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Random n-SuperHyperGraphs: A Probabilistic Model and Generation Algorithm for Hierarchical Networks

Takaaki Fujita[1,*], Florentin Smarandache[2]

[1] *Independent Researcher, Tokyo, Japan*
[2] *University of New Mexico, Gallup Campus, NM 87301, USA*

Abstract. Hypergraphs generalize classical graphs by allowing hyperedges to join any nonempty subset of vertices [1]. Superhypergraphs extend this idea by iterating the powerset operation, producing nested layers that capture hierarchical and self-referential relationships among vertex collections [2]. While random graph and hypergraph models form edges independently with probability p, they do not account for higher-order, multi-scale, or nested dependencies often observed in real-world networks. We introduce the *random superhypergraph* $\mathrm{SuHG}^{(n)}(V_0, p)$, defined on the n-fold powerset of a base set V_0. We give a concise mathematical formulation, derive key properties (including expectation, variance, concentration, and substructure thresholds), and present efficient algorithms for generation. This framework provides a unified, probabilistic model for complex systems with layered, uncertain connectivity.

2020 Mathematics Subject Classifications: 05C65

Key Words and Phrases: Hypergraph, superhypergraph, random graph, random hypergraph, random superhypergraph

1. Introduction

1.1. Graph, Hypergraph, and Superhypergraph Theory

Graph theory formalizes the study of structures consisting of vertices connected by edges, serving as a fundamental tool for modeling pairwise relationships or interactions [3–5]. Its applications span data mining, algorithm design, artificial intelligence, neural networks, quantum information, and chemistry (e.g., [6–10]). However, classical graphs often have limitations in representing hierarchical network concepts and multiway interactions. To address these challenges, the concepts of *Hypergraph* and *Superhypergraph* have been actively studied in recent years.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6835
Email addresses: `Takaaki.fujita060@gmail.com` (T. Fujita), `smarand@unm.edu` (F. Smarandache)

A *hypergraph* extends this framework by allowing edges (hyperedges) to join any nonempty subset of vertices, thereby capturing higher-order associations [11–15]. Hypergraphs are known as a natural generalization of graphs and, like graphs, have been studied in a wide range of applications [16–21]. Building on this, a *Superhypergraph* applies recursive powerset operations to form nested layers of vertices and edges, enabling the representation of hierarchical or multi-level connectivity [2, 22–25]. Superhypergraphs generalize hypergraphs. Such structures have proven valuable for modeling complex, layered systems in numerous domains [26, 27].

Table 1 highlights the principal differences between graphs, hypergraphs, and superhypergraphs. Here, n is assumed to be a finite positive integer. Moreover, numerous graph algorithms for Graphs, Hypergraphs, and Superhypergraphs have also been extensively studied (e.g., [28–31]).

Concept	**Notation**	**Edge Type**	**Extension Mechanism**
Graph	$G = (V, E)$	$E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$	Connects exactly two vertices.
Hypergraph	$H = (V, E)$	$E \subseteq \mathrm{PS}(V) \setminus \{\varnothing\}$	Joins any nonempty subset of vertices.
Superhypergraph	$\mathrm{SuHG}^{(n)} = (V_0, V, E)$	$V \subseteq \mathrm{PS}^n(V), E \subseteq \mathrm{PS}^n(V)$	Applies an n-fold powerset for nested structure.

Table 1: Comparison of Graph, Hypergraph, and Superhypergraph

($\mathrm{PS}(V)$ denotes the powerset of V, i.e. the set of all subsets of V. $\mathrm{PS}^n(V)$ is the n-fold iterated powerset, defined by $\mathrm{PS}^1(V) = \mathrm{PS}(V)$ and $\mathrm{PS}^{k+1}(V) = \mathrm{PS}(\mathrm{PS}^k(V))$ for $k \geq 1$.)

1.2. Random Graph Theory

Graphs are also extensively used from a probabilistic perspective (cf.[32–34]). A *random graph* is a graph in which each possible edge between n nodes appears independently with a fixed probability[35–38]. A *random hypergraph* generalizes this by including each possible k-element subset of nodes as a hyperedge independently with probability p[39–43]. Studying random graph theory provides insights into the emergence of global network properties from local randomness, informs phase transitions, percolation phenomena, and has applications in network science, epidemiology, and algorithmic analysis, bridging probability theory and combinatorics to model complex systems(cf.[44]). Algorithmic research on random graphs has also been conducted[45].

But random graph models assume independent edge formation, ignore higher-order correlations, lack hierarchical structure, and cannot capture multi-scale interactions or nested dependencies present in real-world networks.

Motivation and Our Contributions From the above, research on random graphs, SuperHyperGraphs, and hypergraphs is of great importance. However, research on Super-

HyperGraphs is still in its early stages, and investigations into random SuperHyperGraphs have not yet been advanced.

In this paper, we investigate the notion of a *random SuperHyperGraph* and provide a concise mathematical exploration of its structure. Additionally, we present a simple algorithm for generating random SuperHyperGraphs. Table 2 highlights the principal differences between Random Graph, Random k-Hypergraph, and Random n-Superhypergraph. Unifying graphs and hypergraphs, Random n-Superhypergraphs enable modeling hierarchical, multi-scale structures, capture nested high-order interactions in complex systems, and support probabilistic analysis of layered networks.

Model	**Notation**	**Edge Domain**	**Inclusion Rule**
Random Graph	$G(n,p)$	All 2-element subsets of $V = \{1,\ldots,n\}$	Each $\{u,v\}$ appears independently with probability p.
Random k-Hypergraph	$H_k(n,p)$	All k-element subsets of $V = \{1,\ldots,n\}$	Each $e \subset V$ with $\lvert e\rvert = k$ appears independently with probability p.
Random n-SuperHyperGraph	$\mathrm{SuHG}^{(n)}(V_0,p)$	All nonempty subsets of $V^{(n)} = \mathrm{PS}^n(V_0)$, i.e. $\mathrm{PS}(V^{(n)}) \setminus \{\varnothing\}$	Each potential hyperedge $e \subseteq V^{(n)}$, $e \neq \varnothing$, is included independently with probability p.

Table 2: Comparison of Random Graph, Random k-Hypergraph, and Random n-SuperHyperGraph

Structure of This Paper This paper is structured as follows. In Section 2, we introduce the definitions and fundamental concepts of SuperHyperGraphs and random graphs. Section 3 presents concrete examples and investigates the mathematical properties of random SuperHyperGraphs. In Section 4, we examine algorithms for generating and analyzing random SuperHyperGraphs. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Preliminaries

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. Unless otherwise specified, all graphs considered in this paper are assumed to be finite.

2.1. SuperHyperGraphs

A *SuperHyperGraph* extends the notion of a hypergraph by applying the powerset operator repeatedly, thereby modeling hierarchical, nested, and self-referential connections among vertices and edges [2, 46, 47]. Below we give precise definitions.

Definition 1 (Base Set)**.** *Let S be a nonempty set of* elements*. We call S the* base set*. All subsequent constructions originate from S.*

Definition 2 (Powerset)**.** *[48, 49] For any set S, its* powerset *is*

$$\mathrm{PS}(S) \;=\; \{\, A \mid A \subseteq S \,\},$$

the collection of all subsets of S, including $\varnothing$ and S itself.

The n-th iterated powerset is obtained by applying the powerset operation to a set repeatedly n times, producing nested subsets[50–52].

