFuzzy Set generalizes the (m,n)-SuperhyperFuzzy notion by accepting h inputs and producing k outputs, where the inputs are m-level nested subsets of a base set and the outputs are n-level nested fuzzy–degree sets.

Definition 22 ((h,k)-ary (m,n)-SuperhyperFuzzy Set)**.** *Let X be a nonempty set and let $m,n,h,k \in \mathbb{N}$ with $m,n,h,k \ge 1$. Define*

$$\mathcal{P}^0(X) = X, \qquad \mathcal{P}^r(X) = \mathcal{P}\big(\mathcal{P}^{r-1}(X)\big) \quad (r \ge 1),$$

and analogously $\mathcal{P}^r([0,1])$ for the unit interval. An (h,k)-ary (m,n)-SuperhyperFuzzy Set *on X is a mapping*

$$\mu^{(m,n)}_{(h,k)} :\; \big(\mathcal{P}^m(X)\big)^h \longrightarrow \big(\mathcal{P}^n([0,1])\big)^k,$$

which to each input tuple $\mathbf{A} = (A_1,\ldots,A_h) \in (\mathcal{P}^m(X))^h$ assigns an output tuple $\mu^{(m,n)}_{(h,k)}(\mathbf{A}) = (B_1,\ldots,B_k)$ with $B_j \in \mathcal{P}^n([0,1])$ for every $1 \le j \le k$. We additionally require nonemptiness at the outer level*: for each $\mathbf{A}$ and each j, $B_j \neq \emptyset$. Equivalently, writing the coordinate maps as $\mu^{(m,n)}_j : (\mathcal{P}^m(X))^h \to \mathcal{P}^n([0,1])$, we demand $\mu^{(m,n)}_j(\mathbf{A}) \in \mathcal{P}^n([0,1]) \setminus \{\emptyset\}$ for all $\mathbf{A}$ and all j.*

Example 21 (Smart building comfort control as a $(2,2)$-ary $(1,1)$-SuperhyperFuzzy Set)**.** *Let $X = \{\mathrm{Room1}, \mathrm{Room2}, \mathrm{Room3}\}$ and fix $m = n = 1$, $h = k = 2$. Thus $\mathcal{P}^1(X) = \mathcal{P}(X)$ and $\mathcal{P}^1([0,1]) = \mathcal{P}([0,1])$, so the domain is $D = \mathcal{P}(X) \times \mathcal{P}(X)$ and the codomain is $C = \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$. Interpret an input $(A_1,A_2) \in D$ as: A_1 is the set of occupied rooms, and A_2 the set of rooms with windows open. Define the following concrete mapping:*

$$\mu^{(1,1)}_{(2,2)}(A_1,A_2) = \big(B_{\mathrm{comfort}}(A_1,A_2),\; B_{\mathrm{energy}}(A_1,A_2)\big),$$

where, with the shorthand

$$\theta(A_1, A_2) := \frac{|A_1 \cap A_2|}{\max\{1, |A_1|\}}, \qquad \omega(A_2) := \frac{|A_2|}{|X|},$$

we set

$$B_{\text{comfort}}(A_1, A_2) = \big[\, 0.60 + 0.30\,\theta(A_1, A_2), \quad 0.70 + 0.25\,\theta(A_1, A_2) \,\big] \subset [0, 1],$$

$$B_{\text{energy}}(A_1, A_2) = \big[\, 0.20 + 0.40\,\omega(A_2), \quad 0.30 + 0.45\,\omega(A_2) \,\big] \subset [0, 1].$$

Thus, if $A_1 = \{\text{Room1}, \text{Room2}\}$ *and* $A_2 = \{\text{Room2}\}$*, then* $\theta = 1/2$ *and* $\omega = 1/3$*, hence*

$$B_{\text{comfort}} = [0.60 + 0.15,\ 0.70 + 0.125] = [0.75, 0.825],$$

$$B_{\text{energy}} = [0.20 + 0.133\overline{3},\ 0.30 + 0.15] = [0.333\overline{3}, 0.45],$$

so one may choose, for instance, $\mu^{(1,1)}_{(2,2)}(A_1, A_2) = \big([0.75, 0.825],\ [0.333\overline{3}, 0.45]\big) \in C$*. This implements a concrete* $(2,2)$*-ary* $(1,1)$*-SuperhyperFuzzy Set with explicitly computed fuzzy outputs.*

Example 22 (Two–symptom, two–disease risk assessment)**.** *Let* $X = \{\text{Fever}, \text{Cough}, \text{Fatigue}\}$*, again with* $m = n = 1$ *and* $h = k = 2$*. Given two symptom–subsets* $(A_1, A_2) \in \mathcal{P}(X) \times \mathcal{P}(X)$*, define*

$$\mu^{(1,1)}_{(2,2)}(A_1, A_2) = \big(B_{\text{DisA}}(A_1),\ B_{\text{DisB}}(A_2)\big),$$

where

$$B_{\text{DisA}}(A_1) = \big[\, 0.30 + 0.20\,\tfrac{|A_1|}{3}, \quad 0.50 + 0.30\,\tfrac{|A_1|}{3} \,\big],$$

$$B_{\text{DisB}}(A_2) = \big\{\, 0.25 + 0.15\,\tfrac{|A_2|}{3}, \quad 0.35 + 0.20\,\tfrac{|A_2|}{3} \,\big\}.$$

For instance, $\mu^{(1,1)}_{(2,2)}\big(\{\text{Fever}, \text{Cough}\}, \{\text{Fatigue}\}\big) = \big([0.30 + 0.20 \cdot \tfrac{2}{3},\ 0.50 + 0.30 \cdot \tfrac{2}{3}],\ \{0.25 + 0.15 \cdot \tfrac{1}{3},\ 0.35 + 0.20 \cdot \tfrac{1}{3}\}\big) = \big([0.433\overline{3}, 0.70],\ \{0.30, 0.416\overline{6}\}\big)$.

Theorem 23. *If* $h = k = 1$*, then any* (h, k)*-ary* (m, n)*-SuperhyperFuzzy Set reduces to an ordinary* (m, n)*-SuperhyperFuzzy Set.*

Proof. Let $\mu^{(m,n)}_{(1,1)} : (\mathcal{P}^m(X))^1 \to (\mathcal{P}^n([0,1]))^1$ be given. Define the canonical bijections

$$\phi_h : (\mathcal{P}^m(X))^1 \longrightarrow \mathcal{P}^m(X), \quad \phi_h((A)) = A,$$

$$\phi_k : (\mathcal{P}^n([0,1]))^1 \longrightarrow \mathcal{P}^n([0,1]), \quad \phi_k((B)) = B.$$

Both ϕ_h and ϕ_k are bijective with inverses $A \mapsto (A)$ and $B \mapsto (B)$, respectively. Define

$$\widetilde{\mu} := \phi_k \circ \mu^{(m,n)}_{(1,1)} \circ \phi_h^{-1} :\ \mathcal{P}^m(X) \longrightarrow \mathcal{P}^n([0,1]).$$

For any $A \in \mathcal{P}^m(X)$ we have $\phi_h^{-1}(A) = (A)$ and thus

$$\widetilde{\mu}(A) = \phi_k\big(\mu^{(m,n)}_{(1,1)}((A))\big) = B,$$

where $\mu^{(m,n)}_{(1,1)}((A)) = (B)$ with $B \in \mathcal{P}^n([0,1])$ nonempty by definition of an (h,k)-ary (m,n)-SuperhyperFuzzy Set. Hence $\widetilde{\mu}$ is a well-defined map $\mathcal{P}^m(X) \to \mathcal{P}^n([0,1])$ with nonempty outer values; that is, an ordinary (m,n)-SuperhyperFuzzy Set.

Theorem 24 (Fixing inputs)**.** *Fix indices $1 \le i_1 < \cdots < i_r \le h$ and elements $A_{i_j} \in \mathcal{P}^m(X)$. Define $\mu_{\text{fix}} : (\mathcal{P}^m(X))^{h-r} \to (\mathcal{P}^n([0,1]))^k$ by inserting the fixed A_{i_j} in the corresponding coordinates and applying $\mu^{(m,n)}_{(h,k)}$. Then μ_{fix} is an $(h-r,k)$-ary (m,n)-SuperhyperFuzzy Set.*

Proof. Write $\mathbf{A}^{\text{free}} = (A_\ell)_{\ell \in I^{\text{free}}} \in (\mathcal{P}^m(X))^{h-r}$, where $I^{\text{free}} = \{1,\ldots,h\} \setminus \{i_1,\ldots,i_r\}$ (ordered increasingly). Define the *insertion operator*

$$\iota : \ (\mathcal{P}^m(X))^{h-r} \longrightarrow (\mathcal{P}^m(X))^h, \qquad \iota(\mathbf{A}^{\text{free}}) = \mathbf{B} = (B_1,\ldots,B_h),$$

by

$$B_u = \begin{cases} A_{i_j} & \text{if } u = i_j \text{ for some } j, \\ A_\ell & \text{if } u = \ell \in I^{\text{free}}. \end{cases}$$

Then set $\mu_{\text{fix}} := \mu^{(m,n)}_{(h,k)} \circ \iota$. For any $\mathbf{A}^{\text{free}}$, $\iota(\mathbf{A}^{\text{free}}) \in (\mathcal{P}^m(X))^h$; hence $\mu^{(m,n)}_{(h,k)}$ yields a k-tuple $\big(\mu_1^{(m,n)}(\iota(\mathbf{A}^{\text{free}})),\ldots,\mu_k^{(m,n)}(\iota(\mathbf{A}^{\text{free}}))\big)$ with each coordinate in $\mathcal{P}^n([0,1])$ and nonempty. Therefore μ_{fix} is a well-defined $(h-r,k)$-ary (m,n)-SuperhyperFuzzy Set.

Theorem 25 (Projection to a subset of outputs)**.** *Let $1 \le j_1 < \cdots < j_s \le k$. The coordinate projection*

$$\mu_{\text{proj}} : \ (\mathcal{P}^m(X))^h \longrightarrow (\mathcal{P}^n([0,1]))^s, \qquad \mu_{\text{proj}}(\mathbf{A}) = \big(\mu_{j_1}^{(m,n)}(\mathbf{A}),\ldots,\mu_{j_s}^{(m,n)}(\mathbf{A})\big),$$

is an (h,s)-ary (m,n)-SuperhyperFuzzy Set.

Proof. Let $\mathrm{pr}_J : (\mathcal{P}^n([0,1]))^k \to (\mathcal{P}^n([0,1]))^s$ be the coordinate projection to $J = \{j_1,\ldots,j_s\}$. Then $\mu_{\text{proj}} = \mathrm{pr}_J \circ \mu^{(m,n)}_{(h,k)}$. Since each output coordinate $\mu_j^{(m,n)}(\mathbf{A})$ is a nonempty element of $\mathcal{P}^n([0,1])$, their s-tuple lies in $(\mathcal{P}^n([0,1]))^s$ with nonempty coordinates. Hence μ_{proj} has the required type.

Theorem 26 (Pointwise union)**.** *If $\mu,\nu : (\mathcal{P}^m(X))^h \to (\mathcal{P}^n([0,1]))^k$ are (h,k)-ary (m,n)-SuperhyperFuzzy Sets, then the map $(\mu \cup \nu)(\mathbf{A}) = \big(\mu_1(\mathbf{A}) \cup \nu_1(\mathbf{A}),\ldots,\mu_k(\mathbf{A}) \cup \nu_k(\mathbf{A})\big)$ is again an (h,k)-ary (m,n)-SuperhyperFuzzy Set.*

Proof. Fix $\mathbf{A} \in (\mathcal{P}^m(X))^h$ and a coordinate $1 \le j \le k$. By hypothesis, $\mu_j(\mathbf{A}), \nu_j(\mathbf{A}) \in \mathcal{P}^n([0,1])$ and are nonempty. Recall that $\mathcal{P}^n([0,1])$ consists of sets of level $n-1$ objects; the union of two sets at the same level is again a set at that level. Hence $\mu_j(\mathbf{A}) \cup \nu_j(\mathbf{A}) \in \mathcal{P}^n([0,1])$, and it is nonempty because the union of two nonempty sets is nonempty. Doing this for each j produces a k-tuple in $(\mathcal{P}^n([0,1]))^k$ with all coordinates nonempty, as required.

Theorem 27 (Nested α-cuts)**.** *For $\alpha \in [0,1]$, define*

$$C_\alpha = \left\{ \mathbf{A} \in (\mathcal{P}^m(X))^h \;\middle|\; \exists j \; \exists t \in \mu_j^{(m,n)}(\mathbf{A}) \text{ with } t \ge \alpha \right\}.$$

If $0 \le \alpha_1 < \alpha_2 \le 1$, then $C_{\alpha_2} \subseteq C_{\alpha_1}$.

Proof. Let $\mathbf{A} \in C_{\alpha_2}$. By definition, there exists an index j and a value $t \in \mu_j^{(m,n)}(\mathbf{A})$ such that $t \ge \alpha_2$. Since $\alpha_2 > \alpha_1$, we have $t \ge \alpha_2 > \alpha_1$, hence the same j and t witness that $\mathbf{A} \in C_{\alpha_1}$. Therefore $C_{\alpha_2} \subseteq C_{\alpha_1}$.

Theorem 28 (Compatibility with surjections)**.** *Let $f : X \to Y$ be surjective. Define $f^{-1}_{(1)} : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $f^{-1}_{(1)}(B) = \{x \in X : \; f(x) \in B\}$ and recursively*

$$f^{-1}_{(r+1)}(\mathcal{B}) = \left\{ f^{-1}_{(r)}(B) \;\middle|\; B \in \mathcal{B} \right\} \qquad (r \ge 1).$$

Given $\mu^{(m,n)}_{(h,k)}$ on X, the pushforward

$$(f_*\mu)^{(m,n)}_{(h,k)} : \; (\mathcal{P}^m(Y))^h \longrightarrow (\mathcal{P}^n([0,1]))^k,$$

$$(f_*\mu)^{(m,n)}_{(h,k)}(A'_1, \dots, A'_h) = \mu^{(m,n)}_{(h,k)}\big(f^{-1}_{(m)}(A'_1), \dots, f^{-1}_{(m)}(A'_h)\big),$$

is an (h,k)-ary (m,n)-SuperhyperFuzzy Set on Y.

