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On Generalised f-Projection Operator over Nonconvex Set

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Abstract. This work examines the generalized f-projection operators π_S^f. By proving the local Lipschitz continuity of the generalized projection operator π_S^f for S nonempty closed sets that are not necessarily convex, and using convex subdifferential $\partial^{\text{con}} f$, the Fréchet subdifferential $\partial^F f(x)$, and the Clarke subdifferential, we extend many properties of π_S to π_S^f.

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1. Introduction

For X uniformly convex and uniformly smooth Banach spaces, Alber presented the generalized projection operators π_S. Additionally, he thoroughly examined their characteristics and offered a number of applications for the generalized projections, including approximating the solution to variational-inequalities (see [1]). Li explored the second direction in [2], who expanded the definition of π_S to where X is a reflexive Banach space. Also, examined some of its properties and applied them to the solution of variational-inequalities. Since Lie and Alber's research was based on the ideas that using V-functional, and S is a closed, and convex subset of reflexve Banach space. There have been studies to generalize projection operators by either accepting S as not necessarily convex or by extending V-functional to V^f-functional. In [3], K Wu and N Huang presented the generalized f-projection operator π_S^f, which is an extension of the generalized projection operator. It was demonstrated that π_S^f is well-defined for reflexive Banach spaces by providing certain properties, where they used the FanKKM Theorem to look into the existence of solutions to a few variational-inequality issues as an application of their findings.
In their study in [4], M. Bounkhel and R. Al-Yusof used the generalized projection operator π_S to introduce the new generalized proxmal normal cone in reflexive smooth Banach spaces with S not necessarily convex. However, begin with M. Bounkhel (see [5])

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he examined the existence of the generalized projection operator π_S for S set that are not necessarily convex. In uniformly convex and smooth Banach spaces, the author demonstrates that π_S may be empty for a nonconvex closed set. Nevertheless, he demonstrated that the set of points $x^* \in X^*$ s.t $\pi_S(x^*) \neq \phi$ is dense in X^* for closed nonconvex sets. In [6], the properties of the generalized projection on nonconvex sets in reflexive smooth Banach spaces with the smooth dual norm are further examined. The local Lipschtz continuity of π_S for S not necessarily convex was established by M. Bounkhel and M. Bachar. Additionally, they proved that many of the features of π_S on open subsets in X^* are equivalent. Initiated in the nonconvex case by Bounkhel in [5–7], the present work continues the study of the π^f_S. By proving the local Lipschtz continuity of the generalized projection operator π^f_S for S nonempty closed sets not necessarily convex, and extending many properties of π_S to π^f_S on open subsets in X^*.

2. Mathematical Preliminaries

Let X^* be a topological dual space of a Banach space X. In X^* and X, the closed unit balls are $\mathbb{B}*$ and $\mathbb{B}$, respectively. For definitions and some results concerning uniformly smooth, uniformly convex Banach spaces, and strictly convex spaces, we refer to ([8]),[9]).

For $J : X \rightrightarrows X^*$, called the normalzed duality mapping defined by

$$J(u) = \{u^* \in X^* : \langle u^*, u\rangle = \|u^*\|.\|u\| = \|u^*\|^2 = \|u\|^2\}.$$

It is clear that $\|J(u)\|$ is the norm defined on X^*, and $\|u\|$ is the norm defined on X. These are a few of the $J(x)$ map's features. Consider [10] or [11] for more details.

(i) if X smooth Banach spaces, then J continuous operator.

(ii) J is a single valued mapping, whenever X^* is strictly convex .

(iii) J is a single valued mapping, whenever X is reflexive smooth Banach space.

We now review a number of crucial terms and symbols that are necessary for our work. We begin with the widely recognized concepts of the convex subdifferential $\partial^{\text{con}} f$, the Fréchet subdifferential $\partial^F f(x)$, and the Clarke subdifferential (see [12]).

(i) Let $u \in X$ and consider f to be a convex and continuous function defined on X. The convex subdifferential of f at u is given by the set:

$$\partial^{\text{con}} f(u) = \{u^* \in X^* \mid \langle u^*, u - v\rangle \geq f(u) - f(v), \forall v \in X\}.$$

(ii) We say that $u^* \in \partial^F f(u)$ iff, for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t

$$\langle u^*, v - u\rangle \leq f(v) - f(u) + \epsilon\|u - v\|, \quad \forall v \in u + \delta B.$$

Moreover, if f is a convex extended real-valued functional that is lower semi-continuous and defined at $u \in S$, then the Fréchet normal cone of a nonempty closed set S is given by

$$N^F(S,u) = \{u^* \in X^* \mid \forall \epsilon > 0, \exists \delta > 0 \text{ s.t } \langle u^*, v-u\rangle \leq \epsilon \|u-v\|, \forall v \in u+\delta B\}.$$

(iii) We say that $u^* \in \partial^C f(u)$ iff

$$\langle u^*, u\rangle \leq \lim_{t\to 0, v\to u} sup\, t^{-1}[f(v+tu) - f(v)], \quad u \in X.$$