Definition 3 (n-th Iterated Powerset)**.** *(cf.[53–55]) Let $n \in \mathbb{N}$. Define recursively*

$$\mathrm{PS}^1(S) = \mathrm{PS}(S),$$

$$\mathrm{PS}^{k+1}(S) = \mathrm{PS}\big(\mathrm{PS}^k(S)\big) \quad (k \geq 1).$$

Then $\mathrm{PS}^n(S)$ is the n-th iterated powerset *of S. If one wishes to exclude the empty set at each stage, write $\mathrm{PS}^*_1(S) = \mathrm{PS}(S) \setminus \{\varnothing\}$ and*

$$\mathrm{PS}^*_{k+1}(S) = \mathrm{PS}\big(\mathrm{PS}^*_k(S)\big) \setminus \{\varnothing\}.$$

Example 1 (Hierarchical Meal Planning)**.** *Let*

$$S = \{Eggs,\ Bread,\ Butter,\ Jam\}$$

be basic breakfast ingredients. Then:

- $\mathrm{PS}^1(S)$ *lists all single-meal combinations (e.g. $\{Eggs, Bread\}$).*
- $\mathrm{PS}^2(S) = \mathrm{PS}(\mathrm{PS}^1(S))$ *collects weekly menus, each a set of daily combinations.*
- *More generally, $\mathrm{PS}^n(S)$ for $n \geq 3$ represents n-level meal schedules (e.g. monthly plans).*

This illustrates how iterated powersets encode multi-scale planning: single meals $\to$ weekly menus $\to$ monthly schedules, and so on.

Definition 4 (Hypergraph)**.** *[1, 56] A* hypergraph *$H = (V(H), E(H))$ consists of:*

- *A nonempty set $V(H)$ of vertices.*
- *A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.*

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Definition 5 (n-SuperHyperGraph). *(cf. [2]) Let V_0 be a finite nonempty* base set*. For $k \in \mathbb{N}_0$ define the iterated powerset*

$$\mathrm{PS}^0(V_0) := V_0, \qquad \mathrm{PS}^{k+1}(V_0) := \mathrm{PS}\big(\mathrm{PS}^k(V_0)\big),$$

and write $\mathrm{PS}^*(X) := \mathrm{PS}(X) \setminus \{\varnothing\}$ for the collection of all nonempty subsets of X.

*Fix $n \in \mathbb{N}$. An n-*SuperHyperGraph *over V_0 is a pair*

$$\mathrm{SuHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathrm{PS}^n(V_0) \qquad and \qquad E \subseteq \mathrm{PS}^*(V).$$

*Elements of V are called n-*supervertices*; elements $e \in E$ (each a nonempty subset of V) are called n-*superedges*.*

Example 2 (Small company as a 2-SuperHyperGraph). *Let the base set of employees be*

$$V_0 = \{\text{Hiroko}, \text{Yutaka}, \text{Tae}, \text{Shinya}\}.$$

Choose two project teams (elements of $\mathrm{PS}^1(V_0)$):

$$g_1 = \{\text{Hiroko}, \text{Tae}\}, \qquad g_2 = \{\text{Yutaka}, \text{Shinya}\}.$$

Consider the following 2-supervertex set (a subset of $\mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0))$):

$$V = \big\{ \{g_1\},\ \{g_2\},\ \{g_1, g_2\} \big\} \ \subseteq\ \mathrm{PS}^2(V_0).$$

Define superedges as nonempty subsets of V:

$$E = \big\{ \{\{g_1\}, \{g_1, g_2\}\},\ \{\{g_2\}, \{g_1, g_2\}\} \big\} \ \subseteq\ \mathrm{PS}^*(V).$$

Then $\mathrm{SuHG}^{(2)} = (V, E)$ is a valid 2-SuperHyperGraph. Informally, $\{g_1\}$ and $\{g_2\}$ are "single–team" supervertices, $\{g_1, g_2\}$ is a "department–level" supervertex, and each superedge joins a team to the department–level node.

Example 3 (Social network as a 3-SuperHyperGraph). *Let the base set of users be*

$$V_0 = \{\text{Hiroko}, \text{Yutaka}, \text{Emiko}, \text{Kenji}, \text{Takashi}\}.$$

Pick some friend groups $f_i \in \mathrm{PS}^1(V_0)$, e.g.

$$f_1 = \{\text{Hiroko}, \text{Yutaka}\}, \quad f_2 = \{\text{Yutaka}, \text{Emiko}, \text{Takashi}\}, \quad f_3 = \{\text{Emiko}, \text{Kenji}\}, \quad f_4 = \{\text{Hiroko}, \text{Takashi}\}.$$

Form communities $C_j \in \mathrm{PS}^2(V_0) = \mathrm{PS}(\mathrm{PS}(V_0))$, for instance

$$C_1 = \{f_1, f_2\}, \qquad C_2 = \{f_2, f_3, f_4\}, \qquad C_3 = \{f_1, f_4\}.$$

Select 3-supervertices in $\mathrm{PS}^3(V_0) = \mathrm{PS}(\mathrm{PS}^2(V_0))$:

$$V = \big\{\ \{C_1\},\ \{C_2\},\ \{C_1, C_2\},\ \{C_2, C_3\}\ \big\} \ \subseteq\ \mathrm{PS}^3(V_0).$$

Define superedges as nonempty subsets of V; for example,

$$E = \big\{ \{\{C_1, C_2\}, \{C_2, C_3\}\} \big\} \ \subseteq\ \mathrm{PS}^*(V).$$

Then $\mathrm{SuHG}^{(3)} = (V, E)$ is a valid 3-SuperHyperGraph: the chosen superedge links two meta–community supervertices.

2.2. Random graph and Random hypergraph

Random graphs and hypergraphs provide probabilistic frameworks for modeling networks and higher-order relationships. In the classical Erdős–Rényi model, each potential edge between n vertices is included independently with probability p [35, 38, 57, 58]. The hypergraph analogue extends this by treating every k-element subset of the vertex set as a hyperedge, each included independently with probability p [39–41, 43]. The relevant definitions are provided as follows.

Definition 6 (Erdős–Rényi Random Graph $G(n,p)$)**.** *(cf.[35, 38]) Let $n \in \mathbb{N}$ and $p \in [0,1]$. The* Erdős–Rényi random graph $G(n,p)$ *is the probability space whose sample space consists of all simple undirected graphs on the vertex set $V = \{1,2,\dots,n\}$. Each of the $\binom{n}{2}$ possible edges is included independently with probability p. Equivalently, for any fixed graph $G = (V,E)$,*

$$\Pr\bigl(G(n,p) = G\bigr) = p^{|E|}\,(1-p)^{\binom{n}{2}-|E|}.$$

Example 4 (Erdős–Rényi Random Graph in a Social Network)**.** *Suppose we have n users on a social platform, labeled $V = \{1,2,\dots,n\}$. We model friendships by the random graph $G(n,p)$ where each of the $\binom{n}{2}$ possible pairs $\{i,j\}$ becomes a friendship edge independently with probability p. Concretely, if*

$$E = \bigl\{\{i,j\} : X_{ij} = 1\bigr\}, \quad X_{ij} \sim \text{Bernoulli}(p),$$

then

$$\Pr\bigl(G(n,p) = (V,E)\bigr) = p^{|E|}\,(1-p)^{\binom{n}{2}-|E|}.$$

Here $\mathbb{E}[|E|] = p\binom{n}{2}$ gives the expected number of friendships.

