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Applications and Theoretical Foundations of Best Proximity Points in Generalized Interpolative Proximal Contractions

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Abstract. This paper investigates optimal solutions for best proximity points through the framework of generalized interpolative proximal contractions. We introduce a new method that uses interpolation techniques to handle a wider class of mappings by expanding the concepts of classical proximal contraction. In the absence of a precise solution, best proximity point theorems investigate the existence of such best proximity points for approximate solutions to the fixed point problem. This article aims to develop the best proximity point theorems for contractive non-self mappings via interpolation to generate global optimal approximate solutions to particular fixed point equations. In addition to demonstrating the existence of the optimal proximity points, iterative techniques are also offered to locate such optimal approximative solutions. We illustrate the utility of our findings with a few instances. The value of our research is illustrated with a few examples and applications.
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1. Introduction

Best proximity points have widespread applications in optimization, economics, and various engineering disciplines, where exact fixed points are elusive, and optimal approximations are sought. Future research may extend these concepts to more complex structures, such as partial metric spaces or ordered metric spaces, broadening the scope and

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applicability of these results. In optimization and fixed point theory, best proximity points are crucial when dealing with non-self mappings where fixed points do not exist. The classical Banach contraction principle has seen various extensions to accommodate different contractions and more general settings. This note focuses on a specific generalization: generalized interpolative proximal contractions.

Best proximity point theory is an area of mathematical analysis and optimization that focuses on finding points in one set that are closest to points in another set when a contractive mapping is involved. In metric fixed point theory, the concept of best proximity points plays a crucial role, particularly when dealing with mappings that do not necessarily have fixed points. This note delves into the best proximity points for generalized interpolative proximal contractions, an important class of mappings in metric spaces.

The fundamentals of interpolative contraction are the product of distances with exponents that satisfy certain conditions. The prominent mathematician Erdal Karapinar first used the word "interpolative contraction" in his paper [1], which was published in 2018. The following is the definition of an interpolative contraction:

A self-mapping $\mathfrak{P}$, defined on a metric space $(\mathfrak{W}, \vartheta)$, satisfying the following inequality

$$\vartheta(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}) \leq \mathfrak{k}\left(\vartheta(\mathfrak{b}, \mathfrak{m})\right)^{\nu}, \forall \mathfrak{b}, \mathfrak{m} \in \mathfrak{W}, \tag{1}$$

is called an interpolative contraction, where $\nu \in (0,1]$ and $\mathfrak{k} \in [0,1)$. Since $\nu = 1$, $\mathfrak{P}$ is a Banach contraction. If $\mathfrak{P}$ defined on a metric space $(\mathfrak{W}, \vartheta)$ the followings axioms are holds:

$$\begin{aligned}
\vartheta(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}) &\leq \mathfrak{k}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{b})\right)^{\nu}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m})\right)^{1-\nu},\\
\vartheta(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}) &\leq \mathfrak{k}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m})\right)^{\nu}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m})\right)^{1-\nu},\\
\vartheta(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}) &\leq \mathfrak{k}\left(\vartheta(\mathfrak{b}, w)\right)^{\eta}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{b})\right)^{\nu}\left(\vartheta(\mathfrak{m}, \mathfrak{P}\mathfrak{m})\right)^{1-\nu-\eta},\ \nu+\eta<1\\
\vartheta(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}) &\leq \mathfrak{k}\left(\vartheta(\mathfrak{b}, w)\right)^{\nu}\left(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{b})\right)^{\eta}\left(\vartheta(\mathfrak{m}, \mathfrak{P}\mathfrak{m})\right)^{\gamma}\\
&\quad \left(\frac{1}{2}(\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m}) \cdot \vartheta(\mathfrak{m}, \mathfrak{P}\mathfrak{b}))\right)^{1-\eta-\nu-\gamma},\ \nu+\eta+\gamma<1
\end{aligned}$$

for all $\mathfrak{b}, \mathfrak{m} \in \mathfrak{W}$, then $\mathfrak{P}$ is called Kannan type interpolative contraction, Chatterjea type interpolative contraction, Ćirić-Reich-Rus type interpolative contraction and Hardy Rogers type interpolative contractions, respectively. Through interpolation, numerous complex and conventional contractions have lately been reexamined (see [2–6] and references therein).

The proximal contraction principle appeared in [7]. Proinov [8](2020) offered various fixed-point theorems that built on previous work in [9]. First, Erdal Karapinar introduced the idea of interpolation contraction in his work [1] published in 2018, then Proinov gave the second idea in his paper [8] published in 2020.

Best proximity points are optimal points that minimize the distance between two sets when fixed points may not exist. A *best proximity point* theorem achieves a global minimum of $\vartheta(\mathfrak{b}, \mathfrak{P}(\mathfrak{b}))$ by specifying an approximate solution $\mathfrak{b}$ of the fixed point equation

$\mathfrak{P}(\mathfrak{b}) = \mathfrak{b}$ to satisfy the requirement that $\vartheta(\mathfrak{b}, \mathfrak{P}(\mathfrak{b})) = \vartheta(\mathfrak{C}, \mathfrak{D})$ because the distance between any element $\mathfrak{b}$ in $\mathfrak{C}$ and its image $\mathfrak{P}(\mathfrak{b})$ in $\mathfrak{D}$ is at least the distance between the sets $\mathfrak{C}$ and $\mathfrak{D}$. Recently, Altun and Taşdemir [10] have utilized the interpolative proximal contraction to produce some *best proximity point* theorems.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be metric space subsets that are non-empty. Finding an element $\mathfrak{b}$ in $\mathfrak{C}$ that is as close to $\mathfrak{P}(\mathfrak{b})$ in $\mathfrak{D}$ as possible, is of great interest, since a non-self mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ need not have a fixed point. In other words, it is considered to find an approximation solution $\mathfrak{b}$ in $\mathfrak{C}$ such that the error $\vartheta(\mathfrak{b}, \mathfrak{P}(\mathfrak{b}))$ is smallest, where ϑ is the distance function, if the fixed point equation $\mathfrak{P}(\mathfrak{b}) = \mathfrak{b}$ has no exact solution. In fact, best proximity point theorems look into the possibility of such best proximity point for approximate solutions to the fixed point equation $\mathfrak{P}(\mathfrak{b}) = \mathfrak{b}$ in the absence of a precise solution. In order to produce global optimal approximate solutions to some fixed point equations, this article aims to establish best proximity point theorems for contractive non-self mappings via interpolation. Iterative strategies are also provided to find such ideal approximative solutions in addition to proving the presence of best proximity points. Also, we extend the results appeared in [8, 10] by introducing $(\mathfrak{J}, \pounds)$-interpolative proximal contractions, which generalize and establishing the optimal proximity point theorems for them. To ascertain the generalized interpolative proximal contractions that produce interpolative proximal contractions and proximal contractions as special cases. We extend classical fixed point results by considering a generalized class of contractions known as generalized interpolative proximal contractions and provide conditions under which best proximity points are guaranteed. The interpolative proximal contraction introduced in [10] are generalized by the $(\mathfrak{J}, \pounds)$-interpolative proximal contraction. We look for various conditions on the functions $\mathfrak{J}, \pounds$ to prove the presence of best proximity points of $(\mathfrak{J}, \pounds)$-proximal contraction, $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction, $(\mathfrak{J}, \pounds)$-Hardy Rogers type interpolative proximal contraction. We also show non-trivial examples and applications are given to demonstrate the usefulness of our results.

The structure of the paper is as follows: we begin by reviewing key concepts and preliminary results related to proximal contractions and best proximity points. We then introduce the concept of generalized interpolative proximal contractions and establish the necessary theoretical foundations. Subsequent sections are dedicated to proving essential theorems and demonstrating their practical implications through detailed examples. Finally, we explore applications of our findings in optimization and equilibrium problems, showcasing the practical significance and potential impact of our research.

By advancing the theoretical framework of best proximity points and providing practical solutions to complex problems, this paper aims to contribute to both the academic literature and real-world applications.

2. Preliminaries

Let $(\mathfrak{W}, \vartheta)$ be a metric space and $\mathfrak{C}, \mathfrak{D}$ be two subsets of $(\mathfrak{W}, \vartheta)$. The following information is needed throughout this paper.

$$\begin{aligned}\vartheta(\mathfrak{C}, \mathfrak{D}) &= \inf\{\vartheta(\mathfrak{b}, \mathfrak{m}) : \mathfrak{b} \in \mathfrak{C} \wedge \mathfrak{m} \in \mathfrak{D}\}.\\ \mathfrak{C}_0 &= \{\mathfrak{b} \in \mathfrak{C} : \vartheta(\mathfrak{b}, \mathfrak{m}) = \vartheta(\mathfrak{C}, \mathfrak{D}) \text{ for some } \mathfrak{m} \in \mathfrak{D}\}.\\ \mathfrak{D}_0 &= \{\mathfrak{m} \in \mathfrak{D} : \vartheta(\mathfrak{b}, \mathfrak{m}) = \vartheta(\mathfrak{C}, \mathfrak{D}) \text{ for some } \mathfrak{b} \in \mathfrak{C}\}.\end{aligned}$$

Definition 1. *[11] Let $(\mathfrak{W}, \vartheta)$ be a metric space and $\mathfrak{C}, \mathfrak{D}$ be any nonvoid subsets of $\mathfrak{W}$. A mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ is said to be a proximal contraction, if there exists a real number $\mathfrak{k} \in [0, 1)$ such that*

$$\left.\begin{array}{l}\vartheta(\mathfrak{b}_1, \mathfrak{P}(\mathfrak{m}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D})\\ \vartheta(\mathfrak{b}_2, \mathfrak{P}(\mathfrak{m}_2)) = \vartheta(\mathfrak{C}, \mathfrak{D})\end{array}\right\} \Rightarrow \vartheta(\mathfrak{b}_1, \mathfrak{b}_2) \leq \mathfrak{k}\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)$$

for all $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$ and $\mathfrak{b}_1 \neq \mathfrak{b}_2$.