Definition 1. *Let X be a Banach space with dual space X^* and let $f : X \to \mathbb{R} \cup \{\infty\}$ is proper function. Then we define $V^f : X^* \times X \to \mathbb{R} \cup \{\infty\}$ as*

$$V^f(u^*, u) = \|u^*\|^2 + \|u\|^2 - 2\langle u^*, u\rangle + f(u), \quad u^* \in X^*, u \in X.$$

It's straight to demonstrate that

$$f(u) + (\|u^* - u\|)^2 \leq V^f(u^*, u) \leq f(u) + (\|u^* + u\|)^2. \tag{1}$$

Definition 2. *Given a reflexive Banach space X. Let S be a nonempty closed subset of X. We denote by $M_{f,S}(x^*) := \inf_{v\in S} V^f(x^*, v)$, and we define the operator $\pi_S^f : X^* \rightrightarrows X$ as*

$$\pi_S^f(u^*) = \Big\{u \in S : V^f(u^*, u) = M_{f,S}(u^*)\Big\} \quad \forall u^* \in X^*.$$

This operator is called a generalized f-projection.

If $f(x) = 0$ for every $x \in X$, then $\pi_S^f(x^*)$ coincides with the generalized projection $\pi_S(x^*)$, which was introduced and analyzed in Alber [1] and Li [2] for closed convex sets and by [5, 6] for nonempty closed (and not necessarily convex) sets. We would want to draw attention to the fact that, in some situations, it is possible to find a function $f(x) \neq 0$ for some $x \in X$ s.t $\pi_S(x^*) = \pi_S^f(x^*)$, as demonstrated in the example that follows.

Example 1. *Jinlu Li [2] proved in the example (1.2) that for $X = l_1$, and $X^* = l_\infty$ the $\pi_S(0) = co\{u_1, u_2\}$, where $0 = (0,0,0,\cdots)$, $u_1 = (1,1,0,0,\cdots), u_2 = (1,0,1,0,\cdots)$, and $u_3 = (2,0,0,1,0,\cdots) \in L_1$ with $S = co\{u_1,u_2,u_3\}$. For us we define $f(u) = \|u\|^2$ for all $u \in L_1$. Then $V^f(0,u_1) = V^f(0,u_2) = 8$, $V^f(0,u_3) = 18$. Now let $\mu \in [0,1]$ and $v \in co\{u_1,u_2\}$. Then $v = \mu u_1 + (1-\mu)u_2$ with*

$$V^f(0,v) = 2\|(1,\mu,1-\mu,0,\cdots)\|^2 = 8.$$

Now for any $y \in S$ and $\mu_j \in [0,1]$; $\quad j = 1,2,3$ with $\sum_{j=1}^{3} \mu_j = 1$, then $y = \sum_{j=1}^{3} \mu_j u_j$. Hence

$$V^f(0,y) = 2\|(1+\mu_3, \mu_1, \mu_2, \mu_3, 0,\cdots)\|^2 = 2(2+\mu_3)^2 \geq 8.$$

We can observe from the aforementioned inequality that $V^f(0,y) = 8$ iff $\mu_3 = 0$; this implies $y \in co\{u_1,u_2\}$. Therefore we get

$$\pi_S^f(0) = \{v \in S : V^f(0,v) = \inf_{u\in S} V^f(0,u) = 8\} = co\{u_1,u_2\} = \pi_S(0)$$

Furthermore, Bounkhel (see [5]) shows that $\pi_S(x^*) = \phi$ by using an example considering nonconvex closed sets S in Banach spaces that are uniformly smooth and uniformly convex. Using a similar method and by taking $X = l_p$ with $(p \geq 1)$, $0 = (0,0,0,\dots) \in l_p$ and let

$$S = \{s_1, s_2, \dots, s_m, \dots\} \quad ; s_m = (0,0,\dots,1+\frac{1}{m},\dots)$$

define f as $f(x) = \|x\|$ the projection $\pi_S^f(u^*)$ may in fact be empty for nonconvex closed sets S in a uniformly smoth and uniformly convex Banach space. This shows that the generalized f-projection over nonconvex sets may be empty even if the function f is convex continuous and the space X is smooth reflexive. Additionally, it recently demonstrated that, whenever the space X is taken to be a reflexive Banach space with the smooth dual norm, the set of points u^* in X^* with generalized f-projection is dense in X^* (see Theorem 2.1 in [7]).

Now, we consider the vector space of all convergent sequences of real numbers denoted by C. That equipped with the norm $\|u\|_\infty = \sup_n |u_n|$, and The dual space $C^* = l_1$ (Note that l_1 is neither reflexive nor strictly convex).
If $u = (u_0, u_1, \dots) \in \ell_1$,, and $\lambda = (\lambda_0, \lambda_1, \dots) \in C$. Then the duality pairing of C^*, C is given by

$$\langle u, \lambda \rangle = u_0 \lim_{n\to\infty} \lambda_n + \sum_{i=1}^{\infty} u_i \lambda_i.$$

Note that C_0 denotes the closed subspace of C that contains all convergent real sequences with limit zero. (For additional details, see[13]).
Using C, C_0, ℓ_1 one can show that if the Banach space is not reflexive, π_S^f may be empty for some elements $x^* \in X^*$ even for $f(u) \neq 0$.(Take note of my example, which uses the same reasoning as example 2.6 [14] for π_S).