Proof. We first show by induction on $r \ge 1$ that $f^{-1}_{(r)} : \mathcal{P}^r(Y) \to \mathcal{P}^r(X)$ is well defined. For $r = 1$, this is the usual preimage map on subsets: if $B \subseteq Y$, then $f^{-1}_{(1)}(B) \subseteq X$, so $f^{-1}_{(1)}(B) \in \mathcal{P}(X) = \mathcal{P}^1(X)$. Assume $f^{-1}_{(r)}$ is defined on $\mathcal{P}^r(Y)$ with values in $\mathcal{P}^r(X)$. If $\mathcal{B} \in \mathcal{P}^{r+1}(Y)$, then $\mathcal{B}$ is a set of elements $B \in \mathcal{P}^r(Y)$, and by induction $f^{-1}_{(r)}(B) \in \mathcal{P}^r(X)$ for each such B. Therefore the set $\{ f^{-1}_{(r)}(B) \mid B \in \mathcal{B}\}$ belongs to $\mathcal{P}(\mathcal{P}^r(X)) = \mathcal{P}^{r+1}(X)$, which establishes the inductive step.

Now fix $(A'_1, \dots, A'_h) \in (\mathcal{P}^m(Y))^h$. By the previous paragraph, $f^{-1}_{(m)}(A'_i) \in \mathcal{P}^m(X)$ for each i, so the h-tuple $\big(f^{-1}_{(m)}(A'_1), \dots, f^{-1}_{(m)}(A'_h)\big)$ lies in $(\mathcal{P}^m(X))^h$. Applying $\mu^{(m,n)}_{(h,k)}$ yields a k-tuple in $(\mathcal{P}^n([0,1]))^k$ with nonempty coordinates (by the defining nonemptiness property of $\mu^{(m,n)}_{(h,k)}$). Therefore $(f_*\mu)^{(m,n)}_{(h,k)}$ is a well-defined (h,k)-ary (m,n)-SuperhyperFuzzy Set on Y.

3.2.2. (h, k)-ary (m, n)-SuperhyperNeutrosophic Set

An (h, k)-ary (m, n)-SuperhyperNeutrosophic Set accepts h inputs taken from the m-th nonempty iterated powerset of a base set and returns k outputs that are elements of the n-th nonempty iterated powerset of the unit cube $[0, 1]^3$ of neutrosophic triples. In this way, it models multi-ary interactions between hierarchical (nested) collections of objects and produces hierarchical collections of neutrosophic assessments—each assessment being a triple $(T, I, F) \in [0, 1]^3$ of truth, indeterminacy, and falsity degrees satisfying $0 \leq T + I + F \leq 3$.

Definition 23 ((h, k)-ary (m, n)-SuperhyperNeutrosophic Set)**.** *Let X be a nonempty set, and let $m, n, h, k \in \mathbb{N}$ with $m, n, h, k \geq 1$. An (h, k)-ary (m, n)-SuperhyperNeutrosophic Set on X is a mapping*

$$\tilde{A}^{(m,n)}_{(h,k)} : \ \big(\tilde{\mathcal{P}}_m(X)\big)^h \ \longrightarrow \ \big(\tilde{\mathcal{P}}_n([0, 1]^3)\big)^k$$

with the following property: for every input h-tuple $(A_1, \ldots, A_h) \in (\tilde{\mathcal{P}}_m(X))^h$ the image is a k-tuple $(C_1, \ldots, C_k)$ with each $C_j \in \tilde{\mathcal{P}}_n([0, 1]^3)$; in particular, every base-level element of each C_j is a triple $(T, I, F) \in [0, 1]^3$ satisfying

$$0 \ \leq \ T + I + F \ \leq \ 3.$$

Nonemptiness is enforced at every nested level by the use of $\tilde{\mathcal{P}}_\bullet$.

Example 23 (Unary case $(h, k) = (1, 1)$)**.** *Let $X = \{a, b\}$ and take $m = n = 1$. Then $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X) = \big\{\{a\}, \{b\}, \{a, b\}\big\}$ and $\tilde{\mathcal{P}}_1([0, 1]^3) = \tilde{\mathcal{P}}([0, 1]^3)$. A map*

$$\tilde{A}^{(1,1)}_{(1,1)} : \ \tilde{\mathcal{P}}(X) \ \longrightarrow \ \tilde{\mathcal{P}}([0, 1]^3)$$

is given, for instance, by

$$\tilde{A}^{(1,1)}_{(1,1)}(\{a\}) = \{(0.6, 0.2, 0.1)\}, \quad \tilde{A}^{(1,1)}_{(1,1)}(\{b\}) = \{(0.4, 0.3, 0.2)\},$$

$$\tilde{A}^{(1,1)}_{(1,1)}(\{a, b\}) = \{(0.7, 0.1, 0.1), (0.8, 0.0, 0.1)\}.$$

Each triple lies in $[0, 1]^3$ and satisfies $T + I + F \leq 3$, as required.

Example 24 (Credit & Liquidity Risk as a $(2, 2)$-ary $(1, 1)$-system)**.** *Consider banking indicators*

$$X = \{\text{DebtToIncome}, \ \text{CreditScore}, \ \text{MarketVolatility}, \ \text{InterestRate}\},$$

and set $m = n = 1$, $h = k = 2$. The domain and codomain are

$$D = \tilde{\mathcal{P}}(X) \times \tilde{\mathcal{P}}(X),$$

$$C = \tilde{\mathcal{P}}([0, 1]^3) \times \tilde{\mathcal{P}}([0, 1]^3).$$

Given $(A_1, A_2) \in D$*—say*

$$A_1 = \{\text{DebtToIncome}, \text{CreditScore}\}$$

and

$$A_2 = \{\text{MarketVolatility}, \text{InterestRate}\}$$

—define

$$\tilde{A}^{(1,1)}_{(2,2)}(A_1, A_2) = \big(C_{\text{default}}, C_{\text{liquidity}}\big),$$

with

$$C_{\text{default}} = \{(0.70, 0.20, 0.10), (0.65, 0.25, 0.10)\},$$

$$C_{\text{liquidity}} = \{(0.40, 0.40, 0.20)\}.$$

All triples satisfy $0 \leq T + I + F \leq 3$*. Thus* $\tilde{A}^{(1,1)}_{(2,2)}$ *is a concrete* $(2,2)$*-ary* $(1,1)$*-SuperhyperNeutrosophic Set.*

Theorem 29 (Relation to classical and fuzzy cases)**.** *Every* (h, k)*-ary* (m, n)*-SuperhyperNeutrosophic Set generalizes both (i) the classical* (m, n)*-SuperhyperNeutrosophic Set (when* $h = k = 1$*) and (ii) the* (h, k)*-ary* (m, n)*-SuperhyperFuzzy Set (by projecting triples to their truth components).*

Proof. Let $\tilde{A} = \tilde{A}^{(m,n)}_{(h,k)} : (\tilde{\mathcal{P}}_m(X))^h \to (\tilde{\mathcal{P}}_n([0,1]^3))^k$.

(i) Classical case. Assume $h = k = 1$. Then the Cartesian powers collapse via the canonical bijections

$$\iota_h : \ (\tilde{\mathcal{P}}_m(X))^1 \xrightarrow{\cong} \tilde{\mathcal{P}}_m(X),$$

$$\iota_k : \ (\tilde{\mathcal{P}}_n([0,1]^3))^1 \xrightarrow{\cong} \tilde{\mathcal{P}}_n([0,1]^3),$$

given by $\iota_h(A) = A$ and $\iota_k(C) = C$. Under these identifications, $\tilde{A}$ is precisely a map

$$\tilde{A} : \ \tilde{\mathcal{P}}_m(X) \longrightarrow \tilde{\mathcal{P}}_n([0,1]^3),$$

whose values are nonempty n-level neutrosophic sets. This is exactly the classical (m, n)-SuperhyperNeutrosophic definition.

(ii) Fuzzy case via truth–projection. Define $\pi_T : [0,1]^3 \to [0,1]$ by $\pi_T(T, I, F) = T$. We extend π_T to all nesting levels by recursion on $r \geq 1$:

$$\Pi^{(1)}_T : \tilde{\mathcal{P}}_1([0,1]^3) \to \tilde{\mathcal{P}}_1([0,1]), \quad \Pi^{(1)}_T(S) := \{\ \pi_T(t) \mid t \in S\ \},$$

and, for $r \geq 1$,

$$\Pi^{(r+1)}_T : \ \tilde{\mathcal{P}}_{r+1}([0,1]^3) \to \tilde{\mathcal{P}}_{r+1}([0,1]),$$

$$\Pi^{(r+1)}_T(\mathcal{S}) := \{\ \Pi^{(r)}_T(S) \mid S \in \mathcal{S}\ \}.$$

Nonemptiness is preserved at each step: if $S \neq \emptyset$, then $\Pi_T^{(1)}(S) \neq \emptyset$; if $\mathcal{S} \neq \emptyset$, then $\Pi_T^{(r+1)}(\mathcal{S}) \neq \emptyset$ because it collects the nonempty images of members of $\mathcal{S}$. Put $\Pi_T := \Pi_T^{(n)}$ and define

$$\widehat{\Pi}_T : \ (\tilde{\mathcal{P}}_n([0,1]^3))^k \to (\tilde{\mathcal{P}}_n([0,1]))^k, \qquad \widehat{\Pi}_T(C_1, \dots, C_k) := (\Pi_T(C_1), \dots, \Pi_T(C_k)).$$

Then $\widehat{\Pi}_T \circ \tilde{A}$ maps any $\mathbf{A} \in (\tilde{\mathcal{P}}_m(X))^h$ to a k-tuple of nonempty n-level subsets of $[0,1]$, i.e. an (h,k)-ary (m,n)-SuperhyperFuzzy Set.

Theorem 30 (Fixing some inputs)**.** *Fix indices* $1 \leq i_1 < \cdots < i_r \leq h$ *and super-elements* $A_{i_j} \in \tilde{\mathcal{P}}_m(X)$. *The map*

$$\tilde{A}_{\text{fix}} : \ \big(\tilde{\mathcal{P}}_m(X)\big)^{h-r} \longrightarrow \big(\tilde{\mathcal{P}}_n([0,1]^3)\big)^k,$$

defined by inserting the fixed A_{i_j} *into the corresponding coordinates of an* $(h-r)$*-tuple and then applying* $\tilde{A}$*, is an* $(h-r,k)$*-ary* (m,n)*-SuperhyperNeutrosophic Set.*

Proof. Let $I = \{1, \dots, h\}$ and $J = \{i_1, \dots, i_r\}$. For a free-input tuple $\mathbf{B} = (B_\ell)_{\ell \in I \setminus J} \in (\tilde{\mathcal{P}}_m(X))^{h-r}$, define the *insertion* map

$$\iota(\mathbf{B}) = \mathbf{A} = (A_1, \dots, A_h) \in (\tilde{\mathcal{P}}_m(X))^h, \qquad A_u = \begin{cases} A_{i_j}, & u \in J \text{ with } u = i_j, \\ B_u, & u \in I \setminus J. \end{cases}$$

Then ι is well-defined and $\iota(\mathbf{B})$ has every coordinate in $\tilde{\mathcal{P}}_m(X)$. Set $\tilde{A}_{\text{fix}} := \tilde{A} \circ \iota$. For any $\mathbf{B}$, $\tilde{A}_{\text{fix}}(\mathbf{B}) = \tilde{A}(\iota(\mathbf{B}))$ is a k-tuple whose jth coordinate lies in $\tilde{\mathcal{P}}_n([0,1]^3)$ by definition of $\tilde{A}$. Thus $\tilde{A}_{\text{fix}}$ is an $(h-r,k)$-ary (m,n)-SuperhyperNeutrosophic Set.

Theorem 31 (Projection to selected outputs)**.** *Let* $1 \leq j_1 < \cdots < j_s \leq k$. *The coordinate projection*

$$\tilde{A}_{\text{proj}}(\mathbf{A}) = \big(C_{j_1}, \dots, C_{j_s}\big), \qquad \textit{where} \ \ \tilde{A}(\mathbf{A}) = (C_1, \dots, C_k),$$

defines an (h,s)*-ary* (m,n)*-SuperhyperNeutrosophic Set* $\tilde{A}_{\text{proj}} : (\tilde{\mathcal{P}}_m(X))^h \to (\tilde{\mathcal{P}}_n([0,1]^3))^s$.

Proof. Define $P_J : (\tilde{\mathcal{P}}_n([0,1]^3))^k \to (\tilde{\mathcal{P}}_n([0,1]^3))^s$ by $P_J(C_1, \dots, C_k) = (C_{j_1}, \dots, C_{j_s})$. Each $C_{j_r} \in \tilde{\mathcal{P}}_n([0,1]^3)$ and is nonempty, so P_J is well-defined. Put $\tilde{A}_{\text{proj}} := P_J \circ \tilde{A}$ to obtain a map with domain $(\tilde{\mathcal{P}}_m(X))^h$ and codomain $(\tilde{\mathcal{P}}_n([0,1]^3))^s$, as required.

Theorem 32 (Pointwise union)**.** *If* $\tilde{A}_1, \tilde{A}_2 : (\tilde{\mathcal{P}}_m(X))^h \to (\tilde{\mathcal{P}}_n([0,1]^3))^k$ *are* (h,k)*-ary* (m,n)*-SuperhyperNeutrosophic Sets, then*

$$(\tilde{A}_1 \cup \tilde{A}_2)(\mathbf{A}) \ := \ \big(\tilde{A}_1(\mathbf{A})_j \cup \tilde{A}_2(\mathbf{A})_j\big)_{j=1}^k$$

is again an (h,k)*-ary* (m,n)*-SuperhyperNeutrosophic Set.*

Proof. We prove a general lemma by induction on $r \geq 1$:

Lemma. If $U, V \in \tilde{\mathcal{P}}_r(S)$, then $U \cup V \in \tilde{\mathcal{P}}_r(S)$.

$r = 1$: U, V are nonempty subsets of S, hence $U \cup V$ is a nonempty subset of S, i.e. in $\tilde{\mathcal{P}}_1(S)$.

$r \to r+1$: U, V are nonempty families of r-level elements. Thus $U \cup V$ is a nonempty family whose members all lie in $\tilde{\mathcal{P}}_r(S)$; hence $U \cup V \in \tilde{\mathcal{P}}_{r+1}(S)$.