It is easy to observe that a self-mapping that is a proximal contraction of the first kind reduces to a contraction.

Definition 2. *[11] Let $(\mathfrak{W}, \vartheta)$ be a metric space and $\mathfrak{C}, \mathfrak{D}$ be any nonvoid subsets of $\mathfrak{W}$. A mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ is said to be a proximal contraction of second kind, if there exists a real number $\mathfrak{k} \in [0, 1)$ such that*

$$\left.\begin{array}{l}\vartheta(\mathfrak{b}_1, \mathfrak{P}(\mathfrak{m}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D})\\ \vartheta(\mathfrak{b}_2, \mathfrak{P}(\mathfrak{m}_2)) = \vartheta(\mathfrak{C}, \mathfrak{D})\end{array}\right\} \Rightarrow \vartheta(\mathfrak{P}\mathfrak{b}_1, \mathfrak{P}\mathfrak{b}_2) \leq \mathfrak{k}\vartheta(\mathfrak{P}\mathfrak{m}_1, \mathfrak{P}\mathfrak{m}_2)$$

for all $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$ and $\mathfrak{P}\mathfrak{b}_1 \neq \mathfrak{P}\mathfrak{b}_2$.

For a self-mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{C}$ to be a proximal contraction of second kind, it needs to satisfy the following inequality:

$$\vartheta\left(\mathfrak{P}^2\mathfrak{m}_1, \mathfrak{P}^2\mathfrak{m}_2\right) \leq \mathfrak{k}\vartheta(\mathfrak{P}\mathfrak{m}_1, \mathfrak{P}\mathfrak{m}_2), \text{ for all } \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}.$$

Remark 1. *Every contrs is a proximal contraction of the second kind but the converse is not true. Let be a metric space Indeed, the mapping $\mathfrak{P} : [0, 1] \to [0, 1]$ defined by*

$$\mathfrak{P}(\mathfrak{b}) = \begin{cases} 0 & \text{if } \mathfrak{b} \text{ is rational} \\ \\ 1 & \text{otherwise} \end{cases}$$

is a proximal contraction of the second kind but not a contrs in $(\mathbb{R}, \vartheta)$.

Definition 3. *[10] Let $(\mathfrak{W}, \vartheta)$ be a metric space and $\mathfrak{C}, \mathfrak{D}$ be any nonvoid subsets of $\mathfrak{W}$. We say that $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$, if every sequence $\{\mathfrak{b}_n\}$ in $\mathfrak{D}$ satisfying the following condition*

$$\vartheta(\mathfrak{m}, \mathfrak{b}_n) \to \vartheta(\mathfrak{m}, \mathfrak{D})$$

for some $\mathfrak{m} \in \mathfrak{C}$, has a convergent sub-sequence.

Definition 4. *[10] Let $(\mathfrak{W},\vartheta)$ be a metric space and $\mathfrak{C},\mathfrak{D}$ be nonvoid subsets of $\mathfrak{W}$. An element $\mathfrak{b}^*$ in $\mathfrak{C}$ is called a best proximity point of the mapping $\mathfrak{P}:\mathfrak{C}\to\mathfrak{D}$, if it satisfies the equation:*

$$\vartheta\left(\mathfrak{b}^*,\mathfrak{P}\mathfrak{b}^*\right)=\vartheta\left(\mathfrak{C},\mathfrak{D}\right).$$

3. Main Results

In this section, we define $(\mathfrak{J},\pounds)$-proximal contraction and show that it generalizes proximal contraction 1. We prove the existence of the *best proximity points* of $(\mathfrak{J},\pounds)$-proximal contraction and $(\mathfrak{J},\pounds)$-interpolative proximal contraction in a complete metric space.

3.1. $(\mathfrak{J},\pounds)$-proximal contraction

Let $(\mathfrak{W},\vartheta)$ be a complete metric space, and $\mathfrak{C},\mathfrak{D}$ are subsets of $\mathfrak{W}$. A mapping $\mathfrak{P}:\mathfrak{C}\to\mathfrak{D}$ is said to be a $(\mathfrak{J},\pounds)$-proximal contraction if

$$\left.\begin{array}{l}\vartheta\left(\mathfrak{b}_1,\mathfrak{P}\mathfrak{m}_1\right)=\vartheta\left(\mathfrak{C},\mathfrak{D}\right)\\ \vartheta\left(\mathfrak{b}_2,\mathfrak{P}\mathfrak{m}_2\right)=\vartheta\left(\mathfrak{C},\mathfrak{D}\right)\end{array}\right\}\Rightarrow\mathfrak{J}\left(\vartheta\left(\mathfrak{b}_1,\mathfrak{b}_2\right)\right)\leq\pounds\left(\vartheta\left(\mathfrak{m}_1,\mathfrak{m}_2\right)\right)\tag{2}$$

for all $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{m}_1,\mathfrak{m}_2\in\mathfrak{C}$ with $\mathfrak{b}_1\neq\mathfrak{b}_2$, where $\mathfrak{J},\pounds:\mathbb{R}^+\to\mathbb{R}$ are two mappings.

Example 1. *Let $\mathfrak{W}=\mathbb{R}^2$ and define the function $\vartheta:\mathfrak{W}\times\mathfrak{W}\to[0,\infty)$ by*

$$\vartheta((\mathfrak{b},\mathfrak{m}),(\mathfrak{u},\mathfrak{v}))=|\mathfrak{b}-\mathfrak{u}|+|\mathfrak{m}-\mathfrak{v}|\text{ for all }(\mathfrak{b},\mathfrak{m}),(\mathfrak{u},\mathfrak{v})\in\mathfrak{W}.$$

Then $(\mathfrak{W},\vartheta)$ is a metric space. Let $\mathfrak{C},\mathfrak{D}$ be the subsets of $\mathfrak{W}$ defined by

$$\mathfrak{C}=\{(0,\mathfrak{m});0\leq\mathfrak{m}\leq1\},\ \ \mathfrak{D}=\{(1,\mathfrak{m});0\leq\mathfrak{m}\leq1\},\ \text{then }\vartheta(\mathfrak{C},\mathfrak{D})=1.$$

Define the functions $\mathfrak{J},\pounds:\mathbb{R}^+\to\mathbb{R}$ by

$$\mathfrak{J}(\mathfrak{z})=\mathfrak{z}\text{ and }\pounds(\mathfrak{z})=\mathfrak{z}-\frac{\mathfrak{z}^2}{2},\mathfrak{z}\in\mathbb{R}^+.$$

Define the mapping $\mathfrak{P}:\mathfrak{C}\to\mathfrak{D}$ by $\mathfrak{P}((0,r))=(1,r-\frac{r^2}{2})$ for all $(0,r)\in\mathfrak{C}$. We show that $\mathfrak{P}$ is a $(\mathfrak{J},\pounds)$-proximal contraction. For $\mathfrak{b}=(0,\mathfrak{b}_1)$, $\mathfrak{u}=(0,\mathfrak{b}_2)$ and $\mathfrak{m}_1=(0,\mathfrak{a}_1)$, $\mathfrak{m}_2=(0,\mathfrak{a}_2)$ (let $\mathfrak{a}_1>\mathfrak{a}_2$), we have

$$\vartheta(\mathfrak{b},\mathfrak{P}\mathfrak{m}_1)=\vartheta(\mathfrak{C},\mathfrak{D})\tag{3}$$
$$\vartheta(\mathfrak{u},\mathfrak{P}\mathfrak{m}_2)=\vartheta(\mathfrak{C},\mathfrak{D}).\tag{4}$$

We note that the equations (3) and (4) can further be simplified to have the following information:

$$\mathfrak{b}_1=\mathfrak{a}_1-\frac{\mathfrak{a}_1^2}{2},$$

$$\mathfrak{b}_2 = \mathfrak{a}_2 - \frac{\mathfrak{a}_2^2}{2}.$$

This implies that

$$\begin{aligned}\mathfrak{J}(\vartheta(\mathfrak{b},\mathfrak{u})) &= \mathfrak{J}(\vartheta((0,\mathfrak{b}_1),(0,\mathfrak{b}_2))) = (\mid 0-0\mid + \mid \mathfrak{b}_1-\mathfrak{b}_2\mid) \\ &\leq (\mathfrak{a}_1-\mathfrak{a}_2) - \frac{1}{2}(\mathfrak{a}_1-\mathfrak{a}_2)^2 \\ &= \vartheta(\mathfrak{m}_1,\mathfrak{m}_2) - \frac{1}{2}(\vartheta(\mathfrak{m}_1,\mathfrak{m}_2))^2 = \pounds(\vartheta(\mathfrak{m}_1,\mathfrak{m}_2))\end{aligned}$$

This shows that $\mathfrak{P}$ *is a* $(\mathfrak{J},\pounds)$*-proximal contraction. Next, we show that it is not a proximal contraction. Since*

$$\begin{aligned}\vartheta(\mathfrak{b},\mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C},\mathfrak{D}) \\ \vartheta(\mathfrak{u},\mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C},\mathfrak{D}).\end{aligned}$$

If there exists $\mathfrak{k}\in(0,1)$ *such that*

$$\vartheta(\mathfrak{b},\mathfrak{u}) \leq \mathfrak{k}\vartheta(\mathfrak{m}_1,\mathfrak{m}_2).$$

Then,

$$\begin{aligned}\vartheta((0,\mathfrak{b}_1),(0,\mathfrak{b}_2)) &\leq \mathfrak{k}\vartheta((0,\mathfrak{a}_1),(0,\mathfrak{a}_2)) \\ (\mid 0-0\mid + \mid \mathfrak{b}_1-\mathfrak{b}_2\mid) &\leq \mathfrak{k}(\mid 0-0\mid + \mid \mathfrak{a}_1-\mathfrak{a}_2\mid) \\ \mathfrak{a}_1 - \frac{\mathfrak{a}_1^2}{2} - \mathfrak{a}_2 + \frac{\mathfrak{a}_2^2}{2} &\leq \mathfrak{k}(\mathfrak{a}_1-\mathfrak{a}_2) \\ 1+\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2} &\leq \mathfrak{k}.\end{aligned}$$

This is a contradiction. Hence, $\mathfrak{P}$ *is not a proximal contraction.*

The following lemmas are integral part of this paper and have an impact on further investigations.

