EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 4, Article Number 6837
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Further Study on R-Sets Operator in Acyclic Fashion

Salihah Thabet Alwadani[1,*]

[1] *Mathematics, Yanbu Industrial College, The Royal Commission for Jubail and Yanbu, Yanbu, Saudi Arabia*

Abstract. We provide a distinct and detailed proof of the weak convergence of the acyclic Douglas–Rachford iteration to a point whose nearest-point projections onto each of the N convex sets coincide. Our analysis shows that the cyclic Douglas–Rachford operator is asymptotically regular, that its fixed-point set coincides with the intersection of the individual fixed-point sets when this intersection is nonempty, and that the iteration converges weakly to such a point. Special cases highlight when the method coincides with alternating projections and when it diverges from von Neumann's scheme.

2020 Mathematics Subject Classifications: 47H09, 47H05, 47A06, 90C25

Key Words and Phrases: Nonexpansive mapping, acyclic Douglas–Rachford method, cyclic Douglas–Rachford operator, fixed point theory, convex analysis, projection algorithms, alternating projections, weak convergence, asymptotic regularity

1. Introduction

The Douglas Rachford algorithm is a very popular splitting technique for finding a zero of the sum of two maximally monotone operators. It is also used to solve the *convex feasibility problem*. That is, given convex subsets $C_1, C_2, \dots, C_m$ and $C = \cap C_i \neq \emptyset$,

$$\text{Fid} \quad x \in C \tag{1}$$

For more information about feasibility problems, we refer the reader to [1] which provides a thorough treatment of feasibility problems, especially in Hilbert spaces, which are common in signal processing, image recovery, and optimization. It dicusses projection methods, such as Douglas -Rachford algorithm and alternating projections, which are standard techniques used to solve feasibility problems involving convex sets. See also [2–4] for more details, where [2] presents projection methods and their use in solving large-scale feasibility problems, particulary in applications such as image reconstruction and medical imaging. [3] provides a unified treatment of algorithms for feasibility and

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6837
Email addresses: salihah.s.alwadani@gmail.com (S. Th. Alwadani)

inverse problems where [4] focuses on feasibility problems in signal processing . It explores how projection-based algrithms can be used to recover signal that satisfy multiple constraints represented as convex sets.

Throughout this paper, we shall assume that

$$X = \mathcal{H} \text{ is a real Hilbert space with the product } \langle \cdot, \cdot \rangle \text{ and induced norm } \| \cdot \| \tag{2}$$

In this paper, we provide a different, detailed proof to the weak convergence of Acyclic Douglas-Rachford iteration schema to a point whose nearest point projections onto each of the N sets coincide using the assumption in (2) and convex analysis. This paper is distributed as follows: Section 2 presents standard material and basic facts and collects some useful properties from convex analysis and algebra, which are useful in our later proofs. We designate all of the known results as facts with explicit references. In Section 3, we visit the Cyclic Douglas–Rachford iteration scheme that is defined in [5] and show that even with the case $N = 2$ the the Cyclic Douglas–Rachford iteration is different from the Douglas–Rachford iteration see Proposition 2, Example 1, and Example 2. The main results are in Section 4, where can be summarized as follows:

- We show that the Cyclic Douglas–Rachford operator $T_{[C_1 C_2 \ldots C_N]}$ is asymptotically regular, see Section 4.
- Section 4 shows that the fixed point sets of the Cyclic Douglas–Rachford operator are equal to the intersection of the individual fixed point sets of the individual operators under the assumption that the intersection is not empty.
- The cyclic Douglas–Rachford iteration converges weakly to a point in the fixed point sets of the Cyclic Douglas–Rachford operator, see Theorem 1 for more details.
- Proposition 4 illustrates that if the initial point belongs to the first set, then the Cyclic Douglas-Rachford method coincides with the alternating projection method. Additionally, if the Cyclic Douglas–Rachford schema defined on to two closed affine subspaces C_2 and C_2 is equal to the averaged of T_{C_1,C_2} and T_{C_2,C_1}, see Lemma 1 for more details.
- Example 3 indicates that if $x_0 \notin C_1$, then the cyclic Douglas–Rachford iteration need not coincide with von Neumann's alternating projection method.

2. Background

Recall

$$X = \mathcal{H} \quad \text{is a real Hilbert space with the product } \langle \cdot, \cdot \rangle \text{ and induced norm } \| \cdot \|.$$

The identity operator on $\mathcal{H}$ is denoted by Id. Let $C \subset \mathcal{H}$ is closed and convex set, the projector onto the set C is the mapping $\mathrm{P}_C \colon \mathcal{H} \to C$ defined as,

$$\mathrm{P}_C := \underset{c \in C}{\operatorname{argmin}} \|x - c\| = \{\, z \in C : \ \|x - z\| = \inf_{c \in C} \ \|x - c\| \}, \qquad \text{for all } x \in \mathcal{H} \tag{3}$$

The reflector with respect to the set C is a set valued mapping $\mathrm{P}_C\colon \mathcal{H} \to \mathcal{H}$ defined as,

$$R_C := \mathrm{P}_C + (\mathrm{P}_C - \mathrm{Id}) = 2\,\mathrm{P}_C - \mathrm{Id}, \qquad \text{for all } x \in \mathcal{H} \tag{4}$$

Let $T\colon \mathcal{H} \to \mathcal{H}$ be an operator. Then a fixed point of T is a point $x \in \mathcal{H}$ that map a point to itself. That is, $Tx = x$. The set of fixed points of the operator T is denoted by $\operatorname{Fix} T$, i.e.,

$$\operatorname{Fix} T := \{x \in \mathcal{H}\colon T(x) = x\} \neq \varnothing, \qquad \text{for all } x \in \mathcal{H} \tag{5}$$

Definition 1. [1, Definition 4.1] Let C be a nonempty, closed and convex subset of $\mathcal{H}$. Let $T\colon C \to \mathcal{H}$ then T is;