Apply the lemma with $S = [0,1]^3$ and $r = n$. For each input $\mathbf{A}$ and coordinate j, both $\tilde{A}_1(\mathbf{A})_j$ and $\tilde{A}_2(\mathbf{A})_j$ lie in $\tilde{\mathcal{P}}_n([0,1]^3)$ and are nonempty, so their union does as well. Moreover, every base-level element of the union is still a triple (T, I, F) with $0 \leq T+I+F \leq 3$ (the constraint is preserved under set union). Therefore $(\tilde{A}_1 \cup \tilde{A}_2)(\mathbf{A}) \in (\tilde{\mathcal{P}}_n([0,1]^3))^k$ for all $\mathbf{A}$.

Theorem 33 (Pointwise intersection)**.** *With the same hypotheses, define*

$$(\tilde{A}_1 \cap \tilde{A}_2)(\mathbf{A}) \ :=\ \big(\tilde{A}_1(\mathbf{A})_j \cap \tilde{A}_2(\mathbf{A})_j\big)_{j=1}^{k}.$$

If every coordinate-wise intersection is nonempty, then $(\tilde{A}_1 \cap \tilde{A}_2)$ is an (h,k)-ary (m,n)-SuperhyperNeutrosophic Set.

Proof. We use the companion lemma (proved by induction on $r \geq 1$):

Lemma. If $U, V \in \tilde{\mathcal{P}}_r(S)$ and $U \cap V \neq \emptyset$, then $U \cap V \in \tilde{\mathcal{P}}_r(S)$.

$r = 1$: U, V are nonempty subsets of S. If $U \cap V \neq \emptyset$, then $U \cap V$ is a nonempty subset of S, i.e. in $\tilde{\mathcal{P}}_1(S)$.

$r \to r+1$: U, V are nonempty families of r-level elements. If $U \cap V \neq \emptyset$, every element of $U \cap V$ is itself in $\tilde{\mathcal{P}}_r(S)$, and $U \cap V$ is nonempty; hence $U \cap V \in \tilde{\mathcal{P}}_{r+1}(S)$.

Apply the lemma with $S = [0,1]^3$ and $r = n$. The nonemptiness hypothesis ensures each coordinate-wise intersection lies in $\tilde{\mathcal{P}}_n([0,1]^3)$. Neutrosophic constraints are pointwise and are preserved by set-theoretic intersection. Therefore $(\tilde{A}_1 \cap \tilde{A}_2)$ has the desired codomain.

Theorem 34 (Neutrosophic λ-cuts)**.** *For $\lambda = (\alpha, \beta, \gamma) \in [0,1]^3$, define the cut*

$$C_\lambda \ =\ \Big\{ \mathbf{A} \in (\tilde{\mathcal{P}}_m(X))^h \ \Big|\ \exists j,\ \exists (T,I,F) \in \tilde{A}(\mathbf{A})_j \ \textit{with}\ T \geq \alpha,\ I \leq \beta,\ F \leq \gamma \Big\}.$$

If $\lambda' = (\alpha', \beta', \gamma')$ satisfies $\alpha' \geq \alpha$, $\beta' \leq \beta$, and $\gamma' \leq \gamma$, then $C_{\lambda'} \subseteq C_\lambda$.

Proof. Take any $\mathbf{A} \in C_{\lambda'}$. By definition, there exist an index j and a triple $(T, I, F) \in \tilde{A}(\mathbf{A})_j$ with $T \geq \alpha'$, $I \leq \beta'$, $F \leq \gamma'$. Since $\alpha' \geq \alpha$, $\beta' \leq \beta$, and $\gamma' \leq \gamma$, the same triple satisfies $T \geq \alpha$, $I \leq \beta$, $F \leq \gamma$, so $\mathbf{A} \in C_\lambda$. Thus $C_{\lambda'} \subseteq C_\lambda$.

Theorem 35 (Truth-projection yields fuzzy structure)**.** *Let $\pi_T : [0,1]^3 \to [0,1]$ be $\pi_T(T,I,F) = T$ and extend it levelwise to $\Pi_T : (\tilde{\mathcal{P}}_n([0,1]^3))^k \to (\tilde{\mathcal{P}}_n([0,1]))^k$. Then $\Pi_T \circ \tilde{A}$ is an (h,k)-ary (m,n)-SuperhyperFuzzy Set.*

Proof. Define Π_T recursively as in the proof of the Relation Theorem: base level $n = 1$ by taking elementwise truth components, and for $n > 1$ by applying the previous level to each member of a nonempty family. Nonemptiness and the nesting level are preserved at each step. Hence for every input $\mathbf{A}$, $(\Pi_T \circ \tilde{A})(\mathbf{A})$ is a k-tuple of nonempty n-level subsets of $[0, 1]$, i.e. an element of $(\tilde{\mathcal{P}}_n([0, 1]))^k$. Therefore $\Pi_T \circ \tilde{A}$ satisfies the definition of an (h, k)-ary (m, n)-SuperhyperFuzzy Set.

Theorem 36 (Functoriality under surjections)**.** *Let $f : X \to Y$ be surjective. Define recursively the nonempty preimage maps $f^{-1}_{(r)} : \tilde{\mathcal{P}}_r(Y) \to \tilde{\mathcal{P}}_r(X)$ by*

$$f^{-1}_{(1)}(B) := \{x \in X : \ f(x) \in B\} \quad (B \in \tilde{\mathcal{P}}(Y)), \qquad f^{-1}_{(r+1)}(\mathcal{B}) := \big\{ f^{-1}_{(r)}(B) \ \big| \ B \in \mathcal{B} \big\}.$$

Then the pushforward

$$(f_*\tilde{A})(B_1, \dots, B_h) \ := \ \tilde{A}\big(f^{-1}_{(m)}(B_1), \dots, f^{-1}_{(m)}(B_h)\big), \qquad (B_i \in \tilde{\mathcal{P}}_m(Y)),$$

defines an (h, k)-ary (m, n)-SuperhyperNeutrosophic Set on Y.

Proof. We show by induction on $r \geq 1$ that $f^{-1}_{(r)}$ maps $\tilde{\mathcal{P}}_r(Y)$ into $\tilde{\mathcal{P}}_r(X)$ and preserves nonemptiness.

$r = 1$: Let $B \in \tilde{\mathcal{P}}(Y)$ be nonempty. Pick $y \in B$; surjectivity gives $x \in X$ with $f(x) = y$, hence $x \in f^{-1}_{(1)}(B)$ and $f^{-1}_{(1)}(B) \in \tilde{\mathcal{P}}(X)$.

$r \to r + 1$: Let $\mathcal{B} \in \tilde{\mathcal{P}}_{r+1}(Y)$ be nonempty. For each $B \in \mathcal{B}$, the inductive hypothesis yields $f^{-1}_{(r)}(B) \in \tilde{\mathcal{P}}_r(X)$. Since $\mathcal{B} \neq \emptyset$, the collection $\{f^{-1}_{(r)}(B) \mid B \in \mathcal{B}\}$ is a nonempty family in $\tilde{\mathcal{P}}_r(X)$, i.e. an element of $\tilde{\mathcal{P}}_{r+1}(X)$.

Now let $(B_1, \dots, B_h) \in (\tilde{\mathcal{P}}_m(Y))^h$. Then $f^{-1}_{(m)}(B_i) \in \tilde{\mathcal{P}}_m(X)$ for each i, so $\tilde{A}$ applies to yield a k-tuple in $(\tilde{\mathcal{P}}_n([0, 1]^3))^k$ with nonempty coordinates. Hence $f_*\tilde{A}$ is well-defined and has the required domain and codomain.

Theorem 37 (Reduction to the classical unary case)**.** *If $h = k = 1$, then an (h, k)-ary (m, n)-SuperhyperNeutrosophic Set $\tilde{A}$ reduces to the classical (m, n)-SuperhyperNeutrosophic mapping $\tilde{A} : \tilde{\mathcal{P}}_m(X) \to \tilde{\mathcal{P}}_n([0, 1]^3)$.*

Proof. The canonical bijections $(\tilde{\mathcal{P}}_m(X))^1 \cong \tilde{\mathcal{P}}_m(X)$ and $(\tilde{\mathcal{P}}_n([0, 1]^3))^1 \cong \tilde{\mathcal{P}}_n([0, 1]^3)$ identify the (h, k)-ary map with a unary map of the stated type. No additional properties are needed, since nonemptiness and neutrosophic constraints are part of $\tilde{A}$'s codomain by definition.

3.2.3. (h, k)-ary (m, n)-SuperhyperPlithogenic Set

An (h, k)-ary (m, n)-SuperhyperPlithogenic Set assigns to each h–tuple of m–level (nested) parameter–subsets a k–tuple of n–level sets of attribute–based membership vectors in $[0, 1]^s$, together with a quantified, symmetric degree of contradiction on attribute values.

For a nonempty set S, write

$$\mathcal{P}_{+}^{0}(S) := S, \qquad \mathcal{P}_{+}^{r+1}(S) := \{\, T \subseteq \mathcal{P}_{+}^{r}(S) \mid T \neq \varnothing \,\} \quad (r \geq 0),$$

so $\mathcal{P}_{+}^{1}(S)$ is the collection of all nonempty subsets of S, $\mathcal{P}_{+}^{2}(S)$ is the collection of all nonempty families of nonempty subsets of S, and so on. On $[0,1]^s$ we use the componentwise order: for $x, y \in [0,1]^s$,

$$x \preceq y \quad \Longleftrightarrow \quad x_i \leq y_i \text{ for all } i = 1, \ldots, s.$$

Definition 24 ((h,k)-ary (m,n)-SuperhyperPlithogenic Set)**.** *Let P be a nonempty base set of parameters, fix integers $m, n, h, k \in \mathbb{N}$ with $m, n, h, k \geq 1$, and fix dimensions $s, t \in \mathbb{N}$ with $s, t \geq 1$. Let v be an attribute with value set P_v (nonempty). Define*

$$D := \big(\mathcal{P}_{+}^{m}(P)\big)^{h}, \qquad C := \big(\mathcal{P}_{+}^{n}([0,1]^s)\big)^{k}.$$

An (h,k)-ary (m,n)-SuperhyperPlithogenic Set (with respect to v) is a quintuple

$$SHP_{pl}^{(h,k)}(m,n) \;=\; \Big(D,\; v,\; P_v,\; \tilde{pdf}_v^{(m,n)},\; pCF\Big),$$

where:

(i) $\tilde{pdf}_v^{(m,n)} : D \times P_v \to C$ *(the* Hyper Degree of Appurtenance*) assigns, to each* $(\mathbf{A}, a)$ *with* $\mathbf{A} = (A_1, \ldots, A_h) \in D$ *and* $a \in P_v$*, a k–tuple*

$$\tilde{pdf}_v^{(m,n)}(\mathbf{A}, a) = (B_1, \ldots, B_k) \in C,$$

where each $B_j \in \mathcal{P}_{+}^{n}([0,1]^s)$ is a nonempty, level–n collection of membership vectors $x \in [0,1]^s$;

(ii) $pCF : P_v \times P_v \to [0,1]^t$ *(the* Degree of Contradiction*) is reflexive and symmetric:*

$$pCF(a,a) = \mathbf{0} \in [0,1]^t, \qquad pCF(a,b) = pCF(b,a) \;\; \textit{for all } a, b \in P_v.$$

Example 25 (Smartphone selection: $(2,2)$-ary $(1,1)$ case)**.** *Let* $P = \{\text{iPhone}, \text{Galaxy}, \text{Pixel}\}$ *and the attribute*

$$v = \mathit{BatteryLifeCategory}, \qquad P_v = \{\mathit{Low}, \mathit{Medium}, \mathit{High}\}.$$

Fix $m = n = 1$, $h = k = 2$, $s = 1$, so

$$D = \mathcal{P}_{+}^{1}(P) \times \mathcal{P}_{+}^{1}(P) = \mathcal{P}(P) \times \mathcal{P}(P)$$

and

$$C = \mathcal{P}([0,1]) \times \mathcal{P}([0,1])$$

. Interpret an input $\mathbf{A} = (A_1, A_2) \in D$ *as:* A_1 = *phones within the preferred price range;* A_2 = *phones from preferred brands. Define, for the value* $a = High$*,*

$$\tilde{pdf}_v^{(1,1)}\big((\{\text{iPhone}, \text{Galaxy}\}, \{\text{iPhone}, \text{Pixel}\}), a\big)$$
$$= \big(\,[0.7, 0.9],\ \{0.8\}\,\big),$$

and for $a = Medium$*,*

$$\tilde{pdf}_v^{(1,1)}\big((\{\text{iPhone}, \text{Galaxy}\}, \{\text{iPhone}, \text{Pixel}\}), a\big)$$
$$= \big(\,\{0.5, 0.6\},\ [0.4, 0.6]\,\big).$$

Here the first coordinate encodes the fuzzy membership to "High battery life", while the second aggregates a secondary satisfaction score. Set $pCF(Low, High) = 1$*,* $pCF(Medium, Low) = 0.5$*,* $pCF(a, a) = 0$*, and extend by symmetry. Then* $SHP_{pl}^{(2,2)}(1, 1)$ *models two–stage, attribute–based assessments with quantified contradictions.*

Example 26 (Customer pricing preference: (1, 1)-ary (1, 1) case)**.** *Let* $P = \{\text{Laptop}, \text{Smartphone}\}$ *and* $v = PricingTier$ *with* $P_v = \{Budget, Premium\}$*. Fix* $m = n = h = k = s = t = 1$*. Then* $D = \mathcal{P}(P)$ *and* $C = \mathcal{P}([0, 1])$*. Define*

$$\begin{aligned}
\tilde{pdf}_v^{(1,1)}(\{\text{Laptop}\}, Premium) &= [0.70, 0.90],\\
\tilde{pdf}_v^{(1,1)}(\{\text{Laptop}\}, Budget) &= \{0.40, 0.50\},\\
\tilde{pdf}_v^{(1,1)}(\{\text{Smartphone}\}, Premium) &= \{0.80\},\\
\tilde{pdf}_v^{(1,1)}(\{\text{Smartphone}\}, Budget) &= [0.30, 0.45],\\
\tilde{pdf}_v^{(1,1)}(\{\text{Laptop}, \text{Smartphone}\}, Premium) &= [0.65, 0.85],\\
\tilde{pdf}_v^{(1,1)}(\{\text{Laptop}, \text{Smartphone}\}, Budget) &= \{0.35, 0.55\}.
\end{aligned}$$

