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Tau Approach for the Time-Fractional Diffusion Equation Using Certain Chebyshev Polynomials

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Abstract. This research article presents a numerical technique for solving the time-fractional diffusion equation (TFDE). The approach uses certain Chebyshev polynomials as basis functions. These polynomials are particular cases of the generalized Gegenbauer polynomials. The operational matrices of integer and fractional derivatives are used, along with the tau method, for the spatial and temporal discretization. Hence, the problem with its underlying conditions is converted into a system of equations that can be handled. We investigate the convergence of the double Chebyshev expansion and derive rigorous error bounds. Numerical examples show the method's superior accuracy and efficiency over some existing methods in the literature.

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Key Words and Phrases: Time-fractional diffusion equation, Chebyshev polynomials, tau method, convergence analysis

1. Introduction

The fractional differentiation extends the standard ordinary differentiation. Fractional differential equations (FDEs), in contrast to ordinary differential equations (DEs), are systems that include memory effects and long-range temporal dependencies that are well-represented by FDEs. This is particularly useful when they have to explain complex real-life occurrences. For instance, they model anomalous diffusion and stress-strain correlations in viscoelastic materials in the physical sciences. Using FDEs, we can better

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understand how past conditions impact current processes in biological systems, such as the spread of illness or changes in population size. Control theory, signal analysis, and electrical circuit modeling with memory components are other engineering fields that benefit from FDEs. They are also employed in environmental studies to represent the flow in a porous or fractured material and in finance to illustrate market linkages spanning several years. Due to their increased modeling capabilities and broad applicability, FDEs have been a hot topic in scientific computing and applied mathematics. Some applications of FDEs can be found in [1–4].

Due to the non-availability of analytic solutions for most FDEs, it is necessary to design effective numerical solutions for these equations. There are several numerical approaches for solving FDEs. The authors in [5] analyzed a numerical algorithm to treat the fractional Lane-Emden equations based on using the Laplace transform. The authors in [6] treated a nonlinear system of FDES utilizing the Adomian decomposition method. The authors in [7] used the differential transform method for analyzing fractional dynamical systems. In [8], the authors designed a numerical scheme to treat the fractional Burgers' equation, which used the convolved Fermat polynomials as basis functions, enabling precise and computationally efficient solutions. The authors in [9] implemented a numerical approach for the time-fractional generalized Kawahara equation using the eighth-kind CPs. In [10], a shifted Jacobi collocation scheme was developed to solve multidimensional time-fractional telegraph equations. In [11], the authors investigated a (3+1)-dimensional fractional coupled Burgers' equation using a four-dimensional natural transform combined with the Adomian decomposition method. In addition, in [12], the authors presented a numerical scheme with adaptive step sizing to solve general FDEs. In [13], some analytical solutions of certain time-fractional heat order was treated.

Modeling several physical phenomena across several fields still depends much on the time diffusion equation. Recent studies have used sophisticated computational methods and investigated new areas, therefore broadening their relevance. In computational mathematics, several numerical approaches have been developed to solve the various forms of the diffusion equations more effectively. The authors in [14] proposed a kernel-based approach to treat the TFDE. In [15], a finite difference–collocation scheme was designed for the TFDE, which combines the robustness of finite difference schemes with the advantages of using the collocation method. The authors in [16] used iterative methods for the TFDE. The authors in [17] offered a difference scheme for solving the TFDE. In [18], the authors analyzed a numerical approach to the TFDE. Hosoya polynomial was utilized in [19] to handle a class of TFDE, contributing an algebraic perspective to the numerical analysis. A numerical scheme based on B-splines was presented in [20] for addressing the TFDE. The authors in [21] used an efficient method for distributed-order TFDE. In [22], an exponential-sum-approximation technique was followed to tackle variable-order TFDE. The authors in [23] proposed analytical solutions for both time-fractional diffusion and convection–diffusion equations. Other contributions regarding some types of the TFDE can be found in [24–27].

Chebyshev polynomials (CPs) have vital roles in several branches of the applied sciences. In particular, they are extremely significant in areas like approximation theory, numerical analysis, and applied mathematics. This versatility is due to their exceptional approximation properties; see, for example, [28, 29]. These polynomials are widely used in spectral and pseudospectral methods to solve various differential and integral equations. The most used kinds of CPs are the first and second kinds; see, for example, [30–33]. Other publications were devoted to the use of the third and fourth kinds. These four kinds are all particular ones of the classical Jacobi polynomials [34]. In addition to the classical Jacobi polynomials, there are other types of CPs. Masjed-Jamei, in his Ph.D. thesis [35], introduced two other trigonometric polynomials; he called them CPs of the fifth and sixth kinds. Various papers employed these polynomials; see, for example, [36, 37]. In addition, there are two other types of CPs, namely, CPs of the seventh and eighth kinds. They are specific polynomials of the generalized Gegenbauer polynomials. In [38], the seventh kind of CPs was used to solve the fractional diffusion equation, while the authors of [39] used the eighth kind of CPs to solve the nonlinear time-fractional generalized Kawahara equation. Other CPs were proposed and used to handle the FitzHugh-Nagumo equation in [40].

Spectral methods are numerical techniques employed to solve different types of differential equations (DEs). The philosophy of applying this method is based on expanding the solution in terms of global basis functions such as trigonometric functions (Fourier series), orthogonal polynomials, or any other complete function systems. Spectral methods are able to attain exponential or "spectral" convergence rates, in contrast to the algebraic convergence that is produced by classic finite difference or finite element approaches that rely on local approximations. Because of this, they arise in fields like quantum physics, fluid dynamics, and wave propagation. For some applications of spectral methods, one can refer to [41–44] . Galerkin, collocation (or pseudospectral), and tau are the main spectral methods. The collocation approach, on the other hand, uses discrete places—the collocation points—to ensure that the differential equation is fulfilled. These collocation points are often chosen to represent the roots of orthogonal polynomials; see [45–49]. In the Galerkin method, we use basis functions and enforce the residual of the equation to be orthogonal to them; see, for example [50–54], The tau approach is more flexible than the Galerkin approach in choosing the basis functions, since there are no restrictions for the choice of them, and the underlying conditions are assumed to be as constraints; see, for example, [55–58].

The main objective of this article is to use certain CPs, which are particular polynomials of the generalized Gegenbauer polynomials, to solve the TFDE by applying the tau method. Theoretical and numerical studies of the errors obtained are given.

The main contribution of the paper can be summarized in the following points:

- Constructing the integer and fractional derivatives of the Chebyshev polynomials.
- Applying the spectral tau method for spatial and temporal discretization.
- Investigating the convergence and error analysis of the proposed expansion.
- Testing the numerical algorithm by presenting some numerical results supported by some comparisons.

The advantages of the proposed algorithm can be listed as follows:

- Obtaining highly accurate approximations using a few terms of the basis functions.
- Reducing the equation governed by its conditions into a system of equations that can be efficiently handled.
- The capability of extending the tau approach for treating other types of FDEs.

The paper follows the following structure: In the next section, we present the necessary mathematical preliminaries, including some properties of Caputo's fractional derivative, and also some formulas concerned with the specific CPs that we will employ. Section 3 is interested in analyzing in detail the spectral Tau method for the TFDE using the proposed CPs. In Section 4, a thorough error analysis of the proposed Chebyshev expansion is conducted. Furthermore, we derive theoretical bounds for the double expansion. Section 5 ensures the applicability and accuracy of the presented algorithm through displaying several numerical experiments supported by some comparisons. Finally, concluding remarks are reported in Section 6.

2. Some fundamentals and formulas

This section is confined to displaying an overview of the fractional calculus. Furthermore, an account of the generalized Gegenbauer polynomials is given. Particular CPs that will be used as basis functions are introduced. Some of their formulas that are pivotal to deriving our proposed numerical algorithm are also provided.

2.1. Caputo's fractional derivative

Definition 1. *[1], A fractional-order derivative, according to Caputo, is defined as*

$$D_t^{\nu}\psi(s) = \frac{1}{\Gamma(k-\nu)}\int_0^s (s-t)^{k-\nu-1}\psi^{(k)}(t)dt, \quad \nu>0, \quad s>0,\ k-1<\nu\le k, \quad k\in\mathbb{N}. \tag{1}$$

For D_t^{ν} with $k-1<\nu\le k, \quad k\in\mathbb{N}$, the following identities hold:

$$D_t^{\nu} C = 0, \quad C \text{ is a constant}, \tag{2}$$

$$D_t^{\nu}\, s^k = \begin{cases} 0, & \text{if } k\in\mathbb{N}_0 \quad and \quad k<\lceil\nu\rceil, \\ \frac{(k)!}{\Gamma(k-\nu+1)}\, s^{k-\nu}, & \text{if } k\in\mathbb{N}_0 \quad and \quad k\ge\lceil\nu\rceil, \end{cases} \tag{3}$$

where $\mathbb{N}=\{1,2,...\}$ and $\mathbb{N}_0=\{0,1,2,\ldots\}$, and $\lceil\nu\rceil$ is the ceiling function.