(i) nonexpansive on C if it is Lipschitz continuous with constant 1, i.e.,

$$(\forall x \in C)(\forall y \in C) \quad \|Tx - Ty\| \leq \|x - y\|; \tag{6}$$

(ii) firmly nonexpansive if

$$(\forall x \in C)(\forall y \in C) \ \ \|Tx - Ty\|^2 + \|\big(\mathrm{Id} - T\big)x - \big(\mathrm{Id} - T\big)y\| \leq \|x - y\|^2; \tag{7}$$

(iii) quasinonexpansive if T is Fejér montone with respect to $\operatorname{Fix} T$, i.e.,

$$(\forall x \in C)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\| \leq \|x - y\|; \tag{8}$$

(iv) strictly quasinonexpansive if

$$(\forall x \notin \operatorname{Fix} T)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\| < \|x - y\|; \tag{9}$$

(v) α- avaraged for $\alpha \in (0, 1)$, if there exisits anonexpansive operator $N\colon C \to \mathcal{H}$ such that

$$T = (1 - \alpha)\,\mathrm{Id} + \alpha N \tag{10}$$

Definition 2. [6, Definition 4.8-1] A sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space is said to be *convergent (strongly convergent or convergent in the norm)* if there is an $x^* \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0$$

This is written

$$\lim_{n\to\infty} x_n = x^*,$$

or simply as

$$x_n \to x^*.$$

Definition 3. [6, Definition 4.8-2] A sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space is said to be *weakly convergent* if there is an $x^* \in \mathcal{H}$ such that for every bounded linear functional f on $\mathcal{H}$,

$$\lim_{n\to\infty} f(x_n) = f(x^*).$$

This is written

$$x_n \rightharpoonup x^*.$$

Definition 4. Let $T\colon \mathcal{H} \to \mathcal{H}$. We recall that T is *asymptotically regular* if $T^n x - T^{n+1}x \to 0$, in norm, for all $x \in \mathcal{H}$.

Definition 5. [7, Fact 3.52]Let $C_1, C_2, \ldots, C_n$ be closed and convex subset of $\mathcal{H}$ with $\bigcap\limits_{i=1}^{n} C_i \neq \varnothing$. The Douglas-Rachford operator associated with the ordered tuple $\big(C_1, C_2, \ldots, C_n\big)$ is

$$T_{C_1,C_2,\ldots,C_n} := \frac{1}{2}\big(\operatorname{Id} + R_{C_n}R_{C_{n-1}}\ldots R_{C_2}R_{C_1}\big).$$

For $n = 2$, let C_1 and C_2 closed and convex subset of $\mathcal{H}$ with $C_1 \cap C_2 \neq \varnothing$. The Douglas-Rachford operator associated with the ordered pair $\big(C_1, C_2\big)$ is:

$$T_{C_1,C_2} := \frac{1}{2}\big(\operatorname{Id} + R_{C_2}R_{C_1}\big), \tag{11}$$

and the generated sequence $\big(x_n\big)_{n\in\mathbb{N}}$ is

$$\big(\forall n \in \mathbb{N}\big) \quad x_{n+1} = T_{C_1,C_2}x_n \quad \text{where} \quad x_0 \in \mathcal{H},$$

also called the (DRA) sequence. For more information about Douglas Rachford algorithm you can see [8] where the original theoretical foundation of the Dougals- Rachford algorithm for monotone operator splitting. A comprehensive analysis linking the Dougals-Rachford method to the proximal point algorithm is provided in [9]. In 2004, Bauschke, Combettes, and Luke analyze the application of Douglas-Rachford to convex feasibility and best approximation problems. see [10] In 2005 Combettes and Wajs introduce proximal splitting methods that are closely related to and extend the Douglas-Rachford algorithm, see [11]. 6 years later Combettes and Pesquet applies Douglas-Rachford and related algorithms to signal processing and inverse problems, see [12]. In 2017, Bauschke and Combettes comes up with a textbook-level comprehensive treatment of the Douglas-Rachford method and its role in convex feasibility and optimization, see [1].

Proposition 1. Let C_1 and C_2 be a nonempty closed convex subsets of $\mathcal{H}$. Then P_{C_1} is firmly nonexpansive, R_{C_1} is nonexpansive, $N = R_{C_2}R_{C_1}$ is nonexpansive and $T_{C_1,C_2} := \frac{1}{2}\big(\operatorname{Id} + R_{C_2}R_{C_1}\big)$ is firmly nonexpansive.

Proof. See [1, Lemma 222]. ■

Definition 6. [1, Definition 5.1] Let C be a nonempty subset of $\mathcal{H}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Then $(x_n)_{n\in\mathbb{N}}$ is *Fejér monotone* with respect to C if

$$\big(\forall x \in C\big)\ \big(\forall n \in \mathbb{N}\big) \quad \|x_{n+1} - x\| \leq \|x_n - x\|.$$

[1, Proposition 5.7] Let $(x_n)_n \in N$ be a sequence in $\mathcal{H}$ and let C be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $(x_n)_n \in N$ is Fejér monotone with respect to C. Then the shadow sequence $(\mathrm{P}\, x_n)_n \in N$ converges strongly to a point in C.

[1, Corollary 5.8] Let $(x_n)_n \in N$ be a sequence in $\mathcal{H}$, let C be a nonempty closed convex subset of $\mathcal{H}$, and let $x \in C$. Suppose that $(x_n)_n \in N$ is Fejér monotone with respect to C and that $x_n \rightharpoonup x$. Then $\mathrm{P}_C\, x_n \to x$.