Definition 7 (Uniform Random Graph $G(n,m)$)**.** *(cf.[35, 38]) Let $n,m \in \mathbb{N}$ with $0 \le m \le \binom{n}{2}$. The* uniform random graph $G(n,m)$ *is the probability space of all simple graphs on $V = \{1,2,\dots,n\}$ having exactly m edges, each graph being equally likely. Thus*

$$\Pr\bigl(G(n,m) = G\bigr) = \begin{cases} \dfrac{1}{\binom{\binom{n}{2}}{m}}, & |E(G)| = m, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 8 (k-Uniform Random Hypergraph $H_k(n,p)$)**.** *Let $n,k \in \mathbb{N}$ with $1 \le k \le n$ and $p \in [0,1]$. The k-uniform random hypergraph $H_k(n,p)$ is the probability space whose sample space consists of all k-uniform hypergraphs on $V = \{1,2,\dots,n\}$, where each of the $\binom{n}{k}$ possible k-element subsets is included independently with probability p. Equivalently, for any fixed hypergraph $H = (V,E)$ with $|e| = k$ for all $e \in E$,*

$$\Pr\bigl(H_k(n,p) = H\bigr) = p^{|E|}\,(1-p)^{\binom{n}{k}-|E|}.$$

Example 5 (k-Uniform Random Hypergraph for Research Teams)**.** *Let n researchers form collaborative teams of size k. Label them $V = \{1,2,\dots,n\}$. In the k-uniform random* *hypergraph $H_k(n,p)$, each possible team $e \in \binom{V}{k}$ is included as a hyperedge independently with probability p. Thus for any fixed hypergraph (V,E) with $|E| = m$,*

$$\Pr\big(H_k(n,p) = (V,E)\big) = p^m\,(1-p)^{\binom{n}{k}-m}.$$

This models the random formation of m research groups, each comprising exactly k collaborators.

Example 6 (k-Uniform Random Hypergraph in Academic CoAuthorship)**.** *Consider a set of n researchers in a department, labeled*

$$V = \{1,2,\dots,n\}.$$

We wish to model the random formation of collaborative research groups of size k. In this scenario, each possible subset $\{i_1,i_2,\dots,i_k\} \subseteq V$ of size k represents a potential co-authorship team. In the k-uniform random hypergraph model $H_k(n,p)$, each of the $\binom{n}{k}$ possible k-member subsets is included independently as a hyperedge with probability p.

Concretely, let

$$E \;=\; \big\{\, e \subseteq V : |e| = k \text{ and } X_e = 1\big\},$$

$$X_e \sim \mathrm{Bernoulli}(p),$$

where $X_e = 1$ indicates that the researchers in e form a coauthored paper. Then for any fixed hypergraph (V,E) with $|e| = k$ for all $e \in E$,

$$\Pr\big(H_k(n,p) = (V,E)\big) \;=$$

$$p^{|E|}\,\big(1-p\big)^{\binom{n}{k}-|E|}.$$

Thus, the random hypergraph $H_k(n,p)$ captures the probability that exactly $|E|$ distinct groups of k researchers collaborate on papers, where each group forms independently with probability p.

3. Main Results: Random n-SuperHyperGraph

In this section, we discuss, as one of the results of this paper, the definition and application examples of Random n-SuperHyperGraph.

3.1. Definition of Random n-SuperHyperGraph

A Random n-SuperHyperGraph is a probabilistic model selecting each element of n-th iterated powerset as hyperedges independently with probability p. The definition of the Random n-SuperHyperGraph is given as follows.

Definition 9 (Random n-SuperHyperGraph $\mathrm{SuHG}^{(n)}(V_0, p)$)**.** *Let V_0 be a finite base set with $|V_0| = N$, let $n \in \mathbb{N}$, and fix $p \in [0, 1]$. Define*

$$V^{(n)} \;=\; \mathrm{PS}^n(V_0), \quad M_n \;=\; \big|V^{(n)}\big|,$$

$$Q_n \;=\; 2^{M_n} - 1.$$

We view $\mathrm{SuHG}^{(n)} = (V^{(n)}, E)$ as a hypergraph whose vertex set is $V^{(n)}$ and whose edge set E is a collection of nonempty subsets of $V^{(n)}$:

$$E \;\subseteq\; \mathrm{PS}\big(V^{(n)}\big) \setminus \{\varnothing\}.$$

In the random *model $\mathrm{SuHG}^{(n)}(V_0, p)$, each of the Q_n possible nonempty subsets $e \subseteq V^{(n)}$ is included in E independently with probability p. Thus for any fixed $E \subseteq \mathrm{PS}(V^{(n)}) \setminus \{\varnothing\}$,*

$$\Pr\big(\mathrm{SuHG}^{(n)}(V_0, p) = (V^{(n)}, E)\big)$$

$$=\; p^{|E|}\,(1-p)^{\,Q_n - |E|}\,.$$

3.2. Real-World Examples of Random SuperHyperGraphs

We begin by exploring real-world scenarios where Random SuperHyperGraphs are applicable. Several concrete examples are presented below.

Example 7 (Random 2-SuperHyperGraph: Company Task-Force Formation)**.** *Let the base set of employees be*

$$V_0 = \{\text{Hiroko, Yutaka, Tae, Shinya}\}.$$

Then

$$\mathrm{PS}^1(V_0) = \{\{\mathrm{Hiroko}, \mathrm{Tae}\},\ \{\mathrm{Yutaka}, \mathrm{Shinya}\},\ \dots\}$$

lists all two-person teams. Denote

$$\mathrm{TeamXXX} = \{\mathrm{Hiroko}, \mathrm{Tae}\},$$

$$\mathrm{TeamYYY} = \{\mathrm{Yutaka}, \mathrm{Shinya}\}.$$

Next,

$$\begin{aligned} &\mathrm{PS}^2(V_0) \\ &= \mathrm{PS}\big(\mathrm{PS}^1(V_0)\big) \\ &= \{\{\mathrm{TeamXXX}\},\ \{\mathrm{TeamYYY}\}, \\ &\quad \{\mathrm{TeamXXX}, \mathrm{TeamYYY}\}\}. \end{aligned}$$

We take this as the supervertex set:

$$V^{(2)} = \mathrm{PS}^2(V_0).$$

In our corrected random model, each potential hyperedge is a nonempty subset of $V^{(2)}$*. Let*

$$\mathcal{E} = \mathrm{PS}\big(V^{(2)}\big) \setminus \{\varnothing\}, \quad Q_2 = 2^{|V^{(2)}|} - 1 = 7.$$

In $\mathrm{SuHG}^{(2)}(V_0, p)$*, each* $e \in \mathcal{E}$ *is included independently with probability* p*. Thus for any fixed* $E \subseteq \mathcal{E}$*,*

$$\Pr\big(\mathrm{SuHG}^{(2)}(V_0, p) = (V^{(2)}, E)\big) = p^{|E|}\,(1-p)^{\,Q_2 - |E|}.$$

For example, the edge $\{\{\mathrm{TeamXXX}\}, \{\mathrm{TeamXXX}, \mathrm{TeamYYY}\}\}$ *connects the team* TeamXXX *with the department-level supervertex* $\{\mathrm{TeamXXX}, \mathrm{TeamYYY}\}$*.*