2.2. An account on generalized Gegenbauer polynomials

We give an account of the generalized Gegenbauer polynomials $G_k^{(\mu,\beta)}(x)$. They are orthogonal on $[-1,1]$ regarding $w(x) = (1-x^2)^{\mu-\frac{1}{2}}\,|x|^{2\,\beta}$. They can be represented as [59, 60]

$$G_k^{(\mu,\beta)}(x) = \begin{cases} \dfrac{(\mu+\beta)_{\frac{k}{2}}}{\left(\beta+\frac{1}{2}\right)_{\frac{k}{2}}}\,P_{\frac{k}{2}}^{\left(\mu-\frac{1}{2},\beta-\frac{1}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ is even,}\\ \dfrac{(\mu+\beta)_{\frac{k+1}{2}}}{\left(\beta+\frac{1}{2}\right)_{\frac{k+1}{2}}}\,xP_{\frac{k-1}{2}}^{\left(\mu-\frac{1}{2},\beta+\frac{1}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ is odd,}\end{cases} \tag{4}$$

where $(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}$ denotes the Pochhammer symbol and $P_k^{(\mu,\beta)}(x)$ are the standard Jacobi polynomials.
The orthogonality relation for $G_k^{(\mu,\beta)}(x)$ is

$$\int_{-1}^{1} w(x)\,G_k^{(\mu,\beta)}(x)\,G_m^{(\mu,\beta)}(x)\,dx = \begin{cases} h_k^{\mu,\beta}, & k=m,\\ 0, & k\neq m,\end{cases} \tag{5}$$

where $h_k^{\mu,\beta}$ is given by

$$h_k^{\mu,\beta} = \frac{\left(\Gamma\left(\beta+\frac{1}{2}\right)\right)^2}{(k+\mu+\beta)\left(\Gamma(\mu+\beta)\right)^2}\begin{cases} \dfrac{\Gamma\left(\mu+\frac{k+1}{2}\right)\Gamma\left(\frac{k}{2}+\mu+\beta\right)}{\left(\frac{k}{2}\right)!\,\Gamma\left(\frac{1+k}{2}+\beta\right)}, & k \text{ even,}\\ \dfrac{\Gamma\left(\frac{k}{2}+\mu\right)\Gamma\left(\frac{k}{2}+\mu+\beta+\frac{1}{2}\right)}{\left(\frac{k-1}{2}\right)!\,\Gamma\left(\frac{k}{2}+\beta+1\right)}, & k \text{ odd.}\end{cases} \tag{6}$$

Remark 1. *Several particular polynomials, including CPs, are particular ones of the generalized polynomials $G_k^{(\mu,\beta)}(x)$, see [40]. The authors in the same paper introduced certain CPs as particular ones of $G_k^{(\mu,\beta)}(x)$, and derived some formulas regarding them.*

2.3. Some properties and formulas of a certain kind of CPs

The authors in [40] proposed the polynomials $\mathcal{N}_k(x) = G_k^{2,1}(x)$. They are defined as

$$\mathcal{N}_k(x) = \begin{cases} \dfrac{(3)_{\frac{k}{2}}}{\left(\frac{3}{2}\right)_{\frac{k}{2}}}\,P_{\frac{k}{2}}^{\left(\frac{3}{2},\frac{1}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ even,}\\ \dfrac{x(3)_{\frac{k+1}{2}}}{\left(\frac{3}{2}\right)_{\frac{k+1}{2}}}\,P_{\frac{k-1}{2}}^{\left(\frac{3}{2},\frac{3}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ odd.}\end{cases} \tag{7}$$

$\{\mathcal{N}_k(x)\}_{k\geq 0}$ are orthogonal on $[-1,1]$ in the sense that

$$\int_{-1}^{1} x^2\left(1-x^2\right)^{3/2}\,\mathcal{N}_k(x)\,\mathcal{N}_j(x)\,dx = \mathfrak{h}_k, \tag{8}$$

where

$$\mathfrak{h}_n = \frac{\pi}{128} \begin{cases} (k+2)\,(k+4), & \text{if } k=j, \quad k \text{ even}, \\ (k+1)\,(k+5), & \text{if } k=j, \quad k \text{ odd}, \\ 0, & \text{if } k \neq j. \end{cases} \tag{9}$$

The following two lemmas give the explicit analytic form of $\mathcal{N}_k(x)$ and its inversion formula.

Lemma 1. *[40] For every positive integer s, the polynomials $\mathcal{N}_s(x)$ have the following expression:*

$$\mathcal{N}_s(x) = \sum_{m=0}^{\left\lfloor \frac{s}{2} \right\rfloor} \frac{(-1)^m\,(3)_{s-m}}{m!\,\left(\left\lfloor \frac{s}{2} \right\rfloor - m\right)!\,\left(\frac{3}{2}\right)_{-m+\left\lfloor \frac{1+s}{2} \right\rfloor}}\, x^{s-2m}. \tag{10}$$

Lemma 2. *[40] For every positive integer s, x^s can be expanded as*

$$x^s = \sum_{m=0}^{\left\lfloor \frac{s}{2} \right\rfloor} \frac{(s-2\,m+3)\,\left\lfloor \frac{s}{2} \right\rfloor!\,\left(\frac{3}{2}\right)_{\left\lfloor \frac{s+1}{2} \right\rfloor}}{m!\,(3)_{s-m+1}}\, \mathcal{N}_{s-2\,m}(x). \tag{11}$$

Now, we give the high-order derivatives expression for the polynomials $G_k^{(\mu,\beta)}(x)$ in terms of their original polynomials.

Theorem 1. *[40] The m-th derivative: $\dfrac{d^k\,\mathcal{N}_s(x)}{d\,x^k}$ can be represented as*

$$\frac{d^m\,\mathcal{N}_s(x)}{d\,x^m} = \sum_{r=0}^{\left\lfloor \frac{s-m}{2} \right\rfloor} \mathcal{G}_{r,s}^m\,\mathcal{N}_{s-m-2r}(x), \;\; s \geq m, \tag{12}$$

where

$$\mathcal{G}^{m}_{r,s}=2^{m}\begin{cases}\dfrac{(1+s-m)(3+s-2r-m)(s+2)!}{(s+1)!r!(s-r-m+3)!}\times\\ {}_4F_3\left(\begin{matrix}-r,-\frac{1}{2}-\frac{s}{2},\frac{1}{2}-\frac{s}{2}+\frac{m}{2},-3-s+r+m\\ -2-s,\frac{1}{2}-\frac{s}{2},-\frac{1}{2}-\frac{s}{2}+\frac{m}{2}\end{matrix}\middle|1\right), & s\ \textit{even}, m\ \textit{even},\\[2ex] \dfrac{(-1-s+m)(-3-s+2r+m)(s+1)!}{r!\,(s-r-m+3)!}\times\\ {}_4F_3\left(\begin{matrix}-r,-1-\frac{s}{2},\frac{1}{2}-\frac{s}{2}+\frac{m}{2},-3-s+r+m\\ -2-s,-\frac{s}{2},-\frac{1}{2}-\frac{s}{2}+\frac{m}{2}\end{matrix}\middle|1\right), & s\ \textit{odd}, m\ \textit{odd},\\[2ex] \dfrac{(-2-s+m)(-3-s+2r+m)(s+1)!}{r!\,(s-r-m+3)!}\times\\ {}_4F_3\left(\begin{matrix}-r,-1-\frac{s}{2},-\frac{s}{2}+\frac{m}{2},-3-s+r+m\\ -2-s,-\frac{s}{2},-1-\frac{s}{2}+\frac{m}{2}\end{matrix}\middle|1\right), & s\ \textit{odd}, m\ \textit{even},\\[2ex] \dfrac{(2+s-m)(3+s-2r-m)(s+2)!}{(1+s)\,r!\,(s-r-m+3)!}\times\\ {}_4F_3\left(\begin{matrix}-r,-\frac{1}{2}-\frac{s}{2},-\frac{s}{2}+\frac{m}{2},-3-s+r+m\\ -2-s,\frac{1}{2}-\frac{s}{2},-1-\frac{s}{2}+\frac{m}{2}\end{matrix}\middle|1\right), & s\ \textit{even}, m\ \textit{odd}.\end{cases} \tag{13}$$

2.4. The shifted CPs

Now, we introduce the corresponding shifted polynomials on $[0,1]$ to the polynomials $\mathcal{N}_k(x)$. We denote them by $\mathcal{H}^*_k(x)$ and define them as

$$\mathcal{H}^*_k(x)=\mathcal{N}_k(2\,x-1). \tag{14}$$

All the formulas of the polynomials $\mathcal{N}_k(x)$ can be transformed to give their counterparts on $[0,1]$. From (8), the set $\{\mathcal{H}^*_k(x)\}_{k\geq 0}$ is an orthogonal set on $[0,1]$ with

$$\int_0^1 \mathcal{H}^*_n(x)\,\mathcal{H}^*_m(x)\,w^*(x)\,dx=\frac{1}{16}\,\mathfrak{h}_n, \tag{15}$$

with $w^*(x)=(1-2x)^2(x(1-x))^{3/2}$, and $\mathfrak{h}_n$ is defined in (9).