3. The Cyclic Douglas- Rachford method

In order to solve the feasibility problem (1), where C_i are closed and convex subsets of $\mathcal{H}$ with nonempty intersection, we employ the Cyclic Douglas–Rachford iteration scheme that generates a sequence $(x_n)_{n\in\mathbb{N}}$ by

$$(\forall n \in \mathbb{N}) \;\; x_{n+1} = T_{[C_1C_2\cdots C_{\mathbb{N}}]}x_n \tag{12}$$

and where

$$T_{[C_1C_2\ldots C_N]} := T_{C_N,C_1}T_{C_{N-1},C_N}\ldots T_{C_2,C_3}T_{C_1,C_2}. \tag{13}$$

Proposition 2. Assume $N = 2$, let C_1 and C_2 closed and convex subset of $\mathcal{H}$ with $C_1 \cap C_2 \neq \varnothing$. Recall (12), (13), and (11). Then

$$T_{[C_1C_2]} \neq T_{C_1,C_2}.$$

Proof. Let $N = 2$, C_1 and C_2 closed and convex subset of $\mathcal{H}$ with $C_1 \cap C_2 \neq \varnothing$. Using (12), (13), and (11) gives

$$T_{[C_1C_2]} := T_{C_2,C_1}T_{C_1,C_2} \tag{14}$$

$$= \Big(\frac{\mathrm{Id} + R_{C_1}R_{C_2}}{2}\Big)\Big(\frac{\mathrm{Id} + R_{C_2}R_{C_1}}{2}\Big) \tag{15}$$

Observe that

$$T_{[C_1C_2]} \neq T_{C_1,C_2}.$$

■

Example 1. Suppose that $X = \mathbb{R}^2$, $C_1 = \mathbb{R} \times \{0\}$ and $C_2 = \{x \in \mathbb{R}^2 \mid \|x - 3\| \leq 3\}$. Then $\bigcap_{i=1}^{2} C_i \neq \varnothing$, and for starting point $x_0 \in \,]-\infty, 1[\, \times \{1\}$, the DRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (C_1, C_2) satisfies $(\forall n \in \{2, 3, \cdots\})\; x_n = (0, n)$ and $\mathrm{P}_{C_1} x_n = (0, 0) \in \bigcap_{i=1}^{2} C_i$. The CDRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (C_1, C_2) will converge to $(0, 0)$. See Fig. 1 for an illustration, created with `GeoGebra` [13].

Example 2. Suppose that $X = \mathbb{R}^2$, $C_1 = \mathbb{R} \cdot (1, 1)$, $C_2 = \{0\} \times \mathbb{R}$ and $C_3 = \mathbb{R} \cdot (1, -1)$. Then the 3-set Douglas-Rachford sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (C_1, C_2, C_3) will fail to converge to a point $(0, 0) \in \bigcap_{i=1}^{3} C_i$. The CDRA sequence $(x_n)_{n\in\mathbb{N}}$ with respect to (C_1, C_2, C_3) will converge to $x^* = (0, 0) \in \bigcap_{i=1}^{3} C_i$. See Fig. 2 for an illustration, created with `GeoGebra` [13].

The notation employed in this paper is standard and closely aligned with that in [14], [7], [15], and [16].

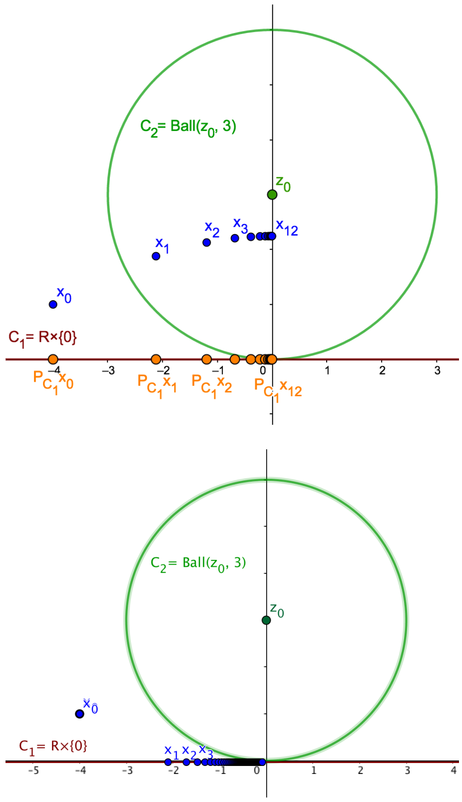

Figure 1: An illustration for Example 1 with the starting point $x_0 = (-4, 1)$. In the left, the DRA sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x = (0, 2.4) \notin \bigcap_{i=1}^{2} C_i$. However, the shadew sequence $P_{C_1}(x_n)$ converges to $(0, 0) = \bigcap_{i=1}^{2} C_i$. In the right, the CDRA sequece $(x_n)_{n\in\mathbb{N}}$ converges to $(0, 0) = \bigcap_{i=1}^{2} C_i$.

4. Main Results

Let $T_i \colon \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive, for each i. Recall from (13) that

$$T_{[C_1C_2\ldots C_N]} \colon = T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_2,C_3} T_{C_1,C_2}$$

with $\operatorname{Fix} T_{[C_1C_2\ldots C_N]} \neq \varnothing$. Then

$$T_{[C_1C_2\ldots C_N]} \quad \text{is asymptotically regular.}$$

Proof. From (13) and Proposition 1 we have $T_{C_i,C_{i+1}}$ is firmly nonexpansive for all i. We also have, $\varnothing \neq \operatorname{Fix} T_{[C_1C_2\ldots C_N]}$. Let $y \in \operatorname{Fix} T_{[C_1C_2\ldots C_N]}$ then;

$$\begin{aligned}\|Tx_n - Ty\|^2 &\overset{\dagger}{\leq} \|T_{C_{N-1},C_N} \cdots T_{C_2,C_3} T_{C_1,C_2} x_n - T_{C_{N-1},C_N} \cdots T_{C_2,C_3} T_{C_1,C_2} y\|^2 \\ &\quad - \|(\operatorname{Id} - T_{C_N,C_1}) x_n - (\operatorname{Id} - T_{C_N,C_1}) y\|^2\end{aligned}$$

† By using the definition of $T_{[C_1C_2\cdots C_N]}$ and the fact that $(\forall i)$ $T_{C_i,C_{i+1}}$ is firmly nonexpansive.