Example 2. *Let* $f(u) = 4$ *and* $u^* = (0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots) \in \ell_1$. *Then* $\pi_{C_0}^f(u^*) = \phi$.

Proof : We define $\lambda_n \in C_0$ for every positive integer n such that its first n components are two and all others are 0. Then $\|u^*\| = (0, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots) = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$, $\|\lambda_n\| = \|(2,2,\dots,2,0,0,\dots)\| = 2$, and so

$$\begin{aligned} \langle u, \lambda_n \rangle &= x_0 \lim_{n\to\infty} \lambda_n + \sum_{i=1}^{\infty} u_i \lambda_i \\ &= 0 \times 2 + \sum_{i=1}^{n} \frac{1}{2^{i-1}} \\ &= 4(1 - 2^{-n+1}). \end{aligned}$$

Which implies

$$V^f(u^*, \lambda_n) = 4 - 8(1 - 2^{-n+1}) + 8 \to 4, \quad as, \quad n \to \infty \tag{2}$$

Now we will prove $V^f(u^*, \lambda) > 4$, for all $\lambda \in C_0$ so that the $inf_{\mu \in C_0} V^f(u^*, \mu)$ by 2 not exists . Suppose that $V^f(u^*, \lambda) = 4$. Then by (1)

$$\begin{aligned}&(\|u^*\| - \|\lambda\|)^2 + f(\lambda) \leq V^f(u^*, \lambda) \leq (\|u^*\| + \|\lambda\|)^2 + f(\lambda)\\ &\Leftrightarrow (\|u^*\| - \|\lambda\|)^2 + 4 \leq 4 \leq (\|u^*\| + \|\lambda\|)^2 + 4\\ &\Leftrightarrow 2 = \|u^*\| = \|\lambda\|\end{aligned}$$

Hence $-2 \leq \lambda_n \leq 2$ for $n = 1, 2, \ldots$. But $lim_{n\to\infty}\lambda_n = 0$, so there are infinitely many n such that $\lambda_n < 2$. That implies

$$\begin{aligned}V^f(u^*, \lambda) &= \|u^*\| - 2\langle u^*, \lambda\rangle + \|\lambda\|^2 + f(\lambda)\\ &= 4 - 2\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\lambda_i + 8\\ &> 4 - 2\sum_{i=1}^{\infty} \frac{1}{2^{i-1}}2 + 8 = 4\end{aligned}$$

which is a contraction to our assumption $V^f(u^*, \lambda) = 4$. Therefor $\pi^f_{C_0}(u^*) = \phi$.

These observations force us to assume that the space X complies with this assumption going onward; in other words, we will presume that X is a reflexive Banach space with a smooth dual norm for the remainder of the work.

3. On $\pi^f_S(x^*)$ for S nonconvex set

In this section, we generalized results related to generalized projection on closed nonconvex sets π_S to π^f_S on closed nonconvex sets with f is a proper, lower semi-continuous function (Unless we specified otherwise,). So we start with the following lemma that is analog with Lemma 2.1 [15] for metric projection, proposition 1.3 in [16] for Hilbert spaces, lemma 2.2 [6] for generalized projection on closed nonconvex sets and Recently this lemma proved for Generalized (f, λ)-projection operator see [7].

Lemma 1. *Let X^* be a smooth dual norm space of a reflexive Banach space X. Then for every $\beta \in (0, 1)$ and for all $u \in \pi^f_S(J(v))$, we get $\pi^f_S((1-\beta)J(u) + \beta J(v)) = \{u\}$.*

Proof : For proving $u \in \pi^f_S((1-\beta)J(u) + \beta J(v))$ proceeds like Theorem 3.1 Part 3 in [7] by taking the constant λ defined there, equal half. Unfortunately, we can't proceed only with $\lambda = \frac{1}{2}$ for

the uniqueness part, so we assume that $u_0 \in \pi^f_S((1-\beta)J(u) + \beta J(v))$, where $u \neq u_0$. Thus, we have two cases:
case1: $f(u_0) - f(u) \geq 0$, we get

$$V^f((1-\beta)J(u) + \beta J(v), u) = V^f((1-\beta)J(u) + \beta J(v), u_0).$$

So

$$V((1-\beta)J(u)+\beta J(v),u)-V((1-\beta)J(u)+\beta J(v),u_0)=f(u_0)-f(u)\geq\beta[f(u_0)-f(u)].$$

On the other hand

$$\begin{aligned}&V((1-\beta)J(u)+\beta J(v),u)-V((1-\beta)J(u)+\beta J(v),u_0)\\&=\beta\|u\|^2+(1-\beta)\|u\|^2-\beta\|u_0\|^2-(1-\beta)\|u_0\|^2-2(1-\beta)\langle J(u),u\rangle+2(1-\beta)\langle J(u),u_0\rangle\\&\quad-2\beta\langle J(v),u\rangle+2\beta\langle J(v),u_0\rangle\\&=\beta\left(V(J(v),u)-V(J(v),u_0)\right)-(1-\beta)V(J(u),u_0).\end{aligned}$$