Theorem 2. *The mth-derivative of $\mathcal{H}^*_j(x)$ can be expressed as*

$$\frac{d^m\,\mathcal{H}^*_s(x)}{d\,x^m}=\sum_{r=0}^{\lfloor\frac{s-m}{2}\rfloor}\tilde{\mathcal{B}}^m_{r,s}\,\mathcal{H}^*_{s-m-2r}(x),\ \ s\geq m, \tag{16}$$

with

$$\tilde{\mathcal{B}}^{k}_{r,j} = 2^{k}\, \mathcal{G}^{k}_{r,j},$$

where $\mathcal{G}^{k}_{r,j}$ *is given in* (13).

Proof. Replacing x by $(2\,x-1)$ in Theorem 1, we get the result in (16).

3. Tau procedure for the TFDE

Consider the following TFDE [20, 61]:

$$D_t^{\zeta}\lambda(x,t) - \beta\,\lambda_{xx}(x,t) = g(x,t), \quad 0<\zeta\le 1, \tag{17}$$

governed by the following conditions:

$$\lambda(x,0) = \rho_1(x), \quad 0<x<1, \tag{18}$$

$$\lambda(0,t) = \rho_2(t), \quad \lambda(1,t) = \rho_3(t), \quad 0<t<\tau, \tag{19}$$

where β is a positive constant, $\rho_1(x)$, $\rho_2(t)$, and $\rho_3(t)$ are known continuous functions, and $g(x,t)$ is the source term.
If we let

$$\mathcal{P}^{\mathcal{M}} = \text{span}\{\mathcal{H}^*_i(x)\,\mathcal{H}^*_j(t) : 0\le i,j\le \mathcal{M}\},$$

then; any function $\lambda^{\mathcal{M}}(x,t)\in\mathcal{P}^{\mathcal{M}}$ can be represented as

$$\lambda^{\mathcal{M}}(x,t) = \sum_{i=0}^{\mathcal{M}}\sum_{j=0}^{\mathcal{M}} c_{ij}\,\mathcal{H}^*_i(x)\,\mathcal{H}^*_j(t) = \boldsymbol{\mathcal{H}}^{*}(x)\,\boldsymbol{C}\,\boldsymbol{\mathcal{H}}^{*}(t)^T, \tag{20}$$

where $\boldsymbol{\mathcal{H}}^{*}(t)^T = [\mathcal{H}^*_0(t), \mathcal{H}^*_1(t), \ldots, \mathcal{H}^*_{\mathcal{M}}(t)]^T$, and $\boldsymbol{C} = (c_{ij})_{0\le i,j\le\mathcal{M}}$ is the $(\mathcal{M}+1)^2$-dimensional matrix of unknowns.
The residual $\mathbf{R}_{\mathcal{M}}(x,t)$ of Eq (17) is given by

$$\mathbf{R}_{\mathcal{M}}(x,t) = D_t^{\zeta}\lambda^{\mathcal{M}}(x,t) - \beta\,\lambda^{\mathcal{M}}_{xx}(x,t) - g(x,t). \tag{21}$$

If we apply the tau method, then we get

$$\int_0^1\int_0^1 \mathbf{R}_{\mathcal{M}}(x,t)\,\mathcal{H}^*_r(x)\,\mathcal{H}^*_s(t))w^*(x)\,w^*(t)\,dx\,dt = 0, \quad 1\le r\le\mathcal{M}-1, \quad 1\le s\le\mathcal{M}. \tag{22}$$

Now, consider the following matrices:

$$\boldsymbol{G} = (g_{r,s})_{(\mathcal{M}-1)\times\mathcal{M}}, \qquad g_{r,s} = \int_0^1\int_0^1 g(x,t)\,\mathcal{H}^*_r(x)\,\mathcal{H}^*_s(t)\,w^*(x)\,w^*(t)\,dx\,dt, \tag{23}$$

$$\boldsymbol{\mathcal{Z}} = (z_{i,r})_{(\mathcal{M}+1)\times(\mathcal{M}-1)}, \qquad z_{i,r} = \int_0^1 \mathcal{H}^*_i(x)\,\mathcal{H}^*_r(x)\,w^*(x)\,dx, \tag{24}$$

$$\boldsymbol{\mathcal{B}} = (b_{j,s})_{(\mathcal{M}+1)\times\mathcal{M}}, \qquad b_{j,s} = \int_0^1 \mathcal{H}_j^*(t)\, \mathcal{H}_s^*(t)\, w^*(t)\, dt, \tag{25}$$

$$\boldsymbol{\mathcal{Y}} = (y_{i,r})_{(\mathcal{M}+1)\times(\mathcal{M}-1)}, \qquad y_{i,r} = \int_0^1 \frac{d^2\, \mathcal{H}_i^*(x)}{d\, x^2}\, \mathcal{H}_r^*(x)\, w^*(x)\, dx, \tag{26}$$

$$\boldsymbol{\mathcal{K}} = (k_{j,s})_{(\mathcal{M}+1)\times\mathcal{M}}, \qquad k_{j,s} = \int_0^1 D_t^{\zeta} \mathcal{H}_j^*(t)\, \mathcal{H}_s^*(t)\, w^*(t)\, dt. \tag{27}$$

Therefore, Eq. (22) can be rewritten as

$$\sum_{i=0}^{\mathcal{M}} \sum_{j=0}^{\mathcal{M}} c_{ij}\, z_{i,r}\, k_{j,s} - \beta \sum_{i=0}^{\mathcal{M}} \sum_{j=0}^{\mathcal{M}} c_{ij}\, y_{i,r}\, b_{j,s} = f_{r,s}, \quad 0 \le r \le \mathcal{M}-2, \quad 0 \le s \le \mathcal{M}-1, \tag{28}$$

or in the following matrix form:

$$\boldsymbol{\mathcal{Z}}^T\, \boldsymbol{C}\, \boldsymbol{\mathcal{K}} - \beta\, \boldsymbol{\mathcal{Y}}^T \boldsymbol{C}\, \boldsymbol{\mathcal{B}} = \boldsymbol{G}.$$

In addition, the conditions in (18) and (19) lead to the following equations:

$$\sum_{i=0}^{\mathcal{M}} \sum_{j=0}^{\mathcal{M}} c_{ij}\, a_{i,r}\, \mathcal{H}_j^*(0) = \int_0^1 \rho_1(x)\, \mathcal{H}_r^*(x)\, w^*(x)\, dx, \quad 0 \le r \le \mathcal{M}, \tag{29}$$

$$\sum_{i=0}^{\mathcal{M}} \sum_{j=0}^{\mathcal{M}} c_{ij}\, a_{j,s}\, \mathcal{H}_i^*(0) = \int_0^1 \rho_2(t)\, \mathcal{H}_s^*(t)\, w^*(t)\, dt, \quad 0 \le s \le \mathcal{M}-1, \tag{30}$$

$$\sum_{i=0}^{\mathcal{M}} \sum_{j=0}^{\mathcal{M}} c_{ij}\, a_{j,s}\, \mathcal{H}_i^*(1) = \int_0^1 \rho_3(t)\, \mathcal{H}_s^*(t)\, w^*(t)\, dt, \quad 0 \le s \le \mathcal{M}-1. \tag{31}$$

The resultant algebraic system of equations, consisting of Eqs. (28)-(31), has an order of $(\mathcal{M}+1)^2$ and may be solved using an appropriate method.

Now, we give an explicit form for the elements of the matrices $\boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{Y}}$.

