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Extending $\mathcal{F}$-Contraction Theory: Fixed Points in Triple-Controlled S-Metric Spaces

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Abstract. This paper explores the landscape of fixed point theory by introducing two novel classes of contraction mappings: the $(\alpha_s, \nu_s, (Q,h)\text{-}\mathcal{F})$-contraction and the $(\alpha_s, \eta_s, \nu_s, (Q,h)\text{-}\mathcal{F})$-contraction, defined within the rich structure of triple-controlled S-metric type spaces. These mappings are constructed using a blend of α_s- and η_s-admissibility, ν_s-subadmissibility, and a pair of upper-class functions (Q,h), integrated with Wardowski's powerful $\mathcal{F}$-contraction approach. Our results significantly extend the classical $(\alpha_s\text{-}\mathcal{F})$-contraction framework by proving the existence and uniqueness of fixed points under these generalized settings. Furthermore, we derive meaningful corollaries by specifying various (Q,h) pairs, illustrating the versatility and depth of the proposed theory and its contribution to the advancement of fixed point results in generalized metric environments.

2020 Mathematics Subject Classifications: 47H10, 54H25

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1. Introduction

The study of fixed point theory has a rich and influential history, tracing back to the foundational work of Banach (1922) [1], whose pioneering contributions to metric fixed point theory culminated in the celebrated Banach Contraction Principle. This cornerstone result has inspired extensive research across diverse branches of mathematics and numerous scientific fields, including computer science, engineering, physics, and economics. By its very nature, fixed point theory draws upon ideas from topology, analysis, and geometry

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to investigate the existence and uniqueness of fixed points of mappings—an endeavor that has generated profound theoretical insights as well as practical applications.

Over the decades, the Banach Fixed Point Theorem has undergone significant generalizations, particularly through the development of new classes of metric spaces. For instance, Bakhtin [2] introduced the concept of b-metric spaces, which was subsequently extended into extended b-metric spaces [3]. Researchers further expanded this framework to include controlled metric spaces [4], and later to more complex structures such as double and triple controlled metric spaces [5], [6], [7], [8], [9],[10].

In parallel, Sedghi et al. [11] introduced the concept of S-metric spaces as a broad generalization of classical metric spaces. Although Sedghi et al. later demonstrated in [12] that every standard metric d induces an S-metric and conversely, that a standard metric can be defined from an S-metric, researchers have continued to work extensively with S-metric spaces (see [13], [14]). This is because many fixed point results admit clearer and more natural formulations within the S-metric framework. Moreover, S-metric spaces unify and generalize several other structures, such as G-metrics, and admissibility conditions, as well as contraction mappings may behave differently in S-metric settings, yielding genuinely new fixed point results. These developments have further stimulated generalizations, leading to structures such as S_b-metric spaces [15], [16] partial S metric space [17], extended S metric space [18], and controlled S-metric spaces [19], [20], and recently, controlled orthogonal S-metric spaces [21], [22]. Collectively, these advances underscore the dynamic evolution of fixed point theory, which continues to broaden in scope and enrich its applications.

Building on this tradition, recent research has focused on both new spaces and novel contraction conditions. For example, Azmi [20] introduced triple controlled S-metric type spaces and established corresponding fixed point results, thereby extending the framework of controlled metric-type spaces. This illustrates how the theory has gradually shifted from classical metric settings to highly generalized structures to address broader classes of mappings.

A parallel stream of research has centered on the development of generalized contractions. A milestone in this direction was the introduction of the $\mathcal{F}$-contraction by Wardowski (2012) [23], which generated significant interest. Subsequent studies have examined $\mathcal{F}$-contraction mappings in various contexts—for instance, multi-valued $\mathcal{F}$ contraction [24], [25], modified $\mathcal{F}$-contraction [26]. Karapinar et al. provided a nice survey on $\mathcal{F}$-contractions [27]. For broader treatments and further generalizations of $\mathcal{F}$-contractions, we refer to [28], [29], [30], and [31].

Additionally, Samet et al. [32] introduced the concept of α-admissible mappings in metric spaces, which was later developed by Gopal et al. [33] into (α-$\mathcal{F}$)-contractive mappings in 2016. These mappings have proven crucial in extending fixed point theorems to various complete metric spaces (see also [34], [35], [36]). Subsequently, Priyobarta et al. generalized the idea to α_s-admissible mappings in S-metric spaces [37]. More recently, Azmi [20] advanced this concept by defining such mappings in the context of triple controlled S-metric spaces and deriving new fixed point results.