And so

$$\beta\left(V(J(v),u)-V(J(v),u_0)\right)-(1-\beta)V(J(u),u_0)\geq\beta[f(u_0)-f(u)].$$

Hence,

$$\begin{aligned}(1-\beta)V(J(u),u_0)&\leq\beta\left(V(J(v),u)+f(u)-V(J(v),u_0)-f(u_0)\right)\\&=\beta\left(V^f(J(v),u)-V^f(J(v),u_0)\right)\leq 0.\end{aligned}$$

Since $u\in\pi_S^f(J(v))$, and so $V^f(J(v),u)\leq V^f(J(v),u_0)$. which implies that $V(J(u),u_0)\leq 0$ and hence $V(J(u),u_0)=0$, and so $u=u_0$.

case2: $f(u)-f(u_0)\geq 0$, we get

$$V^f((1-\beta)J(u)+\beta J(v),u)=V^f((1-\beta)J(u)+\beta J(v),u_0).$$

So

$$V((1-\beta)J(u)+\beta J(v),u_0)-V((1-\beta)J(u)+\beta J(v),u)=f(u)-f(u_0)\geq\beta[f(u)-f(u_0)].$$

proceed like case1, we get the same result, and this ends our prove □

Now, using concepts from [17] and [6]. We provide a necessary and sufficient condition for the existence and uniqueness of the $M_{f,S}(x^*)$ in the following lemma, in terms of $\partial^F M_{f,S}(x^*)$.

Lemma 2. *Let x^* be an element of smooth dual norm space X^* of reflexive Banach space X. Then $\pi_S^f(x^*)$ is exists and unique, whenever $\partial^F M_{f,S}(x^*)\neq\phi$. Also, we get that $\partial^F M_{f,S}(x^*)=\{2(J^*x^*-\pi_S^f(x^*))\}$.*

Proof : Let $\partial^F M_{f,S}(x^*)\neq\phi$ and let $x_1\in\partial^F M_{f,S}(x^*)$. Then for every $\epsilon>0$ and applying the concept of ∂^F to obtain, $\exists\delta>0$ such that given any $u^*\in B_*$ and any $\mu\in(0,\delta)$, we get:

$$\langle x_1;x^*+\mu u^*-x^*\rangle=\langle x_1;\mu u^*\rangle\leq M_{f,S}(x^*+\mu u^*)-M_{f,S}(x^*)+\epsilon\mu.$$

Therefore, we can find a sufficiently large $\beta \in \mathbb{N}$ s.t:

$$\langle x_1; \beta^{-1}u^* \rangle \leq M_{f,S}(x^* + \beta^{-1}u^*) - M_{f,S}(x^*) + \epsilon\beta^{-1} \quad ,\forall u^* \in B_*. \tag{3}$$

Using definition of $M_{f,S}$, $\exists v_n \in S$ for all $n \geq 1$.

$$M_{f,S}(x^*) \leq V^f(x^*, v_n) \leq M_{f,S}(x^*) + \frac{1}{n^2}. \tag{4}$$

Hence, for sufficiently large $\beta \in \mathbb{N}$, we can combine inequalities (3) and (4) for all $u^* \in B_*$.

$$\begin{aligned}\langle x_1; \beta^{-1}u^* \rangle &\leq V^f(x^* + \beta^{-1}u^*; v_\beta) - V^f(x^*, v_\beta) + \beta^{-2} + \beta^{-1}\epsilon \\ &\leq \|x^* + \beta^{-1}u^*\|^2 - \|x^*\|^2 - 2\langle \beta^{-1}u^*; v_\beta \rangle + \beta^{-2} + \beta^{-1}\epsilon.\end{aligned}$$

Thus

$$\langle x_1 + 2v_\beta; u^* \rangle \leq \beta\|x^* + \beta^{-1}u^*\|^2 - \beta\|x^*\|^2 + \beta^{-1} + \epsilon \quad \forall u^* \in B_*.$$

For β large enough, and $\nabla^F \|.\|^2(x^*) = 2J^*x^*$, we can write

$$\sup_{u^* \in B_*} \beta\big[\|x^* + \beta^{-1}u^*\|^2 - \|x^*\|^2 - \langle 2J^*x^*; \beta^{-1}u^* \rangle\big] \leq \epsilon.$$

Therefore, for β large enough

$$\|x_1 - 2(J^*x^* - v_\beta)\|| = \sup_{u^* \in B_*} \langle x_1 - 2(J^*x^* - v_\beta); u^* \rangle \leq 3\epsilon. \tag{5}$$

We conclude that this sequence $(v_\beta)_\beta$ converges to $v := J^*x^* - \frac{1}{2}x_1$ by keeping in mind that the sequence(v_β) can be selected arbitrarily and that for arbitrary positive ϵ, the aforementioned inequality (5) is holds. (when β is sufficiently large). Additionally, since $V^f(x^*, v_n) \to M_{f,S}(x^*)$, we conclude that $v \in \pi_S^f(x^*)$. Now let $\bar{x} \in \pi_S^f(x^*)$. Then carrying on as previously we get $\bar{x} := J^*x^* - \frac{1}{2}x_1 = v.\square$

From Lemma (2), it is clearly that if $\pi_S^f(u^*) \neq \phi$, then $M_{f,S}(u^*) \subset \{2(J^*u^* - u)\}$ for all $u \in \pi_S^f(u^*)$.
In the upcoming lemma, we will show that $M_{f,S}$ is locally Lipschtz continuous.