Example 8 (Random 2-SuperHyperGraph: Social Media Community Formation)**.** *On a platform with users*

$$V_0 = \{\text{Hiroko, Yutaka, Tae, Shinya, Eve}\},$$

the level-1 powerset $\mathrm{PS}^1(V_0)$ *lists all friend groups (e.g.* $g_1 = \{\mathrm{Hiroko}, \mathrm{Yutaka}\}$*,* $g_2 = \{\mathrm{Yutaka}, \mathrm{Tae}, \mathrm{Eve}\}$*, ...). The level-2 powerset* $\mathrm{PS}^2(V_0)$ *collects all candidate communities:*

$$V^{(2)} = \mathrm{PS}^2(V_0)$$

$$= \{\,\{g_i\} : g_i \in \mathrm{PS}^1(V_0)\} \cup$$

$$\{\{g_i, g_j\} : g_i, g_j \in \mathrm{PS}^1(V_0),\, i \neq j\} \,\cup\, \dots.$$

Define the set of all nonempty hyperedges $\mathcal{E} = \mathrm{PS}(V^{(2)}) \setminus \{\varnothing\}$*. In the random model* $\mathrm{SuHG}^{(2)}(V_0, p)$*, each* $e \in \mathcal{E}$ *appears independently with probability* p*. Concretely, for any chosen* $E \subseteq \mathcal{E}$*,*

$$\Pr\big(\mathrm{SuHG}^{(2)}(V_0, p) = (V^{(2)}, E)\big)$$

$$= p^{|E|}\,(1-p)^{\,|\mathcal{E}| - |E|}.$$

An example hyperedge $\{\,\{g_1\}, \{g_1, g_2\}\}$ *connects a single friend group* g_1 *with a larger community* $\{g_1, g_2\}$*.*

Example 9 (Random 3-SuperHyperGraph: Manufacturing Product Lines)**.** *Let basic components be* $\{\,\mathrm{Bolt}, \mathrm{Nut}, \mathrm{Plate}, \mathrm{Gear}\}$*. Then* PS^1 *lists all modules (e.g.* $\{\mathrm{Bolt}, \mathrm{Nut}\}$*),* PS^2 *lists all subassemblies (e.g.* $\{Module_1, Module_2\}$*), and* PS^3 *lists all products (e.g.* $\{SubAsm\}$*). Setting*

$$V^{(3)} = \mathrm{PS}^3(V_0), \quad \mathcal{E} = \mathrm{PS}(V^{(3)}) \setminus \{\varnothing\},$$

the random model $\mathrm{SuHG}^{(3)}(V_0, p)$ *includes each* $e \in \mathcal{E}$ *independently with probability* p*:*

$$\Pr\big(\mathrm{SuHG}^{(3)}(V_0, p) = (V^{(3)}, E)\big) = p^{|E|}\,(1-p)^{\,|\mathcal{E}| - |E|}.$$

Here a hyperedge such as $\{\{SubAsm\}, \{SubAsm, OtherAsm\}\}$ *links different product variants.*

Example 10 (Social Networks: Overlapping Community Formation). *Let*

$$V_0 = \{\text{Alice, Bob, Carol, Dave, Eve}\}.$$

$\mathrm{PS}^1(V_0)$ *lists all friend circles (e.g.* $g_1 = \{\text{Alice}, \text{Bob}\}$*,* $g_2 = \{\text{Bob}, \text{Carol}, \text{Eve}\}$*, ...). Then*

$$\mathrm{PS}^2(V_0) = \mathrm{PS}\big(\mathrm{PS}^1(V_0)\big)$$

collects all candidate communities, for instance

$$\mathrm{Comm}_1 = \{g_1, g_2\}$$

,

$$\mathrm{Comm}_2 = \{g_2, g_3\}$$

, etc. Set

$$V^{(2)} = \mathrm{PS}^2(V_0),$$

$$\mathcal{E} = \mathrm{PS}(V^{(2)}) \setminus \{\varnothing\}, \quad Q_2 = 2^{|V^{(2)}|} - 1.$$

In the random model $\mathrm{SuHG}^{(2)}(V_0, p)$*, each* $e \in \mathcal{E}$ *is included independently with probability* p*. Thus, for any fixed hyperedge set* $E \subseteq \mathcal{E}$*,*

$$\Pr\big(\mathrm{SuHG}^{(2)}(V_0, p) = (V^{(2)}, E)\big) = p^{|E|}(1-p)^{\,Q_2 - |E|}.$$

Here each selected e *represents an active overlapping community of friend circles at a given time.*

Example 11 (Biology: Protein Complexes, Pathways, and Systems). *Let*

$$V_0 = \{\text{A, B, C, D}\}$$

be proteins. Then

$$\mathrm{PS}^1(V_0) = \{\{\mathrm{A}, \mathrm{B}\}, \{\mathrm{B}, \mathrm{C}\}, \{\mathrm{C}, \mathrm{D}\}, \dots\}$$

lists all protein complexes, e.g. $\mathrm{Cpx}_1 = \{\mathrm{A}, \mathrm{B}\}$*,* $\mathrm{Cpx}_2 = \{\mathrm{B}, \mathrm{C}\}$*. Next,*

$$\mathrm{PS}^2(V_0) = \mathrm{PS}\big(\mathrm{PS}^1(V_0)\big)$$

lists all pathways (collections of complexes), e.g. $\mathrm{Path}_1 = \{\mathrm{Cpx}_1, \mathrm{Cpx}_2\}$*,* $\mathrm{Path}_2 = \{\mathrm{Cpx}_2, \mathrm{Cpx}_3\}$*. Finally,*

$$\mathrm{PS}^3(V_0) = \mathrm{PS}\big(\mathrm{PS}^2(V_0)\big)$$

lists all systems (collections of pathways), for example $\mathrm{Sys}_\alpha = \{\mathrm{Path}_1, \mathrm{Path}_2\}$*.*

Set

$$V^{(3)} = \mathrm{PS}^3(V_0), \quad \mathcal{E} = \mathrm{PS}(V^{(3)}) \setminus \{\varnothing\}, \quad Q_3 = 2^{|V^{(3)}|} - 1.$$

In $\mathrm{SuHG}^{(3)}(V_0, p)$*, each potential system* $e \in \mathcal{E}$ *is active independently with probability* p*. Hence for any chosen edge-set* $E \subseteq \mathcal{E}$*,*

$$\Pr\big(\mathrm{SuHG}^{(3)}(V_0, p) = (V^{(3)}, E)\big) = p^{|E|}(1-p)^{\,Q_3 - |E|}.$$

Each selected e *models a functional biological system formed by randomly activated pathways of protein complexes.*

3.3. Properties and Theorems of Random SuperHyperGraphs

This subsection presents the fundamental properties and theoretical results related to random SuperHyperGraphs.

Theorem 1 (Generalization of $G(N,p)$ and $H_k(N,p)$)**.** *The random superhypergraph* $\mathrm{SuHG}^{(n)}(V_0,p)$ *simultaneously extends:*

(i) the Erdős–Rényi graph $G(N,p)$, and

(ii) the uniform k-hypergraph $H_k(N,p)$.

Proof. First note that when $n=1$,

$$V^{(1)} = \mathrm{PS}^1(V_0) = \mathrm{PS}(V_0), \quad Q_1 = 2^N - 1.$$

Thus $\mathrm{SuHG}^{(1)}(V_0,p)$ is a random hypergraph on the N-element vertex set V_0 whose potential edges are all nonempty subsets of V_0.