With $pCF(Budget, Premium) = pCF(Premium, Budget) = 0.8$ *and* $pCF(a, a) = 0$*, the structure records fuzzy membership to pricing tiers and their quantified contradiction.*

Example 27 (Filtered product satisfaction: (2, 2)-ary (1, 1) with vector degrees)**.** *Let* $P = \{Laptop, Tablet, Smartphone\}$*, and fix* $m = n = 1$*,* $h = k = 2$*,* $s = 2$*,* $t = 1$*. Let* $v = Satisfaction$ *with* $P_v = \{Low, Medium, High\}$*. For the input* $\big(\{Laptop, Tablet\}, \{Premium\}\big)$ *at* $a = High$*, set*

$$\tilde{pdf}_v^{(1,1)}\Big(\big(\{Laptop, Tablet\}, \{Premium\}\big), High\Big)$$
$$= \big(\ \{(0.85, 0.80), (0.88, 0.82)\},\ \{(0.90, 0.85)\}\ \big),$$

where each $(x, y) \in [0, 1]^2$ *denotes (reviewScore, returnRateScore).*

Let $pCF(High, Low) = 1$*,* $pCF(High, Medium) = 0.6$*,* $pCF(Medium, Low) = 0.4$*,* $pCF(a, a) = 0$*, with symmetry.*

Example 28 (Hierarchical sensor–analytics confidence: $(2,2)$-ary $(3,3)$)**.** *Let* $P = \{\text{Temp}, \text{Pressure}\}$, $m = n = 3$*,* $h = k = 2$*,* $s = 2$*,* $t = 1$*, and* $v = ConfidenceType$ *with* $P_v = \{Low, Medium, High\}$*. Then* $D = \mathcal{P}_+^3(P) \times \mathcal{P}_+^3(P)$ *and* $C = (\mathcal{P}_+^3([0,1]^2))^2$*. Pick*

$$A = \big\{ \{\{\text{Temp}\}, \{\text{Temp}, \text{Pressure}\}\},\ \{\{\text{Pressure}\}\} \big\},$$

$$B = \big\{ \{\{\text{Pressure}\}, \{\text{Temp}, \text{Pressure}\}\} \big\},$$

and define, at $\mathbf{A} = (A, B)$ *and* $a = High$*,*

$$\tilde{pdf}_v^{(3,3)}(\mathbf{A}, a) \ = \ (E_1, E_2),$$

where

$$E_1 = \big\{\{(0.90, 0.85), (0.92, 0.88)\},\ \{(0.88, 0.80)\}\big\},$$
$$E_2 = \big\{\{(0.75, 0.70)\},\ \{(0.78, 0.73), (0.80, 0.75)\}\big\}.$$

Set $pCF(Low, High) = 0.9$*,* $pCF(Low, Medium) = 0.5$*,* $pCF(Medium, High) = 0.6$*,* $pCF(a, a) = 0$*, with symmetry.*

Example 29 (Urban Route Planning as a $(2,2)$-ary $(2,2)$-SuperhyperPlithogenic Set)**.** ***Setting.*** *Consider an urban mobility platform that recommends public–transport routes. Let the base parameter set be*

$$P = \{CostCap,\ TimeWindow,\ MaxTransfers,\ EmissionCap\}.$$

Fix $(m, n, h, k) = (2, 2, 2, 2)$ *and dimensions* $s = 2$*,* $t = 1$*. We interpret the two membership coordinates* $x = (x_1, x_2) \in [0,1]^2$ *as*

$$x_1 = \text{(commuter satisfaction)}, \qquad x_2 = \text{(operational reliability)}.$$

The two output coordinates $(k = 2)$ *will represent two stakeholder views:*

$$\text{output } 1 = \text{commuter view}, \qquad \text{output } 2 = \text{operator view}.$$

Domain and codomain. *The input space is*

$$D = \big(\mathcal{P}_+^2(P)\big)^2,$$

so each input is a pair $\mathbf{A} = (A_1, A_2)$ *of* level–2 hyper–parameters*, with* $A_i \in \mathcal{P}_+^2(P) = \{\ \Gamma \subseteq \mathcal{P}_+^1(P) \mid \Gamma \neq \varnothing\ \}$*. The output space is*

$$C = \big(\mathcal{P}_+^2([0,1]^2)\big)^2,$$

so each output coordinate is a nonempty family (level–2) of nonempty subsets (level–1) of $[0,1]^2$*.*

Attribute and its values. *Let the plithogenic attribute be*

$$v = RouteQuality, \qquad P_v = \{Eco,\ Fast,\ Cheap\}.$$

We quantify incompatibility between attribute values via the degree–of–contradiction

$$pCF : P_v \times P_v \to [0,1], \quad pCF(a,a) = 0, \quad pCF(a,b) = pCF(b,a),$$

specified by

$$pCF(Eco, Fast) = 0.6, \quad pCF(Eco, Cheap) = 0.4, \quad pCF(Fast, Cheap) = 0.5.$$

A concrete input. *Interpret* A_1 *as a* policy/constraint cluster *and* A_2 *as a* user–preference cluster. *Choose*

$$A_1 = \Big\{ \{CostCap, EmissionCap\},\ \{TimeWindow\} \Big\},$$
$$A_2 = \Big\{ \{MaxTransfers\},\ \{TimeWindow, EmissionCap\} \Big\},$$

so $\mathbf{A} = (A_1, A_2) \in D$.

Hyper Degree of Appurtenance (HDAF). *For each attribute value* $a \in P_v$, *the map*

$$\tilde{pdf}_v^{(2,2)} : D \times P_v \longrightarrow C$$

returns a pair $\big(B_1^{(a)}, B_2^{(a)}\big)$ *with* $B_j^{(a)} \in \mathcal{P}_+^2([0,1]^2)$. *At the input* $\mathbf{A}$ *above, define:*

Eco:

$$B_1^{(Eco)} = \{\ \{(0.88, 0.84), (0.86, 0.83)\},\ \{(0.82, 0.80)\}\ \},$$
$$B_2^{(Eco)} = \{\ \{(0.78, 0.75)\},\ \{(0.81, 0.79)\}\ \}.$$

Fast:

$$B_1^{(Fast)} = \{\ \{(0.90, 0.72)\},\ \{(0.85, 0.70), (0.83, 0.69)\}\ \},$$
$$B_2^{(Fast)} = \{\ \{(0.76, 0.68)\}\ \}.$$

Cheap:

$$B_1^{(Cheap)} = \{\ \{(0.80, 0.77)\},\ \{(0.74, 0.73)\}\ \},$$
$$B_2^{(Cheap)} = \{\ \{(0.88, 0.65)\}\ \}.$$

Each $B_j^{(a)}$ *is a level–2 object: a nonempty* family *of nonempty* sets *of two–dimensional membership vectors* $(satisfaction, reliability) \in [0,1]^2$. *The first coordinate (*j=1*) aggregates commuter–side appraisals; the second (*j=2*) aggregates operator–side appraisals. The plithogenic nature appears in the simultaneous presence of multiple attribute values* a *together with the quantified contradictions* pCF *between them.*

The choice (A_1, A_2) *prioritizes emission limits, moderate cost caps, and few transfers within a usable time window. Consequently, Eco achieves high* (x_1, x_2) *on both outputs* *(good satisfaction without sacrificing reliability), whereas Fast increases satisfaction but lowers reliability for the operator; Cheap is relatively reliable for the operator yet yields only moderate commuter satisfaction. The contradiction matrix pCF quantifies these trade–offs (e.g., Eco vs. Fast at 0.6), enabling plithogenic decision aggregation across attribute values. Thus, all components*

$$\Big(D,\ v,\ P_v,\ \tilde{pdf}_v^{(2,2)},\ pCF\Big)$$

instantiate a concrete $(2,2)$-ary $(2,2)$-SuperhyperPlithogenic Set for multi–criteria route planning.

Theorem 38 (Unary reduction)**.** *Every (m,n)-SuperhyperPlithogenic Set is the special case $h = k = 1$ of Definition 24.*

Proof. By Definition 24, the multi–ary structure is

$$SHP_{pl}^{(h,k)}(m,n) = \big(D,\ v,\ P_v,\ \tilde{pdf}_v^{(m,n)},\ pCF\big), \quad D = (\mathcal{P}_+^m(P))^h,\ C = (\mathcal{P}_+^n([0,1]^s))^k.$$

Set $h = k = 1$. There are canonical bijections

$$\Phi : (\mathcal{P}_+^m(P))^1 \to \mathcal{P}_+^m(P), \qquad \Psi : \big(\mathcal{P}_+^n([0,1]^s)\big)^1 \to \mathcal{P}_+^n([0,1]^s),$$

given by $\Phi(A) = A$ and $\Psi(B) = B$. Under Φ and Ψ, the map

$$\tilde{pdf}_v^{(m,n)} : (\mathcal{P}_+^m(P))^1 \times P_v \longrightarrow \big(\mathcal{P}_+^n([0,1]^s)\big)^1$$

is identified with

$$\tilde{pdf}_v^{(m,n)} :\ \mathcal{P}_+^m(P) \times P_v \longrightarrow \mathcal{P}_+^n([0,1]^s),$$

which is precisely the classical (m,n)-SuperhyperPlithogenic mapping (with the same attribute v, value set P_v, and contradiction function pCF). Reflexivity $pCF(a,a) = \mathbf{0}$ and symmetry $pCF(a,b) = pCF(b,a)$ are unaffected by the identifications. Hence the unary case is exactly the classical one.

Theorem 39 (Fuzzy and neutrosophic specializations)**.** *An (h,k)-ary (m,n)-SuperhyperPlithogenic Set specializes to*

(i) an (h,k)-ary (m,n)-SuperhyperFuzzy Set when $s = 1$, and

(ii) an (h,k)-ary (m,n)-SuperhyperNeutrosophic Set when $s = 3$ (with triples (T,I,F)).

Proof. Let $\tilde{pdf}_v^{(m,n)} : D \times P_v \to C$ with $D = (\mathcal{P}_+^m(P))^h$ and $C = (\mathcal{P}_+^n([0,1]^s))^k$.

(i) If $s = 1$, then $[0,1]^s = [0,1]$ and $C = (\mathcal{P}_+^n([0,1]))^k$. Thus

$$\tilde{pdf}_v^{(m,n)} :\ D \times P_v \longrightarrow \big(\mathcal{P}_+^n([0,1])\big)^k$$

assigns to each input a k–tuple of nonempty, level–n fuzzy degree sets, which is exactly the codomain required by the (h,k)-ary (m,n)-SuperhyperFuzzy structure. The domain and pCF are unchanged, so all axioms remain valid.

(ii) If $s = 3$, then $[0,1]^s = [0,1]^3$ and $C = (\mathcal{P}_+^n([0,1]^3))^k$. Hence

$$\tilde{pdf}_v^{(m,n)} : D \times P_v \longrightarrow \left(\mathcal{P}_+^n([0,1]^3)\right)^k$$

returns k level–n families of neutrosophic triples $(T, I, F) \in [0,1]^3$ (with any standard neutrosophic constraint, e.g. $0 \le T + I + F \le 3$, enforced elementwise when imposed). This is precisely the codomain of the (h,k)-ary (m,n)-SuperhyperNeutrosophic structure.

Theorem 40 (Fixing some inputs)**.** *Fix indices* $1 \le i_1 < \cdots < i_r \le h$ *and super–elements* $A_{i_j} \in \mathcal{P}_+^m(P)$. *Define* $\iota : (\mathcal{P}_+^m(P))^{h-r} \hookrightarrow (\mathcal{P}_+^m(P))^h$ *by inserting the fixed* A_{i_j} *in the positions* i_j*. Then*

$$\tilde{pdf}_{\text{fix}}^{(m,n)}(\mathbf{B}, a) := \tilde{pdf}_v^{(m,n)}(\iota(\mathbf{B}), a)$$

together with pCF yields an $(h-r, k)$*-ary* (m,n)*-SuperhyperPlithogenic Set.*

Proof. Let $\mathbf{B} = (B_\ell)_{\ell \in I^{\text{free}}} \in (\mathcal{P}_+^m(P))^{h-r}$, where $I^{\text{free}} = \{1, \dots, h\} \setminus \{i_1, \dots, i_r\}$. Define the h–tuple $\iota(\mathbf{B}) = (C_1, \dots, C_h)$ by

$$C_u = \begin{cases} A_{i_j}, & u = i_j \text{ for some } j, \\ B_\ell, & u = \ell \in I^{\text{free}}. \end{cases}$$

Thus $\iota(\mathbf{B}) \in (\mathcal{P}_+^m(P))^h$. For any $a \in P_v$,

$$\tilde{pdf}_{\text{fix}}^{(m,n)}(\mathbf{B}, a) = \tilde{pdf}_v^{(m,n)}(\iota(\mathbf{B}), a) \in (\mathcal{P}_+^n([0,1]^s))^k$$

has nonempty coordinates because $\tilde{pdf}_v^{(m,n)}$ does so for every h–tuple. Hence $\tilde{pdf}_{\text{fix}}^{(m,n)}$ is a valid hyper degree function on the smaller domain $(\mathcal{P}_+^m(P))^{h-r} \times P_v$. The contradiction function pCF is defined only on $P_v \times P_v$ and does not depend on the arity of the domain; its axioms (reflexivity, symmetry) are preserved.

Theorem 41 (Projection to fewer outputs)**.** *Let* $1 \le j_1 < \cdots < j_s \le k$*. Define* $\pi_{j_1,\dots,j_s} : C \to \left(\mathcal{P}_+^n([0,1]^s)\right)^s$ *by*

$$\pi_{j_1,\dots,j_s}(B_1, \dots, B_k) := (B_{j_1}, \dots, B_{j_s}).$$

Then $(\mathbf{A}, a) \mapsto \pi_{j_1,\dots,j_s}\big(\tilde{pdf}_v^{(m,n)}(\mathbf{A}, a)\big)$ *(with D and the same pCF) defines an* (h,s)*-ary* (m,n)*-SuperhyperPlithogenic Set.*

Proof. Fix $(\mathbf{A}, a) \in D \times P_v$. Since $\tilde{pdf}_v^{(m,n)}(\mathbf{A}, a) = (B_1, \dots, B_k) \in C$, each B_{j_r} is a nonempty element of $\mathcal{P}_+^n([0,1]^s)$. Therefore

$$\pi_{j_1,\dots,j_s}\big(\tilde{pdf}_v^{(m,n)}(\mathbf{A}, a)\big) = (B_{j_1}, \dots, B_{j_s}) \in \left(\mathcal{P}_+^n([0,1]^s)\right)^s.$$

Nonemptiness and level–n nesting are preserved coordinatewise. The domain D and the contradiction function pCF remain unchanged, so the projected structure satisfies Definition 24 with output arity s.