Lemma 3. *Let X^* be the dual space of the reflexive Banach space X and f is a proper function, and bounded from bellow on X, then $u^* \mapsto V^f(u^*, u)$ is locally Lipschtz on X^*.*

Proof :

Fix $\delta > 0$ and let u^* be an element of X^*. Choose a value ϵ such that $0 < \epsilon < \delta$ and fix two elements v^* and z^* in $u^* + \delta\mathbb{B}*$. By definition of $M_{f,S}$ there exists $r_\epsilon \in S$ such that

$$M_{f,S}(v^*) \leq V^f(v^*, r_\epsilon) < M_{f,S}(v^*) + \epsilon.$$

So

$$\begin{aligned} M_{f,S}(z^*) - M_{f,S}(v^*) &\le V^f(z^*, r_\epsilon) - V^f(v^*, r_\epsilon) + \epsilon \\ &= V(z^*, r_\epsilon) - V(v^*, r_\epsilon) + \epsilon \\ &\le (\|z^*\| + \|v^*\| + 2\|r_\epsilon\|)\|z^* - v^*\| + \epsilon. \end{aligned}$$

Let $r = M_{f,S}(u^*)$. Then $\exists u_\epsilon \in S$ s.t.

$$M_{f,S}(u^*) \le V^f(u^*, u_\epsilon) < M_{f,S}(u^*) + \epsilon.$$

Now we can write

$$\begin{aligned} \|r_\epsilon\| &\le (V(v^*, r_\epsilon))^{\frac{1}{2}} + \|v^*\| \\ &= \left(V^f(v^*, r_\epsilon) - f(r_\epsilon)\right)^{\frac{1}{2}} + \|v^*\| \\ &< (M_{f,S}(v^*) - f(r_\epsilon) + \epsilon)^{\frac{1}{2}} + \|v^*\| \\ &\le (M_{f,S}(v^*) - f(r_\epsilon) + \epsilon)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left(V^f(v^*, u_\epsilon) - f(r_\epsilon) + \epsilon\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le (V(v^*, u_\epsilon) + f(u_\epsilon) - f(r_\epsilon) + \epsilon)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left((\|v^*\| + \|u_\epsilon\|)^2 + f(u_\epsilon) - f(r_\epsilon) + \epsilon\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left(\left(\|u^*\| + \delta + (V(u^*, u_\epsilon))^{\frac{1}{2}} + \|u^*\|\right)^2 + f(u_\epsilon) - f(r_\epsilon) + \epsilon\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left(\left(2\|u^*\| + \delta + (V^f(u^*, u_\epsilon) - f(u_\epsilon))^{\frac{1}{2}}\right)^2 + f(u_\epsilon) - f(r_\epsilon) + \epsilon\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left(\left(2\|u^*\| + \delta + (r + \epsilon - f(u_\epsilon))^{\frac{1}{2}}\right)^2 + f(u_\epsilon) - f(r_\epsilon) + \epsilon\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &\le \left((2\|u^*\| + \delta)^2 + r + 2\epsilon + 2(2\|u^*\| + \delta)(r + \epsilon - f(u_\epsilon))^{\frac{1}{2}} - f(r_\epsilon)\right)^{\frac{1}{2}} + \|u^*\| + \delta. \end{aligned}$$

Since $f(u) \ge L$ for some $L \in \mathbb{R}$, so we can write

$$\begin{aligned} \|r_\epsilon\| &\le \left((2\|u^*\| + \delta)^2 + r + 2\epsilon + 2(2\|u^*\| + \delta)(r + \epsilon - L)^{\frac{1}{2}} - L\right)^{\frac{1}{2}} + \|u^*\| + \delta \\ &:= C_{\delta,\epsilon,u^*}. \end{aligned}$$

Now letting $\epsilon \to 0$ and with $K_{\delta,u^*} =: 2(\|u^*\| + C_{\delta,u^*} + \delta)$ we get

$$|M_{f,S}(z^*) - M_{f,S}(v^*)| \le 2(\|u^*\| + C_{\delta,u^*} + \delta)\|z^* - v^*\| \le K_{\delta,u^*}\|z^* - v^*\|.$$

So the proof is complete □
The Fréchet sub-differentiability of $M_{f,S}$ is equivalent to its Fréchet differentiability, as demonstrated by the following Lemma.

Lemma 4. *Given that x^* an element of smooth dual norm space X^* of reflexive Banach space X. Then, the two statements that follow are equivalent:*

(i) $\partial^F M_{f,S}(x^*) \neq \phi$.

(ii) $M_{f,S}$ *is Fréchet differentiable at* x^*.