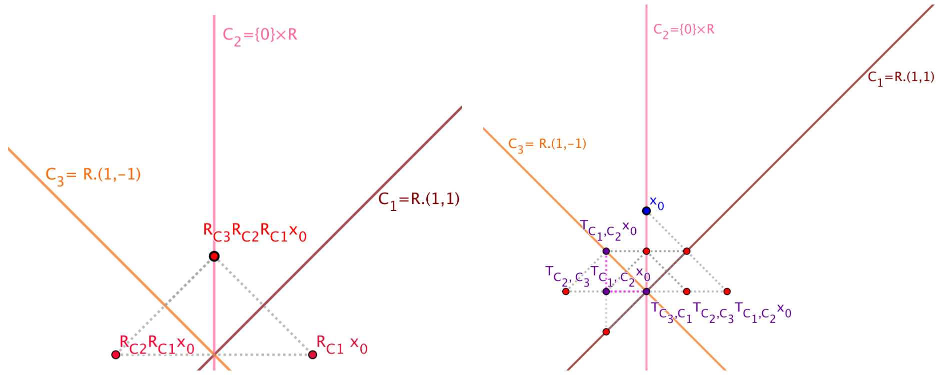

Figure 2: An illustration for Example 2 with the starting point $x_0 = (0, 2)$. The graph in the left side describes the 3- sets DRA iterations which faill to converge to $(0, 0)$. However, the graph in the right side describes the 3- sets CDRA iteration which converges to $(0, 0)$.

$$
\begin{aligned}
&\leq \|T_{C_{N-2},C_{N-1}} \cdots T_{C_2,C_3} T_{C_1,C_2} x_n - T_{C_{N-2},C_{N-1}} \cdots T_{C_2,C_3} T_{C_1,C_2} y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_N,C_1}) x_n - (\mathrm{Id} - T_{C_N,C_1}) y\|^2 \\
&\leq \vdots \\
&\leq \|T_{C_1,C_2} x_n - T_{C_1,C_2} y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \cdots T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \cdots T_{C_N,C_1} y\|^2 \\
&\quad - \cdots - \|(\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_N,C_1}) x_n - (\mathrm{Id} - T_{C_N,C_1}) y\|^2 \\
&\overset{\ddagger}{\leq} \|x_n - y\|^2 - \|(\mathrm{Id} - T_{C_N,C_1}) x_n - (\mathrm{Id} - T_{C_N,C_1}) y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} y\|^2 \\
&\quad - \cdots \\
&\quad - \|(\mathrm{Id} - T_{C_2,C_3}) T_{C_3,C_4} \cdots T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_2,C_3}) T_{C_3,C_4} \cdots T_{C_N,C_1} y\|^2 \\
&\quad - \|(\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \cdots T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \cdots T_{C_N,C_1} y\|^2
\end{aligned}
\tag{16}
$$

Therefore, $(x_n)_{n \in \mathbb{N}}$ is Fejér montone with respect to $\mathrm{Fix} T_{[C_1 C_2 \ldots C_N]}$ and,

$$(\mathrm{Id} - T_{C_N,C_1}) x_n - (\mathrm{Id} - T_{C_N,C_1}) y \to 0 \tag{17}$$

$$(\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_{N-1},C_N}) T_{C_N,C_1} y \to 0 \tag{18}$$

$$\vdots \tag{19}$$

$$(\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \cdots T_{C_N,C_1} x_n - (\mathrm{Id} - T_{C_1,C_2}) T_{C_2,C_3} \ldots T_{C_N,C_1} y \to 0 \tag{20}$$

Adding (17) - - (20), we obtain $x_n - T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_1,C_2} x_n \to 0$ ■

‡Because T_{C_1,C_2} is firmly nonexpansive which means it is nonexpansive.

Let $T_{C_i,C_{i+1}}:\mathcal{H}\to\mathcal{H}$ be firmly nonexpansive for each i and recall from (13) that

$$T_{[C_1C_2\ldots C_N]} = T_{C_N,C_1}\ldots T_{C_2,C_3}T_{C_1,C_2}.$$

If $\bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}} \neq \varnothing$, then

$$\operatorname{Fix} T_{[C_1C_2\cdots C_N]} = \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}}.$$

Proof. Since $T_{C_i,C_{i+1}}$ is firmly nonexpansive for each i, then $T_{C_i,C_{i+1}}$ is α- avaraged with $\alpha = \frac{1}{2}$ for each i. Moreover,

$$\varnothing \neq \bigcap_{i=1}^{N} C_i \subseteq \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}}$$

The inclusion $\bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}} \subseteq \operatorname{Fix} T_{[C_1C_2\ldots C_N]}$ is obvious. Now we show that the converse inclusion also holds. When $N = 1$ let $y \in \operatorname{Fix} T_{C_1,C_2}$ and let $x \in \operatorname{Fix} T_{[C_1C_2]} := \operatorname{Fix} T_{C_2,C_1}T_{C_1,C_2}$. Then;

- If $x \in \operatorname{Fix} T_{C_1,C_2} \Rightarrow T_{C_1,C_2}x = x$. Therefore, $T_{C_2,C_1}x = T_{C_2,C_1}T_{C_1,C_2}x = x \in \operatorname{Fix} T_{C_1,C_2}$. Therefore, under this case we have $\operatorname{Fix} T_{[C_1C_2]} \subseteq \operatorname{Fix} T_{C_1,C_2}$.
- If $T_{C_1,C_2}x \in \operatorname{Fix} T_{C_2,C_1} \Rightarrow T_{C_1,C_2}x = T_{C_2,C_1}T_{C_1,C_2}x = x \in \operatorname{Fix} T_{C_1,C_2} \Rightarrow \operatorname{Fix} T_{[C_1C_2]} \subseteq \operatorname{Fix} T_{C_1,C_2}$.
- Let $x \notin \operatorname{Fix} T_{C_2,C_1}$ and $T_{C_2,C_1}x \notin \operatorname{Fix} T_{C_1,C_2}$. Since T_{C_1,C_2} and T_{C_2,C_1} are firmly nonexpansive and by [1, Corollary 2.15] they are strictly quasinonexpansive. Therefore, $\|x-y\| = \|T_{C_2,C_1}T_{C_1,C_2}x - y\| < \|T_{C_1,C_2}x - y\| < \|x-y\|$ which is not true. Therefore, $\operatorname{Fix} T_{C_2,C_1}T_{C_1,C_2} = \operatorname{Fix} T_{C_1,C_2}$.