Theorem 42 (Closure under pointwise union). *If $\tilde{pdf}, \tilde{pdf}' : D \times P_v \to C$ are hyper degree functions (with the same pCF), then*

$$(\tilde{pdf} \cup \tilde{pdf}')(\mathbf{A}, a) \ := \ \tilde{pdf}(\mathbf{A}, a) \ \cup \ \tilde{pdf}'(\mathbf{A}, a)$$

(where the union is taken coordinatewise in C) defines another hyper degree function, hence an (h, k)-ary (m, n)-SuperhyperPlithogenic Set.

Proof. Fix $(\mathbf{A}, a)$. Write $\tilde{pdf}(\mathbf{A}, a) = (B_1, \ldots, B_k)$ and $\tilde{pdf}'(\mathbf{A}, a) = (B'_1, \ldots, B'_k)$, with $B_j, B'_j \in \mathcal{P}^n_+([0,1]^s)$. We show by induction on $n \geq 1$ that $B_j \cup B'_j \in \mathcal{P}^n_+([0,1]^s)$.

Base $n = 1$: B_j and B'_j are nonempty subsets of $[0,1]^s$, so $B_j \cup B'_j$ is a nonempty subset of $[0,1]^s$, i.e. an element of $\mathcal{P}^1_+([0,1]^s)$.

Inductive step: suppose the claim holds for level n. Let $n+1$. Then $B_j, B'_j \subseteq \mathcal{P}^n_+([0,1]^s)$ are nonempty families of level–n elements. Their union $B_j \cup B'_j$ is nonempty (as a union of two nonempty sets) and every element of it still lies in $\mathcal{P}^n_+([0,1]^s)$. Hence $B_j \cup B'_j \in \mathcal{P}^{n+1}_+([0,1]^s)$.

Applying this coordinatewise yields $(\tilde{pdf} \cup \tilde{pdf}')(\mathbf{A}, a) \in C$ with nonempty coordinates. The domain D, attribute data (v, P_v), and pCF are unchanged, so $(\tilde{pdf} \cup \tilde{pdf}')$ is a valid hyper degree function.

Theorem 43 (Closure under pointwise intersection). *With the same hypotheses as Theorem 42, define*

$$(\tilde{pdf} \cap \tilde{pdf}')(\mathbf{A}, a) \ := \ \tilde{pdf}(\mathbf{A}, a) \ \cap \ \tilde{pdf}'(\mathbf{A}, a)$$

(coordinatewise). If every coordinatewise intersection is nonempty, this again defines a valid hyper degree function.

Proof. Fix $(\mathbf{A}, a)$ and j. Assume $B_j \cap B'_j \neq \varnothing$, where $B_j, B'_j \in \mathcal{P}^n_+([0,1]^s)$. We prove by induction on $n \geq 1$ that $B_j \cap B'_j \in \mathcal{P}^n_+([0,1]^s)$.

Base $n = 1$: $B_j, B'_j \subseteq [0,1]^s$ are nonempty and $B_j \cap B'_j \neq \varnothing$ by hypothesis, so $B_j \cap B'_j \in \mathcal{P}^1_+([0,1]^s)$.

Inductive step: suppose the claim holds for level n. For level $n+1$, $B_j, B'_j \subseteq \mathcal{P}^n_+([0,1]^s)$ are nonempty families. The intersection $B_j \cap B'_j$ is a family of level–n elements; by the nonemptiness hypothesis it belongs to $\mathcal{P}^{n+1}_+([0,1]^s)$.

Thus each coordinate intersection is (by assumption) nonempty and of the correct level. Collecting the k coordinates gives $(\tilde{pdf} \cap \tilde{pdf}')(\mathbf{A}, a) \in C$, so the result is a valid hyper degree function.

Theorem 44 (Nested $\boldsymbol{\alpha}$-cuts). *Fix $a \in P_v$ and $\alpha \in [0,1]^s$. Define*

$$C^a_\alpha \ := \ \left\{ \ \mathbf{A} \in D \ \middle| \ \exists j \in \{1, \ldots, k\} \ \exists x \in \left[\tilde{pdf}^{(m,n)}_v(\mathbf{A}, a)\right]_j \ \text{ with } \ x \succeq \alpha \ \right\},$$

where $\succeq$ is the componentwise order on $[0,1]^s$. If $\alpha', \alpha \in [0,1]^s$ with $\alpha' \succeq \alpha$, then $C^a_{\alpha'} \subseteq C^a_\alpha$.

Proof. Let $\mathbf{A} \in C^a_{\alpha'}$. Then there exist a coordinate j and a vector $x \in [\tilde{pdf}_v^{(m,n)}(\mathbf{A}, a)]_j$ such that $x \succeq \alpha'$. Since $\alpha' \succeq \alpha$, transitivity of $\succeq$ implies $x \succeq \alpha$. Hence the same j and x witness $\mathbf{A} \in C^a_\alpha$. Therefore $C^a_{\alpha'} \subseteq C^a_\alpha$, i.e. the family $\{C^a_\alpha\}_\alpha$ is nested decreasing in the threshold α.

Theorem 45 (Compatibility with surjective mappings)**.** *Let $f : P \to Q$ be surjective. For $r \geq 1$ define recursively*

$$f^{-1}_{(1)}(B) := \{\, x \in P \mid f(x) \in B \,\} \quad (B \in \mathcal{P}^1_+(Q)), \qquad f^{-1}_{(r+1)}(\mathcal{B}) := \{\, f^{-1}_{(r)}(B) \mid B \in \mathcal{B} \,\}.$$

Given $\tilde{pdf}_v^{(m,n)} : D_P \times P_v \to C$ on P, set

$$\tilde{pdf}_{v;Q}^{(m,n)}(\mathbf{B}, a) \;:=\; \tilde{pdf}_v^{(m,n)}\big(f^{-1}_{(m)}(\mathbf{B}), a\big), \qquad \mathbf{B} \in D_Q := (\mathcal{P}^m_+(Q))^h.$$

Then $\big(D_Q,\, v,\, P_v,\, \tilde{pdf}_{v;Q}^{(m,n)},\, pCF\big)$ is an (h,k)-ary (m,n)-SuperhyperPlithogenic Set on Q.

Proof. **Step 1 (well-definedness and nonemptiness of $f^{-1}_{(r)}$).** We prove by induction on $r \geq 1$ that $f^{-1}_{(r)} : \mathcal{P}^r_+(Q) \to \mathcal{P}^r_+(P)$ is well defined and preserves nonemptiness.

Base $r = 1$: If $B \in \mathcal{P}^1_+(Q)$, then $B \neq \varnothing$. Surjectivity of f ensures $\exists x \in P$ with $f(x) \in B$, hence $f^{-1}_{(1)}(B) \neq \varnothing$ and $f^{-1}_{(1)}(B) \subseteq P$, so $f^{-1}_{(1)}(B) \in \mathcal{P}^1_+(P)$.

Inductive step: suppose $f^{-1}_{(r)}$ maps $\mathcal{P}^r_+(Q)$ into $\mathcal{P}^r_+(P)$ and preserves nonemptiness. Let $\mathcal{B} \in \mathcal{P}^{r+1}_+(Q)$ be nonempty. For each $B \in \mathcal{B}$, $f^{-1}_{(r)}(B) \in \mathcal{P}^r_+(P)$ and the family $\{\, f^{-1}_{(r)}(B) \mid B \in \mathcal{B} \,\}$ is nonempty. Hence $f^{-1}_{(r+1)}(\mathcal{B}) \in \mathcal{P}^{r+1}_+(P)$.

Step 2 (type checking for the pushforward). Take any $\mathbf{B} = (B_1, \ldots, B_h) \in D_Q = (\mathcal{P}^m_+(Q))^h$. By Step 1, $f^{-1}_{(m)}(B_i) \in \mathcal{P}^m_+(P)$ for every i, so $f^{-1}_{(m)}(\mathbf{B}) := (f^{-1}_{(m)}(B_1), \ldots, f^{-1}_{(m)}(B_h)) \in D_P$. Therefore

$$\tilde{pdf}_{v;Q}^{(m,n)}(\mathbf{B}, a) = \tilde{pdf}_v^{(m,n)}\big(f^{-1}_{(m)}(\mathbf{B}), a\big) \in C = (\mathcal{P}^n_+([0,1]^s))^k$$

with nonempty coordinates. The attribute v, its value set P_v, and the contradiction function pCF are unchanged. Hence all clauses of Definition 24 hold on Q.

Theorem 46 (Reduction to the classical case)**.** *If $h = k = 1$, then $\tilde{pdf}_v^{(m,n)} : \mathcal{P}^m_+(P) \times P_v \to \mathcal{P}^n_+([0,1]^s)$ is exactly the classical (m,n)-SuperhyperPlithogenic Set.*

Proof. As in the proof of Unary reduction, apply the canonical identifications

$$(\mathcal{P}^m_+(P))^1 \cong \mathcal{P}^m_+(P), \qquad \big(\mathcal{P}^n_+([0,1]^s)\big)^1 \cong \mathcal{P}^n_+([0,1]^s),$$

to rewrite the multi–ary map on $D \times P_v$ into the unary map

$$\tilde{pdf}_v^{(m,n)} :\; \mathcal{P}^m_+(P) \times P_v \longrightarrow \mathcal{P}^n_+([0,1]^s).$$

All structural data (v, P_v, pCF) are preserved verbatim, and nonemptiness/level constraints in the codomain are unchanged. This is precisely the classical definition.

4. Additional Result: (h,k)-ary (m,n)-SuperhyperSoft Set and (h,k)-ary (m,n)-SuperhyperRough Set

Although SuperHyperSoft Sets and SuperHyperRough Sets have already been studied in existing literature, their formulations exhibit slight conceptual differences from the framework of SuperHyperUncertain Sets presented above. Therefore, in this paper, we deliberately reconsider and refine their definitions to better suit the context and objectives of the present study. It is important to note that the existing definitions of SuperHyperSoft Sets and SuperHyperRough Sets are, of course, mathematically correct.

Note that soft Sets assign parameters to subsets of a universe, while k-ary Soft Sets extend this mapping to tuples. In contrast, (h,k)-ary (m,n)-SuperhyperSoft Sets embed hierarchical multi-uncertainty using iterated powersets and superhyperoperations, capturing complex interactions across multiple structural levels. Rough Sets approximate subsets by lower and upper bounds under an indiscernibility relation. (h,k)-ary (m,n)-SuperhyperRough Sets extend this by encoding hierarchical, multi-ary superhyperstructures, enabling refined layered approximations, multi-level granularity, and advanced attribute interactions for complex uncertain domains.

4.1. (h,k)-ary (m,n)-SuperhyperSoft Set

For a nonempty set X, write

$$\mathcal{P}_+(X) := \{\, A \subseteq X \mid A \neq \varnothing \,\}$$

for the *nonempty powerset*. Define the *rth nonempty iterated powerset* recursively by

$$\widetilde{\mathcal{P}}_0(X) := X, \qquad \widetilde{\mathcal{P}}_{r+1}(X) := \mathcal{P}_+\big(\widetilde{\mathcal{P}}_r(X)\big) \quad (r \geq 0).$$

We use the componentwise inclusion order on Cartesian powers: for $\mathbf{A} = (A_1, \dots, A_h),\ \mathbf{A}' = (A'_1, \dots, A'_h) \in (\widetilde{\mathcal{P}}_m(S))^h$, write $\mathbf{A} \subseteq \mathbf{A}'$ if $A_i \subseteq A'_i$ for all i.

In classical soft set theory [8, 69], one fixes a parameter set S and uses a mapping $F : S \to \mathcal{P}(U)$. To capture hierarchical parameter groupings and multi-output responses, we lift parameters to the mth iterated nonempty powerset and allow h inputs (interacting parameter aggregates), while the outputs are lifted to the nth iterated nonempty powerset and replicated in k coordinates.

Definition 25 ((h,k)-ary (m,n)-SuperhyperSoft Set)**.** *Let U be a nonempty universe and S a nonempty parameter set. Fix integers $m, n, h, k \geq 1$. An* (h,k)-ary (m,n)-SuperhyperSoft Set *over U (with respect to S) is a mapping*

$$F : \ \big(\widetilde{\mathcal{P}}_m(S)\big)^h \ \longrightarrow \ \big(\widetilde{\mathcal{P}}_n(U)\big)^k.$$

For every input $\mathbf{A} = (A_1, \dots, A_h) \in (\widetilde{\mathcal{P}}_m(S))^h$ the value

$$F(\mathbf{A}) = \big(B_1, \dots, B_k\big)$$

is a k-tuple with each $B_j \in \widetilde{\mathcal{P}}_n(U)$ (in particular, each B_j is nonempty and n-nested). When $m = n = h = k = 1$ this reduces (up to the conventional treatment of the empty set) to the classical soft set.

Example 30 (Unary case)**.** *Let $U = \{u_1, u_2, u_3\}$ and $S = \{a, b\}$. Then $\widetilde{\mathcal{P}}_1(S) = \{\{a\}, \{b\}, \{a, b\}\}$ and $\widetilde{\mathcal{P}}_1(U) = \{\, B \subseteq U \mid B \neq \varnothing \,\}$. For $m = n = h = k = 1$, an $(1,1)$-ary $(1,1)$-SuperhyperSoft Set is a map $F : \widetilde{\mathcal{P}}_1(S) \to \widetilde{\mathcal{P}}_1(U)$, e.g.*

$$F(\{a\}) = \{u_1, u_2\}, \qquad F(\{b\}) = \{u_2, u_3\}, \qquad F(\{a, b\}) = \{u_1, u_2, u_3\}.$$

If one increases h and k, inputs become tuples of parameter-aggregates and outputs become tuples of nonempty U-subsets (or, for $n > 1$, of nested nonempty families of such).