Proof : From (2) → (1) clear, so we merely need to demonstrate the opposite implication. Suppose $\partial^F M_{f,S}(x^*) \neq \phi$ and let $z \in \partial^F M_{f,S}(x^*)$. By lemma (2) we get $z = 2(J^*x^* - v)$ with $v \in \pi_S^f(x^*)$. For some $\epsilon > 0$. Using concept of ∂^F, $\exists \beta_1 > 0$, and so $\mu \in (0, \beta_1)$ and all $u^* \in B_*$, we get

$$\langle 2(J^*x^* - v); \mu u^* \rangle \leq M_{f,S}(x^* + \mu u^*) - M_{f,S}(x^*) + \epsilon\mu.$$

Which implies

$$\mu^{-1} \left[M_{f,S}(x^* + \mu u^*) - M_{f,S}(x^*) \right] - \langle 2(J^*x^* - v); u^* \rangle \geq -\epsilon; \quad \forall \mu \in (0, \beta_1), \forall u^* \in B_*.$$

Also, by definition of $M_{f,S}$ we have

$$\mu^{-1} \left[M_{f,S}(x^* + \mu u^*) - M_{f,S}(x^*) \right] \leq \mu^{-1} \left[V^f(x^* + \mu u^*, y) - V^f(x^*, y) \right].$$

Since $\nabla^F V^f(\cdot; v) = 2(J^*x^* - v)$, there exists $\beta_2 > 0$ such that for any $\mu \in (0, \beta_2)$ and for all $u^* \in B_*$ we have.

$$\left| \mu^{-1} \left[V^f(x^* + \mu u^*, y) - V^f(x^*, y) \right] - \langle 2(J^*x^* - v), u^* \rangle \right| \leq \epsilon.$$

And so

$$\mu^{-1} \left[M_{f,S}(x^* + \mu u^*) - M_{f,S}(x^*) \right] - \langle 2(J^*x^* - v); u^* \rangle \leq \epsilon.$$

Hence

$$\left| \mu^{-1} \left[M_{f,S}(x^* + \mu u^*) - M_{f,S}(x^*) \right] - \langle 2(J^*x^* - v); u^* \rangle \right| \leq \epsilon, \quad \mu \in (0, \beta), \forall u^* \in B_*.$$

With $\beta = min\{\beta_1, \beta_2\}$, since $\epsilon > 0$ is arbitrary, so

$$\nabla^F M_{f,S}(x^*) = 2(J^*x^* - v).$$

Thus, our proof is finished. □

The $\|.\| - \|.\|$ continuity of π_S^f are related to the continuou $\nabla^F M_{f,S}$ by the following lemma.

Lemma 5. *Given that U^* is an open set of smooth dual norm space X^* of reflexive Banach space X. Then, the two claims that follow are equivalent:*

(i) The function $M_{f,S}$ is C^1 on U^*.

(ii) The operator π_S^f is single-valued and $\|.\| - \|.\|$ continuous on U^*.

proof : (1) $\Rightarrow$ (2) Assuming that $M_{f,S}$ is C^1 on U^*, π_S^f is single-value functional on U^* and for every $u^* \in U^*$, $\pi_S^f(u^*) = J^*u^* - \frac{1}{2}\nabla^F M_{f,S}(u^*)$ by lemma (4). Since J^* and $\nabla^F M_{f,S}$ are $\|.\| - \|.\|$ continuous, so π_S^f is $\|.\| - \|.\|$ continuous on U^*.
(2) $\Rightarrow$ (1) First, we observe that $M_{f,S}$ is locally Lipschtz on U^* (by lemma 3) hence according to Mordukhovich-Shao (see [18]), for all $u^* \in U^*$ we get

$$\partial^C M_{f,S}(u^*) = \bar{co}\{weak\text{-}\lim_{v^* \to u^*} sup \partial^F M_{f,S}(v^*)\} = \bar{co}\{weak\text{-}lim u_n : u_n \in \partial^F M_{f,S}(u_n^*); u_n^* \to u^*\}.$$

Now by lemma (2) for any $u_n^* \to u^*$ with $u_n \in \partial^F M_{f,S}(u_n^*)$ we have $x_n = 2(J^*u^* - \pi_S^f(u_n^*))$. Therefore

$$\partial^C M_{f,S}(u^*) = \bar{co}\{weak - lim u_n : u_n = 2(J^*u_n^* - \pi_S^f(u_n^*)), u_n^* \to u^*\}.$$

We now obtain $\partial^C M_{f,S}(u^*) = \{2(J^*u^* - \pi_S^f(u^*)\}$ using the $\|.\| - \|.\|$ continuity of π_S^f and J^* on X^*. As a result, $M_{f,S}$ will definitely be continuously Gâteaux differentiable on U^*. We may finally detect that it is C^1 on U^* since $M_{f,S}$ is locally Lipschtz on U^*.□

The following lemma proves that, whenever theπ_S^f is single valued on reflexive Banach spaces with the Kadec condition(i.e. For any sequence $(u_n) \rightharpoonup u$ weakly in X with $\|u_n\| \to \|u\|$, then $\|u_n - u\| \to 0$, see [8]), its norm-to-weak continuity and $\|.\| - \|.\|$ continuity are equivalent. Its proof is based on a concept from Lemma 5.1 proof in [15] for metric projection, and lemma 2.8 [6] for generalized projection operator.