Hypothesis induction assumption: for $n \geq 2$ the result holds up to N operators. We have $\operatorname{Fix} T_{[C_1C_2\cdots C_{N-1}]} = \bigcap_{i=1}^{N} \operatorname{Fix} T_{C_i,C_{i+1}}$. Then show the results hold for $N+1$ operators. Let $S_1 = T_{C_{N-1},C_N}\ldots T_{C_2,C_3}T_{C_1,C_2}$ and let $S_2 = T_{C_N,C_1}$ because S_2 is quasinonexpansive with $\operatorname{Fix} S_2 = \operatorname{Fix} T_{C_N,C_1}$ and by the induction hypothesis we have, $\operatorname{Fix} S_1 = \bigcap_{i=1}^{N} \operatorname{Fix} T_{C_i,C_{i+1}}$. Therefore, by the fact that $S_1S_2 = T_{C_N,C_1}T_{C_{N-1},C_N}\cdots T_{C_2,C_3}T_{C_1,C_2}$ is strictly quasinonexpansive then ,

$$\operatorname{Fix} T_{C_N,C_1}T_{C_{N-1},C_N}\ldots T_{C_2,C_3}T_{C_1,C_2} = \operatorname{Fix} S_1S_2 = \operatorname{Fix} S_1 \cap S_2 = \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}}$$

■

Theorem 1. Let $T_{C_i,C_{i+1}}:\mathcal{H}\to\mathcal{H}$ be firmly nonexpansive for each i, with $\bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}} \neq \varnothing$. Then, for any $x_0 \in \mathcal{H}$, the sequence

$$T^n_{[C_1C_2\ldots C_N]}x_0 \rightharpoonup x \in \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i,C_{i+1}}.$$

Proof. First, show that every weak cluster point x of $(x_n)_{\in\mathbb{N}}$ lies in Fix $T_{[C_1C_2...C_N]}$. By Section 4 $(x_n)_{n\in\mathbb{N}}$ is Fejér montone with respect to Fix $T_{[C_1C_2...C_N]}$, it is bounded. Let x be a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$. Then, there exists a subsequence x_{n_k} of x_n such that $x_{n_k} \rightharpoonup x$.

$$\begin{aligned}
\left\|x - T_{[C_1...C_N]}x\right\|^2 &= \left\|x_{n_k} - T_{[C_1...C_N]}x\right\|^2 - \|x_{n_k} - x\|^2 - 2\left\langle x_{n_k} - x, x - T_{[C_1...C_N]}x\right\rangle \\
&= \left\|x_{n_k} - T_{[C_1...C_N]}x + T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 - \|x_{n_k} - x\|^2 \\
&\quad - 2\left\langle x_{n_k} - x, x - T_{[C_1...C_N]}x\right\rangle \\
&= \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 + 2\left\langle x_{n_k} - T_{[C_1...C_N]}x_{n_k}, T_{[C_1...C_N]}x_{n_k} - T_{[C_1\cdots C_N]}x\right\rangle \\
&\quad + \left\|T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x\right\|^2 - \|x_{n_k} - x\|^2 - 2\left\langle x_{n_k} - x, x - T_{[C_1...C_N]}x\right\rangle \\
&\overset{\S}{\leq} \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 + 2\left\langle x_{n_k} - T_{[C_1...C_N]}x_{n_k}, T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x\right\rangle \\
&\quad + \|x_{n_k} - x\|^2 - \|x_{n_k} - x\|^2 - 2\left\langle x_{n_k} - x, x - T_{[C_1...C_N]}x\right\rangle \\
&= \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 + 2\left\langle x_{n_k} - T_{[C_1...C_N]}x_{n_k}, T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x\right\rangle \\
&\quad - 2\left\langle x_{n_k} - x, x - T_{[C_1...C_N]}x\right\rangle
\end{aligned}$$

Note that;

$$\begin{aligned}
\left\|\left\langle x_{n_k} - T_{[C_1...C_N]}x_{n_k}, T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x\right\rangle\right\|^2 &\overset{\P}{\leq} \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 \left\|T_{[C_1...C_N]}x_{n_k} - T_{[C_1...C_N]}x\right\|^2 \\
&\overset{\|}{\leq} \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 \|x_{n_k} - x\|^2 \\
&= \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 \Big(\|x_{n_k}\|^2 - 2\langle x_{n_k}, x\rangle \\
&\qquad + \|x\|^2\Big) \\
&\leq \left\|x_{n_k} - T_{[C_1...C_N]}x_{n_k}\right\|^2 \Big(\|x_{n_k}\|^2 - 2\|x_{n_k}\|\|x\| \\
&\qquad + \|x\|^2\Big)
\end{aligned}$$

Therefore, $\sup\|x_{n_k}\| = M < \infty$. Also, $x_{n_k} \rightharpoonup x$. Then

$$\|x\| = \lim |\langle x_{n_k}, x\rangle| \leq \underline{\lim}\, \|x_{n_k}\| < \infty.$$

---

§Follows from the nonexpansiveness of $T_{[C_1...C_N]}$.
¶Follows from Cauchy-Schwarz inequality $|\langle x, y\rangle| \leq \|x\|\|y\|$.
‖Follows from the nonexpansiveness of $T_{[C_1...C_N]}$.