(1) Recovery of $G(N,p)$. If we restrict attention to those random edges of size exactly 2, i.e.

$$E_2 \;=\; \{\, e \in E : |e| = 2\},$$

then $\bigl(V_0, E_2\bigr)$ is exactly an instance of the Erdős–Rényi random graph $G(N,p)$, since each 2-element subset of V_0 appears independently with probability p.

(2) Recovery of $H_k(N,p)$. More generally, restricting to edges of size k,

$$E_k \;=\; \{\, e \in E : |e| = k\},$$

yields the k-uniform random hypergraph $H_k(N,p)$, because each k-subset of V_0 is included independently with probability p.

In both cases, the marginal distributions and independence assumptions coincide with the classical models, so $\mathrm{SuHG}^{(n)}(V_0,p)$ indeed generalizes $G(N,p)$ and $H_k(N,p)$.

Theorem 2 (Expectation of Number of Superedges)**.** *Let V_0 be a finite set, $n \in \mathbb{N}$, and write*

$$M_n = \bigl|\mathrm{PS}^n(V_0)\bigr|,$$
$$\mathcal{E} = \mathrm{PS}\bigl(\mathrm{PS}^n(V_0)\bigr) \setminus \{\varnothing\},$$
$$Q_n = |\mathcal{E}| = 2^{M_n} - 1.$$

In the random model $\mathrm{SuHG}^{(n)}(V_0,p)$*, the expected number of superedges is*

$$\mathbb{E}\bigl[|E|\bigr] \;=\; p\,Q_n.$$

Proof. For each potential hyperedge $e \in \mathcal{E}$, let

$$X_e = \mathbf{1}_{\{e \in E\}}$$

be its indicator. Since each e is included independently with probability p, we have $\mathbb{E}[X_e] = p$. Hence

$$|E| \;=\; \sum_{e \in \mathcal{E}} X_e, \quad \mathbb{E}\big[|E|\big] = \sum_{e \in \mathcal{E}} \mathbb{E}[X_e] = Q_n \, p.$$

Theorem 3 (Variance of Number of Superedges)**.** *Under the same notation,*

$$\operatorname{Var}\big[|E|\big] \;=\; p(1-p)\, Q_n.$$

Proof. Since the X_e are independent Bernoulli(p) variables,

$$\operatorname{Var}\big[|E|\big] = \sum_{e \in \mathcal{E}} \operatorname{Var}(X_e)$$

$$= \sum_{e \in \mathcal{E}} p(1-p) = Q_n \, p(1-p).$$

Theorem 4 (Chernoff Concentration)**.** *For any $\varepsilon \in (0,1)$,*

$$\Pr\Big(\big||E| - p\, Q_n\big| > \varepsilon\, p\, Q_n\Big)$$

$$< 2 \exp\Big(-\frac{\varepsilon^2\, p\, Q_n}{3}\Big).$$

Proof. This follows from the multiplicative Chernoff bound applied to the sum $\sum_{e \in \mathcal{E}} X_e$ of independent indicators, noting that $\mathbb{E}[|E|] = p\, Q_n$.

Theorem 5 (Substructure Appearance Threshold)**.** *Let $H = (V_H, E_H)$ be a fixed n-superhypergraph with $|V_H| = s$ and $|E_H| = t$. Write*

$$M_n = \big|\mathrm{PS}^n(V_0)\big|,$$

$$(M_n)_s = M_n\,(M_n - 1)\cdots(M_n - s + 1).$$

In $\mathrm{SuHG}^{(n)}(V_0, p)$*, let X_H be the number of* labeled *embeddings of H. Then*

$$\mathbb{E}[X_H] \;=\; (M_n)_s\, p^{\,t}.$$

In particular, if $p \ll M_n^{-\,s/t}$ then $\mathbb{E}[X_H] \to 0$ and hence with high probability no copy of H appears; while if $p \gg M_n^{-\,s/t}$ then $\mathbb{E}[X_H] \to \infty$ and with high probability at least one copy appears.

Proof. Each labeled embedding of H into $\mathrm{PS}^n(V_0)$ is a choice of an ordered s-tuple of distinct supervertices, of which there are $(M_n)_s$ possibilities, together with the requirement that each of the t specified hyperedges of H appears. Since each potential hyperedge is included independently with probability p, the probability that a given labeled embedding is present is p^t. Linearity of expectation then yields $\mathbb{E}[X_H] = (M_n)_s\, p^t$. The "0–infinity" dichotomy follows by comparing p^t to $(M_n)_s^{-1}$.

Theorem 6 (Generalization of $G(N,p)$ and $H_k(N,p)$)**.** *The random superhypergraph* $\mathrm{SuHG}^{(n)}(V_0,p)$ *recovers:*

- *The Erdős–Rényi graph* $G(N,p)$ *by restricting to hyperedges of size* 2 *when* $n = 1$.
- *The* k*-uniform hypergraph* $H_k(N,p)$ *by restricting to hyperedges of size* k *when* $n = 1$.

Proof. When $n = 1$, we have $V^{(1)} = \mathrm{PS}(V_0)$ and each nonempty subset of V_0 is a potential hyperedge. Restricting to those of size r yields exactly the model in which each r-subset of V_0 appears independently with probability p, i.e. $G(N,p)$ if $r = 2$ and $H_k(N,p)$ if $r = k$.

Theorem 7 (Central Limit Theorem for Number of Hyperedges)**.** *Let*

$$Q_n = 2^{M_n} - 1, \quad |E| \sim \mathrm{Binomial}(Q_n, p).$$

As $Q_n \to \infty$,

$$\frac{|E| - p\,Q_n}{\sqrt{p(1-p)\,Q_n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Since $|E|$ is the sum of Q_n independent Bernoulli(p) indicators, the classical central limit theorem applies directly.

Theorem 8 (Strong Law of Large Numbers)**.** *In the same notation,*

$$\frac{|E|}{Q_n} \xrightarrow{\text{a.s.}} p \quad (Q_n \to \infty).$$

Proof. By Kolmogorov's strong law for independent, identically distributed Bernoulli trials, the sample proportion converges almost surely to the common success probability p.

Theorem 9 (Distribution of Fixed-Size Hyperedges)**.** *Let* $V^{(n)}$ *be the set of* n*-supervertices with* $|V^{(n)}| = M_n$*, and let*

$$\mathcal{E} \;=\; \mathrm{PS}\big(V^{(n)}\big) \setminus \{\varnothing\}$$

be the set of all potential hyperedges. For any integer r *with* $1 \le r \le M_n$*, set*

$$M_{n,r} \;=\; \binom{M_n}{r}, \quad Y_r \;=\; \big|\{\, e \in E : |e| = r\}\big|.$$

Then

$$Y_r \;\sim\; \mathrm{Binomial}\big(M_{n,r},\, p\big),$$

with $\mathbb{E}[Y_r] = p\,M_{n,r}$ *and* $\mathrm{Var}(Y_r) = p(1-p)\,M_{n,r}$.

Proof. There are $\binom{M_n}{r} = M_{n,r}$ possible hyperedges of size r, each included independently with probability p. Hence the count Y_r follows a Binomial$(M_{n,r}, p)$ distribution, from which the stated expectation and variance follow by standard formulas.