Lemma 1. [8] Let $\{\mathfrak{b}_n\}$ be a *sequence* in $(\mathfrak{W},\vartheta)$ verifying $\lim_{n\to\infty}\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})=0$. If the *sequence* $\{\mathfrak{b}_n\}$ is not Cauchy, then there are sub*seq*uences $\{\mathfrak{b}_{n_\mathfrak{k}}\}$, $\{\mathfrak{b}_{m_\mathfrak{k}}\}$ and $\mathfrak{p}>0$ such that

$$\lim_{\mathfrak{k}\to\infty}\vartheta(\mathfrak{b}_{n_\mathfrak{k}+1},\mathfrak{b}_{m_\mathfrak{k}+1}) = \mathfrak{p}+. \tag{5}$$

$$\lim_{\mathfrak{k}\to\infty}\vartheta(\mathfrak{b}_{n_\mathfrak{k}},\mathfrak{b}_{m_\mathfrak{k}}) = \vartheta(\mathfrak{b}_{n_\mathfrak{k}+1},\mathfrak{b}_{m_\mathfrak{k}}) = \vartheta(\mathfrak{b}_{n_\mathfrak{k}},\mathfrak{b}_{m_\mathfrak{k}+1}) = \mathfrak{p}. \tag{6}$$

Lemma 2. *[8]*Let $\mathfrak{J}$: $(0,\infty)\to R$ be a function. Then the statements *(i)* − *(iii)* are equivalent:

(i) $\inf_{\mathfrak{z}>\varepsilon}\mathfrak{J}(\mathfrak{z}) > -\infty$ for every $\varepsilon>0$.

(ii) $\lim_{\mathfrak{z}\to\varepsilon+}\inf \mathfrak{J}(\mathfrak{z}) > -\infty$ for every $\varepsilon > 0$.

(iii) $\lim_{n\to\infty} \mathfrak{J}(\mathfrak{z}_n) = -\infty$ implies that $\lim_{n\to\infty} \mathfrak{z}_n = 0$.

Lemma 3. *Let $\{\mathfrak{b}_n\}$ be a sequence in $(\mathfrak{W}, \vartheta)$ obeying the equation $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$. Suppose that the mapping and $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ satisfying the condition (2). If $\mathfrak{J}, \pounds : (0,\infty) \to \mathbb{R}$ are such that*

(1) $\limsup_{\mathfrak{z}\to\epsilon+} \pounds(\mathfrak{z}) < \mathfrak{J}(\in +)$ *for any* $\epsilon > 0$.

Then $\{\mathfrak{b}_n\}$ is Cauchy.

Proof. Consider *sequence* $\{\mathfrak{b}_n\}$ is not Cauchy, then by Lemma 1, then two subsequences $\{\mathfrak{b}_{n_\mathfrak{k}}\}$, $\{\mathfrak{b}_{m_\mathfrak{k}}\}$ of $\{\mathfrak{b}_n\}$ and $\epsilon > 0$ such that the equations (5) and (6) hold. By (5), we get that $\vartheta(\mathfrak{b}_{n_\mathfrak{k}+1}, \mathfrak{b}_{m_\mathfrak{k}+1}) > \epsilon$. Since, for $\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{m_\mathfrak{k}}, \mathfrak{b}_{m_\mathfrak{k}+1}, \mathfrak{b}_{n_\mathfrak{k}+1} \in \mathfrak{C}$, we have

$$\begin{aligned} \vartheta(\mathfrak{b}_{n_\mathfrak{k}+1}, \mathfrak{P}(\mathfrak{b}_{m_\mathfrak{k}})) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \\ \vartheta(\mathfrak{b}_{m_\mathfrak{k}+1}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } \mathfrak{k} \geq 1. \end{aligned}$$

Thus, by (2), we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_{n_\mathfrak{k}+1}, \mathfrak{b}_{m_\mathfrak{k}+1})) \leq \pounds(\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{m_\mathfrak{k}})), \text{ for any } \mathfrak{k} \geq 1.$$

For if $\mathfrak{c}_\mathfrak{k} = \vartheta(\mathfrak{b}_{n_\mathfrak{k}+1}, \mathfrak{b}_{m_\mathfrak{k}+1})$ and $j_\mathfrak{k} = \vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{m_\mathfrak{k}})$, we have

$$\mathfrak{J}(\mathfrak{c}_\mathfrak{k}) \leq \pounds(j_\mathfrak{k}), \text{ for any } \mathfrak{k} \geq 1. \tag{7}$$

By (5) and (6), we have $\lim_{\mathfrak{k}\to\infty} \mathfrak{c}_\mathfrak{k} = \epsilon+$ and $\lim_{\mathfrak{k}\to\infty} j_\mathfrak{k} = \epsilon$. By (7), we get that

$$\mathfrak{J}(\epsilon+) = \lim_{\mathfrak{k}\to\infty} \mathfrak{J}(\mathfrak{c}_\mathfrak{k}) \leq \lim \sup_{\mathfrak{k}\to\infty} \pounds(j_\mathfrak{k}) \leq \limsup_{c\to\epsilon} \pounds(c). \tag{8}$$

This is a contradiction to the assumption (1). Consequently, $\{\mathfrak{b}_n\}$ is a Cauchy *sequence* in $\mathfrak{C}$.

Theorem 1. *Let $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ be a $(\mathfrak{J}, \pounds)$-proximal contraction defined on a complete metric space $(\mathfrak{W}, \vartheta)$ and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$. If*

(i) $\mathfrak{J}$ *is non-decreasing function and* $\limsup_{t\to\epsilon+} \pounds(t) < \mathfrak{J}(\epsilon+)$ *for any* $\epsilon > 0$.

(ii) $\mathfrak{C}_0$ *is non-void subset of* $\mathfrak{C}$ *such that* $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.

Then $\mathfrak{P}$ has a best proximity point.

Proof. *Let $\mathfrak{b}_0 \in \mathfrak{C}_0$. Since $\mathfrak{P}(\mathfrak{b}_0) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{b}_1 \in \mathfrak{C}_0$ such that, $\vartheta(\mathfrak{b}_1, \mathfrak{P}(\mathfrak{b}_0)) = \vartheta(\mathfrak{C}, \mathfrak{D})$. Also we have $\mathfrak{P}(\mathfrak{b}_1) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, so, there exist $\mathfrak{b}_2 \in \mathfrak{C}_0$ such that $\vartheta(\mathfrak{b}_2, \mathfrak{P}(\mathfrak{b}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D})$. Then $\mathfrak{C}_0$ implies to have a sequence $\{\mathfrak{b}_n\} \subseteq \mathfrak{C}_0$ such that*

$$\vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \in \mathbb{N}. \tag{9}$$

If $\exists\ n \in \mathbb{N}$ such that $\mathfrak{b}_n = \mathfrak{b}_{n+1}$, then by (9), then $\mathfrak{b}_n$ is a best proximity point of the mapping $\mathfrak{P}$. If $\mathfrak{b}_{n-1} \neq \mathfrak{b}_n\ \forall\ n \in \mathbb{N}$, then by (9), we have

$$\begin{aligned} \vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \\ \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \geq 1. \end{aligned}$$

Thus, by (2), we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) \leq \pounds(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n)), \text{ for all } \mathfrak{b}_{n-1}, \mathfrak{b}_n, \mathfrak{b}_{n+1} \in \mathfrak{C}.$$

Let $\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = \theta_n$, we have

$$\mathfrak{J}(\theta_n) \leq \mathfrak{F}(\theta_{n-1}) < \mathfrak{J}(\theta_{n-1}). \tag{10}$$

Since $\mathfrak{J}$ is non-decreasing, so, by (10), we have $\theta_n < \theta_{n-1}$ for all $n \in N$. If $\theta > 0$, so that, by (10), we obtain the following:

$$\mathfrak{J}(\theta+) = \lim_{n\to\infty} \mathfrak{J}(\theta_n) \leq \lim_{n\to\infty} \mathfrak{F}(\theta_{n-1}) \leq \lim_{t\to\theta+} \sup \mathfrak{F}(t).$$