Therefore, $\left(\|x_{n_k}\|^2 - 2\|x_{n_k}\|\|x\| + \|x\|^2\right) < \infty$. Hence, taking the limit as $n \to \infty$, we have

$$\|x_{n_k} - T_{[C_1 \dots C_N]} x_{n_k}\|^2 \left(\|x_{n_k}\|^2 - 2\|x_{n_k}\|\|x\| + \|x\|^2\right) \to 0$$

Moreover, $x_{n_k} - T_{[C_1 \dots C_N]} x_{n_k} \to 0$ because $T_{[C_1 \dots C_N}$ is asympotically regular by Section 4. Also, by Definition 3, $\left\langle x_{n_k} - x, x - T_{[C_1 \dots C_N]} x \right\rangle = \left\langle x_{n_k}, x - T_{[C_1 \dots C_N]} x \right\rangle - \left\langle x, x - T_{[C_1 \dots C_N]} x \right\rangle = 0$. Therefore,

$$x - T_{[C_1 \dots C_N]} x = 0$$

Next, show that $(x_n)_{n \in \mathbb{N}}$ cannot have two distinct weak sequential cluster points in $\operatorname{Fix} T_{[C_1 \dots C_N]}$. Let x and y be weak sequential cluster points of $(x_n)_{n \in \mathbb{N}} \in \operatorname{Fix} T_{[C_1 \dots C_N]}$, say $x_{n_k} \rightharpoonup x$ and $x_{n_l} \rightharpoonup y$

Then, by monotonicity the sequences $(\|x_n - x\|^2)_{n \in \mathbb{N}}$ and $(\|x_n - y\|^2)_{n \in \mathbb{N}}$ converge. Since

$$\|x - y\|^2 = \|x_n - y\|^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - y\rangle$$

$$\begin{aligned} \Rightarrow \|x - y\|^2 + 2\langle x_n, x - y\rangle &= \|x_n - y\|^2 - \|x_n - x\|^2 + 2\langle x_n, x - y\rangle \\ &\quad - 2\langle x_n - x, x - y\rangle \\ &= \|x_n - y\|^2 - \|x_n - x\|^2 + 2\langle x, x - y\rangle \end{aligned}$$

$$\begin{aligned} 2\langle x_n, x - y\rangle &= \|x_n - y\|^2 - \|x_n - x\|^2 + 2\langle x, x - y\rangle - \|x - y\|^2 \\ &= \|x_n - y\|^2 - \|x_n - x\|^2 + 2\langle x, x - y\rangle - \langle x - y, x - y\rangle \\ &= \|x_n - y\|^2 - \|x_n - x\|^2 + \|x\|^2 - \|y\|^2 \end{aligned}$$

$$(\forall n \in \mathbb{N}) \quad 2\langle x_n, x - y\rangle = \|x_n - y\|^2 - \|x_n - x\|^2 + \|x\|^2 - \|y\|^2 \tag{21}$$

$(\langle x_n, x - y\rangle)_{n \in \mathbb{N}}$ converges as well. Let $\langle x_n, x - y\rangle \to m$. Taking the limit along (x_{n_k}) and (x_{n_l}) repectively, we have

$$m = \langle x, x - y\rangle = \langle y, x - y\rangle$$

Therefore,

$$\|x - y\|^2 = 0.$$

Hence, $T^n_{[C_1 C_2 \dots C_N]} x_0 \rightharpoonup x \in \operatorname{Fix} T_{[C_1 \dots C_N]}$ and from Section 4

$$T^n_{[C_1 C_2 \dots C_N]} x_0 \rightharpoonup x \in \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i, C_{i+1}}.$$

Finally, by using Section 2 and Section 2, we get that shadow sequence $(P_{\operatorname{Fix} T_{[C_1 \dots C_N]}})_{n \in \mathbb{N}}$ converges strongly to a point in $x \in \operatorname{Fix} T_{[C_1 \dots C_N]} = \bigcap_{i=1}^{N+1} \operatorname{Fix} T_{C_i, C_{i+1}}$. ■

Proposition 3. Let $C_1, C_2 \cdots C_N \subseteq \mathcal{H}$ be closed and convex set with non empty intersection. Recall from Definition 5 that

$$T_{C_1,C_2,\ldots,C_n} := \frac{1}{2}\big(\operatorname{Id} + R_{C_n} R_{C_{n-1}} \ldots R_{C_2} R_{C_1}\big).$$

If $x \in C_i$, then

$$T_{C_i,C_{i+1}} x = \mathrm{P}_{C_{i+1}} x. \tag{22}$$

Proof. Let $x \in C_i$, then

$$\begin{aligned} T_{C_i,C_{i+1}} x &= 2^{-1}\big(x + R_{C_{i+1}} R_{C_i} x\big) \\ &\overset{**}{=} 2^{-1}\big(x + R_{C_{i+1}} x\big) \\ &= 2^{-1}\big(x + 2\,\mathrm{P}_{C_{i+1}} x - x\big) \quad \text{by (4)} \\ &= \mathrm{P}_{C_{i+1}} x, \end{aligned}$$

as required. ∎

Lemma 1. Let C_1 and C_2 be two closed affine subspaces. Recall from (13) that

$$T_{[C_1 C_2 \ldots C_N]} := T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_2,C_3} T_{C_1,C_2}.$$