Theorem 3. *The elements $z_{i,r}$, $b_{j,s}$, $y_{i,r}$, and $k_{j,s}$ have the following forms:*

$$z_{i,r} = \int_0^1 \mathcal{H}_i^*(x)\, \mathcal{H}_r^*(x)\, w^*(x)\, dx = \frac{1}{16}\, \mathfrak{h}_i, \tag{32}$$

$$b_{j,s} = \int_0^1 \mathcal{H}_j^*(t)\, \mathcal{H}_s^*(t)\, w^*(t)\, dt = \frac{1}{16}\, \mathfrak{h}_j, \tag{33}$$

$$y_{i,j} = \int_0^1 \frac{d^2 \mathcal{H}_i^*(x)}{d\theta^2} \mathcal{H}_j^*(x)\, w^*(x)\, dx = \frac{1}{16} \sum_{r=0}^{\left\lfloor \frac{i-2}{2} \right\rfloor} \tilde{\mathcal{B}}_{r,i}^2\, \mathfrak{h}_{i-k-2r}, \tag{34}$$

and

$$
\begin{aligned}
k_{j,s} &= \int_0^1 D_t^{\zeta}\mathcal{H}_j^*(t)\mathcal{H}_s^*(t)\, w^*(t)\, dt \\
&= \sum_{r=1}^{j}\sum_{m=0}^{j}\sum_{p=0}^{s}\sum_{n=0}^{s} \frac{2^r r!(-1))^{\frac{j-m}{2}}(-1)^{m-r} a(j+m)\binom{m}{r}(3)_{\frac{j+m}{2}}}{\left(\frac{j-m}{2}\right)!\Gamma(r-\alpha+1)\left(\frac{3}{2}\right)_{\frac{m-j}{2}+\left\lfloor\frac{j+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(-j+m+2)+\left\lfloor\frac{j}{2}\right\rfloor\right)}\times \\
&\frac{3\sqrt{\pi}2^{p-2}(-1)^{n-p}(-1)^{\frac{s-n}{2}} a(n+s)\binom{n}{p}(3)_{\frac{n+s}{2}}\left(\alpha^2-\alpha(2p+2r+1)+(p+r)^2+p+r+5\right)}{\frac{s-n}{2}!\left(\frac{3}{2}\right)_{\frac{n-s}{2}+\left\lfloor\frac{s+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(n-s+2)+\left\lfloor\frac{s}{2}\right\rfloor\right)}\times \\
&\frac{\Gamma\left(p+r-\alpha+\frac{5}{2}\right)}{\Gamma(p+r-\alpha+7)}.
\end{aligned}
\tag{35}
$$

Proof. With the direct application of the orthogonality relation (15), we can easily acquire the elements $z_{i,r}$ and $b_{j,s}$. With the direct application of (15) with (16), we can easily obtain the elements $y_{i,j}$.
Now, we will compute $k_{j,s}$,
Based on Eq. (10), the power form representation of $\mathcal{N}_j(t)$ as can be rewritten as

$$
\mathcal{N}_j(t) = \sum_{r=0}^{j} \frac{(-1)^{\frac{j-r}{2}} a_{j+r}\,(3)_{j-\frac{j-r}{2}}}{\left(\frac{j-r}{2}\right)!\,\left(\frac{3}{2}\right)_{\left\lfloor\frac{j+1}{2}\right\rfloor-\frac{j-r}{2}}\,\Gamma\left(-\frac{1}{2}(j-r)+\left\lfloor\frac{j}{2}\right\rfloor+1\right)}\, t^r, \tag{36}
$$

where

$$
a_r = \begin{cases} 1, & \text{if } r \text{ even}, \\ 0, & \text{otherwise.} \end{cases} \tag{37}
$$

and hence, $\mathcal{H}_j^*(t)$ given in (14) may be represented as

$$
\mathcal{H}_j^*(t) = \sum_{r=0}^{j} \frac{(-1)^{\frac{j-r}{2}} a_{j+r}\,(3)_{j-\frac{j-r}{2}}}{\left(\frac{j-r}{2}\right)!\,\left(\frac{3}{2}\right)_{\left\lfloor\frac{j+1}{2}\right\rfloor-\frac{j-r}{2}}\,\Gamma\left(-\frac{1}{2}(j-r)+\left\lfloor\frac{j}{2}\right\rfloor+1\right)}\,(2\,t-1)^r, \tag{38}
$$

the last equation can be rewritten after using the relation $(2\,t-1)^2 = \sum_{n=0}^{r}(2\,t)^n(-1)^{r-n}\binom{r}{n}$, expanding, rearranging and collecting similar terms as

$$
\mathcal{H}_j^*(t) = \sum_{r=0}^{j}\sum_{m=0}^{j} \frac{2^r\,(-1)^{\frac{j-m}{2}}\,(-1)^{m-r}\,a_{j+m}\binom{m}{r}(3)_{\frac{j+m}{2}}}{\left(\frac{j-m}{2}\right)!\,\left(\frac{3}{2}\right)_{\frac{m-j}{2}+\left\lfloor\frac{j+1}{2}\right\rfloor}\,\Gamma\left(\frac{1}{2}(-j+m+2)+\left\lfloor\frac{j}{2}\right\rfloor\right)}\, t^r. \tag{39}
$$

Therefore, the identity of fractional Caputo derivative (3) along with Eq. (39) enables us to write

$$D_t^{\zeta}\mathcal{H}_j^*(t) = \sum_{r=1}^{j}\sum_{m=0}^{j} \frac{2^r r!(-1)^{\frac{j-m}{2}}\,(-1)^{m-r}\,a_{j+m}\binom{m}{r}(3)_{\frac{j+m}{2}}}{\left(\frac{j-m}{2}\right)!\,\Gamma(r-\alpha+1)\left(\frac{3}{2}\right)_{\frac{m-j}{2}+\left\lfloor\frac{j+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(-j+m+2)+\left\lfloor\frac{j}{2}\right\rfloor\right)}\,t^{r-\zeta}. \tag{40}$$

Now, $k_{j,s}$ is given by

$$\begin{aligned} k_{j,s} &= \int_0^1 D_t^{\zeta}\mathcal{H}_j^*(t)\mathcal{H}_s^*(t)\,w^*(t)\,dt \\ &= \sum_{r=1}^{j}\sum_{m=0}^{j} \frac{2^r r!(-1)^{\frac{j-m}{2}}\,(-1)^{m-r}\,a_{j+m}\binom{m}{r}(3)_{\frac{j+m}{2}}}{\left(\frac{j-m}{2}\right)!\,\Gamma(r-\alpha+1)\left(\frac{3}{2}\right)_{\frac{m-j}{2}+\left\lfloor\frac{j+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(-j+m+2)+\left\lfloor\frac{j}{2}\right\rfloor\right)}\times \\ &\quad \int_0^1 t^{r-\zeta}\mathcal{H}_s^*(t)\,w^*(t)\,dt \\ &= \sum_{r=1}^{j}\sum_{m=0}^{j} \frac{2^r r!(-1)^{\frac{j-m}{2}}\,(-1)^{m-r}\,a_{j+m}\binom{m}{r}(3)_{\frac{j+m}{2}}}{\left(\frac{j-m}{2}\right)!\,\Gamma(r-\alpha+1)\left(\frac{3}{2}\right)_{\frac{m-j}{2}+\left\lfloor\frac{j+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(-j+m+2)+\left\lfloor\frac{j}{2}\right\rfloor\right)} \\ &\quad \times \sum_{p=0}^{s}\sum_{n=0}^{s} \frac{2^p(-1)^{n-p}i^{s-n}a(n+s)\binom{n}{p}(3)_{\frac{n+s}{2}}}{\frac{s-n}{2}!\,\left(\frac{3}{2}\right)_{\frac{n-s}{2}+\left\lfloor\frac{s+1}{2}\right\rfloor}\Gamma\left(\frac{1}{2}(n-s+2)+\left\lfloor\frac{s}{2}\right\rfloor\right)} \int_0^1 (1-2\,t)^2(t\,(1-t))^{3/2}t^{-\alpha+p+r}\,dt. \end{aligned} \tag{41}$$

Finally, after evaluating the integral on the right-hand side of the previous equation, we get the desired result (35).

4. Error bound

In this part, we analyze the errors in the proposed polynomial expansion in detail. It is planned to prove four theorems.

- The first theorem provides an upper bound for the truncation error.
- The second theorem, provides a maximum estimate for the m-th derivative of truncation error with respect to the variable x.
- The third theorem provides a maximum estimate for the fractional derivative of the truncation error with respect to the variable t
- The last theorem shows $\|\mathbf{R}_{\mathcal{M}}(x,t)\|_{L^2_{\omega}(\Omega)}$ for sufficiently high $\mathcal{M}$, will be small enough, where $\omega = w^*(x)\,w^*(t)$.