This defies presumption (i), hence, $\theta = 0$ and $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$. Now (i) and Lemma 3, we conclude that $\{\mathfrak{b}_n\}$ is a Cauchy sequence. Since $(\mathfrak{W}, \vartheta)$ is a complete metric space and $\mathfrak{C}$ is a closed subset of $\mathfrak{W}$. Then there exists $\mathfrak{b}^* \in \mathfrak{C}$, such that $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}^*) = 0$. Moreover,

$$\begin{aligned} \vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{C}, \mathfrak{D}) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}^*, \mathfrak{D}). \end{aligned}$$

Therefore, $\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) \to \vartheta(\mathfrak{b}^*, \mathfrak{D})$ as $n \to \infty$. Since $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$, there exists a subsequence $\{\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})\}$ of $\{\mathfrak{P}(\mathfrak{b}_n)\}$. Such that $\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}}) \to \mathfrak{m}^* \in \mathfrak{D}$ as $\mathfrak{k} \to \infty$. Thus, by solving the following equation with $\mathfrak{k} \to \infty$,

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \tag{11}$$

we have,

$$\vartheta(\mathfrak{b}^*, \theta^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Since, $l^* \in \mathfrak{C}_0$, so, $\mathfrak{P}(\mathfrak{b}^*) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$ and $\mathfrak{p} \in \mathfrak{C}_0$

$$\vartheta(\mathfrak{p}, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{12}$$

Now, (11) and (12), by (2) we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p})) \leq \mathfrak{b}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)) < \mathfrak{J}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)), \text{ for all } \mathfrak{k} \in \mathbb{N}.$$

Since, $\mathfrak{J}$ is non-decreasing function, so, we have

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}) < \vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)$$

Thus, as $\mathfrak{k} \to \infty$, we have $\vartheta(\mathfrak{b}^*, \mathfrak{p}) = 0$ or $\mathfrak{b}^* = \mathfrak{p}$. Finally, by (12) we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

Theorem 2. Let $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ be a $(\mathfrak{J}, \pounds)$-proximal contraction defined on a complete *metric space* $(\mathfrak{W}, \vartheta)$ and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is approximately compact *with respect to* $\mathfrak{C}$. If

(i) $\mathfrak{J}$ is non-decreasing and $\{\mathfrak{J}(t_n)\}$ and $\{\pounds(t_n)\}$ are convergent *seq*uence such that $\lim_{n\to\infty} \mathfrak{J}(t_n) = \lim_{n\to\infty} \pounds(t_n)$, then $\lim_{n\to\infty} t_n = 0$.

(ii) $\mathfrak{C}_0$ is non-empty subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.

Then $\mathfrak{P}$ admits a *best proximity point.*

Proof. As in the proof of Theorem *1, we have*

$$\mathfrak{J}(\theta_n) \leq \pounds(\theta_{n-1}) < \mathfrak{J}(\theta_{n-1}). \tag{13}$$

By (13), then $\{\mathfrak{J}(\theta_n)\}$ is a strictly decreasing sequence. We have two cases here; either the sequence $\{\mathfrak{J}(\theta_n)\}$ is bounded below or Not. If $\{\mathfrak{J}(\theta_n)\}$ is not bounded below, then

$$\inf_{\theta_n > \varepsilon} \mathfrak{J}(\theta_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

From lemma 2, indicated that $\theta_n \to 0$ as $n \to \infty$. Second, the sequence $\{\mathfrak{J}(\theta_n)\}$ is convergent if it is bounded below. The sequence $\{\pounds(\theta_n)\}$ likewise converges by (13), and, both have the same limit. Using (i), we have $\lim_{n\to\infty} \theta_n = 0$, for any sequence $\{\mathfrak{b}_n\}$ in $\mathfrak{C}$. Now, the rest of the proof aligns with the methodology outlined in Theorem 1, we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

Example 2. *Let* $\mathfrak{W} = \mathbb{R}^2$ *and define the function* $\vartheta : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ *by*

$$\vartheta((\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v})) = |\mathfrak{b} - \mathfrak{u}| + |\mathfrak{m} - \mathfrak{v}| \textit{ for all } (\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v}) \in \mathfrak{W}.$$

Then $(\mathfrak{W}, \vartheta)$ *is a complete metric space. Let* $\mathfrak{C}, \mathfrak{D}$ *be the subsets of* $\mathfrak{W}$ *defined by*

$$\mathfrak{C} = \{(0, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\}, \ \ \mathfrak{D} = \{(1, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\}, \ \textit{then } \vartheta(\mathfrak{C}, \mathfrak{D}) = 1.$$

Here $\mathfrak{C}_0 = \mathfrak{C}$ *and* $\mathfrak{D}_0 = \mathfrak{D}$*. Define the mapping* $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ *by* $\mathfrak{P}((0, r)) = (1, \frac{r}{2})$ *for all* $(0, r) \in \mathfrak{C}$*. Thus* $\mathfrak{P}(\mathfrak{C}_0) = \mathfrak{D}_0$*. Define the functions* $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$ *by*

$$\mathfrak{J}(\mathfrak{b}) = 2\mathfrak{b} \textit{ and } \pounds(\mathfrak{b}) = \mathfrak{b};\ \mathfrak{b} \in \mathbb{R}^+.$$

As $\mathfrak{J}(\mathfrak{b}) > \pounds(\mathfrak{b})$ *for every* $\mathfrak{b} \geq t > 0$*. Also* $\lim_{s \to \varepsilon^+} \mathfrak{J}(\mathfrak{b}) > \lim_{\mathfrak{b} \to \varepsilon^+} \sup \pounds(\mathfrak{b})$*. We need to check whether* $\mathfrak{P}$ *is a* $(\mathfrak{J}, \pounds)$*-proximal contraction or not.*

For $\mathfrak{u}_1 = (0, \mathfrak{b})$, $\mathfrak{u}_2 = (0, \mathfrak{m})$ and $\mathfrak{v}_1 = (0, 2\mathfrak{b})$,$\mathfrak{v}_2 = (0, 2\mathfrak{m})$

$$\begin{aligned} \vartheta(\mathfrak{u}_1, \mathfrak{P}\mathfrak{v}_1) &= \vartheta((0, \mathfrak{b}), \mathfrak{P}(0, 2\mathfrak{b})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \\ \vartheta(\mathfrak{u}_2, \mathfrak{P}\mathfrak{v}_2) &= \vartheta((0, \mathfrak{m}), \mathfrak{P}(0, 2\mathfrak{m})) = \vartheta(\mathfrak{C}, \mathfrak{D}). \end{aligned}$$

This implies that,

$$\mathfrak{J}(\vartheta(\mathfrak{u}_1, \mathfrak{u}_2)) \leq \pounds(\vartheta(\mathfrak{v}_1, \mathfrak{v}_2))$$

Therefore, the $(\mathfrak{J}, \pounds)$-proximal contraction is fulfilled. Also, $(0, 0)$ is the best proximity point of the mapping $\mathfrak{P}$. Hence, all the conditions of the Theorem 1 are hold.

3.2. $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction

Let $(\mathfrak{W}, \vartheta)$ be a complete metric space, and $\mathfrak{C}, \mathfrak{D}$ be a pair of nonvoid subsets of $\mathfrak{W}$. Let $\mathfrak{J}, \pounds : (0, \infty) \to \mathbb{R}$ be two functions. A mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ is said to be a $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction if there exist $\alpha, \beta \in (0, 1)$; $\alpha + \beta < 1$ satisfying

$$\left.\begin{array}{l} \vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) = \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) = \vartheta(\mathfrak{C}, \mathfrak{D}) \end{array}\right\} \Rightarrow \mathfrak{J}(\vartheta(\mathfrak{b}_1, \mathfrak{b}_2)) \leq \pounds\left((\vartheta(\mathfrak{m}_1, \mathfrak{m}_2))^{\alpha} (\vartheta(\mathfrak{m}_1, \mathfrak{b}_1))^{\beta} (\vartheta(\mathfrak{m}_2, \mathfrak{b}_2))^{1-\alpha-\beta}\right), \tag{14}$$

for all distinct $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$.

The following example shows that Ćirić-Reich-Rus type interpolative proximal contracion generalizes Ćirić-Reich-Rus type interpolative proximal contraction.

Example 3. *Let* $\mathfrak{W} = \mathbb{R}$ *and* $(\mathfrak{W}, \vartheta)$ *be a usual metric space. Let* $\mathfrak{C} = \{1, 2, 3, 4, 5\}$, $\mathfrak{D} = \{1, 2, 3, 4, 5, 6, 7\}$ *and define* $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ *by* $\mathfrak{P}(\mathfrak{b}) = \mathfrak{b} + 1$*. Then, for* $\ell_1, \ell_2, \nu_1, \nu_2 \in \mathfrak{C}$,

$$\begin{array}{l} \vartheta(\ell_1, \mathfrak{P}(\mathfrak{v}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\ell_2, \mathfrak{P}(\mathfrak{v}_2)) = \vartheta(\mathfrak{C}, \mathfrak{D}) \end{array}.$$

Let $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$*, and suppose that the following inequality holds:*

$$(\vartheta(\ell_1, \ell_2)) \leq \lambda (\vartheta(\mathfrak{v}_1, \mathfrak{v}_2))^{\alpha} (\vartheta(\mathfrak{v}_1, \ell_1))^{\beta} (\vartheta(\mathfrak{v}_2, \ell_2))^{1-\alpha-\beta}.$$