Lemma 6. *Given that U^* is an open set in X^*, where X Banach space is reflexive with the Kadec condition. Suppose that π_S^f is a single-valued function on U^*, and that $f : X \to \mathbb{R} \cup \{\infty\}$ is proper continuous function. Then, π_S^f is $\|.\| - \|.\|$ continuous on U^* iff it is $\|.\| - to - weak$ continuous on U^*.*

Proof : π_S^f is $\|.\| - \|.\|$ continuous on U^* follows naturally from the $\|.\| - to - weak$ continuity on U^*. All we have to do is demonstrate the opposite. Let $\pi_S^f(u_n^*) \to \pi_S^f(u^*)$ weakly in X, and $lim\|u_n^* - x^*\| = 0$. Then, by Lipschtz continuity of $M_{f,S}$ we have

$$V^f(u_n^*, \pi_S^f(u_n^*)) = M_{f,S}(u_n^*) \to M_{f,S}(u^*) = V^f(x^*, \pi_S^f(u^*)).$$

Now notice that:

$$\begin{aligned}\|\pi_S^f(u_n^*)\|^2 - \|\pi_S^f(u^*)\|^2 = &\Big[V^f(u_n^*, \pi_S^f(u_n^*)) - \|u_n^*\|^2 + 2\langle u_n^*; \pi_S^f(u_n^*)\rangle - f(\pi_S^f(u_n^*))\Big] \\ &- \Big[V^f(u^*, \pi_S^f(u^*)) - \|u^*\|^2 + 2\langle u^*; \pi_S^f(u^*)\rangle - f(\pi_S^f(u^*)\Big]\end{aligned}$$

$$= \left[V^f(u_n^*, \pi_S^f(u_n^*)) - V^f(u^*, \pi_S^f(u^*))\right] + \left[\|u^*\|^2 - \|u_n^*\|^2\right]$$
$$+ 2\left[\langle u_n^*; \pi_S^f(u_n^*)\rangle - \langle u^*; \pi_S^f(u^*)\rangle\right] + \left[f(\pi_S^f(u^*) - f(\pi_S^f(u_n^*)\right].$$

By continuity of f, we get that $|\pi_S^f(u_n^*)\| \to \|\pi_S^f(u^*)\|$ and hence $\|\pi_S^f(u_n^*) - \pi_S^f(u^*)\| \to 0$ is guaranteed since the space X has Kadec property, this concluding the proof □.

The $\nabla^F M_{f,S}$ and continuous Fréchet differentiability of $M_{f,S}$ are equivalent on reflexive Banach spaces with the Kadec condition. This is shown by the following lemma, based on a concept from Lemma 4.2 in [15] and Lemma 2.9 in [6].

Lemma 7. *Given U^* as an open set in X^*, with X a reflexive Banach space with a smooth dual norm and the Kadec condition, and f is proper, convex, and Fréchet differentiable on U^*, the two assertions that follow are equivalent:*

(i) $M_{f,S}$ is C^1 in U^*.

(ii) $M_{f,S}$ is Fréchet differentiable on U^*.

Proof : From (1) → (2) clear, We need only to prove (2) → (1). Let u_n^* be a sequence that converges to u^* in X. We want to demonstrate that $\nabla^F M_{f,S}(u_n^*) \to \nabla^F M_{f,S}(u^*)$. Notice that

$$M_{f,S}(v^*) = \inf_{s\in S} V^f(v^*, s) = \inf_{s\in S}\{\|v^*\|^2 - 2\langle v^*, s\rangle + \|s\|^2 + f(s)\}$$
$$= \|v^*\|^2 - \sup_{s\in S}\{-\|s\|^2 + 2\langle v^*, s\rangle - f(s)\}.$$

The function f is convex, and it is clear that both functions

$$v^* \mapsto V_S(v^*) = \sup_{s\in S}\{-\|s\|^2 + 2\langle v^*, s\rangle - f(s)\}.$$

$M_{f,S}(v^*) = \|v^*\|^2 + V_S(v^*)$ are convex. f is Fréchet differentiable. Then, V_S is functional and is both convex and Fréchet differentiable. So the derivative of V_S is norm-to-weak continuous (see[19]), which implies $\nabla^F V_S(u_n^*)$ converges weakly to $\nabla^F V_S(u^*)$, which means that $\nabla^F M_{f,S}(u_n^*)$ approaching $\nabla^F M_{f,S}(u^*)$ weakly. Lemma 4 allows us to write $\nabla^F M_{f,S}(u_n^*) = 2(J^*u_n^* - \pi_S^f(u_n^*))$ and $\nabla^F M_{f,S}(u^*) = 2(J^*u^* - \pi_S^f(u^*))$. Thus, $\pi_S^f(u_n^*)$ converges weakly to $\pi_S^f(u^*)$. By 6 we get the Strong convergence of $\pi_S^f(u_n^*)$ to $\pi_S^f(u^*)$.The continuity of $\nabla^F M_{f,S}$ has been established, Which brings the proof to an end. □

The theorem below establishes an equivalence between the properties of π_S^f and $M_{f,S}$ on an open subset of X^*.

Theorem 1. *Let X be a reflexive Banach space with smooth dual norm and Kadec property. Let U^* be a subset of X^* that is open. Let f is proper, convex, continuous, bounded from bellow, and Fréchet differentiable function. The statements that follow are equivalent.*

(i) $M_{f,S}$ *is* C^1 *in* U^*.