Example 31 (Personalized movie recommendations)**.** *Let $U = \{M_1, \dots, M_6\}$ be a catalog and $S = \{\text{Comedy}, \text{Action}, \text{Drama}\}$. Choose $m = 2$, $n = 1$, $h = 2$, $k = 2$. Then*

$$\widetilde{\mathcal{P}}_1(S) = \mathcal{P}_+(S), \qquad \widetilde{\mathcal{P}}_2(S) = \mathcal{P}_+\big(\mathcal{P}_+(S)\big),$$

so each element of $\widetilde{\mathcal{P}}_2(S)$ is a nonempty collection *of nonempty genre-subsets. Set*

$$D = \big(\widetilde{\mathcal{P}}_2(S)\big)^2, \qquad C = \big(\widetilde{\mathcal{P}}_1(U)\big)^2.$$

Pick

$$\gamma^{(1)} = \big\{\{\text{Comedy}\}, \{\text{Action}, \text{Drama}\}\big\}, \qquad \gamma^{(2)} = \big\{\{\text{Action}\}, \{\text{Drama}, \text{Comedy}\}\big\} \in \widetilde{\mathcal{P}}_2(S).$$

A $(2,2)$-ary $(2,1)$-SuperhyperSoft map $F : D \to C$ may assign

$$F\big(\gamma^{(1)}, \gamma^{(2)}\big) = \big(B_1, B_2\big), \qquad B_1 = \{M_1, M_3, M_5\}, \quad B_2 = \{M_2, M_4\},$$

interpreted as primary and secondary recommendations driven by two hierarchical genre profiles.

Example 32 (University Course Planning as a $(2,2)$-ary $(2,1)$-SuperhyperSoft Set)**.** *Let the* universe of objects *be the catalog of next-term courses*

$$U = \{\mathit{CS101}, \mathit{CS102}, \mathit{MATH201}, \mathit{STAT210}, \mathit{AI310}, \mathit{NLP320}, \mathit{SYS220}, \mathit{HCI230}\}.$$

Let the parameter set *encode requirement tags and scheduling constraints:*

$$S = \{\text{Req-Core}, \text{Req-Math}, \text{Req-AI}, \text{Req-Systems},$$
$$\text{Time-Morning}, \text{Time-Afternoon}, \text{Mode-InPerson}, \text{No-Friday}\}.$$

We choose $m = 2$, $n = 1$, $h = 2$, $k = 2$. Thus the domain *is $D = (\widetilde{\mathcal{P}}_2(S))^2$, where*

$$\widetilde{\mathcal{P}}_1(S) = \{A \subseteq S \mid A \neq \varnothing\},$$
$$\widetilde{\mathcal{P}}_2(S) = \{\, \Gamma \subseteq \widetilde{\mathcal{P}}_1(S) \mid \Gamma \neq \varnothing \,\}.$$

The codomain *is $(\widetilde{\mathcal{P}}_1(U))^2$, i.e., ordered pairs of nonempty subsets of U.*

Interpret the two input coordinates as:

- *coordinate 1: a* cluster of degree requirements *(hyper–parameter on S),*
- *coordinate 2: a* cluster of personal constraints/preferences *(hyper–parameter on S).*

Consider the concrete hyper–parameters

$$\Gamma_{\text{req}} = \big\{\{\text{Req-Core}\},\ \{\text{Req-Math}, \text{Req-AI}\}\big\} \in \widetilde{\mathcal{P}}_2(S),$$

$$\Gamma_{\text{pref}} = \big\{\{\text{Time-Afternoon}\},\ \{\text{Mode-InPerson}\},\ \{\text{No-Friday}\}\big\} \in \widetilde{\mathcal{P}}_2(S).$$

An (h,k)-ary (m,n)-SuperhyperSoft Set is a mapping

$$F:\ D = (\widetilde{\mathcal{P}}_2(S))^2 \longrightarrow (\widetilde{\mathcal{P}}_1(U))^2,$$

$$(\Gamma_{\text{req}}, \Gamma_{\text{pref}}) \longmapsto \big(B_1, B_2\big),$$

where $B_1, B_2 \subseteq U$ are nonempty. For the above input, define (one valid instantiation)

$B_1 = \{\mathit{CS101}, \mathit{MATH201}, \mathit{AI310}\}$

(primary plan: meets core/math/AI while fitting afternoon, in–person, no–Friday),

$B_2 = \{\mathit{STAT210}, \mathit{NLP320}, \mathit{HCI230}\}$

(contingency plan under the same preferences).

Hence

$$F\big(\Gamma_{\text{req}}, \Gamma_{\text{pref}}\big) = \big(B_1, B_2\big) \in \big(\widetilde{\mathcal{P}}_1(U)\big)^2.$$

The nested, set–of–subsets form of Γ_{req} and Γ_{pref} lets a program advisor encode hierarchical *requirement/preference groupings (e.g., "satisfy* Req-Core and *at least one of* {Req-Math, Req-AI}*") together with scheduling clusters (e.g., "Afternoon* and *In–Person* and *No–Friday"). The two outputs B_1, B_2 provide a primary and a backup recommendation set, each a nonempty element of $\widetilde{\mathcal{P}}_1(U)$. This realizes, in a real advising scenario, a $(2,2)$-ary $(2,1)$-SuperhyperSoft Set in the sense of the definition above.*

Theorem 47 (Classical case as a special instance)**.** *Every classical soft set $F_{\text{cl}} : S \to \mathcal{P}(U)$ is naturally represented by an (h,k)-ary (m,n)-SuperhyperSoft Set with $m = n = h = k = 1$.*

Proof. Set $m = n = h = k = 1$. The domain of a $(1,1)$-ary $(1,1)$-SuperhyperSoft Set is $\widetilde{\mathcal{P}}_1(S) = \mathcal{P}_+(S)$ (nonempty subsets of S) and its codomain is $\widetilde{\mathcal{P}}_1(U) = \mathcal{P}_+(U)$ (nonempty subsets of U).

Step 1 (embedding singletons). Define the canonical injection

$$\iota_S : S \longrightarrow \mathcal{P}_+(S), \qquad \iota_S(s) := \{s\}.$$

Although ι_S is not surjective (multielement parameter blocks are not hit), it embeds the *classical* parameter space into the *level–1* parameter space on which SuperhyperSoft sets act.

Step 2 (lifting the codomain to nonempty subsets). If $F_{\text{cl}}(s) \neq \varnothing$ for all $s \in S$, define

$$L_U : \mathcal{P}(U) \longrightarrow \mathcal{P}_+(U), \qquad L_U(B) := B \ \ \text{(which lies in } \mathcal{P}_+(U)).$$

If some $F_{\text{cl}}(s)$ may be empty, one may either:

- (a) adjoin a dummy element $\star \notin U$ and lift to $U^\star := U \sqcup \{\star\}$ by $L_{U^\star}(B) := B$ if $B \neq \varnothing$, and $L_{U^\star}(\varnothing) := \{\star\}$, or
- (b) relax the codomain to $\mathcal{P}(U)$ (dropping the "nonempty" requirement).

To stay within the present definition $\widetilde{\mathcal{P}}_1(U) = \mathcal{P}_+(U)$, we adopt (a).
Step 3 (a canonical extension to $\mathcal{P}_+(S)$). Define the *union–lift* $F : \mathcal{P}_+(S) \to \mathcal{P}_+(U^\star)$ by

$$F(A) \;:=\; L_{U^\star}\Big(\bigcup_{s\in A} F_{\text{cl}}(s) \Big).$$

Then F is well defined and nonempty-valued: if $\bigcup_{s\in A} F_{\text{cl}}(s) \neq \varnothing$, we keep that set; otherwise $L_{U^\star}$ returns $\{\star\}$.
Step 4 (agreement with the classical soft set on singletons). For $s \in S$,

$$F(\{s\}) \;=\; L_{U^\star}\big(F_{\text{cl}}(s)\big) \;=\; \begin{cases} F_{\text{cl}}(s), & \text{if } F_{\text{cl}}(s) \neq \varnothing, \\ \{\star\}, & \text{if } F_{\text{cl}}(s) = \varnothing. \end{cases}$$

Hence, modulo the harmless convention of representing the classical empty image by $\{\star\}$, the mapping F reproduces F_{cl} on the embedded copy $\iota_S(S)$ of S, and extends it canonically to all nonempty parameter blocks via union. (One could just as well choose the *intersection–lift* $A \mapsto \bigcap_{s\in A} F_{\text{cl}}(s)$ if conjunctive semantics are desired.)

Therefore $F : (\widetilde{\mathcal{P}}_1(S))^1 \to (\widetilde{\mathcal{P}}_1(U^\star))^1$ is a $(1,1)$-ary $(1,1)$-SuperhyperSoft Set that represents the given classical soft set.

Theorem 48 (Fixing some parameters)**.** *Fix indices $1 \leq i_1 < \cdots < i_r \leq h$ and elements $A_{i_j} \in \widetilde{\mathcal{P}}_m(S)$. Define*

$$F_{\text{fix}} : \; \big(\widetilde{\mathcal{P}}_m(S)\big)^{h-r} \longrightarrow \big(\widetilde{\mathcal{P}}_n(U)\big)^k$$

by inserting the fixed A_{i_j} into the corresponding coordinates of each $(h-r)$-tuple and then applying F. Then F_{fix} is an $(h-r,k)$-ary (m,n)-SuperhyperSoft Set.

Proof. Let $I = \{1,\ldots,h\}$ and $I^{\text{free}} := I \setminus \{i_1,\ldots,i_r\}$. Define the *insertion map*

$$\iota_{\text{fix}} : \; \big(\widetilde{\mathcal{P}}_m(S)\big)^{h-r} \longrightarrow \big(\widetilde{\mathcal{P}}_m(S)\big)^h$$

as follows: for $\mathbf{B} = (B_\ell)_{\ell\in I^{\text{free}}}$, put $\iota_{\text{fix}}(\mathbf{B}) = \mathbf{A}$ with

$$A_u := \begin{cases} A_{i_j}, & u = i_j \text{ for some } j, \\ B_\ell, & u = \ell \in I^{\text{free}}. \end{cases}$$

This is well defined and injective. Set $F_{\text{fix}} := F \circ \iota_{\text{fix}}$. Since F takes values in $(\widetilde{\mathcal{P}}_n(U))^k$, so does F_{fix}. Thus F_{fix} is a mapping with domain $(\widetilde{\mathcal{P}}_m(S))^{h-r}$ and codomain $(\widetilde{\mathcal{P}}_n(U))^k$, i.e., an $(h-r,k)$-ary (m,n)-SuperhyperSoft Set.

Theorem 49 (Projection to selected outputs). *Let $1 \le j_1 < \cdots < j_s \le k$. The coordinate projection*

$$\pi_{j_1,\ldots,j_s} : \left(\widetilde{\mathcal{P}}_n(U)\right)^k \longrightarrow \left(\widetilde{\mathcal{P}}_n(U)\right)^s, \qquad (B_1,\ldots,B_k) \mapsto (B_{j_1},\ldots,B_{j_s}),$$

composed with F, yields an (h,s)-ary (m,n)-SuperhyperSoft Set.

Proof. By definition, each $B_{j_r} \in \widetilde{\mathcal{P}}_n(U)$ is nonempty and level–n. Hence $\pi_{j_1,\ldots,j_s}$ maps $(\widetilde{\mathcal{P}}_n(U))^k$ into $(\widetilde{\mathcal{P}}_n(U))^s$, preserving nonemptiness coordinatewise. Therefore $\pi_{j_1,\ldots,j_s} \circ F : (\widetilde{\mathcal{P}}_m(S))^h \to (\widetilde{\mathcal{P}}_n(U))^s$ is an (h,s)-ary (m,n)-SuperhyperSoft Set.

Theorem 50 (Pointwise union). *If $F_1, F_2 : (\widetilde{\mathcal{P}}_m(S))^h \to (\widetilde{\mathcal{P}}_n(U))^k$ are (h,k)-ary (m,n)-SuperhyperSoft Sets, then*

$$(F_1 \cup F_2)(\mathbf{A}) := \left(F_1(\mathbf{A})_j \cup F_2(\mathbf{A})_j\right)_{j=1}^k$$

is again an (h,k)-ary (m,n)-SuperhyperSoft Set.

Proof. Fix $\mathbf{A}$ and a coordinate j. We prove by induction on n that $F_1(\mathbf{A})_j \cup F_2(\mathbf{A})_j \in \widetilde{\mathcal{P}}_n(U)$.
Base $n = 1$. Then $F_i(\mathbf{A})_j \in \mathcal{P}_+(U)$, so each is a nonempty subset of U. The union of two nonempty subsets of U is nonempty, hence lies in $\mathcal{P}_+(U) = \widetilde{\mathcal{P}}_1(U)$.
Step $n \Rightarrow n{+}1$. Now $F_i(\mathbf{A})_j \in \widetilde{\mathcal{P}}_{n+1}(U) = \mathcal{P}_+(\widetilde{\mathcal{P}}_n(U))$, i.e., each is a nonempty family of level–n elements. Their union is a nonempty family of level–n elements and therefore belongs to $\mathcal{P}_+(\widetilde{\mathcal{P}}_n(U)) = \widetilde{\mathcal{P}}_{n+1}(U)$.

Doing this in each coordinate $j = 1,\ldots,k$ gives the claim.

Theorem 51 (Pointwise intersection). *With the same hypotheses, define*

$$(F_1 \cap F_2)(\mathbf{A}) := \left(F_1(\mathbf{A})_j \cap F_2(\mathbf{A})_j\right)_{j=1}^k.$$

If each intersection is nonempty, then $(F_1 \cap F_2)$ is an (h,k)-ary (m,n)-SuperhyperSoft Set.

Proof. Fix $\mathbf{A}$ and j, and assume $F_1(\mathbf{A})_j \cap F_2(\mathbf{A})_j \neq \varnothing$. Induct on n.
Base $n = 1$. Two nonempty subsets of U that meet have a nonempty intersection, which lies in $\mathcal{P}_+(U) = \widetilde{\mathcal{P}}_1(U)$.
Step $n \Rightarrow n{+}1$. Here $F_i(\mathbf{A})_j$ are nonempty *families* of level–n elements. A nontrivial intersection of such families is again a nonempty family of level–n elements; hence it lies in $\mathcal{P}_+(\widetilde{\mathcal{P}}_n(U)) = \widetilde{\mathcal{P}}_{n+1}(U)$.