Then

$$\begin{aligned}
(\vartheta(4,2)) &\leq \lambda (\vartheta(3,1))^{\alpha} (\vartheta(3,4))^{\beta} (\vartheta(1,2))^{1-\alpha-\beta} \\
(2) &\leq \lambda((2)^{\alpha} (1)^{\beta} (1)^{1-\alpha-\beta}) \\
(2) &\leq \lambda((2)^{\frac{1}{2}} (1)^{\frac{1}{3}} (1)^{1-\frac{1}{2}-\frac{1}{3}}) \\
(2) &\leq \lambda((2)^{\frac{1}{2}} (1)^{\frac{1}{3}} (1)^{0.166}) \\
\frac{2}{1.4142} &\leq \lambda \ (\textit{a contradiction to } \lambda \in (0,1)).
\end{aligned}$$

This shows that $\mathfrak{P}$ *is not a Ćirić-Reich-Rus type interpolative proximal contraction, however, for the functions* $\mathfrak{J}, \pounds : (0, \infty) \to \mathbb{R}$ *defined by*

$$\mathfrak{J}(\ell) = \begin{cases} \ell + 1 \text{ for } \ell = 2 \\ \ell + 4 \text{ for } \ell \neq 2 \end{cases} \qquad \pounds(\ell) = \begin{cases} \frac{\ell}{2} \text{ for } \ell = 2 \\ \ell + 3 \text{ for } \ell \neq 2 \end{cases}$$

the mapping $\mathfrak{P}$ *satisfies* $(\mathfrak{J}, \mathfrak{b})$*-Ćirić-Reich-Rus type interpolative proximal contraction. Indeed,*

$$\begin{aligned}
\mathfrak{J}(\vartheta(\mathfrak{b}_1, \mathfrak{b}_2)) &\leq \mathfrak{F}\left((\vartheta(\mathfrak{m}_1, \mathfrak{m}_2))^{\alpha} (\vartheta(\mathfrak{m}_1, \mathfrak{b}_1))^{\beta} (\vartheta(\mathfrak{m}_2, \mathfrak{b}_2))^{1-\alpha-\beta}\right) \\
\textit{implies } \mathfrak{J}(2) &\leq \mathfrak{F}(1.4142) \\
3 &\leq 4.4142.
\end{aligned}$$

Theorem 3. *Let* $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ *be an* $(\mathfrak{J}, \pounds)$*-interpolativw Ćirić-Reich-Rus proximal contraction defined on a complete metric space* $(\mathfrak{W}, \vartheta)$ *and* $\mathfrak{C}, \mathfrak{D}$ *be nonvoid, closed subsets of* $\mathfrak{W}$ *such that* $\mathfrak{D}$ *is approximately compact with respect to* $\mathfrak{C}$*. If*

(i) $\mathfrak{J}$ *is non-decreasing function and for any* $\epsilon > 0$

$$\limsup_{t \to \epsilon+} \pounds(t) < \mathfrak{J}(\epsilon+).$$

(ii) $\mathfrak{C}_0$ *is nonvoid subset of* $\mathfrak{C}$ *such that* $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.

Then $\mathfrak{P}$ *has a best proximity point.*

Proof. Let $\mathfrak{b}_0 \in \mathfrak{C}_0$. Since $\mathfrak{P}(\mathfrak{b}_0) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{b}_1 \in \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{b}_1, \mathfrak{P}(\mathfrak{b}_0)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Also, $\mathfrak{P}(\mathfrak{b}_1) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{b}_2 \in \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{b}_2, \mathfrak{P}(\mathfrak{b}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Continuing this process, we construct a *sequence* $\{\mathfrak{b}_n\}$ in $\mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) = \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \in \mathbb{N}. \tag{15}$$

Now, if there exists some $n \in \mathbb{N}$ such that $\mathfrak{b}_n = \mathfrak{b}_{n+1}$, then by (15), then $\mathfrak{b}_n$ is a best proximity point of the mapping $\mathfrak{P}$. If $\mathfrak{b}_n \neq \mathfrak{b}_{n+1}$ for all $n \in \mathbb{N}$ and using (15), we have

$$\vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) = \vartheta(\mathfrak{C}, \mathfrak{D}),$$

and

$$\vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) = \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \geq 1.$$

Thus, by (14), we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) \leq \pounds((\vartheta(\mathfrak{b}_{n-1,}\mathfrak{b}_n))^{\alpha} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{1-\alpha-\beta}) \tag{16}$$

for all distinct $\mathfrak{b}_{n-1}, \mathfrak{b}_n, \mathfrak{b}_{n+1} \in \mathfrak{C}$. Since, $\pounds(t) < \mathfrak{J}(t)$ for all $t > 0$, by (16), we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) < \mathfrak{J}((\vartheta(\mathfrak{b}_{n-1,}\mathfrak{b}_n))^{\alpha} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{1-\alpha-\beta}).$$

Since, $\mathfrak{J}$ is a non-decreasing function, then

$$\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) < (\vartheta(\mathfrak{b}_{n-1,}\mathfrak{b}_n))^{\alpha+\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{1-\alpha-\beta}.$$

This implies that

$$(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})^{\alpha+\beta}) < (\vartheta(\mathfrak{b}_{n-1,}\mathfrak{b}_n))^{\alpha+\beta}.$$

This implies $\theta_n < \theta_{n-1}$ for all $n \in \mathbb{N}$. Suppose on contrary that $\theta > 0$, so that, by (16), we have:

$$\mathfrak{J}(\theta+) = \lim_{n\to\infty} \mathfrak{J}(\theta_n) \leq \lim_{n\to\infty} \pounds\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right) \leq \lim_{t\to\theta+} \sup \pounds(t).$$

This defies presumption (i), hence, $\theta = 0$ and $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$. Now, (i) and Lemma 3, we conclude that $\{\mathfrak{b}_n\}$ is a Cauchy *sequence*. Since $(\mathfrak{W}, \vartheta)$ is a complete *metric space* and $\mathfrak{C}$ is a closed subset of $\mathfrak{W}$. Then there exists $\mathfrak{b}^* \in \mathfrak{C}$, such that $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}^*) = 0$. Further,

$$\begin{aligned}
\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) \\
&\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{C}, \mathfrak{D}) \\
&\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}^*, \mathfrak{D}).
\end{aligned}$$

Therefore, $\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) \to \vartheta(\mathfrak{b}^*, \mathfrak{D})$ as $n \to \infty$. Since $\mathfrak{D}$ is approximately compact *with respect to* $\mathfrak{C}$, there exists a sub*sequence* $\{\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})\}$ of $\{\mathfrak{P}(\mathfrak{b}_n)\}$. Such that $\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}}) \to \mathfrak{m}^* \in \mathfrak{D}$ as $\mathfrak{k} \to \infty$. Thus, by solving the following equation with $\mathfrak{k} \to \infty$,

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \tag{17}$$

we have,

$$\vartheta(\mathfrak{b}^*, y^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Since, $\mathfrak{b}^* \in \mathfrak{C}_0$, so, $\mathfrak{P}(\mathfrak{b}^*) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$ and there exists $\mathfrak{p} \in \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{p}, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{18}$$

Now, (17) and (18), by (14) we have

$$\begin{aligned} \mathfrak{J}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p})) &\leq \pounds\left((\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*))^{\alpha} \cdot (\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}))^{\beta} \cdot (\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{1-\alpha-\beta}\right) \\ &< \mathfrak{J}\left((\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*))^{\alpha} \cdot (\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}))^{\beta} \cdot (\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{1-\alpha-\beta}\right), \text{ for all } \mathfrak{k} \in \mathbb{N}. \end{aligned}$$

Since, $\mathfrak{J}$ is non-decreasing function, so, we have

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}) < (\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*))^{\alpha} \cdot (\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}))^{\beta} \cdot (\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{1-\alpha-\beta}, \text{ for all } \mathfrak{k} \in \mathbb{N}.$$

Thus, as $\mathfrak{k} \to \infty$, we have $\mathfrak{b}^* = \mathfrak{p}$. Finally, by (18) we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

Theorem 4. *Let $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ be an $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction defined on a complete metric space $(\mathfrak{W}, \vartheta)$ and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$. If*

(i) $\mathfrak{J}$ is non-decreasing and $\{\mathfrak{J}(t_n)\}$ and $\{\mathfrak{b}(t_n)\}$ are convergent sequences such that $\lim_{n\to\infty} \mathfrak{J}(t_n)$ then $\lim_{n\to\infty} t_n = 0$.

(ii) $\mathfrak{C}_0$ is non-void subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.

Then $\mathfrak{P}$ has a best proximity point.

Proof. As in the proof of Theorem 3, we have

$$\mathfrak{J}(\theta_n) \leq \pounds\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right) < \mathfrak{J}\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right) \tag{19}$$

By (19), we have $\{\mathfrak{J}(\theta_n)\}$ is strictly decreasing *sequence*. We have two cases here; either the *sequence* $\{\mathfrak{J}(\theta_n)\}$ is bounded below or Not. If $\{\mathfrak{J}(\theta_n)\}$ is not bounded below, then

$$\inf_{\theta_n > \varepsilon} \mathfrak{J}(\theta_n) > -\infty \text{ for every } \varepsilon > 0,\ n \in \mathbb{N}$$

Lemma 2, indicates that $\theta_n \to 0$ as n approaches to ∞. Second, the sequence $\{\mathfrak{J}(\theta_n)\}$ is convergent if it is bounded below. The sequence $\{\mathfrak{b}(\theta_n)\}$ likewise cgs by (25), and, both have the same limit. For each *sequence* $\{\mathfrak{b}_n\}$ in $\mathfrak{C}$ we have $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$ according to (i) .Now, Theorem 3, we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}\mathfrak{b}^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

Note that, if $\mathfrak{P}$ is a self-mapping defined on $\mathfrak{C}$, then best proximity point is a fixed point of $\mathfrak{P}$.