Motivated by these developments, this paper introduces the concept of $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})$-contraction mappings tailored for triple-controlled S-metric spaces. These mappings rely on α_s- and η_s-admissible mappings, ν_s-subadmissible mappings, a set of upper-class functions (Q, h), and Wardowski's $\mathcal{F}$-contraction. Building on the $(\alpha_s - \mathcal{F})$-contraction framework proposed by Azmi [20], we demonstrate the existence and uniqueness of fixed points in complete triple-controlled S-metric spaces and provide an example to illustrate our findings. Furthermore, we derive meaningful corollaries by specifying various (Q, h) pairs, illustrating the versatility and depth of the proposed theory and its contribution to the advancement of fixed point results in generalized metric environments.

2. Preliminaries

In this section, we will first review some key results and definitions of S-metric spaces introduced by Sedghi et al. [11].

Definition 1. *[11] Let $X \neq \emptyset$, and let $\mathcal{S} : X^3 \to [0, +\infty)$ be a mapping satistfying the following conditions, for all $x, y, z \in X$ and $a \in X$:*

(i) $\mathcal{S}(x, y, z) = 0$ *iff* $x = y = z$,

(ii) $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$.

Then, $(X, \mathcal{S})$ is called an S- metric space.

Definition 2. *[15] Let $X \neq \emptyset$ and $b \geq 1$. Let $\mathcal{S} : X^3 \to [0, \infty)$ be a mapping satisfying the following conditions, for all $x, y, z \in X$:*

(i) $\mathcal{S}(x, y, z) = 0$ *iff* $x = y = z$;

(ii) $\mathcal{S}(x, y, y) = \mathcal{S}(y, x, x)$

(iii) $\mathcal{S}(x, y, z) \leq b[\mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)]$.

The pair $(X, \mathcal{S})$ is called an S_b- metric space.

Definition 3. *[16] Let $\mathcal{S} : X^3 \to [0, \infty)$ be a mapping, where X is a non-void set, and consider a function $\theta : X^3 \to [1, +\infty)$, satisfying the following conditions, for all $x, y, z \in X$:*

(i) $\mathcal{S}(x, y, z) = 0$ *iff* $x = y = z$;

(ii) $\mathcal{S}(x, y, z) \leq \theta(x, y, z)[\mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)]$.

The pair $(X, \mathcal{S})$ is known as an extended S_b- metric space.

Ekiz et al. [19] presented the concept of controlled S-metric type spaces in the following way.

Definition 4. *[19] Let $\mathcal{S} : X^3 \to [0, +\infty)$ be a mapping, where X be a non-void set and suppose $\alpha : X^2 \to [1, +\infty)$ is a function, such that for all $x, y, z, a \in X$, the following conditions hold:*

(i) $\mathcal{S}(x, y, z) = 0$ *iff* $x = y = z$*;*

(ii) $\mathcal{S}(x, y, z) \leq \alpha(x, a)\mathcal{S}(x, x, a) + \alpha(y, a)\mathcal{S}(y, y, a) + \alpha(z, a)\mathcal{S}(z, z, a)$.

Then, the pair $(X, \mathcal{S})$ is referred to as a controlled S-metric type space.

We will now present the idea of triple controlled S-metric type spaces as introduced by Azmi [20].

Definition 5. *[20] Consider a mapping $\mathcal{S} : X^3 \to [0, +\infty)$ with X being a non-void set, and suppose $\beta, \mu, \gamma : X^2 \to [1, +\infty)$ are functions such that for all $x, y, z, a \in X$, the following conditions are fulfilled:*
$(T1)$ $\mathcal{S}(x, y, z) = 0$ *iff* $x = y = z$*;*
$(T2)$ $\mathcal{S}(x, x, z) = \mathcal{S}(z, z, x)$;, *for all* $x, z \in X$*;*
$(T3)$ $\mathcal{S}(x, y, z) \leq \beta(x, a)\mathcal{S}(x, x, a) + \mu(y, a)\mathcal{S}(y, y, a) + \gamma(z, a)\mathcal{S}(z, z, a)$.
Then, the pair $(X, \mathcal{S})$ is referred to as a triple controlled S-metric type space, abbreviated as $\mathcal{TC}$-S-$\mathcal{MTS}$.

Remark 1. *[20] By setting $\beta = \mu = \gamma$ in Definition 5, we get a controlled S-metric type space, as described in Definition 4. Therefore, the definition of $\mathcal{TC}$-S-$\mathcal{MTS}$ is a generalization of the controlled S-metric type space. Additionally, if we set $\beta = \mu = \gamma = 1$, the definition of $\mathcal{TC}$-S-$\mathcal{MTS}$ turns into an S-metric space, as outlined in Definition 1.*

The following example demonstrates that a triple controlled S-metric type space is different from a controlled S-metric type space [20].