Coordinatewise application yields the result in $(\widetilde{\mathcal{P}}_n(U))^k$.

Theorem 52 (Functoriality under surjections)**.** *Let $f : U \to V$ be surjective. Define recursively maps $f_*^{(1)} : \widetilde{\mathcal{P}}_1(U) \to \widetilde{\mathcal{P}}_1(V)$ by $f_*^{(1)}(B) := f[B] = \{ f(u) \mid u \in B \}$ and, for $r \geq 1$,*

$$f_*^{(r+1)} : \widetilde{\mathcal{P}}_{r+1}(U) \longrightarrow \widetilde{\mathcal{P}}_{r+1}(V), \qquad f_*^{(r+1)}(\mathcal{B}) := \{ f_*^{(r)}(B) \mid B \in \mathcal{B} \}.$$

Then the pushforward

$$f_*F : (\widetilde{\mathcal{P}}_m(S))^h \longrightarrow (\widetilde{\mathcal{P}}_n(V))^k, \qquad f_*F(\mathbf{A}) = \big(f_*^{(n)}(F(\mathbf{A})_j)\big)_{j=1}^k,$$

is an (h,k)-ary (m,n)-SuperhyperSoft Set over V.

Proof. We prove by induction on $r \geq 1$ that $f_*^{(r)}$ is well defined and preserves nonemptiness.

Base $r = 1$. If $B \in \widetilde{\mathcal{P}}_1(U)$ then $B \neq \varnothing$; surjectivity of f implies $f[B] \neq \varnothing$, so $f_*^{(1)}(B) \in \widetilde{\mathcal{P}}_1(V)$.

Step $r \Rightarrow r{+}1$. Let $\mathcal{B} \in \widetilde{\mathcal{P}}_{r+1}(U)$, so $\mathcal{B} \neq \varnothing$ and each $B \in \mathcal{B}$ lies in $\widetilde{\mathcal{P}}_r(U)$. By induction, each $f_*^{(r)}(B) \in \widetilde{\mathcal{P}}_r(V)$; since $\mathcal{B} \neq \varnothing$, the set $\{f_*^{(r)}(B) \mid B \in \mathcal{B}\}$ is a *nonempty* subset of $\widetilde{\mathcal{P}}_r(V)$, hence belongs to $\mathcal{P}_+(\widetilde{\mathcal{P}}_r(V)) = \widetilde{\mathcal{P}}_{r+1}(V)$.

Now, for any input $\mathbf{A} \in (\widetilde{\mathcal{P}}_m(S))^h$, $F(\mathbf{A}) \in (\widetilde{\mathcal{P}}_n(U))^k$; applying $f_*^{(n)}$ to each coordinate yields an element of $(\widetilde{\mathcal{P}}_n(V))^k$. Thus f_*F has the required domain and codomain and is an (h,k)-ary (m,n)-SuperhyperSoft Set over V.

Theorem 53 (Reduction to the classical soft set)**.** *If $m = n = h = k = 1$, then $F : \widetilde{\mathcal{P}}_1(S) \to \widetilde{\mathcal{P}}_1(U)$ corresponds naturally to a classical soft set $F_{\mathrm{cl}} : S \to \mathcal{P}(U)$ (up to the treatment of the empty set).*

Proof. Consider the restriction of F to singleton parameters via $\iota_S : S \to \mathcal{P}_+(S)$, $s \mapsto \{s\}$. Define $F_{\mathrm{cl}}(s) := F(\{s\})$ if $F(\{s\}) \neq \varnothing$ and $F_{\mathrm{cl}}(s) := \varnothing$ otherwise (if one insists on allowing empties in the classical codomain). This recovers a map $F_{\mathrm{cl}} : S \to \mathcal{P}(U)$. Conversely, given F_{cl}, the union– or intersection–lift (as in the proof of the first theorem) produces an $F : \mathcal{P}_+(S) \to \mathcal{P}_+(U)$ whose restriction to singletons coincides with F_{cl} (up to the chosen convention for the empty set). Hence the two notions correspond naturally in the unary, level–1 case.

Theorem 54 (Nested inclusion under antitone monotonicity)**.** *Suppose F is* antitone *in each parameter coordinate, i.e., for every $i \in \{1, \ldots, h\}$ and tuples $\mathbf{A}, \mathbf{A}'$ that coincide in all coordinates except possibly i,*

$$A_i \subseteq A_i' \implies F(\mathbf{A})_j \supseteq F(\mathbf{A}')_j \quad \textit{for all } j = 1, \ldots, k.$$

Then for any $\mathbf{A}, \mathbf{A}' \in (\widetilde{\mathcal{P}}_m(S))^h$ with $\mathbf{A} \subseteq \mathbf{A}'$ coordinatewise, one has

$$F(\mathbf{A})_j \ \supseteq\ F(\mathbf{A}')_j \qquad (j = 1, \ldots, k).$$

Proof. Enumerate the coordinates so that we can move from $\mathbf{A}$ to $\mathbf{A}'$ by changing one coordinate at a time. Define a chain of tuples

$$\mathbf{A}^{(0)} := \mathbf{A}, \quad \mathbf{A}^{(t)} := (A_1', \dots, A_t', A_{t+1}, \dots, A_h) \;\; (1 \le t \le h),$$

so $\mathbf{A}^{(h)} = \mathbf{A}'$ and $\mathbf{A}^{(t-1)}$ and $\mathbf{A}^{(t)}$ differ only in coordinate t, with $A_t \subseteq A_t'$. By antitonicity in coordinate t,

$$F\big(\mathbf{A}^{(t-1)}\big)_j \;\supseteq\; F\big(\mathbf{A}^{(t)}\big)_j \qquad \text{for all } j.$$

Chaining these inclusions for $t = 1, \dots, h$ yields $F(\mathbf{A})_j = F(\mathbf{A}^{(0)})_j \supseteq F(\mathbf{A}^{(h)})_j = F(\mathbf{A}')_j$ for all j, as required.

4.2. (h,k)-ary (m,n)-SuperhyperRough Set

An (h,k)-ary (m,n)-SuperhyperRough Set extends classical rough sets by allowing *multiblock, higher–order* parameterization and *nested* outputs. Concretely, we collect k input *blocks*, each block being an h-tuple of m-level (nonempty) parameter subsets, and we output an n-level (nonempty) subset of the universe. Lower and upper approximations are then taken on the *ground* subset of the universe obtained by flattening the nested output.

Notation (nonempty nested powerset and flattening). For a nonempty set Y and $r \in \mathbb{N}_{\ge 1}$, define recursively

$$\widetilde{\mathcal{P}}_1(Y) := \{\, A \subseteq Y \mid A \neq \varnothing \,\}, \qquad \widetilde{\mathcal{P}}_{r+1}(Y) := \{\, \mathcal{A} \subseteq \widetilde{\mathcal{P}}_r(Y) \mid \mathcal{A} \neq \varnothing \,\}.$$

The *level–r flattening* $\mathrm{Flat}_r : \widetilde{\mathcal{P}}_r(Y) \to \mathcal{P}(Y)$ is defined by

$$\mathrm{Flat}_1(W) := W, \qquad \mathrm{Flat}_{r+1}(\mathcal{W}) := \bigcup_{W \in \mathcal{W}} \mathrm{Flat}_r(W).$$

Thus $\mathrm{Flat}_r(W) \subseteq Y$ for every $W \in \widetilde{\mathcal{P}}_r(Y)$, and $\mathrm{Flat}_r(W_1 \cup W_2) = \mathrm{Flat}_r(W_1) \cup \mathrm{Flat}_r(W_2)$. Moreover, $\mathrm{Flat}_r(W_1 \cap W_2) \subseteq \mathrm{Flat}_r(W_1) \cap \mathrm{Flat}_r(W_2)$.

Definition 26 ((h,k)-ary (m,n)-SuperhyperRough Set)**.** *Let X be a nonempty finite universe and let $R \subseteq X \times X$ be an equivalence relation; write $[x]_R$ for the R-class of x. Let J be a nonempty parameter space (e.g. a Cartesian product of attribute sets). Fix integers $m, n, h, k \ge 1$.*

Input space. *Put*

$$D := \big(\widetilde{\mathcal{P}}_m(J)\big)^h \quad \textit{and} \quad D^k := \underbrace{D \times \cdots \times D}_{k \text{ factors}},$$

so an input to our map is a k-tuple $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$, each

$$\gamma_\ell = (A_{\ell,1}, \dots, A_{\ell,h}) \in D$$

with

$$A_{\ell,i} \in \widetilde{\mathcal{P}}_m(J)$$

.

Output space. *Put* $C := \widetilde{\mathcal{P}}_n(X)$. *Its elements are nonempty level–n families of subsets of* X.

An (h, k)-ary (m, n)-SuperhyperRough Set *is a function*

$$F : D^k \longrightarrow C, \qquad \gamma \longmapsto W := F(\gamma).$$

The rough lower *and* upper *approximations of* W *(relative to* R*) are defined on the* ground *subset* $\mathrm{Flat}_n(W) \subseteq X$ *by*

$$\underline{W} := \{\, x \in X \mid [x]_R \subseteq \mathrm{Flat}_n(W) \,\},$$

$$\overline{W} := \{\, x \in X \mid [x]_R \cap \mathrm{Flat}_n(W) \neq \varnothing \,\}.$$

When $n = 1$, $\mathrm{Flat}_1(W) = W$ *and these are the classical rough approximations of* $W \subseteq X$.

Example 33 (2-ary $(1, 1)$ SuperhyperRough Set)**.** *Let*

$$X = \{u_1, u_2, u_3, u_4\}$$

with R-classes

$$[u_1]_R = [u_2]_R = \{u_1, u_2\}$$

and

$$[u_3]_R = [u_4]_R = \{u_3, u_4\}$$

. Choose $m = n = h = 1$ *and* $k = 2$*. Take any nonempty parameter set* J *(its internal structure is irrelevant here), so* $D = \widetilde{\mathcal{P}}_1(J) \cong J$*, hence* $D^2 \cong J \times J$*, and* $C = \widetilde{\mathcal{P}}_1(X) = \mathcal{P}_+(X)$*. Define* $F : J \times J \to \mathcal{P}_+(X)$ *by*

$$F(a, b) := \{u_1, u_2, u_3\} \quad (\textit{for all } (a, b)).$$

Then $W = F(a, b)$ *has approximations*

$$\underline{W} = \{u_1, u_2\}, \qquad \overline{W} = \{u_1, u_2, u_3, u_4\}.$$

Thus F *is a concrete* 2*-ary* $(1, 1)$*-SuperhyperRough Set.*

Example 34 (Medical diagnosis as a 2-ary $(2, 2)$-SuperhyperRough Set)**.** *Let*

$$X = \{\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4\}$$

and let R *group patients by age:* $[\mathrm{P}_1]_R = [\mathrm{P}_2]_R$*,* $[\mathrm{P}_3]_R = [\mathrm{P}_4]_R$*. Let*

$$J = \{\textit{Fever}, \textit{Cough}\} \times \{\textit{Pain}, \textit{Nausea}\}$$

. Take $m = 2$, $n = 1$, $h = 2$, $k = 2$. Then

$$D = \big(\widetilde{\mathcal{P}}_2(J)\big)^2, \qquad C = \widetilde{\mathcal{P}}_1(X) = \mathcal{P}_+(X).$$

Choose

$$\gamma_1 = \big\{\{(Fever, Pain)\}, \{(Cough, Nausea)\}\big\}, \quad \gamma_2 = \big\{\{(Fever, Nausea)\}\big\}.$$

Define $F(\gamma_1, \gamma_2) := \{\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3\}$. *Then (with* $n = 1$*) the approximations are*

$$\underline{W} = \{\mathrm{P}_1, \mathrm{P}_2\}, \qquad \overline{W} = \{\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4\}.$$

Hence F models a two–block, hierarchical symptom selection yielding a rough decision set.

Example 35 (Household Contact Tracing as a $(2,2)$-ary $(2,1)$-SuperhyperRough Set)**.** ***Universe and indiscernibility.*** *Let the finite universe X be six residents labeled*

$$X = \{A, B, C, D, E, F\}.$$

Let $R \subseteq X \times X$ encode the household *equivalence:*

$$[A]_R = [B]_R = \{A, B\}, \quad [C]_R = [D]_R = \{C, D\}, \quad [E]_R = [F]_R = \{E, F\}.$$

Parameter space and hyperparameters. *Consider the attribute space*

$$J = S_{\text{symp}} \times S_{\text{expo}}, \qquad S_{\text{symp}} = \{Fever, Cough, Asympt.\}, \quad S_{\text{expo}} = \{Close, Casual, Travel\}.$$

Fix $(m, n, h, k) = (2, 1, 2, 2)$. Then

$$\widetilde{\mathcal{P}}_1(J) = \{\, U \subseteq J \mid U \neq \varnothing \,\}, \qquad \widetilde{\mathcal{P}}_2(J) = \{\, \Gamma \subseteq \widetilde{\mathcal{P}}_1(J) \mid \Gamma \neq \varnothing \,\}.$$

The input *space is $D = (\widetilde{\mathcal{P}}_2(J))^2$ (two coordinates), and an input to F is a pair $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in D^2$ with $\gamma_\ell = (A_{\ell,1}, A_{\ell,2})$, $A_{\ell,i} \in \widetilde{\mathcal{P}}_2(J)$. The* output *space is $C = \widetilde{\mathcal{P}}_1(X) = \{\, W \subseteq X \mid W \neq \varnothing \,\}$.*

Interpret the two coordinates in each γ_ℓ as:

- *coordinate 1: a* cluster of symptom patterns*,*
- *coordinate 2: a* cluster of exposure contexts*.*

Choose the concrete hyperparameters

$$A_{1,1} = \big\{\{(Fever, Close)\},\ \{(Cough, Close)\}\big\}, \quad A_{1,2} = \big\{\{(Asympt., Close)\}\big\},$$
$$A_{2,1} = \big\{\{(Fever, Travel)\}\big\}, \qquad A_{2,2} = \big\{\{(Cough, Casual)\},\ \{(Asympt., Travel)\}\big\}.$$

Thus $\gamma_1 = (A_{1,1}, A_{1,2})$ captures high-risk local symptoms/close contacts*, while $\gamma_2 = (A_{2,1}, A_{2,2})$ captures* travel–related or casual exposures*.*

SuperhyperRough mapping. *Define*

$$F : \; D^2 \longrightarrow C = \widetilde{\mathcal{P}}_1(X), \qquad (\gamma_1, \gamma_2) \longmapsto W,$$

and, for the above input, set (one valid epidemiological decision rule)

$$W \; = \; \{A, B, D\}.$$

Intuitively, $\{A, B\}$ are flagged due to close symptomatic contact (household cluster), and D is flagged due to a travel–symptom combination matching $A_{2,1}$.