Then

$$T_{[C_1 C_2]} = 2^{-1}\big(T_{C_1,C_2} + T_{C_2,C_1}\big).$$

Proof. Using (13) with $N = 2$ and (11) give

$$\begin{aligned} T_{[C_1 C_2]} &= T_{C_2,C_1} T_{C_1,C_2} \\ &= 2^{-1}\big(\operatorname{Id} + R_{C_1} R_{C_2}\big) T_{C_1,C_2} \\ &= 2^{-1}\big(T_{C_1,C_2} + R_{C_1} R_{C_2} T_{C_1,C_2}\big) \\ &= 2^{-1}\Big(T_{C_1,C_2} + R_{C_1} R_{C_2}\Big(2^{-1}\big(\operatorname{Id} + R_{C_2} R_{C_1}\big)\Big)\Big) \\ &= 2^{-1}\Big(T_{C_1,C_2} + R_{C_1}\Big(2^{-1}\big(R_{C_2} + R_{C_2} R_{C_2} R_{C_1}\big)\Big)\Big) \\ &\overset{\dagger\dagger}{=} 2^{-1}\Big(T_{C_1,C_2} + 2^{-1}\big(R_{C_1} R_{C_2} + R_{C_1} R_{C_1}\big)\Big) \\ &= 2^{-1}\Big(T_{C_1,C_2} + 2^{-1}\big(R_{C_1} R_{C_2} + \operatorname{Id}\big)\Big) \\ &= 2^{-1}\big(T_{C_1,C_2} + T_{C_2,C_1}\big). \end{aligned}$$

∎

Proposition 4. When $x_0 \in C_1$, the cyclic Douglas- Rachford method coincides with alternating projection method.

** By assumption that $x \in C_i$

†† By using the fact that C_2 is closed an affine therefore $R_{C_2}^2 = \operatorname{Id}$. Similarly for C_1.

Proof. Let $x_0 \in C_1$, then by (13) we have,

$$T_{[C_1 C_2 \ldots C_{N-1} C_N]} x_0 = T_{C_N, C_1} T_{C_{N-1}, C_N} \ldots T_{C_1, C_2} x_0$$

Note:

$$\begin{aligned} T_{C_1,C_2} x_0 &= 2^{-1}\Big(x_0 + R_{C_2}(R_{C_1} x_0)\Big) \\ &= 2^{-1}\Big(x_0 + R_{C_2} x_0\Big) \\ &= 2^{-1}\Big(x_0 + 2P_{C_2} x_0 - x_0\Big) \\ &= P_{C_2} x_0 \in C_2 \end{aligned}$$

$$\begin{aligned} T_{C_2,C_3}(P_{C_2} x_0) &= \frac{1}{2}\Big(P_{C_2} x_0 + R_{C_3}(R_{C2}(P_{C_2} x_0))\Big) \\ &= \frac{1}{2}\Big(P_{C_2} x_0 + R_{C_3}(P_{C_2} x_0)\Big) \\ &= \frac{1}{2}\Big(P_{C_2} x_0 + 2P_{C_3}(P_{C_2} x_0) - P_{C_2} x_0\Big) \\ &= P_{C_3} P_{C_2} x_0 \in C_3 \end{aligned}$$

Keep doing that we have

$$T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_2,C_3} \mathrm{P}_{C_2}\, x0 \overset{(22)}{=} \mathrm{P}_{C_1}\, \mathrm{P}_N \ldots \mathrm{P}_{C_3}\, \mathrm{P}_2\, x_0 \ \in C_1$$

■

The next example indicates that if $x_0 \notin C_1$, then the cyclic Douglas–Rachford iteration need not coincide with von Neumann's alternating projection method.

Example 3. Let $C_1 = \{x \in \mathcal{H} \mid \langle a, x\rangle \le 0\}$, and $C_2 = \{x \in \mathcal{H} \mid \langle a, x\rangle = 0\}$, where $a \in \mathcal{H}$ and $\|a\| = 1$. If $x_0 \notin C_1 \cup C_2$, then $\Big\langle a, T_{[C_1 C_2]} x\Big\rangle \neq 0$.

Proof. The projection to C_1 and C_2, see [1, Example 28.15 and Example 28.16], are

$$\mathrm{P}_{C_1}\, x = \begin{cases} x - \langle a, x\rangle\, a & \text{if } \langle a, x\rangle > 0 \\ x & \text{if } \langle a, x\rangle \le 0 \end{cases}$$

and ,

$$\mathrm{P}_{C_2}\, x = x - \langle a, x\rangle\, a$$

$$T_{C_1,C_2} = 2^{-1}\big(x + R_{C_2} R_{C_1} x\big). \tag{23}$$

$$R_{C_1} x = \begin{cases} x - 2\langle a, x\rangle\, a & \text{if } \langle a, x\rangle > 0 \\ x & \text{if } \langle a, x\rangle \le 0 \end{cases}$$

and,

$$R_{C_2}(R_{C_1}x) = \begin{cases} x & \text{if } \langle a, x \rangle > 0 \\ x - 2\langle a, x \rangle a & \text{if } \langle a, x \rangle \leq 0 \end{cases}$$

Then plugging this result in (23) we get that

$$T_{C_1,C_2}x = \begin{cases} x & \text{if } \langle a, x \rangle > 0 \\ x - \langle a, x \rangle a & \text{if } \langle a, x \rangle \leq 0 \end{cases}$$

Similarly,

$$T_{C_2,C_1}x = \begin{cases} x & \text{if } \langle a, x \rangle > 0 \\ x - \langle a, x \rangle a & \text{if } \langle a, x \rangle \leq 0 \end{cases}$$

Then, by Lemma 1 we have,

$$2\left\langle a, T_{[C_1C_2]}x \right\rangle = 2\left\langle a, 2^{-1}(T_{C_1,C_2} + T_{C_2,C_1})x \right\rangle = \langle a, T_{C_1,C_2}x \rangle + \langle a, T_{C_2,C_1}x \rangle \tag{24}$$

When $\langle a, x \rangle > 0$;

$$\left\langle a, T_{[C_1C_2]} \right\rangle \overset{(24)}{=} \langle a, x \rangle + \langle a, x \rangle = 2\langle a, x \rangle . \tag{25}$$