Remark 2. *The generality of Ćirić-Reich-Rus type* $(\mathfrak{J}, \pounds)$ *interpolative proximal contraction for the particular definitions of the mappings* $\mathfrak{J}, \pounds$ *is demonstrated by the observation that follows.*

1. *Defining* $\pounds(\mathfrak{b}) = \mathfrak{J}(\mathfrak{b}) - \tau$ *for all* $\mathfrak{b} \in (0, \infty)$, *in Theorem 3 and Theorem 4, we obtain the existence of best proximity points of the Ćirić-Reich-Rus type interpolative proximal contractions [12].*
2. *Theorem 3 and Theorem 4, produce the existence of best proximity points of the Ćirić-Reich-Rus interpolative type* $(\tau, \mathfrak{J}_{\mathfrak{P}})$*-proximal contraction if* $\pounds(\mathfrak{b}) = \mathfrak{J}(\mathfrak{b}) - \tau(\mathfrak{b})$ *for all* $\mathfrak{b} \in (0, \infty)$.
3. *Letting* $\mathfrak{J}$ *as an identity mapping and* $\pounds(t) = \lambda t$ *for all* $t > 0$ *and* $\lambda \in (0,1)$, *in Theorem 3 and Theorem 4, we receive the existence of best proximity points of the Ćirić-Reich-Rus type interpolative proximal contraction [10].*
4. *If we define* $\pounds(\mathfrak{b}) = \beta(\mathfrak{b})\mathfrak{b}$ *and* $\mathfrak{J}(\mathfrak{b}) = \mathfrak{b}$ *for all* $\mathfrak{b} > 0$ *and* $\beta : (0, \infty) \to (0, 1)$ *verifying* $\limsup_{\mathfrak{b}\to\mathfrak{p}+} \beta(\mathfrak{b}) < 1$ *for each* $\mathfrak{p} > 0$ *in Theorem 3 and Theorem 4, we receive the existence of best proximity points of the Ćirić-Reich-Rus type interpolative Geraghty's proximal contraction.*
5. *For* $\mathfrak{v} = 0$, *we obtain* $(\mathfrak{J}, \pounds)$*-interpolative Kannan type proximal contraction from (14).*

Example 4. *Let* $\mathfrak{b} = \mathbb{R}^2$ *with Euclidean metric* ϑ *on* $\mathbb{R}^2$ *and* $\mathfrak{C} = \{(\mathfrak{b}, \mathfrak{m}) : \mathfrak{m} = \sqrt[2]{9 - \mathfrak{b}^2}\}$ $\mathfrak{D} = \{(\mathfrak{b}, \mathfrak{m}) : \mathfrak{m} = \sqrt[2]{16 - \mathfrak{b}^2}\}$ *be two subsets of* $\mathfrak{W}$. *Then* $\vartheta(\mathfrak{C}, \mathfrak{D}) = 1$, $\mathfrak{C}_0$ *and* $\mathfrak{D}_0$ *are nonvoid subsets of* $\mathfrak{C}$ *and* $\mathfrak{D}$ *respectively. Define a mapping* $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ *by*

$$\mathfrak{P}(\zeta) = \mathfrak{P}(\mathfrak{b}, \mathfrak{m}) = \begin{cases} \left(\frac{\mathfrak{b}}{2}, \frac{\mathfrak{m}}{2}\right) & \text{for } \mathfrak{b} \geq 0 \\ (-1, 0) & \text{for } \mathfrak{b} < 0 \end{cases}.$$

We note that for $\mathfrak{b} \geq 0$, *there is* $\zeta = (\mathfrak{b}, \mathfrak{m}) \in \mathfrak{C}$ *such that* $\vartheta(\zeta, \mathfrak{P}(\zeta)) = \vartheta(\mathfrak{C}, \mathfrak{D}) = 1$. *The following information shows that* $\mathfrak{P}$ *generalizes the Ćirić-Reich-Rus type interpolative proximal contraction [10]. For* $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$, *we have*

$$\vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) = \vartheta(\mathfrak{C}, \mathfrak{D})$$

$$\vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) \quad = \quad \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Let $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$ with $\alpha+\beta< 1$, and suppose on contrary that $\mathfrak{P}$ satisfies the interpolative Ćirić-Reich-Rus type proximal contraction, then

$$\begin{array}{rcl} \vartheta\left(\mathfrak{b}_1, \mathfrak{b}_2\right) & \leq & \lambda(\vartheta\left(\mathfrak{m}_1, \mathfrak{m}_2\right))^{\alpha}(\vartheta\left(\mathfrak{m}_1, \mathfrak{b}_1\right))^{\beta}(\vartheta\left(\mathfrak{m}_2, \mathfrak{b}_2\right))^{1-\alpha-\beta} \\ \vartheta\left((1,0),(1,2)\right) & \leq & \lambda\left(\vartheta\left(1,2\right),(0,1)\right)^{\alpha}\left(\vartheta\left(2,2\right),(1,0)\right)^{\beta}\left(\vartheta\left(0,4\right),(1,2)\right)^{1-\alpha-\beta}. \end{array}$$

This implies

$$\begin{array}{rcl} \sqrt{(1-1)^2+(2-0)^2} & \leq & \lambda\left(\sqrt[2]{(0-2)^2+(4-2)^2}\right)^{\alpha}(\sqrt[2]{(1-2)^2+(0-2)^2})^{\beta} \\ & & \left(\sqrt[2]{(1-0)^2+(2-4)^2}\right)^{1-\alpha-\beta}] \\ 2 & \leq & \lambda[\left(\sqrt[2]{8}\right)^{\frac{1}{2}}(\sqrt[2]{5})^{\frac{1}{3}}\left(\sqrt[2]{5}\right)^{1-\frac{1}{2}-\frac{1}{3}}] \\ 2 & \leq & \lambda[(1.6817)(1.3076)(1.1435)]. \end{array}$$

This implies that $\lambda \geq 1$, a contradiction. Hence, $\mathfrak{P}$ does not satisfy the intplv Ćirić-Reich-Rus type proximal contraction. However $\mathfrak{P}$ satisfies $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction. Indeed, define the functions $\mathfrak{J}, \pounds : (0,\infty) \to \mathbb{R}$ by

$$\pounds(\mathfrak{b}) = \begin{cases} \frac{\mathfrak{b}}{2} & \text{for } \mathfrak{b}=1 \\ \frac{\mathfrak{b}}{10} & \text{otherwise} \end{cases} \quad \mathfrak{J}(\mathfrak{b}) = \begin{cases} \mathfrak{b} & \text{for } \mathfrak{b}=1 \\ \frac{\mathfrak{b}}{8} & \text{otherwise} \end{cases}.$$

Then $\pounds(t) < \mathfrak{J}(t)$ for all $t > 0$ and satisfies assumption (i) and since

$$\begin{array}{rcl} \mathfrak{J}\left(2\right) & \leq & \pounds\left(2.5145\right) \\ \frac{2}{10} & \leq & \frac{2.5145}{8} \\ 0.2 & \leq & 0.3143, \end{array}$$

so, $\mathfrak{P}$ satisfies $(\mathfrak{J}, \pounds)$-Ćirić-Reich-Rus type interpolative proximal contraction.

3.3. $(\mathfrak{J}, \pounds)$-Hardy Rogers type interpolative proximal contraction

Let $(\mathfrak{W}, \vartheta)$ be a complete metric space, and $\mathfrak{C}, \mathfrak{D}$ be a pair of nonvoid subsets of $\mathfrak{W}$. A mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ is said to be a $(\mathfrak{J}, \pounds)$-interpolative Hardy Rogers type proximal contraction if there exist $\alpha, \beta, \gamma, \delta \in (0,1)$ satisfying $\alpha + \beta + \gamma + \delta < 1$ such that

$$\left.\begin{array}{l} \vartheta\left(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right) \\ \vartheta\left(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right) \end{array}\right\} \Rightarrow \mathfrak{J}\left(\vartheta\left(\mathfrak{b}_1, \mathfrak{b}_2\right)\right) \leq \pounds\left(\begin{array}{c} \vartheta\left(\mathfrak{m}_1, \mathfrak{m}_2\right)^{\alpha}\vartheta\left(\mathfrak{m}_1, \mathfrak{b}_1\right)^{\beta}\vartheta\left(\mathfrak{m}_2, \mathfrak{b}_2\right)^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta\left(\mathfrak{m}_1, \mathfrak{b}_2\right)+\vartheta\left(\mathfrak{m}_2, \mathfrak{b}_1\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right), \tag{20}$$

for all distinct $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$ and $\mathfrak{b}_i \neq \mathfrak{m}_i,\ i \in \{1,2\}$ with $\vartheta\left(\mathfrak{P}\mathfrak{b}, \mathfrak{P}\mathfrak{m}\right) > 0$; $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$ are two functions.