(ii) $M_{f,S}$ *is Fréchet differentiable on* U^*.

(iii) $\partial^F M_{f,S} \neq \phi$ *on* U^*.

(iv) π_S^f *is single-valued and* $\|.\| - to - weak$ *continuous on* U^*.

(v) π_S^f *is single-valued and* $\|.\| - \|.\|$ *continuous on* U^*.

Proof : Combining the previously proven lemmas to prove this theorem is enough.

4. Conclusion

To investigate the π_S^f, this study shows the efficiency of the subdifferential $\partial^{\text{con}} f$, the Fréchet subdifferential $\partial^F f(x)$, and the Clarke subdifferential to extend many properties of π_S to π_S^f In addition to provide a necessary and sufficient condition for the existence and uniqueness and prove locally Lipschtz continuous of the π_S^f. In the future, we intend to introduce a new set defined in terms of the generalized f-projection, to extend many well-known results on the usual V-proximal normal cone in the setting of Banach spaces. Furthermore, we will extend the widely studied notion of prox-regularity by incorporating the flexibility of f-projections. such as in [6, 7].

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

[1] Ya I Alber. Metric and generalized projection operators in banach spaces: properties and applications. *arXiv preprint funct-an/9311001*, 1993.

[2] Jinlu Li. The generalized projection operator on reflexive banach spaces and its applications. *Journal of Mathematical Analysis and Applications*, 306(1):55–71, 2005, https://doi.org/10.1016/j.jmaa.2004.11.007.

[3] Ke-Qing Wu and Nan-Jing Huang. The generalised f-projection operator with an application. *Bulletin of the Australian Mathematical Society*, 73(2):307–317, 2006, https://doi.org/10.1017/S0004972700038892.

[4] M Bounkhel and R Al-Yusof. Proximal analysis in reflexive smooth banach spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 73(7):1921–1939, 2010, https://doi.org/10.1016/j.na.2010.04.077.

[5] Messaoud Bounkhel et al. Generalized projections on closed nonconvex sets in uniformly convex and uniformly smooth banach spaces. *Journal of Function Spaces*, 2015, 2015, https://doi.org/10.1155/2015/478437.

[6] Messaoud Bounkhel and Mostafa Bachar. Generalised-prox-regularity in reflexive smooth banach spaces with smooth dual norm. *Journal of Mathematical Analysis*

and Applications, 475(1):699–729, 2019,
https://doi.org/10.1016/j.jmaa.2019.02.064.

[7] Messaoud Bounkhel. Generalized (f, λ)-projection operator on closed nonconvex sets and its applications in reflexive smooth banach spaces. *AIMS Mathematics*, 8(12):29555–29568, 2023,
https://doi.org/10.3934/math.20231513.

[8] Robert Deville, Gilles Godefroy, and Václav Zizler. *Smoothness and renormings in Banach spaces*. Longman Scientific Technical, Harlow; copublished in the United States with ..., 1993.

[9] J Diestel. Geometry of banach space-selected topics lect. *Notes in Math*, 485, 1975.

[10] Wataru Takahashi. Nonlinear functional analysis. *Fixed Point Theory and its Applications*, 2000.

[11] Mordukhaui Moiseevich Vauinberg, Alexander Libin, and David Louvish. Variational method and method of monotone operators in the theory of nonlinear equations. *(No Title)*, 1973,
https://lccn.loc.gov/73016383.

[12] Messaoud Bounkhel. *Regularity concepts in nonsmooth analysis: theory and applications*, volume 59. Springer Science & Business Media, 2011.

[13] Nelson Dunford and Jacob T Schwartz. *Linear operators, part 1: general theory*, volume 10. John Wiley & Sons, 1988.

[14] Akhtar A Khan, Jinlu Li, and Simeon Reich. Generalized projections on general banach spaces. *arXiv preprint arXiv:2207.09317*, 2022,
https://doi.org/10.48550/arXiv.2207.09317.

[15] Frédéric Bernard, Lionel Thibault, and Nadia Zlateva. Characterizations of prox-regular sets in uniformly convex banach spaces. *Journal of Convex Analysis*, 13(3/4):525, 2006,.

[16] Francis H Clarke, Yuri S Ledyaev, Ronald J Stern, and Peter R Wolenski. *Nonsmooth analysis and control theory*, volume 178. Springer Science & Business Media, 2008.

[17] Jonathan M Borwein and JR Giles. The proximal normal formula in banach space. *Transactions of the American Mathematical Society*, 302(1):371–381, 1987,
https://doi.org/10.1090/S0002-9947-1987-0887515-5.

[18] Boris Mordukhovich and Yongheng Shao. Nonsmooth sequential analysis in asplund spaces. *Transactions of the American Mathematical Society*, 348(4):1235–1280, 1996,
https://doi.org/10.1090/S0002-9947-96-01543-7.

[19] Ivar Ekeland and Gérard Lebourg. Generic fréchet-differentiability and perturbed optimization problems in banach spaces. *Transactions of the American Mathematical Society*, 224(2):193–216, 1976,
https://doi.org/10.1090/S0002-9947-1976-0431253-2.