Lemma 3. *[62] Given that* $\mathrm{m} \geq 1$ *and that* $\mathrm{m}+\mathrm{c} > 1$ *and* $\mathrm{m}+\mathrm{d} > 1$*, respectively, for all constants* c, d*, one has*

$$\frac{\Gamma(\mathrm{m}+\mathrm{c})}{\Gamma(\mathrm{m}+\mathrm{d})} \leq \mathbf{o}_{\mathrm{m}}^{\mathrm{c},\mathrm{d}}\,\mathrm{m}^{\mathrm{c}-\mathrm{d}}, \tag{42}$$

where

$$\mathbf{o}_{\mathrm{m}}^{\mathrm{c,d}} = exp\left(\frac{\mathrm{c}-\mathrm{d}}{2\,(\mathrm{m}+\mathrm{d}-1)} + \frac{1}{12\,(\mathrm{m}+\mathrm{c}-1)} + \frac{(\mathrm{c}-\mathrm{d})^2}{m}\right). \tag{43}$$

Remark 2. *For fixed* c, d. $\mathbf{o}_{\mathrm{m}}^{\mathrm{c,d}}$ *can be written as:*

$$\mathbf{o}_{m}^{\mathrm{c,d}} = 1 + O(\mathrm{m}^{-1}).$$

Theorem 4. *Assume that* $\frac{\partial^{i+j}\,\lambda(x,t)}{\partial\,x^i\,\partial\,t^j} \in \mathbf{C}(\Omega)$, $i, j = 0, 1, 2, \ldots, \mathcal{M}+1$ *and* $\lambda^{\mathcal{M}}(x,t)$ *is the suggested approximate solution belonging to* $\mathcal{P}^{\mathcal{M}}$ *and*

$$\ell_{\mathcal{M}} = \sup_{(x,t)\in\Omega}\left|\frac{\partial^{2\,(\mathcal{M}+1)}\,\lambda(x,t)}{\partial\,x^{\mathcal{M}+1}\,\partial\,t^{\mathcal{M}+1}}\right|. \tag{44}$$

Consequently, this estimate is valid:

$$\|\lambda(x,t) - \lambda^{\mathcal{M}}(x,t)\|_{L^2_\omega(\Omega)} \lesssim \frac{\ell_{\mathcal{M}}}{\mathcal{M}^{\frac{1}{2}}((\mathcal{M}+1)!)^2}, \tag{45}$$

where $q1 \lesssim q_2$ *implies the existence of a constant* n *satisfies* $q_1 \leq n\,q_2$.

Proof. Consider the following Taylor expansion of $\lambda(x,t)$:

$$\chi_{\mathcal{M}}(x,t) = \sum_{i=0}^{\mathcal{M}}\sum_{j=0}^{\mathcal{M}-i}\left(\frac{\partial^{i+j}\,\lambda(x,t)}{\partial\,x^i\,\partial\,t^j}\right)_{(0,0)}\frac{x^i\,t^j}{i!\,j!}, \tag{46}$$

$$\lambda(x,t) - \chi_{\mathcal{M}}(x,t) = \frac{x^{\mathcal{M}+1}\,t^{\mathcal{M}+1}\,\partial^{2\,(\mathcal{M}+1)}\,\lambda(n_1,n_2)}{((\mathcal{M}+1)!)^2\,\partial\,x^{\mathcal{M}+1}\,\partial\,t^{\mathcal{M}+1}}, \quad (n_1,n_2)\in\Omega. \tag{47}$$

Since $\mathrm{z}_{\mathcal{M}}(x,t)$ is the best approximate solution of $\mathrm{z}(x,t)$, we get, in accordance with the best approximation concept:

$$\begin{aligned}\|\lambda(x,t) - \lambda^{\mathcal{M}}(x,t)\|^2_{L^2_\omega(\Omega)} &\leq \|\lambda(x,t) - \chi_{\mathcal{M}}(x,t)\|^2_{L^2_\omega(\Omega)}\\ &= \int_0^1\int_0^1 \frac{{\ell_{\mathcal{M}}}^2\,x^{2\,(\mathcal{M}+1)}\,t^{2\,(\mathcal{M}+1)}}{((\mathcal{M}+1)!)^4}\,\omega\,d\,x\,d\,t\\ &= \frac{{\ell_{\mathcal{M}}}^2\,\pi}{((\mathcal{M}+1)!)^4}\left(\frac{\Gamma\left(2\,\mathcal{M}+\frac{9}{2}\right)}{\Gamma(2\,\mathcal{M}+5)} - 5\,\frac{\Gamma\left(2\,\mathcal{M}+\frac{11}{2}\right)}{\Gamma(2\,\mathcal{M}+6)} + 8\,\frac{\Gamma\left(2\,\mathcal{M}+\frac{13}{2}\right)}{\Gamma(2\,\mathcal{M}+7)}\right.\\ &\qquad\left. -4\,\frac{\Gamma\left(2\,\mathcal{M}+\frac{15}{2}\right)}{\Gamma(2\,\mathcal{M}+8)}\right)^2.\end{aligned} \tag{48}$$

The following estimate may be written using Lemma 3.

$$\|\lambda(x,t) - \lambda^{\mathcal{M}}(x,t)\|_{L^2_\omega(\Omega)} \lesssim \frac{\ell_{\mathcal{M}}}{\mathcal{M}^{\frac{1}{2}}((\mathcal{M}+1)!)^2}. \tag{49}$$

Theorem 5. *Suppose that* $\lambda(x,t)$, $\lambda^{\mathcal{M}}(x,t)$ *and* $\frac{\partial^{i+j}\lambda(x,t)}{\partial x^i\,\partial t^j}$ *meet the assumption of Theorem 4 and*

$$\tau_{\mathcal{M},m}=\sup_{(x,t)\in\Omega}\left|\frac{\partial^{2\,\mathcal{M}-m+2}\,\lambda(x,t)}{\partial\,x^{\mathcal{M}-m+1}\,\partial\,t^{\mathcal{M}+1}}\right|,\quad m\in\mathbb{N},\tag{50}$$

where $\mathbb{N}=\{1,2,\ldots\}$. *Then, we can write*

$$\left\|\frac{\partial^m}{\partial\,x^m}(\,\lambda(x,t)-\lambda^{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}\lesssim\frac{\tau_{\mathcal{M},m}}{(\mathcal{M}\,(\mathcal{M}-m))^{\frac{1}{4}}(\mathcal{M}+1)!\,(\mathcal{M}-m+1)!}.\tag{51}$$

Proof. Assume that $\frac{\partial^m\,\chi_{\mathcal{M}}(x,t)}{\partial\,x^m}$ is the Taylor expansion of $\frac{\partial^m\,\lambda(x,t)}{\partial\,x^m}$ about the point $(0,0)$, then the residual between $\frac{\partial^m\,\lambda(x,t)}{\partial\,x^m}$ and $\frac{\partial^m\,\chi_{\mathcal{M}}(x,t)}{\partial\,x^m}$ can be written as [63]

$$\frac{\partial^m}{\partial\,x^m}(\,\lambda(x,t)-\chi_{\mathcal{M}}(x,t))=\frac{x^{\mathcal{M}-m+1}\,t^{\mathcal{M}+1}\,\partial^{2\,\mathcal{M}-m+2}\,\lambda(\bar{n_1},\bar{n_2})}{(\mathcal{M}+1)!\,(\mathcal{M}-m+1)!\,\partial\,x^{\mathcal{M}-m+1}\,\partial\,t^{\mathcal{M}+1}},\quad(\bar{n_1},\bar{n_2})\in\Omega.\tag{52}$$

Since $\frac{\partial^m\,\lambda^{\mathcal{M}}(x,t)}{\partial\,x^m}$ is the best approximate solution of $\frac{\partial^m\,\lambda(x,t)}{\partial\,x^m}$, then according to the definition of the best approximation, we get

$$\left\|\frac{\partial^m}{\partial\,x^m}\,(\lambda(x,t)-\lambda^{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}\leq\left\|\frac{\partial^m}{\partial\,x^m}\,(\lambda(x,t)-\chi_{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}.\tag{53}$$

We get the desired result by following similar steps to those followed in Theorem 4.