Remark 3. *Defining* $\pounds(\mathfrak{b}) = \mathfrak{J}(\mathfrak{b}) - \tau$ *for all* $\mathfrak{b} \in (0, \infty)$*;* $\pounds(\mathfrak{b}) = \mathfrak{J}(\mathfrak{b}) - \tau(\mathfrak{b})$ *for all* $\mathfrak{b} \in (0, \infty)$*; letting* $\mathfrak{J}$ *is a identity mapping and* $\pounds(t) = \lambda t$ *for all* $t > 0$ *and* $\lambda \in (0, 1)$*;* $\pounds(\mathfrak{b}) = \beta(\mathfrak{b})\mathfrak{b}$ *and* $\mathfrak{J}(\mathfrak{b}) = \mathfrak{b}$ *for all* $\mathfrak{b} > 0$ *and* $\beta : (0, \infty) \to (0, 1)$ *verifying* $\limsup_{\mathfrak{b} \to \mathfrak{p}+} \beta(\mathfrak{b}) < 1$ *for each* $\mathfrak{p} > 0$ *in (20), we obtain the interpolative Hardy Rogers type F-proximal contrs [12]; intplv H-R type* $(\tau, F_{\mathfrak{P}})$*-prox contrs; interpolative Hardy Rogers type proximal contraction [10] and interpolative Hardy Rogers type Geraghty's proximal contraction respectively.*

The following example shows that $(\mathfrak{J}, \pounds)$-Hardy Rogers type interpoative proximal contraction generalizes the Hardy Rogers type interpolative proximal contraction [10].

Example 5. *Let* $\mathfrak{W} = \mathbb{R}$ *and define the function* $\vartheta : \mathfrak{W} \times \mathfrak{W} \to \mathbb{R}$ *by*

$$\vartheta(\mathfrak{b}, \mathfrak{m}) = \mid \mathfrak{b} - \mathfrak{m} \mid$$

Then $(\mathfrak{W}, \vartheta)$ *is a metric space. Let* $\mathfrak{C}, \mathfrak{D}$ *be the subsets of* $\mathfrak{W}$ *defined as*

$$\mathfrak{C} = \{1, 2, 3, 4, 5\}, \mathfrak{D} = \{1, 2, 3, 4, 5, 6, 7\} \text{ then } \vartheta(\mathfrak{C}, \mathfrak{D}) = 0$$

Define the functions $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$

$$\mathfrak{J}(\mathfrak{b}) = \begin{cases} \mathfrak{b} + 1 & \text{for } \mathfrak{b} = 2 \\ \mathfrak{b} + 10 & \text{for } \mathfrak{b} \neq 2 \end{cases} \text{ and } \pounds(\mathfrak{b}) = \left\{ \begin{array}{ll} \frac{\mathfrak{b}}{2} & \text{for } \mathfrak{b} = 2 \\ \mathfrak{b} + 5 & \text{otherwise} \end{array} \right\}$$

Define the mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ by $\mathfrak{P}(\mathfrak{b}) = \mathfrak{b} + 1$ for all $\mathfrak{b} \in \mathfrak{C}$. We show that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-interpolative Hardy Rogers type proximal contraction. For $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{m}_1$, $\mathfrak{m}_2 \in \mathfrak{C}$,and $\alpha = \frac{1}{8}, \beta = \frac{1}{7}, \gamma = \frac{1}{6}$

$$\begin{aligned} \vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \end{aligned}$$

implies

$$\mathfrak{J}(\vartheta(\mathfrak{b}_1, \mathfrak{b}_2)) \leq \pounds\left(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)^{\alpha} \vartheta(\mathfrak{m}_1, \mathfrak{b}_1)^{\beta} \vartheta(\mathfrak{m}_2, \mathfrak{b}_2)^{\gamma} \left(\frac{1}{2}(\vartheta(\mathfrak{m}_1, \mathfrak{b}_2) + \vartheta(\mathfrak{m}_2, \mathfrak{b}_1))\right)^{1-\alpha-\beta-\gamma}\right).$$

This shows that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-Hardy Rogers interpoative type proximal contraction. However, the following calculation shows that it is not an interpolative Hardy Rogers type proximal contraction. We know that

$$\begin{aligned} \vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \end{aligned}$$

If there exists $\mathfrak{k} \in (0, 1)$ such that

$$\vartheta(\mathfrak{b}_1, \mathfrak{b}_2) \leq \mathfrak{k}\left(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)^{\alpha} \vartheta(\mathfrak{m}_1, \mathfrak{b}_1)^{\beta} \vartheta(\mathfrak{m}_2, \mathfrak{b}_2)^{\gamma} \left(\frac{1}{2}(\vartheta(\mathfrak{m}_1, \mathfrak{b}_2) + \vartheta(\mathfrak{m}_2, \mathfrak{b}_1))\right)^{1-\alpha-\beta-\gamma}\right)$$

$$\begin{aligned} 2 &\leq \mathfrak{k}\left((2)^{\frac{1}{8}} (1)^{\frac{1}{7}} (1)^{\frac{1}{6}} \left(\frac{1}{2}(3+1) \right)^{1-\frac{1}{8}-\frac{1}{7}-\frac{1}{6}} \right) \\ 2 &\leq \mathfrak{k}(1.6138), \end{aligned}$$

a contradiction. Hence, $\mathfrak{P}$ is not an interpolative Hardy Rogers type proximal contraction.

Theorem 5. *Let $(\mathfrak{W}, \vartheta)$ be a complete metric space and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$. Let $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ be an $(\mathfrak{J}, \pounds)-$ interpolative Hardy Rogers type proximal contraction. If*

(i) $\mathfrak{J}$ is non-decreasing function and for any $\varepsilon > 0$,

$$\lim_{t\to\varepsilon+} \sup \pounds(t) < \mathfrak{J}(\varepsilon+).$$

(ii) $\mathfrak{C}_0$ is nonvoid subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.

Then $\mathfrak{P}$ has a best proximity point.

Proof. Let $\mathfrak{b}_0 \in \mathfrak{C}_0$. Since $\mathfrak{P}(\mathfrak{b}_0) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exist $\mathfrak{b}_1 \in \mathfrak{C}_0$ such that, $\vartheta(\mathfrak{b}_1, \mathfrak{P}(\mathfrak{b}_0)) = \vartheta(\mathfrak{C}, \mathfrak{D})$.Similarly, for $\mathfrak{P}(\mathfrak{b}_1) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{b}_2 \in \mathfrak{C}_0$ such that $\vartheta(\mathfrak{b}_2, \mathfrak{P}(\mathfrak{b}_1)) = \vartheta(\mathfrak{C}, \mathfrak{D})$. Then $\mathfrak{C}_0$ implies to have a *sequence* $\{\mathfrak{b}_n\} \subseteq \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) = \vartheta(\mathfrak{C}, \mathfrak{D}) \tag{21}$$

If there exists some $n \in \mathbb{N}$ such that $\mathfrak{b}_n = \mathfrak{b}_{n+1}$, then $\mathfrak{b}_n$ is a best proximity point of the mapping $\mathfrak{P}$ (see (21)). Assume that $\mathfrak{b}_{n+1} \neq \mathfrak{b}_n$ for all $n \in \mathbb{N}$, then by (21) we have

$$\begin{aligned} \vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \\ \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \geq 1. \end{aligned}$$

Thus by (20), we have

$$\begin{aligned} \mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) &\leq \pounds\left(\begin{array}{l} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\alpha} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_n, \mathfrak{b}_n))\right)^{1-\alpha-\beta-\gamma} \end{array} \right) \\ \mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) &= \pounds\left(\begin{array}{l} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\alpha} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_{n+1}))\right)^{1-\alpha-\beta-\gamma} \end{array} \right) \\ \mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) &\leq \pounds\left(\begin{array}{l} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\alpha} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n) + \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))\right)^{1-\alpha-\beta-\gamma} \end{array} \right) \\ \mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) &\leq \pounds\left(\begin{array}{l} (\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n))^{\alpha+\beta} (\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n) + \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}))\right)^{1-\alpha-\beta-\gamma} \end{array} \right), \end{aligned}$$

for all distinct $\mathfrak{b}_{n-1}, \mathfrak{b}_n, \mathfrak{b}_{n+1} \in \mathfrak{C}$. Let $\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = \theta_n$. Since, $\pounds(t) < \mathfrak{J}(t)$ for all $t > 0$, so we get

$$\mathfrak{J}(\theta_n) < \mathfrak{J}\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{\gamma}\left(\frac{1}{2}(\theta_n + \theta_{n-1})\right)^{1-\alpha-\beta-\gamma}\right). \tag{22}$$

Assume that for some $n \geq 1$, $\theta_{n-1} < \theta_n$. According to (22), we have $(\theta_n)^{\alpha+\beta} < (\theta_n)^{\alpha+\beta}$ since $\mathfrak{J}$ is non- decreasing. As a result, for every $n \in \mathbb{N}$, we obtain $\theta_n < \theta_{n-1}$. This indicates a strictly decreasing *sequence* $\{\theta_n\}$. As a result, it approaches an element $\theta \geq 0$. Consequently, $\theta = 0$, in case $\theta > 0$, we can derive the following via (22):

$$\mathfrak{J}(\theta+) = \lim_{n\to\infty}\mathfrak{J}(\theta_n) \leq \lim_{n\to\infty}\pounds\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{\gamma}\left(\frac{1}{2}(\theta_n + \theta_{n-1})\right)^{1-\alpha-\beta-\gamma}\right) \leq \lim_{t\to\theta+}\mathfrak{b}(t)$$

This contradicts (i), hence, $\theta = 0$ and $\lim_{n\to\infty}\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$. Now, (i) and lemma 3, we conclude that $\{\mathfrak{b}_n\}$ is a Cauchy *sequence*. Since $(\mathfrak{W}, \vartheta)$ is a complete *metric space* and $\mathfrak{C}$ is a closed subset of $\mathfrak{W}$, so, there exists $\mathfrak{b}^* \in \mathfrak{C}$, such that $\lim_{n\to\infty}\vartheta(\mathfrak{b}_n, \mathfrak{b}^*) = 0$. Moreover,

$$\begin{aligned}
\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) \\
&\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{C}, \mathfrak{D}) \\
&\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}^*, \mathfrak{D}).
\end{aligned}$$