Rough approximations (with $n = 1$). *Since $n = 1$,* $\mathrm{Flat}_1(W) = W$*. The* lower *and* upper *rough approximations of W (relative to R) are*

$$\underline{W} \; = \; \{\, x \in X \mid [x]_R \subseteq W \,\}, \qquad \overline{W} \; = \; \{\, x \in X \mid [x]_R \cap W \neq \varnothing \,\}.$$

Compute using the three households:

$$\begin{aligned}
&[A]_R = [B]_R = \{A, B\} \subseteq W \implies \{A, B\} \subseteq \underline{W},\\
&[C]_R = [D]_R = \{C, D\} \not\subseteq W \text{ but } \{C, D\} \cap W = \{D\} \neq \varnothing,\\
&[E]_R = [F]_R = \{E, F\} \not\subseteq W \text{ and } \{E, F\} \cap W = \varnothing.
\end{aligned}$$

Hence

$$\underline{W} = \{A, B\}, \qquad \overline{W} = \{A, B, C, D\}.$$

Interpretation. *The lower approximation $\underline{W}$ contains residents whose* entire *household is flagged ($\{A, B\}$); they are quarantined with certainty under the household–indiscernibility R. The upper approximation $\overline{W}$ additionally includes residents whose household has* some *flagged member (C is included because D is in W). This example realizes a real–world $(2, 2)$-ary $(2, 1)$-SuperhyperRough Set: multi–level, clustered parameters (m=2, h=2), two input tuples (k=2), and rough reasoning on the ground set X with n=1.*

Theorem 55 (Classical rough sets as a special case)**.** *If $m = h = k = 1$ and $n = 1$, then any classical rough set mapping $F_{\mathrm{cl}} : J \to \mathcal{P}(X)$ is exactly a $(1, 1)$-ary $(1, 1)$-SuperhyperRough Set. More generally, for arbitrary $n \geq 1$, the composition* $\mathrm{Flat}_n \circ F$ *recovers the classical setting on X.*

Proof. When $m = h = k = n = 1$ we have $D^1 = \widetilde{\mathcal{P}}_1(J) \cong J$ and $C = \widetilde{\mathcal{P}}_1(X) = \mathcal{P}_+(X)$. Thus every $F_{\mathrm{cl}} : J \to \mathcal{P}_+(X)$ is a $(1, 1)$-ary $(1, 1)$-SuperhyperRough Set. If $n > 1$, then $F : D^1 \to \widetilde{\mathcal{P}}_n(X)$ and $\mathrm{Flat}_n \circ F : D^1 \to \mathcal{P}(X)$ produces ground subsets on which the classical lower/upper approximations are taken. Hence the generalized model subsumes the classical one.

Theorem 56 (Reduction of input arity)**.** *Fix indices $1 \leq i_1 < \cdots < i_r \leq k$ and blocks $\gamma^*_{i_j} \in D$. Define*

$$\iota : \; D^{k-r} \longrightarrow D^{k}, \qquad \iota(\delta_1, \ldots, \delta_{k-r}) = \boldsymbol{\gamma}$$

by inserting the fixed blocks at slots $i_1, \ldots, i_r$ and the free blocks elsewhere, and put $F_{\mathrm{fix}} := F \circ \iota$. Then $F_{\mathrm{fix}} : D^{k-r} \to \widetilde{\mathcal{P}}_n(X)$ is an $(h, k - r)$-ary (m, n)-SuperhyperRough Set.

Proof. The insertion map ι is well defined; composing with F yields a map into C. Nonemptiness and the definition of $\underline{\cdot}, \overline{\cdot}$ depend only on the image $W = F(\boldsymbol{\gamma})$, which is unaffected by viewing the fixed coordinates as parameters of the new map. Thus the structure and approximations remain valid with arity $k - r$.

Theorem 57 (Union). *If $F_1, F_2 : D^k \to \widetilde{\mathcal{P}}_n(X)$ are (h,k)-ary (m,n)-SuperhyperRough Sets, then*

$$(F_1 \cup F_2)(\boldsymbol{\gamma}) := F_1(\boldsymbol{\gamma}) \ \cup \ F_2(\boldsymbol{\gamma})$$

is again an (h,k)-ary (m,n)-SuperhyperRough Set. Moreover,

$$\mathrm{Flat}_n\big((F_1 \cup F_2)(\boldsymbol{\gamma})\big) = \mathrm{Flat}_n\big(F_1(\boldsymbol{\gamma})\big) \ \cup \ \mathrm{Flat}_n\big(F_2(\boldsymbol{\gamma})\big),$$

hence $\underline{F_1 \cup F_2} = \underline{F_1} \cup \underline{F_2}$ and $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$.

Proof. By induction on n, the union of two nonempty level–n families is nonempty level–n. Thus $(F_1 \cup F_2)(\boldsymbol{\gamma}) \in \widetilde{\mathcal{P}}_n(X)$. The displayed equality follows from the definition of Flat_n and distributivity of ordinary unions. The identities for lower/upper approximations are the classical rough equalities applied to the ground sets $\mathrm{Flat}_n(\cdot)$.

Theorem 58 (Intersection). *With the same hypotheses, define*

$$(F_1 \cap F_2)(\boldsymbol{\gamma}) := F_1(\boldsymbol{\gamma}) \ \cap \ F_2(\boldsymbol{\gamma}).$$

If this intersection is nonempty for all $\boldsymbol{\gamma}$, then $F_1 \cap F_2$ is an (h,k)-ary (m,n)-SuperhyperRough Set. Furthermore,

$$\mathrm{Flat}_n\big((F_1 \cap F_2)(\boldsymbol{\gamma})\big) \ \subseteq \ \mathrm{Flat}_n\big(F_1(\boldsymbol{\gamma})\big) \ \cap \ \mathrm{Flat}_n\big(F_2(\boldsymbol{\gamma})\big),$$

whence $\underline{F_1 \cap F_2} \subseteq \underline{F_1} \cap \underline{F_2}$ and $\overline{F_1 \cap F_2} \subseteq \overline{F_1} \cap \overline{F_2}$. Equality holds in all three inclusions when $n = 1$.

Proof. As for union, the intersection of two level–n families that meets nontrivially is again a nonempty level–n family, by induction on n. The inclusion for Flat_n is immediate from the definition (the base elements surviving the intersection are a subfamily of those present in each factor). The inclusions for rough approximations follow from the monotonicity of $\underline{\cdot}, \overline{\cdot}$ with respect to set inclusion on ground subsets. When $n = 1$, Flat_1 is the identity and all inclusions become equalities.

Theorem 59 (Monotonicity of approximations). *If $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in D^k$ satisfy $F(\boldsymbol{\gamma}') \subseteq F(\boldsymbol{\gamma})$ (in $\widetilde{\mathcal{P}}_n(X)$), then*

$$\underline{F(\boldsymbol{\gamma}')} \ \subseteq \ \underline{F(\boldsymbol{\gamma})}, \qquad \overline{F(\boldsymbol{\gamma}')} \ \subseteq \ \overline{F(\boldsymbol{\gamma})}.$$

Proof. From $F(\boldsymbol{\gamma}') \subseteq F(\boldsymbol{\gamma})$ we get $\mathrm{Flat}_n(F(\boldsymbol{\gamma}')) \subseteq \mathrm{Flat}_n(F(\boldsymbol{\gamma}))$. If $[x]_R \subseteq \mathrm{Flat}_n(F(\boldsymbol{\gamma}'))$, then $[x]_R \subseteq \mathrm{Flat}_n(F(\boldsymbol{\gamma}))$, hence $x \in \underline{F(\boldsymbol{\gamma})}$. Similarly, if $[x]_R \cap \mathrm{Flat}_n(F(\boldsymbol{\gamma}')) \neq \varnothing$, then $[x]_R \cap \mathrm{Flat}_n(F(\boldsymbol{\gamma})) \neq \varnothing$, whence the upper inclusion.

Theorem 60 (Functoriality under quotients)**.** *Let $f : X \to Y$ be a surjection that is constant on R-classes (i.e., $x\,R\,x' \Rightarrow f(x) = f(x')$). Let S be the induced equivalence on Y whose classes are the images $f([x]_R)$ ($x \in X$). Define the level–n direct image*

$$f_*^{(1)} : \widetilde{\mathcal{P}}_1(X) \to \widetilde{\mathcal{P}}_1(Y), \quad B \mapsto f[B],$$

$$f_*^{(r+1)}(\mathcal{B}) := \{\, f_*^{(r)}(B) \mid B \in \mathcal{B} \,\}.$$

Then

$$(f_*F)(\gamma) \;:=\; f_*^{(n)}\big(F(\gamma)\big) \;\in\; \widetilde{\mathcal{P}}_n(Y)$$

defines an (h, k)-ary (m, n)-SuperhyperRough Set on Y. Moreover,

$$\mathrm{Flat}_n\big((f_*F)(\gamma)\big) = f(\mathrm{Flat}_n(F(\gamma)))\,,$$

and for rough approximations with respect to S one has

$$f\big(\,\underline{F(\gamma)}\,\big) \;\subseteq\; \underline{(f_*F)(\gamma)},$$

$$f\big(\,\overline{F(\gamma)}\,\big) \;\subseteq\; \overline{(f_*F)(\gamma)},$$

with equality whenever f is the quotient map $X \to X/R$ followed by a bijection onto Y.

Proof. The construction of $f_*^{(n)}$ preserves nonemptiness by induction on n (surjectivity gives $f[B] \neq \varnothing$ when $B \neq \varnothing$; passing from n to $n+1$ takes nonempty families to nonempty families). The identity $\mathrm{Flat}_n(f_*^{(n)}(W)) = f(\mathrm{Flat}_n(W))$ follows directly from the recursive definitions.

Let $W_X := \mathrm{Flat}_n(F(\gamma)) \subseteq X$ and $W_Y := f(W_X) = \mathrm{Flat}_n((f_*F)(\gamma))$. If $x \in \underline{F(\gamma)}$ then $[x]_R \subseteq W_X$. By construction of S, $f([x]_R) = [f(x)]_S \subseteq f(W_X) = W_Y$, so $f(x) \in \underline{(f_*F)(\gamma)}$, proving the first inclusion. If $x \in \overline{F(\gamma)}$ then $[x]_R \cap W_X \neq \varnothing$, hence $[f(x)]_S = f([x]_R)$ meets $W_Y = f(W_X)$, and $f(x) \in \overline{(f_*F)(\gamma)}$. If, moreover, f is the canonical quotient collapsing R-classes (up to bijection), these inclusions are equalities by standard properties of rough sets on quotient spaces.

5. Conclusion

In this paper, we introduced two new and more general frameworks:
the (m, n)–SuperHyperUncertain Set and the (h, k)–ary (m, n)–SuperHyperUncertain Set. In particular, we formally defined the concepts summarized in Table 6. These frameworks are expected to provide a refined means of representing real–world notions involving hierarchical uncertainty. Modern datasets often exhibit hierarchical, multi–source uncertainty with interacting attributes. The (m, n) and (h, k)–ary SuperHyperUncertain Sets unify classical models, support multi–ary composition, and enable provably consistent and scalable reasoning across multiple levels and modalities.

Future work on the (m, n)-SuperHyperUncertain Set and
the (h, k)-ary (m, n)-SuperHyperUncertain Set includes:

Table 6: Concise overview of (h, k)-ary (m, n)-SuperHyperSet families

Family	**Essence**
(h, k)-ary (m, n)-SuperHyperFuzzy Set	Maps hyper-parameters to fuzzy degrees $[0, 1]$.
(h, k)-ary (m, n)-SuperHyperNeutrosophic Set	Encodes triples (T, I, F) for truth, indeterminacy, falsity.
(h, k)-ary (m, n)-SuperHyperPlithogenic Set	Adds contradiction function pCF to capture attribute conflicts.
(h, k)-ary (m, n)-SuperHyperSoft Set	Extends soft sets with higher-order parameters and multi-outputs.
(h, k)-ary (m, n)-SuperHyperRough Set	Provides hierarchical lower and upper approximations.

- Designing efficient algorithms for constructing and manipulating these SuperHype-rUncertain Sets;
- Investigating in detail the mathematical properties and structural features of the proposed frameworks;
- Exploring potential applications for modeling hierarchical uncertainty in practical systems.

Furthermore, it is our hope that the concepts presented in this work will stimulate further research into their applications in areas such as Graphs, HyperFunctions [70, 71], HyperGraphs [30, 72, 73], and SuperHyperGraphs [32, 74].

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Data Availability

This paper is purely theoretical and does not involve any empirical data. We welcome future empirical studies that build upon and test the concepts presented here.

Ethical Approval

As this work is entirely conceptual and involves no human or animal subjects, ethical approval was not required.

Conflicts of Interest

The authors declare no conflicts of interest in connection with this study or its publication.

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Research Integrity

The authors affirm that, to the best of their knowledge, this manuscript represents their original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

Disclaimer on Computational Tools

No computer-based tools—such as symbolic computation systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq)—were employed in the development, analysis, or verification of the results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Disclaimer on Scope and Accuracy

The theoretical models and concepts proposed in this manuscript have not yet undergone empirical testing or practical deployment. Future work may investigate their utility in applied or experimental contexts. While the authors have taken care to maintain accuracy and provide appropriate citations, inadvertent errors or omissions may remain. Readers are encouraged to consult original references for confirmation and further study.

The authors assert that all mathematical results and justifications included in this work have been carefully reviewed and are believed to be correct. Should any inaccuracies or ambiguities be discovered, the authors welcome constructive feedback and will provide clarification upon request.

The conclusions presented are valid only within the specific theoretical framework and assumptions described in the text. Generalizing these results to other mathematical contexts may require further investigation. All opinions and interpretations expressed herein are solely those of the authors and do not necessarily reflect the views of their respective institutions.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Consent to Publish

All authors have given their consent for submission of this manuscript to the journal.

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