When $\langle a, x \rangle \leq 0$;

$$\left\langle a, T_{[C_1C_2]} \right\rangle \overset{(24)}{=} \langle a, x \rangle - \langle a, x \rangle \|a\|^2 + \langle a, x \rangle - \langle a, x \rangle \|a\|^2 = 0. \tag{26}$$

Therefore (25) and (26) show that if $x_0 \notin C_1 \cup C_2$ then the Douglas- Rachford iterates will not lie in C_1 or C_2. Hence, if $\langle a, x \rangle \nleq 0$, then $\left\langle a, T_{[C_1C_2]} \right\rangle \neq 0$. ■

4.1. A product version of the Cyclic Douglas–Rachford method

Consider the Hilbert space $\mathcal{H}^{\mathcal{N}} = \mathcal{H} \times \mathcal{H} \times \cdots \times \mathcal{H}$. Define two closed and convex subsets C and D of $\mathcal{H}^{\mathcal{N}}$, where $C \cap D \neq \varnothing$, by

$$C := \{(x_1, x_2, \ldots, x_n) \in \mathcal{H}^{\mathcal{N}} \mid x_i \in C_i\},$$

and

$$D := \{(x, x, \ldots, x) \in \mathcal{H}^{\mathcal{N}} \mid x \in \mathcal{H}\}.$$

(1) will be solved by find $x \in (C \cap D) \subseteq \mathcal{H}^{\mathcal{N}}$. The projection on C and D will be

$$\mathrm{P}_C = (\mathrm{P}_{C_1}x_1, \mathrm{P}_{C_2}x_2, \ldots, \mathrm{P}_{C_{\mathcal{N}}}x_{\mathcal{N}}),$$

and

$$\mathrm{P}_D = (\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x_i, \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x_i, \ldots, \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x_i).$$

See [1, Proposition 25.4(iii) and (iv)] for more details.

Lemma 2. The iteration for the product version of the cyclic Douglas-Rachford method will be define as

$$T_{[D\ C]}x = x - \mathrm{P}_D\, x + 2\,\mathrm{P}_D\,\mathrm{P}_C\, T_{D,C}x - \mathrm{P}_C\, T_{D,C}x + \mathrm{P}_C\, R_D x - \mathrm{P}_D\,\mathrm{P}_C\, R_D x. \tag{27}$$

Proof.

Then

$$\begin{aligned} R_D R_C T_{DC}x &= (2\,\mathrm{P}_D - \mathrm{Id})R_C T_{DC}x = 2\,\mathrm{P}_D\, R_C T_{DC}x - R_C T_{D,C}x \\ &= 2\,\mathrm{P}_D(2\,\mathrm{P}_C - \mathrm{Id})T_{DC}x - R_C T_{DC}x \\ &= 4\,\mathrm{P}_D\,\mathrm{P}_C\, T_{DC}x - 2\,\mathrm{P}_D\, T_{DC}x - R_C T_{DC}x \\ &= 4\,\mathrm{P}_D\,\mathrm{P}_C\, T_{DC}x - 2\,\mathrm{P}_D\, T_{DC}x - (2\,\mathrm{P}_C - \mathrm{Id})T_{DC}x \\ &= 4\,\mathrm{P}_D\,\mathrm{P}_C\, T_{DC}x - 2\,\mathrm{P}_D\, T_{DC}x - 2\,\mathrm{P}_C\, T_{DC}x + T_{DC}x \end{aligned} \tag{28}$$

Using the result from (28) in (**??**), we have

$$T_{[D\ C]}x = T_{D,C}x + 2\,\mathrm{P}_D\,\mathrm{P}_C\, T_{D,C}x - \mathrm{P}_D\, T_{D,C}x - \mathrm{P}_C\, T_{D,C}x \tag{29}$$

However,

$$T_{DC}x = 2^{-1}\big(x + R_C R_D x\big) \tag{30}$$

$$= 2^{-1}\big(x + (2\,\mathrm{P}_C - \mathrm{Id})R_D x\big) \tag{31}$$

$$= 2^{-1}\big(x + 2\,\mathrm{P}_C\, R_D x - R_D x\big) \tag{32}$$

$$= x - \mathrm{P}_D\, x - \mathrm{P}_C\, R_D x. \tag{33}$$

Moreover,

$$\begin{aligned} -\,\mathrm{P}_D\, T_{DC}x &= -2^{-1}\,\mathrm{P}_D\, x - 2^{-1}\,\mathrm{P}_D(R_C R_D x) \\ &= -2^{-1}\,\mathrm{P}_D\, x - 2^{-1}\,\mathrm{P}_D\,\Big((2\,\mathrm{P}_C - \mathrm{Id})R_D x\Big) \\ &= -2^{-1}\,\mathrm{P}_D\, x - \mathrm{P}_D\,\mathrm{P}_C\, R_D x + 2^{-1}\,\mathrm{P}_D\, R_D x \\ &= -2^{-1}\,\mathrm{P}_D\, x - \mathrm{P}_D\,\mathrm{P}_C\, R_D x + 2^{-1}\,\mathrm{P}_D(2\,\mathrm{P}_D - \mathrm{Id}) \\ &= -2^{-1}\,\mathrm{P}_D\, x - \mathrm{P}_D\,\mathrm{P}_C\, R_D x + \mathrm{P}_D\, x - 2^{-1}\,\mathrm{P}_D\, x \\ &= -\,\mathrm{P}_D\,\mathrm{P}_C\, R_D x \end{aligned} \tag{34}$$

By (33) and (34), the updated formula for equation (29) will be

$$T_{[D\ C]}x = x - \mathrm{P}_D\, x + 2\,\mathrm{P}_D\,\mathrm{P}_C\, T_{D,C}x - \mathrm{P}_C\, T_{D,C}x + \mathrm{P}_C\, R_D x - \mathrm{P}_D\,\mathrm{P}_C\, R_D x.$$

■

5. Clarification

There is no conflict of interest and there is no data were used to support this study. Moreover, I would like to bring to your attention that the work I am submitting is authored solely by myself.

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