Thus, $\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) \to \vartheta(\mathfrak{b}^*, \mathfrak{D})$ as $n \to \infty$. Since $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$, there exists a sub*sequence* $\{\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})\}$ of $\{\mathfrak{P}(\mathfrak{b}_n)\}$ such that $\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}}) \to \mathfrak{m}^* \in \mathfrak{D}$ as $\mathfrak{k} \to \infty$. Letting $\mathfrak{k} \to \infty$ in the following equation:

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \tag{23}$$

we have,

$$\vartheta(\mathfrak{b}^*, \mathfrak{m}^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Since, $\mathfrak{b}^* \in \mathfrak{C}_0$, so $\mathfrak{P}(\mathfrak{b}^*) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{p} \in \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{p}, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{24}$$

Now, using (20) in association with (21) and (22), for all $\mathfrak{k} \in \mathbb{N}$, we have

$$\begin{aligned}
\mathfrak{J}\left(\vartheta\left(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}\right)\right) &\leq \pounds\begin{pmatrix} (\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*))^{\alpha}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^{\beta}(\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix} \\
&< \mathfrak{J}\begin{pmatrix} (\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*))^{\alpha}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^{\beta}(\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix}.
\end{aligned}$$

By using the monotonicity of $\mathfrak{J}$, for all $\mathfrak{k} \in \mathbb{N}$, we have

$$\vartheta\left(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}\right) \leq (\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*))^{\alpha}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^{\beta}(\vartheta(\mathfrak{b}^*, \mathfrak{p}))^{\gamma}\left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma}.$$

Thus, as $\mathfrak{k} \to \infty$, $\mathfrak{b}^* = \mathfrak{p}$. Finally, by (24) we have

$$\vartheta\left(\mathfrak{b}^*, \mathfrak{P}\left(\mathfrak{b}^*\right)\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

Theorem 6. *Let $(\mathfrak{W}, \vartheta)$ be a complete metric space and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is approximately compact with respect to $\mathfrak{C}$. Let $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ be an $(\mathfrak{J}, \pounds)$-interpolative Hardy Rogers type proximal contraction. If*

(i) *$\mathfrak{J}$ is non-decreasing and $\{\mathfrak{J}(t_n)\}$ and $\{\mathfrak{b}(t_n)\}$ are convergent sequences such that*

$$\lim_{n\to\infty} \mathfrak{J}(t_n) = \lim_{n\to\infty} \mathfrak{b}(t_n),$$

then $\lim_{n\to\infty} t_n = 0$.

(ii) *$\mathfrak{C}_0$ is nonvoid subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.*

Then $\mathfrak{P}$ has a best proximity point.

Proof. The proof aligns with the methodology outlined in Theorem 5, we have

$$\begin{aligned} \mathfrak{J}(\theta_n) &\leq \pounds\left(\left(\theta_{n-1}\right)^{\alpha+\beta}\left(\theta_n\right)^{\gamma}\left(\frac{1}{2}\left(\theta_n + \theta_{n-1}\right)\right)^{1-\alpha-\beta-\gamma}\right) \\ &< \mathfrak{J}\left(\left(\theta_{n-1}\right)^{\alpha+\beta}\left(\theta_n\right)^{\gamma}\left(\frac{1}{2}\left(\theta_n + \theta_{n-1}\right)\right)^{1-\alpha-\beta-\gamma}\right). \end{aligned} \tag{25}$$

We establish that $\{\mathfrak{J}(\theta_n)\}$ is a strictly decreasing *sequence* by (25). This presents two scenarios: either the sequence $\{\mathfrak{J}(\theta_n)\}$ is bounded below, or it is not. If $\{\mathfrak{J}(\theta_n)\}$ is not have a lower bounded, then

$$\inf_{\theta_n > \varepsilon} \mathfrak{J}(\theta_n) > -\infty \text{ for every } \varepsilon > 0,\ n \in \mathbb{N}.$$

Lemma 2, indicates that $\theta_n \to 0$ as n approaches to ∞. Second, the *sequence* $\{\mathfrak{J}(\theta_n)\}$ is convergent if it is bounded below. The *sequence* $\{\mathfrak{b}(\theta_n)\}$ likewise converges by (25), and, both have the same limit. For each *sequence* $\{\mathfrak{b}_n\}$ in $\mathfrak{C}$ we have $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$ according to (i). Now, according to Theorem 5 proof, we have

$$\vartheta\left(\mathfrak{b}^*, \mathfrak{P}\mathfrak{b}^*\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right).$$

Hence, $\mathfrak{b}^*$ is a best proximity point of the mapping $\mathfrak{P}$.

4. Application to integral equations

The theory of integral equations is essentially credited to Fourier's exploration of the theory about fundamentals that bears his name; in fact, this theorem, although it is not consistent with Fourier's perspective, can be understood as a statement of the solution to a specific first-order integral equation. However, Abel, Liouville, and several others who came behind them began to consciously explore incredible integral equations, and many of them came to the realisation that the theory might be necessary. The objective is to use Theorem 1 (for $\mathfrak{C} \subseteq \mathfrak{D}$) to show that the following nonlinear Volterra-type integral equations have a solution.

$$f(k) = \int_0^k H_\varsigma(k, h, f)\, dh, \tag{26}$$

for all $k \in [0, 1]$, $\varsigma \in \Theta$, and H_ς is a function defined on $[0, 1]^2 \times C([0, 1], \mathbb{R}_+)$ to $\mathbb{R}$. We demonstrate that the solution to (26) exists. For $f \in C([0, 1], \mathbb{R}_+)$, define norm as: $\|f\|_\tau = \sup_{k\in[0,1]} |f(k)|\, e^{-\tau k}$, $\tau > 0$. Define

$$\eta_\tau(f, \varkappa) = \left[\sup_{k\in[0,1]} |f(k) - \varkappa(k)|\, e^{-\tau k} \right] = \|f - \varkappa\|_\tau$$

for all $f, \varkappa \in C([0, 1], \mathbb{R}_+)$, with these settings, $(C([0, 1], \mathbb{R}_+), \eta_\tau)$ represents a complete metric space.
Now, we show the following theorem to clarify that the solution of integral equation exists.

Theorem 7. *Suppose that the mapping $H_\varsigma : [0, 1]\times[0, 1]\times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$ is a continuous satisfying:*

$$|H_\varsigma(k, h, f) - H_\varsigma(k, h, c)| \leq \frac{\tau\eta_\tau(f, c)}{\tau\eta_\tau(f, c) + 1} e^{\tau h} \tag{27}$$

for every $h, k \in [0, 1]$ and $f, c \in C([0, 1], \mathbb{R})$. Then, integral equation (26) have at most one solution in $C([0, 1], \mathbb{R}_+)$ or equivalently the associated operator $L_\varsigma : \mathbb{R} \to \mathbb{R}$ defined by

$$(L_\varsigma f)(k) = \int_0^k H_\varsigma(k, h, f)\, dh, \tag{28}$$

admits a best proximity point.

Proof. By (27) and (28), we have the following information.

$$|L_\varsigma f - L_\varsigma \varkappa| = \int_0^k |H_\varsigma(k, h, f) - H_\varsigma(k, h, \varkappa)|\, dh,$$

$$\leq \int_0^k \frac{\tau\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1} e^{\tau h} dh$$

$$\leq \frac{\tau\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1} \int_0^k e^{\tau h} dh$$

$$\leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1} e^{\tau k}.$$

This implies

$$|L_\varsigma f - L_\varsigma \varkappa|\, e^{-\tau k} \leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}.$$

$$\|L_\varsigma f - L_\varsigma \varkappa\|_\tau \leq \frac{\eta_\tau(f,\varkappa)}{\tau\eta_\tau(f,\varkappa)+1}.$$

$$\frac{\tau\eta_\tau(f,\varkappa)+1}{\eta_\tau(f,\varkappa)} \leq \frac{1}{\|L_\varsigma f - L_\varsigma \varkappa\|_\tau}.$$

$$\tau + \frac{1}{\eta_\tau(f,\varkappa)} \leq \frac{1}{\|L_\varsigma f - L_\varsigma \varkappa\|_\tau}.$$

which further implies

$$\tau - \frac{1}{\|L_\varsigma f - L_\varsigma \varkappa\|_\tau} \leq \frac{-1}{\eta_\tau(f,\varkappa)}.$$

So all the conditions of Theorem 1 are satisfied for $\mathfrak{J}(\varkappa) = \frac{-1}{\varkappa}$; $\varkappa > 0$ and $\pounds(\varkappa) = \mathfrak{J}(\varkappa) - \tau$. Hence, the integral equation (26) admits a solution.

5. Conclusion

Generalized interpolative proximal contractions provide a robust framework for solving proximity problems in various mathematical and applied contexts. The established existence and uniqueness results facilitate their practical use, offering significant insights and solutions in various applied mathematics and engineering fields.

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6. Authors' contributions

M.N. tabled the main idea of this paper; K. J. wrote the first draft of this paper; M.N., M. A. and M. D. S. reviewed and prepared the second draft; M. D. S. supervised the project. All authors have read and agreed to the published version of the manuscript.

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The authors declare that they do not have any competing interests. All authors read and approved the final manuscript.

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