Theorem 10 (Monotonicity in p). *For any* $0 \le p_1 < p_2 \le 1$*, one can couple* $\mathrm{SuHG}^{(n)}(V_0, p_1)$ *and* $\mathrm{SuHG}^{(n)}(V_0, p_2)$ *so that, almost surely,*

$$E(p_1) \ \subseteq \ E(p_2),$$

where $E(p_i)$ *is the edge-set under parameter* p_i*.*

Proof. Assign each potential hyperedge $e \in \mathcal{E}$ an independent $U_e \sim \mathrm{Uniform}(0,1)$. Then define

$$E(p_i) = \{\, e \in \mathcal{E} \mid U_e \le p_i\}, \quad i = 1, 2.$$

Since $p_1 < p_2$ implies $\{U_e \le p_1\} \subseteq \{U_e \le p_2\}$ for each e, we have $E(p_1) \subseteq E(p_2)$ almost surely. Marginally, each $E(p_i)$ has the law of $\mathrm{SuHG}^{(n)}(V_0, p_i)$.

Theorem 11 (Vertex Degree Distribution). *Let* $\mathrm{SuHG}^{(n)}(V_0, p) = (V^{(n)}, E)$ *and write* $M_n = |V^{(n)}|$*. For each* $v \in V^{(n)}$*, define its degree*

$$\deg(v) = \big|\{\, e \in E : v \in e\}\big|.$$

Then

$$\deg(v) \ \sim \ \mathrm{Binomial}\big(2^{\,M_n - 1},\, p\big),$$

so

$$\mathbb{E}[\deg(v)] = p\, 2^{\,M_n-1},$$
$$\mathrm{Var}[\deg(v)] = p(1-p)\, 2^{\,M_n-1},$$

and, as $M_n \to \infty$*, the normalized degree* $\big(\deg(v) - p2^{M_n-1}\big)/\sqrt{p(1-p)\,2^{M_n-1}}$ *converges in distribution to* $\mathcal{N}(0,1)$*.*

Proof. Each potential hyperedge $e \subseteq V^{(n)}$ includes v with probability $1/2$ independently of other vertices, so there are 2^{M_n-1} such hyperedges. Since each is present in E with probability p, $\deg(v)$ is Binomial$(2^{M_n-1}, p)$. The stated moments and CLT then follow by standard properties.

Definition 10 (2-Section Graph). *Given* $\mathrm{SuHG}^{(n)}(V_0, p) = (V^{(n)}, E)$*, its* 2-section G_2 *is the simple graph on* $V^{(n)}$ *where* $\{u, v\}$ *is an edge of* G_2 *if and only if there exists* $e \in E$ *with* $\{u, v\} \subseteq e$*.*

Example 12 (2-Section Graph of a 2-SuperHyperGraph). *Recall the* 2*-SuperHyperGraph from the company task-force example:*

$$V^{(2)} = \{\, v_1, v_2, v_3\},$$

where

$$v_1 = \{\text{TeamXXX}\}, \quad v_2 = \{\text{TeamYYY}\},$$
$$v_3 = \{\text{TeamXXX}, \text{TeamYYY}\},$$

and the hyperedge set is

$$E = \big\{\{v_1, v_3\},\ \{v_2, v_3\}\big\}.$$

Its 2-section graph $G_2 = (V^{(2)}, E_2)$ *has the same vertex set* $V^{(2)}$ *and an edge between any two vertices that appear together in some hyperedge. Hence*

$$E_2 = \big\{\{v_1, v_3\},\ \{v_2, v_3\}\big\}.$$

In other words, G_2 *is the simple path*

$$v_1 \ - \ v_3 \ - \ v_2,$$

with v_1 *and* v_2 *both adjacent to* v_3 *but not to each other.*

Theorem 12 (Giant-Component Phase Transition)**.** *Let* G_2 *be the 2-section of* $\mathrm{SuHG}^{(n)}(V_0, p)$*. Define*

$$p_2 \ = \ 1 - \big(1 - p\big)^{2^{M_n - 2}} \ \approx \ p\, 2^{M_n - 2}.$$

- *If* $p_2 = \frac{1-\varepsilon}{M_n}$ *for some fixed* $\varepsilon > 0$*, then with high probability* G_2 *has all connected components of size* $O(\log M_n)$*.*
- *If* $p_2 = \frac{1+\varepsilon}{M_n}$*, then with high probability* G_2 *contains a unique "giant" component of size* $\Theta(M_n)$*, while all other components are* $O(\log M_n)$*.*

Proof. For each pair $\{u, v\} \subset V^{(n)}$, the probability that no hyperedge contains both is $(1-p)^{2^{M_n-2}}$, so in G_2 we have $\Pr(\{u, v\} \in E(G_2)) = p_2$. These events are asymptotically independent. Thus G_2 behaves like an Erdős–Rényi random graph $G(M_n, p_2)$, which exhibits the stated phase transition at $p_2 = 1/M_n$ by classical results.

Theorem 13 (Connectivity Threshold)**.** *With notation as above, if*

$$p_2 \ = \ \frac{\log M_n + c}{M_n}$$

for some constant c*, then*

$$\Pr\big(G_2 \text{ is connected}\big) \ \longrightarrow \ e^{-e^{-c}} \quad (M_n \to \infty).$$

Equivalently, in the original model this occurs when

$$p \ = \ \frac{\log M_n + c}{M_n\, 2^{M_n - 2}}.$$

Proof. Again $G_2 \approx G(M_n, p_2)$. In the classical Erdős–Rényi model, connectivity emerges at $p_2 = (\log M_n + c)/M_n$, with limiting probability $\exp(-e^{-c})$. Translating p_2 back to p gives the claimed threshold.

Algorithm 1 Generate a Random n-SuperHyperGraph

Require: A finite base set V_0, integer $n \geq 1$, probability $p \in [0, 1]$
Ensure: A random n-SuperHyperGraph $\mathrm{SuHG}^{(n)} = (V^{(n)}, E)$

```
PS^0(V_0) ← V_0
for k = 1 to n do
    PS^k(V_0) ← PS(PS^{k−1}(V_0))
end for
V^(n) ← PS^n(V_0)
𝓔 ← PS(V^(n)) \ {∅}
E ← ∅
for each potential hyperedge e ∈ 𝓔 do
    draw u ∼ Uniform(0, 1)
    if u ≤ p then E ← E ∪ {e}
    end if
end for
return SuHG^(n) = (V^(n), E)
```

4. Result: Algorithm for Generating Random SuperHyperGraphs

In this section, we give an algorithm to generate a random n-SuperHyperGraph. See Algorithm 1 for a summary.