Theorem 6. *Suppose that* $D_t^{\zeta}\,\lambda(x,t)\in\mathbf{C}(\Omega)$ *satisfies the conditions of Theorem 4, then the following estimation holds*

$$\left\|D_t^{\zeta}\,(\,\lambda(x,t)-\lambda^{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}\lesssim\frac{\ell_{\mathcal{M}}}{\mathcal{M}^{\frac{1}{4}}\,(\mathcal{M}-\zeta)^{\frac{1}{4}}(\mathcal{M}+1)!\,\Gamma(\mathcal{M}+2-\zeta)}.\tag{54}$$

Proof. According to Eq. (47) and properties of the Caputo operator in (3), one gets

$$\left|D_t^{\zeta}\,(\,\lambda(x,t)-\chi_{\mathcal{M}}(x,t)\,)\right|\leq\frac{x^{\mathcal{M}+1}\,t^{\mathcal{M}+1-\zeta}\,\ell_{\mathcal{M}}}{\Gamma(\mathcal{M}+2-\zeta)\,(\mathcal{M}+1)!},\quad(n_1,n_2)\in\Omega.\tag{55}$$

Taking $\|.\|_{L^2_\omega(\Omega)}$ yields

$$\left\|D_t^{\zeta}\,(\,\lambda(x,t)-\lambda^{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}\leq\int_0^1\int_0^1\frac{{\ell_{\mathcal{M}}}^2\,x^{2\,(\mathcal{M}+1)}\,t^{2\,(\mathcal{M}+1-\zeta)}}{(\Gamma(\mathcal{M}+2-\zeta))^2((\mathcal{M}+1)!)^2}\,\omega\,d\,x\,d\,t.\tag{56}$$

Now, imitating similar steps as in Theorem 4, we get

$$\left\|D_t^{\zeta}\,(\,\lambda(x,t)-\lambda^{\mathcal{M}}(x,t))\right\|_{L^2_\omega(\Omega)}\lesssim\frac{\ell_{\mathcal{M}}}{\mathcal{M}^{\frac{1}{4}}\,(\mathcal{M}-\zeta)^{\frac{1}{4}}(\mathcal{M}+1)!\,\Gamma(\mathcal{M}+2-\zeta)}.\tag{57}$$

Theorem 7. *Assume that $\mathbf{R}_{\mathcal{M}}(x,t)$ be the residual of Eq. (17), then $\|\mathbf{R}_{\mathcal{M}}(x,t)\|_{L^2_\omega(\Omega)}$ will be sufficiently small for the sufficiently large values of $\mathcal{M}$.*

Proof. Eqs. (21) and (17) enables us too write $\mathbf{R}_{\mathcal{M}}(x,t)$ as

$$\mathbf{R}_{\mathcal{M}}(x,t) = D_t^{\zeta}\left(\lambda^{\mathcal{M}}(x,t) - \lambda(x,t)\right) - \beta\,\frac{\partial^2}{\partial\, x^2}\left(\lambda^{\mathcal{M}}(x,t) - \lambda(x,t)\right). \tag{58}$$

If we consider L^2-norm, then using Theorems 4,5 and 6, we get

$$\begin{aligned}\|\mathbf{R}_{\mathcal{M}}(x,t)\|_{L^2_\omega(\Omega)} \lesssim\, & \frac{\ell_{\mathcal{M}}}{\mathcal{M}^{\frac{1}{4}}\,(\mathcal{M}-\zeta)^{\frac{1}{4}}(\mathcal{M}+1)!\,\Gamma(\mathcal{M}+2-\zeta)} \\ & - \beta\,\frac{\tau_{\mathcal{M},2}}{(\mathcal{M}\,(\mathcal{M}-2))^{\frac{1}{4}}(\mathcal{M}+1)!\,(\mathcal{M}-2+1)!}.\end{aligned} \tag{59}$$

For large enough values of $\mathcal{M}$, it is evident from Eq. (59) that $\|\mathbf{R}_{\mathcal{M}}(x,t)\|_{L^2_\omega(\Omega)}$ will be small enough. This concludes the proof of the theorem.

5. Some numerical tests

In this section, we present some numerical examples to demonstrate the accuracy and efficiency of the suggested numerical algorithm by using the absolute error (AE), maximum absolute error (MAE) and L_∞ - error

$$AE = \left|\lambda(x,t) - \lambda^{\mathcal{M}}(x,t)\right|, \tag{60}$$

$$MAE = \max_{x_i}\left|\lambda(x_i,x_j) - \lambda^{\mathcal{M}}(x_i,x_j)\right|, \quad x_i \in \left\{\frac{1}{10},\frac{2}{10},\frac{3}{10},\ldots,1\right\}, \tag{61}$$

$$L_\infty = \max_{(x,t)\in\Omega}\left|\lambda(x,t) - \lambda^{\mathcal{M}}(x,t)\right|. \tag{62}$$

In addition, comparisons with some methods are given.

Example 1. *[64] Consider the following equation:*

$$D_t^{\zeta}\lambda(x,t) - \lambda_{xx}(x,t) = \frac{1}{\Gamma(2-\alpha)}\,t^{1-\alpha}\,x^2\,(1-x) + 2\,t\,(3\,x-1), \quad 0<\alpha\le 1, \tag{63}$$

subject to the initial condition (IC)

$$\lambda(x,0) = 0, \quad 0 < x \le 1, \tag{64}$$

and the homogeneous boundary conditions (HBCs)

$$\lambda(0,t) = \lambda(1,t) = 0, \quad 0 < t \le 1, \tag{65}$$

where $\lambda(x,t) = t\,x^2\,(1-x)$ is the exact solution of this problem.
Table 1 shows the AE at various values of ζ when $\mathcal{M} = 3$. Figure 1 illustrates the AE

Table 1: The AE of Example 1 at $\mathcal{M} = 3$.

(x,t)	$\zeta = 0.2$	CPU time	$\zeta = 0.4$	CPU time	$\zeta = 0.6$	CPU time	$\zeta = 0.8$	CPU time
(0.1,0.1)	3.74914×10^{-14}		8.69194×10^{-15}		8.28406×10^{-15}		1.75789×10^{-14}	
(0.2,0.2)	5.27894×10^{-14}		1.66768×10^{-14}		1.26583×10^{-14}		2.83619×10^{-14}	
(0.3,0.3)	6.04031×10^{-15}		9.24955×10^{-15}		7.10196×10^{-15}		1.20529×10^{-14}	
(0.4,0.4)	6.2942×10^{-14}		1.40582×10^{-14}		6.8695×10^{-16}		1.98938×10^{-14}	
(0.5,0.5)	1.13222×10^{-13}	2.469	4.27158×10^{-14}	2.656	1.77636×10^{-15}	2.999	4.69347×10^{-14}	2.578
(0.6,0.6)	1.07289×10^{-13}		6.05765×10^{-14}		9.96425×10^{-15}		5.43871×10^{-14}	
(0.7,0.7)	5.93969×10^{-14}		5.37348×10^{-14}		1.57374×10^{-14}		4.09117×10^{-14}	
(0.8,0.8)	9.90874×10^{-15}		2.01922×10^{-14}		8.20177×10^{-15}		1.8055×10^{-14}	
(0.9,0.9)	3.08087×10^{-15}		1.78052×10^{-14}		1.01724×10^{-14}		2.45637×10^{-15}	

at $\zeta = 0.5$ and $\zeta = 0.9$ when $\mathcal{M} = 3$. Also, the CPU time (in seconds) for the method is computed in this table. Table 2 gives a comparison of L_2 and L_∞ errors between our method at $\mathcal{M} = 3$ and the method in [64] at $\Delta t = 0.001$ and $\Delta x = 0.015625$.

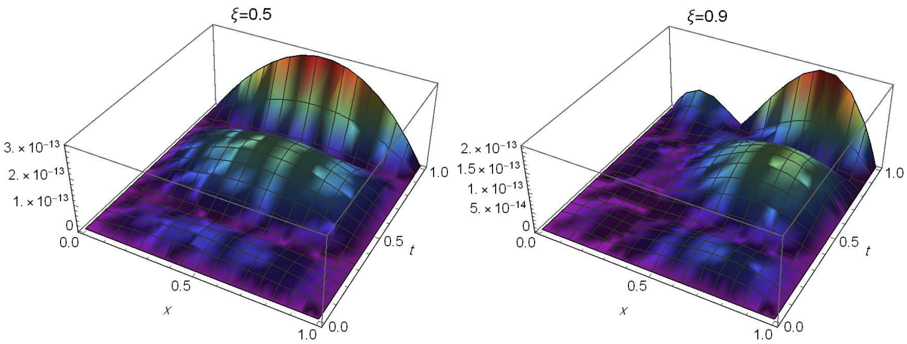

Figure 1: The AE for Example 1 at $\zeta = 0.5$ and $\zeta = 0.9$ when $\mathcal{M} = 3$.

Table 2: Comparison of errors for Problem 1.

Method in [64] at $\Delta t = 0.001$ and $\Delta x = 0.015625$		Our method at $\mathcal{M} = 3$	
L_2 error	L_∞ error	L_2 error	L_∞ error
1.6868×10^{-12}	2.3978×10^{-12}	6.37117×10^{-16}	1.44382×10^{-13}

Example 2. *[20] Consider the following equation:*

$$D_t^{\zeta} \lambda(x,t) - \lambda_{xx}(x,t) = g(x,t), \quad 0 < \zeta \leq 1, \tag{66}$$

controlled by

$$\begin{aligned} &\lambda(x,0) = 0, \quad 0 < x < 1, \\ &\lambda(0,t) = \lambda(1,t) = 0, \quad 0 < t < 1, \end{aligned} \tag{67}$$

where $g(x,t)$ is chosen to meet the exact solution given by $\lambda(x,t) = \sin(\pi\, t)\, \sin(\pi\, x)$. Table 3 gives a comparison of MAE between our method at $\mathcal{M} = 8$ and the method in [20] *at $M = 32$ and $\Delta x = 0.001$. Figures 2 and 3 illustrate the MAE and L_∞-error at various values of $\mathcal{M}$ when $\zeta = 0.5$. Table 4 shows the AE at various values of t when $\zeta = 0.7$ and $\mathcal{M} = 8$. Table 5 shows the MAE and L_∞-error at various values of $\mathcal{M}$ when $\zeta = 0.9$. Also, Table 6 shows the MAE and L_∞-error at various values of $\mathcal{M}$ when $\zeta = 0.3$.*

Table 3: Comparison of the MAE for Example 2.

ζ	Method in [20] ($M = 32$ and $\Delta x = 0.001$)	Our method ($\mathcal{M} = 8$)
0.5	1.10×10^{-3}	3.30215×10^{-6}
0.7	3.21×10^{-3}	3.30734×10^{-6}

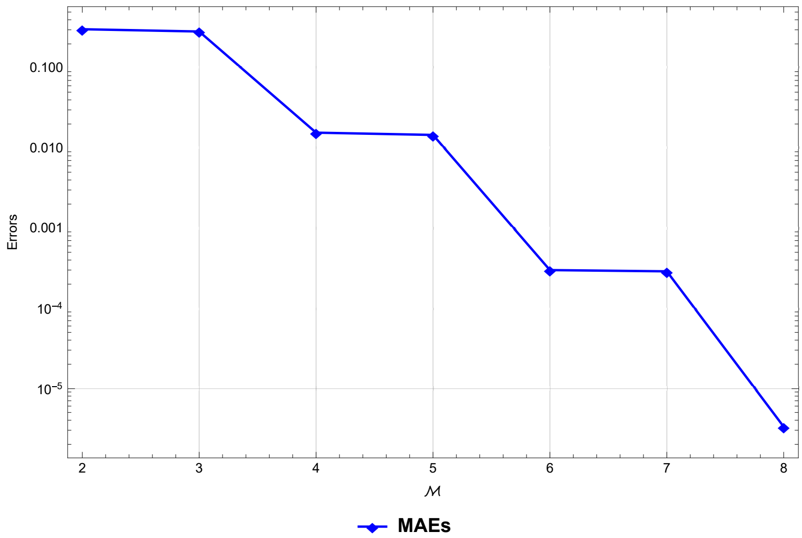

Figure 2: The MAE of Example 2 at various values of $\mathcal{M}$ when $\zeta = 0.5$.

Table 4: The AE of Example 2 at $\zeta = 0.7$, $\mathcal{M} = 8$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	2.98478×10^{-7}	5.82311×10^{-7}	6.6952×10^{-7}	4.94957×10^{-7}
0.2	6.22613×10^{-7}	1.19821×10^{-6}	1.36547×10^{-6}	9.99651×10^{-7}
0.3	9.18475×10^{-7}	1.75345×10^{-6}	1.98639×10^{-6}	1.44529×10^{-6}
0.4	1.33961×10^{-6}	2.48775×10^{-6}	2.76405×10^{-6}	1.96795×10^{-6}
0.5	1.57757×10^{-6}	2.89203×10^{-6}	3.18343×10^{-6}	2.24218×10^{-6}
0.6	1.33959×10^{-6}	2.488×10^{-6}	2.76412×10^{-6}	1.96804×10^{-6}
0.7	9.18451×10^{-7}	1.75373×10^{-6}	1.98646×10^{-6}	1.44539×10^{-6}
0.8	6.22605×10^{-7}	1.19833×10^{-6}	1.36551×10^{-6}	9.99693×10^{-7}
0.9	2.98475×10^{-7}	5.82369×10^{-7}	6.69545×10^{-7}	4.94976×10^{-7}

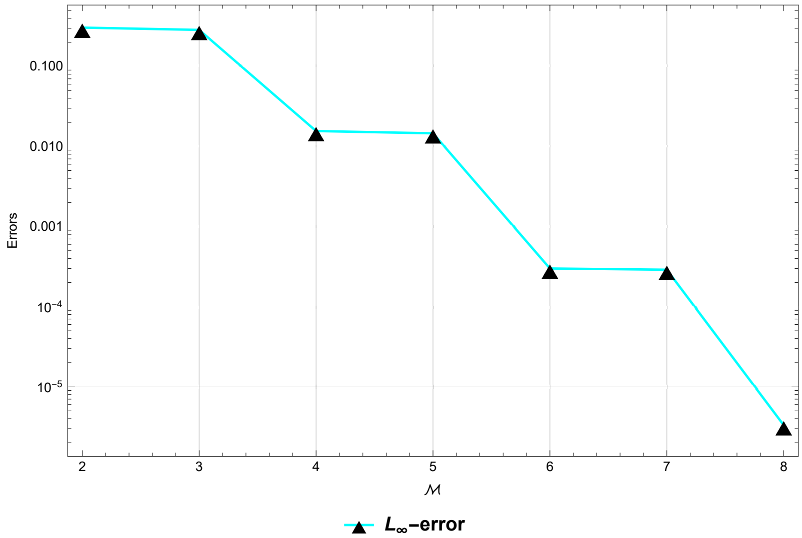

Figure 3: The L_∞-error of Example 2 at various values of $\mathcal{M}$ when $\zeta = 0.5$.

Table 5: The errors of Example 2 at various values of $\mathcal{M}$ when $\zeta = 0.9$.

$\mathcal{M}$	2	4	6	8
MAE	2.90874×10^{-1}	1.55707×10^{-2}	2.98468×10^{-4}	3.31145×10^{-6}
L_∞-error	2.91952×10^{-1}	1.57822×10^{-2}	3.02187×10^{-4}	3.35344×10^{-6}

Table 6: The errors of Example 2 at various values of $\mathcal{M}$ when $\zeta = 0.3$.

$\mathcal{M}$	2	3	4	5	6	7	8
MAE	3.0980×10^{-1}	2.854×10^{-1}	2.8600×10^{-2}	1.4596×10^{-2}	3.0076×10^{-4}	2.8939×10^{-4}	3.3021×10^{-6}
L_∞-error	3.0986×10^{-1}	2.8600×10^{-1}	1.5560×10^{-2}	1.4612×10^{-2}	3.0092×10^{-4}	2.8949×10^{-4}	3.3090×10^{-6}

Example 3. *Consider the following equation:*

$$D_t^{\zeta}\lambda(x,t) - \lambda_{xx}(x,t) = \left(\pi^2\, t^2 + \frac{2\, t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\sin(\pi\, x), \quad 0 < \alpha \le 1, \tag{68}$$

subject to the IC

$$\lambda(x,0) = 0, \quad 0 < x \le 1, \tag{69}$$

and the HBCs

$$\lambda(0,t) = \lambda(1,t) = 0, \quad 0 < t \le 1, \tag{70}$$

where $\lambda(x,t) = t^2\,\sin(\pi\, x)$ is the exact solution of this problem.
Figure 4 illustrates the MAE and L_∞-error at various values of $\mathcal{M}$ when $\zeta = 0.5$. Table 7 shows the AE at various values of t when $\zeta = 0.9$ and $\mathcal{M} = 8$. Table 8 shows the MAE and L_∞-error at various values of $\mathcal{M}$ when $\zeta = 0.9$. Figure 5 illustrates the AE at various values of $\mathcal{M}$ when $\zeta = 0.3$.

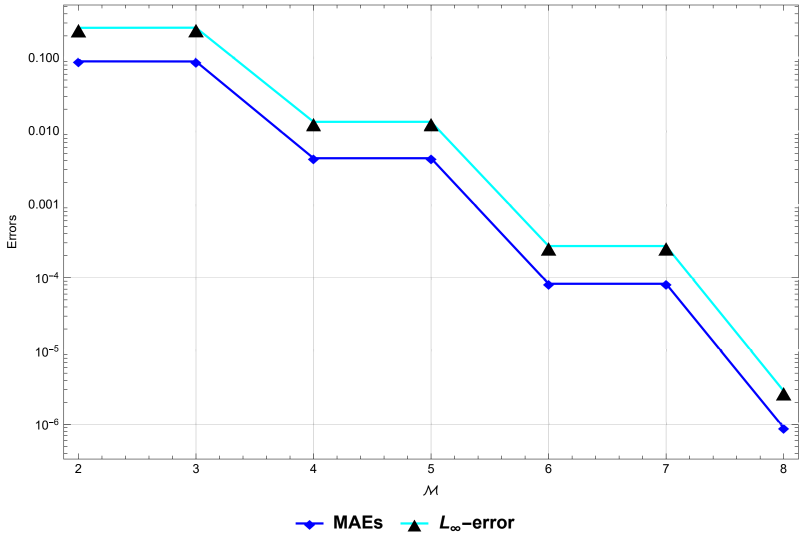

Figure 4: The errors of Example 3 at various values of $\mathcal{M}$ when $\zeta = 0.5$.