Remark 1. *(i)* ***Compute powersets.*** *Starting from V_0, iteratively form $\mathrm{PS}^k(V_0)$ up to $k = n$.*

(ii) ***Define vertices.*** *Set $V^{(n)} = \mathrm{PS}^n(V_0)$ as the supervertex set.*

(iii) ***Define potential edges.*** *Let $\mathcal{E} = \mathrm{PS}(V^{(n)}) \setminus \{\varnothing\}$.*

(iv) ***Random selection.*** *Include each $e \in \mathcal{E}$ in E independently with probability p.*

(v) ***Output.*** *Return the hypergraph $(V^{(n)}, E)$.*

Example 13 (University Curriculum as a 3-SuperHyperGraph)**.** *Let*

$$V_0 = \{\text{Algebra, Calculus,}$$
$$\text{Physics, Chemistry}\}.$$

Then

$$\mathrm{PS}^1(V_0)$$
$$= \{\{\text{Algebra}, \text{Calculus}\},$$
$$\{\text{Physics}, \text{Chemistry}\}, \dots\}$$

represents all courses. *Next,*

$$\mathrm{PS}^2(V_0) = \mathrm{PS}\big(\mathrm{PS}^1(V_0)\big)$$

lists all majors *(each a set of courses), for example* $\mathrm{Major}_A = \{\mathrm{Course}_1, \mathrm{Course}_2\}$*. Finally,*

$$\mathrm{PS}^3(V_0) = \mathrm{PS}\big(\mathrm{PS}^2(V_0)\big)$$

lists all degree programs *(each a collection of majors), e.g.* $\mathrm{Program}_X = \{\mathrm{Major}_A, \mathrm{Major}_B\}$.

In the random model $\mathrm{SuHG}^{(3)}(V_0, p)$*, set*

$$V^{(3)} = \mathrm{PS}^3(V_0),$$

$$\mathcal{E} = \mathrm{PS}(V^{(3)}) \setminus \{\varnothing\}.$$

Each candidate program $P \in \mathcal{E}$ *is included independently with probability* p*. Thus Algorithm 1 produces a random collection of active degree programs each academic year, capturing the hierarchy: topics* $\to$ *courses* $\to$ *majors* $\to$ *programs.*

Theorem 14 (Correctness of GenerateRandomNSuperHyperGraph)**.** *Let* V_0 *be a finite set,* $n \in \mathbb{N}$*, and* $p \in [0, 1]$*. Write*

$$V^{(n)} = \mathrm{PS}^n(V_0),$$

$$M_n = \big|V^{(n)}\big|.$$

Then Algorithm 1 returns each pair $(V^{(n)}, E)$ *with probability*

$$\Pr\big(E \textit{ is chosen}\big) \;=\; p^{|E|}\,(1-p)^{\,M_n - |E|},$$

and the events $\{x \in E\}$ *for* $x \in V^{(n)}$ *are independent Bernoulli(*p*). Hence its output distribution coincides exactly with that of the random model* $\mathrm{SuHG}^{(n)}(V_0, p)$*.*

Proof. After computing $V^{(n)} = \mathrm{PS}^n(V_0)$, the algorithm processes each $x \in V^{(n)}$ once, drawing $u_x \sim \mathrm{Uniform}(0, 1)$ and including x in E precisely if $u_x \le p$. Therefore

$$\Pr(x \in E) = p, \quad \Pr(x \notin E) = 1 - p,$$

and these events are independent over x. For any fixed $E \subseteq V^{(n)}$ with $|E| = m$, exactly the m elements of E must satisfy $u_x \le p$ and the other $M_n - m$ must satisfy $u_x > p$, yielding

$$\Pr(\text{algorithm returns } E) = p^m\,(1-p)^{M_n - m}.$$

This matches the defining law of $\mathrm{SuHG}^{(n)}(V_0, p)$.

Theorem 15 (Time Complexity)**.** *Define* $M_k = |\mathrm{PS}^k(V_0)|$ *for* $0 \le k \le n$*, so* $M_0 = |V_0|$ *and* $M_k = 2^{M_{k-1}}$*. Then Algorithm 1 runs in*

$$T(n) \;=\; \Theta\big(M_n \log M_n\big).$$

Proof. Powerset computation: At each $1 \le k \le n$, listing all $M_k = 2^{M_{k-1}}$ subsets of $\mathrm{PS}^{k-1}(V_0)$ (each of size up to $M_{k-1} = \log_2 M_k$) costs $\Theta(M_k\, M_{k-1}) = \Theta(M_k \log M_k)$. Summing over k is dominated by $k = n$, giving $\Theta(M_n \log M_n)$. *Random selection:* Testing each of the M_n candidates costs $\Theta(M_n)$, which is lower order. Hence $T(n) = \Theta(M_n \log M_n)$.

Theorem 16 (Space Complexity)**.** *With $M_n = |\mathrm{PS}^n(V_0)|$, the algorithm uses*

$$S(n) \;=\; \Theta\big(M_n \,\log M_n\big)$$

space.

Proof. At stage k, storing $\mathrm{PS}^k(V_0)$ of size M_k requires $\Theta(M_k\, M_{k-1})$ bits (each subset encoded in $\Theta(M_{k-1})$ bits). The maximum occurs at $k = n$, giving $\Theta(M_n \log M_n)$. The additional $\Theta(M_n)$ space for the edge set E is asymptotically smaller.

Next, as an example of a graph algorithm that mitigates the aforementioned complexity, we present a size-sampling–based approach. The algorithm is detailed in Algorithm 2.

Algorithm 2 Generate a Random n-SuperHyperGraph by Size-Sampling

Require: A finite base set V_0, integer $n \geq 1$, probability $p \in [0, 1]$
Ensure: A random n-SuperHyperGraph $\mathrm{SuHG}^{(n)} = (V^{(n)}, E)$

Compute $V^{(n)} \leftarrow \mathrm{PS}^n(V_0)$, let $M = |V^{(n)}|$
Initialize $E \leftarrow \varnothing$
for $s = 1$ to M **do**
    $M_s \leftarrow \binom{M}{s}$
    Draw $K_s \sim \mathrm{Binomial}(M_s, p)$
    for $i = 1$ to K_s **do**
        Sample uniformly an s-subset $e \subseteq V^{(n)}$ without replacement
        if $e \notin E$ **then** $E \leftarrow E \cup \{e\}$
        end if
    end for
end for
return $\mathrm{SuHG}^{(n)} = (V^{(n)}, E)$

Remark 2. *This size-sampling approach avoids iterating over all $2^M - 1$ possible hyperedges. Instead, for each edge-size s, it first draws the number K_s of s-hyperedges to include (a $\mathrm{Binomial}(\binom{M}{s}, p)$ variate), then samples exactly K_s distinct s-subsets uniformly at random. When p is small or one restricts to moderate s, the running time and memory scale with $\sum_s K_s \cdot s$ rather than $\sum_s \binom{M}{s}$.*

Example 14 (Size-Sampling in Action)**.** *Let the base set be*

$$V_0 = \{\text{Hiroko, Yutaka}\}.$$

Then the level-1 supervertex set is

$$V^{(1)} = \mathrm{PS}^1(V_0)$$

$$= \{\varnothing,\ \{\text{Hiroko}\},\ \{\text{Yutaka}\},\ \{\text{Hiroko}, \text{Yutaka}\}\},$$

so $M = |V^{(1)}| = 4$. We choose $p = 0.3$. Algorithm 2 proceeds as follows:

- $s = 1$:

$$M_1 = \binom{4}{1} = 4,$$

$$K_1 \sim \text{Binomial}(4, 0.3), \quad K_1 = 1.$$

Sample one 1-subset from $V^{(1)}$, e.g.

$$e_1 = \{\{\text{Yutaka}\}\}.$$

- $s = 2$:

$$M_2 = \binom{4}{2} = 6,$$

$$K_2 \sim \text{Binomial}(6, 0.3), \quad K_2 = 2.$$

Sample two 2-subsets, for instance

$$e_2 = \{\{\text{Hiroko}\}, \{\text{Hiroko}, \text{Yutaka}\}\},$$

$$e_3 = \{\{\text{Yutaka}\}, \{\text{Hiroko}, \text{Yutaka}\}\}.$$

- $s = 3$:

$$M_3 = \binom{4}{3} = 4,$$

$$K_3 \sim \text{Binomial}(4, 0.3), \quad K_3 = 1.$$

Sample one 3-subset, e.g.

$$e_4 = \{\{\text{Hiroko}\}, \{\text{Yutaka}\}, \{\text{Hiroko}, \text{Yutaka}\}\}.$$

- $s = 4$:

$$M_4 = \binom{4}{4} = 1,$$

$$K_4 \sim \text{Binomial}(1, 0.3), \quad K_4 = 0.$$

Hence the sampled hyperedge set is

$$E = \{\, e_1, e_2, e_3, e_4\},$$

and we obtain

$$\text{SuHG}^{(1)} = \big(V^{(1)}, E\big).$$

This concrete run demonstrates how size-sampling generates a random superhypergraph by drawing only $1+2+1+0 = 4$ hyperedges, rather than enumerating all $2^4 - 1 = 15$ possible subsets of $V^{(1)}$.

Theorem 17 (Correctness of Size-Sampling Algorithm)**.** *Algorithm 2 produces a random n-SuperHyperGraph with the same law as $\text{SuHG}^{(n)}(V_0, p)$. In particular, each potential s-hyperedge $e \subseteq V^{(n)}$, $|e| = s$, is included independently with probability p.*

Proof. Fix $s \in \{1, \dots, M\}$ and write $M_s = \binom{M}{s}$. Let $\mathcal{E}_s$ be the collection of all s-subsets of $V^{(n)}$. The algorithm first draws

$$K_s \sim \mathrm{Binomial}(M_s, p),$$

so

$$\Pr(K_s = k) = \binom{M_s}{k} p^k (1-p)^{M_s - k}.$$

Conditioned on $K_s = k$, it chooses uniformly k distinct subsets from $\mathcal{E}_s$. Therefore for any fixed $e \in \mathcal{E}_s$,

$$\begin{aligned} &\Pr\bigl(e \in E \mid K_s = k\bigr) \\ &= \frac{\binom{M_s-1}{k-1}}{\binom{M_s}{k}} = \frac{k}{M_s}. \end{aligned}$$

Hence marginally

$$\begin{aligned} \Pr(e \in E) &= \sum_{k=1}^{M_s} \Pr(K_s = k)\, \frac{k}{M_s} \\ &= \frac{1}{M_s} \sum_{k=1}^{M_s} k \binom{M_s}{k} p^k (1-p)^{M_s-k} \\ &= \frac{M_s\, p}{M_s} = p. \end{aligned}$$

Moreover, for any set $S \subseteq \mathcal{E}_s$ of size m,

$$\begin{aligned} &\Pr\bigl(S \subseteq E,\ \mathcal{E}_s \setminus S \cap E = \varnothing\bigr) \\ &= \binom{M_s}{m} p^m (1-p)^{M_s-m} \frac{1}{\binom{M_s}{m}} = p^m (1-p)^{M_s-m}, \end{aligned}$$

which matches the product measure of independent Bernoulli(p) trials on $\mathcal{E}_s$. Repeating across all sizes s shows that every potential hyperedge is included independently with probability p. Thus the algorithm exactly recovers the distribution of $\mathrm{SuHG}^{(n)}(V_0, p)$.

Theorem 18 (Time and Space Complexity)**.** *Let $M = |V^{(n)}|$ and $M_s = \binom{M}{s}$. Assume that sampling one s-subset uniformly at random (without full enumeration) takes $O(s)$ time. Then Algorithm 2 runs in*

$$O\Bigl(\sum_{s=1}^{M} \bigl(T_{\mathrm{binom}}(M_s, p) + K_s \cdot s\bigr)\Bigr),$$

where $T_{\mathrm{binom}}(M_s, p)$ is the cost of drawing a $\mathrm{Binomial}(M_s, p)$ *variate and K_s is its realized value. In expectation,*

$$\mathbb{E}\bigl[\mathit{time}\bigr]$$

$$= O\Big(\sum_{s=1}^{M} \big(\mathrm{polylog}(M_s) + p\, M_s\ s\big)\Big).$$

Space usage is

$$O\big(M + \sum_{s=1}^{M} K_s \cdot s\big),$$

i.e. $O(M + |E|)$.

Proof. **Time.** Computing $V^{(n)}$ costs $O(M \log M)$ by Theorem 15. For each s, drawing $K_s \sim \mathrm{Binomial}(M_s, p)$ takes $T_{\mathrm{binom}}(M_s, p)$ time (e.g. $O(\log M_s)$ with acceptance–rejection). Sampling K_s distinct s-subsets costs $O(K_s \cdot s)$ if each unranked in $O(s)$ time. Summing yields the stated bound. In expectation, $\mathbb{E}[K_s] = p\, M_s$, giving the second formula.

Space. Storing $V^{(n)}$ uses $O(M \log M)$ bits. The edge set E requires space proportional to the total size of sampled hyperedges, $\sum_s K_s \cdot s$. Ignoring logarithmic factors, total space is $O(M + |E|)$.

5. Conclusions

In this paper, we investigated the notion of a *random SuperHyperGraph* and provided a concise mathematical exploration of its structure. We also included a brief discussion of the associated algorithm. We list below several scientific benefits of random SuperHyperGraphs. These points underscore the potential impact of this research in areas such as computer science and social systems.

- **Unified Framework:** Merges random graphs and hypergraphs into one probabilistic model.
- **Hierarchical Representation:** Encodes multi-level interactions via nested powerset layers.
- **Probabilistic Insights:** Enables study of phase transitions and concentration in layered networks.
- **Algorithmic Foundations:** Supports development of algorithms for complex, probabilistic hierarchies.

To capture richer uncertainty in Random SuperHyperGraphs, we plan to integrate fuzzy-based frameworks. In particular, we will extend the current model to encompass Fuzzy Sets [59], Intuitionistic Fuzzy Sets [60, 61], Vague Sets [62], Hesitant Fuzzy Sets [63, 64], HyperFuzzy Sets [65, 66], Picture Fuzzy Sets [67, 68], Pythagorean fuzzy set [69, 70], Neutrosophic Sets [71, 72], and Plithogenic Sets [27, 73, 74]. We will also develop directed[75, 76], bidirected [77, 78], and multidirected [79, 80] variants, as well as hypergraph and superhypergraph analogues of Quasi-Random Graphs [81, 82] and Semi-Random Graphs [83, 84]. For example, regarding Random SuperHyperGraphs, we expect future investigations to explore extensions that incorporate Fuzzy random variables[85, 86], Neutrosophic random variables[87–89], Plithogenic random variables[90, 91], and related constructs. Finally, we aim to validate and refine these extensions through comprehensive computational experiments.

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Data Availability

This paper is purely theoretical and does not involve any empirical data. We welcome future empirical studies that build upon and test the concepts presented here.

Ethical Approval

As this work is entirely conceptual and involves no human or animal subjects, ethical approval was not required.

Conflicts of Interest

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Research Integrity

The authors affirm that, to the best of their knowledge, this manuscript represents their original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

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No computer-based tools—such as symbolic computation systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq)—were employed in the development, analysis, or verification of the results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the authors.

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I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Consent to Publish

All authors have given their consent for submission of this manuscript to the journal.

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