Table 7: The AE of Example 3 at $\zeta = 0.9$, $\mathcal{M} = 8$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	1.1953×10^{-8}	6.96178×10^{-8}	1.83032×10^{-7}	3.5209×10^{-7}
0.2	2.62526×10^{-8}	1.47194×10^{-7}	3.81809×10^{-7}	7.30076×10^{-7}
0.3	3.98826×10^{-8}	2.18594×10^{-7}	5.63519×10^{-7}	1.44529×10^{-6}
0.4	6.41293×10^{-8}	3.26744×10^{-7}	8.22244×10^{-7}	1.07306×10^{-6}
0.5	7.8752×10^{-8}	3.89168×10^{-7}	9.68425×10^{-7}	1.54566×10^{-6}
0.6	6.41292×10^{-8}	3.26811×10^{-7}	8.22324×10^{-7}	1.80976×10^{-6}
0.7	3.98859×10^{-8}	2.1865×10^{-7}	5.63611×10^{-7}	1.54565×10^{-6}
0.8	2.62589×10^{-8}	1.47195×10^{-7}	3.81857×10^{-7}	7.3006×10^{-7}
0.9	1.19538×10^{-8}	6.96403×10^{-8}	1.83059×10^{-7}	3.52091×10^{-7}

Table 8: The errors of Example 3 at various values of $\mathcal{M}$ when $\zeta = 0.7$.

$\mathcal{M}$	2	4	6	8
MAE	8.7641×10^{-2}	4.11803×10^{-3}	7.9429×10^{-5}	8.71518×10^{-7}
L_∞-error	2.54328×10^{-1}	1.32×10^{-2}	2.66423×10^{-4}	2.94457×10^{-6}

Example 4. *[64] Consider the following equation*

$$D_t^{\zeta}\lambda(x,t) - \lambda_{xx}(x,t) = \left(4\,\pi^2\,t^2 + \frac{2\,t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\sin(2\,\pi\,x), \quad 0 < \alpha \leq 1, \tag{71}$$

subject to the Initial condition (IC):

$$\lambda(x,0) = 0, \quad 0 < x \leq 1, \tag{72}$$

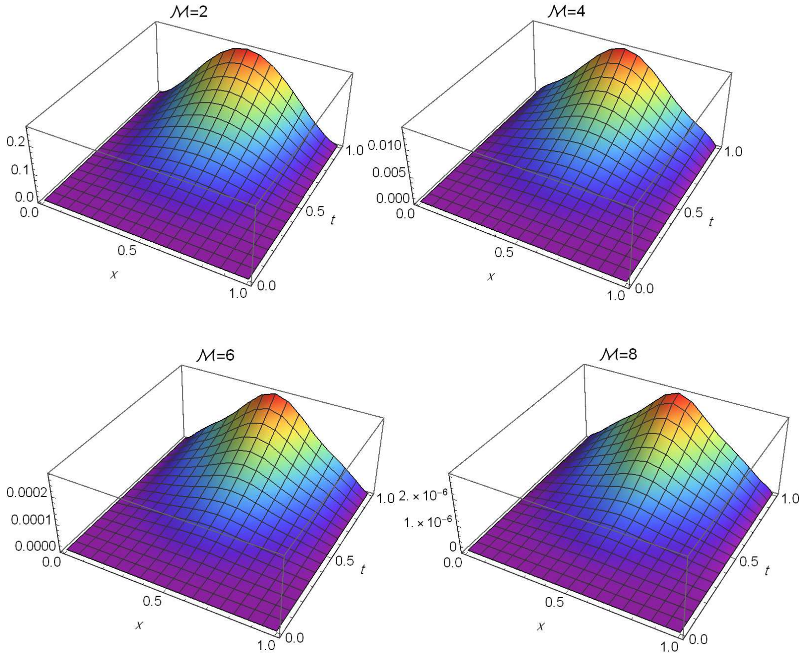

Figure 5: The AE of Example 3 at various values of $\mathcal{M}$ when $\zeta = 0.3$.

and the HBCs

$$\lambda(0,t) = \lambda(1,t) = 0, \quad 0 < t \leq 1, \tag{73}$$

where $\lambda(x,t) = t^2 \sin(2\,\pi\,x)$ *is the exact solution of this problem.*

Table 9 gives a comparison of MAE *between our method at* $\mathcal{M} = 9$ *and method in [64] at* $\Delta t = 0.001$ *and* $M = 64$*. Table 10 displays the AE at* $\mathcal{M} = 9$ *and* $\zeta = 0.5$*. Table 11 reports the* MAE *and* L_∞*-error at various values of* $\mathcal{M}$ *when* $\zeta = 0.8$ *Figure 6 illustrates the AE at various values of* t *when* $\mathcal{M} = 9$ *and* $\zeta = 0.3$*.*

Table 9: Comparison of the MAE for Example 4.

Method in [64] at $\Delta t = 0.001$ and $M = 64$	Our method at $\mathcal{M} = 9$
7.70×10^{-4}	1.66×10^{-5}

Table 11: The errors of Example 4 at various values of $\mathcal{M}$ when $\zeta = 0.8$.

$\mathcal{M}$	3	6	9
MAEs	1.37846×10^{-1}	1.42851×10^{-2}	1.5862×10^{-5}
L_∞-error	2.83552×10^{-1}	3.13749×10^{-2}	4.61017×10^{-5}

Table 10: The AE of Example 4 at $\mathcal{M} = 9$ and $\zeta = 0.5$.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	4.33573×10^{-7}	1.71139×10^{-6}	4.02304×10^{-6}	7.16437×10^{-6}
0.2	7.58102×10^{-7}	3.01934×10^{-6}	7.05377×10^{-6}	1.25795×10^{-5}
0.3	1.16957×10^{-6}	4.53537×10^{-6}	1.06733×10^{-5}	1.89359×10^{-5}
0.4	1.86083×10^{-6}	7.1667×10^{-6}	1.66918×10^{-5}	2.95361×10^{-5}
0.5	2.46676×10^{-8}	9.86345×10^{-8}	5.44645×10^{-8}	1.4437×10^{-8}
0.6	1.81921×10^{-6}	7.3363×10^{-6}	1.66×10^{-5}	2.95617×10^{-5}
0.7	1.1434×10^{-6}	4.64781×10^{-6}	1.06156×10^{-5}	1.89542×10^{-5}
0.8	7.42571×10^{-7}	3.08815×10^{-6}	7.01915×10^{-6}	1.25912×10^{-5}
0.9	4.25343×10^{-7}	1.74765×10^{-6}	4.00476×10^{-6}	7.17045×10^{-6}

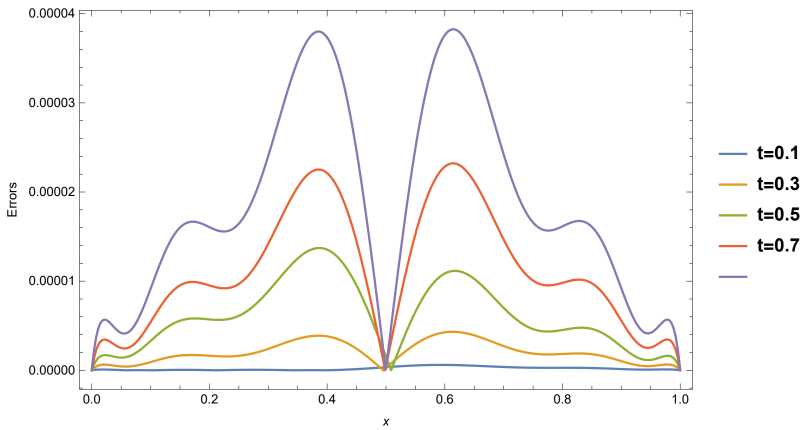

Figure 6: The AE of Example 4 at $\mathcal{M} = 9$ and $\zeta = 0.3$.

6. Concluding remarks

This article presented an effective Tau-based numerical scheme for solving the TFDE based on employing certain CPs, which are particular polynomials of the generalized Gegenbauer polynomials. Our tau algorithm transforms the equation with its initial-boundary equations into an algebraic system of equations that can be numerically solved. The method exhibits excellent approximation capabilities, as evidenced by the low errors in various test problems. The approach used in this paper may be a powerful tool for handling other fractional PDEs in future research. In addition, we anticipate that other generalized CPs as special cases of the generalized Gegenbauer polynomials may be introduced and used along with suitable spectral methods to treat different types of differential equations.

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Declarations

- Conflicts of interest: The authors declare that they have no conflicts of interest.

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