Example 1. *[20] Let $X = \{0, 1, 2\}$, and define $\mathcal{S} : X^3 \to [0, +\infty)$ by*

$$\mathcal{S}(x, y, z) = \begin{cases} 0 & \text{if } \ x = y = z, \\ 1 & \text{if } \ x \neq y \neq z, \\ \frac{3}{2} & \text{if } \ x = y, y \neq z. \end{cases}$$

Let $\beta, \mu, \gamma : X^2 \to [1, +\infty)$ be defined as follows:
$\beta(x, y) = 1 + x + y,$
$\mu(x, y) = 1 + xy,$ *and*
$\gamma(x, y) = 2 + x + y.$

It is evident that $(X, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, since $\beta \neq \mu \neq \gamma$, this indicates that $(X, \mathcal{S})$ is not a controlled S-metric type space.

We recall the concepts of Cauchy and convergent sequences, completeness, and the concept of the open ball in $\mathcal{TC}$-S-$\mathcal{MTS}$, as defined in [20].

Definition 6. *[20] Let $(X, \mathcal{S})$ be a $\mathcal{TC}$-S-$\mathcal{MTS}$ and let $\{x_n\}$ be any sequence in X.*
(1) For $x \in X$ with $\varepsilon > 0$. Then,

$$B(x, \varepsilon) = \{w \in X, \mathcal{S}(w, w, x) < \varepsilon\}, \ \textit{denotes the open ball.}$$

(2) $\{x_n\}$ converges to a point w in X, if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such $\mathcal{S}(x_n, x_n, w) < \varepsilon$ for all $n \geq N$.
(3) $\{x_n\}$ is referred to as a Cauchy sequence if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\mathcal{S}(x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.
(4) The space $(X, \mathcal{S})$ is called complete if every Cauchy sequence in X is convergent.

Lemma 1. *[20] Let $(X, \mathcal{S})$ be a $\mathcal{TC}$-S-$\mathcal{MTS}$ and let $\beta, \mu, \gamma : X^2 \to [1, +\infty)$ be mappings. If the sequence $\{x_n\}$ in X is convergent, then the limit is unique.*

Samet et al. [32] initially presented the class of α_s-admissible mappings. For more details, refer to [38] and [32].

Definition 7. *Let $T : X \longrightarrow X$ be a mapping with X a non-void set, and let $\alpha_s : X \times X \to [0, +\infty)$ be a function. T is called an α_s-admissible, if whenever $\alpha_s(\hat{x}, \hat{y}) \geq 1$ implies $\alpha_s(T\hat{x}, T\hat{y}) \geq 1$, for all $\hat{x}, \hat{y} \in X$.*

On the other hand, Priyobarta et al. [37] extended the class of α_s-admissible mappings in the framework of S-metric space as demonstrated below:

Definition 8. *[37] Consider the mapping $T : X \longrightarrow X$ and let $\alpha_s : X^3 \to [0, +\infty)$ be a function, with X a non-void set. T is referred to as α_s-admissible mapping, if for all $\hat{x}, \hat{y}, \hat{z} \in X$, we have*

$$\alpha_s(\hat{x}, \hat{y}, \hat{z}) \geq 1 \Longrightarrow \alpha_s(T\hat{x}, T\hat{y}, T\hat{z}) \geq 1. \tag{1}$$

Example 2. *[37] Assume $X = [0, +\infty)$, the mappings $T : X \longrightarrow X$, and $\alpha_s : X^3 \to [0, +\infty)$ be defined by $T(u) = 4u$, for all $u \in X$, and $\alpha_s(x, y, z) = e^{\frac{z}{xy}}$ if $x \geq y \geq z$, $x \neq 0, y \neq 0$, and $\alpha_s(x, y, z) = 0$, if $x < y < z$. Then, T is an α_s-admissible mapping.*

Definition 9. *[34] Consider the mapping $T : X \longrightarrow X$ and let $\nu_s : X^3 \to [0, +\infty)$ be a function, with X a non-void set. T is referred to as ν_s-subadmissible mapping, if for all $\hat{x}, \hat{y}, \hat{z} \in X$, we have*

$$\nu_s(\hat{x}, \hat{y}, \hat{z}) \leq 1 \Longrightarrow \nu_s(T\hat{x}, T\hat{y}, T\hat{z}) \leq 1. \tag{2}$$

We will now provide definitions for the functions associated with a subclass of types I and II, as well as for the pairs belonging to the upper class of types I and II. For more information, see [30, 39].

Definition 10. *([30, 39]) We say that the function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is of subclass of type I, if $x \geq 1 \Longrightarrow h(1, y) \leq h(x, y)$, for all $x, y \in \mathbb{R}^+$.*

Next, we present examples of functions belonging to the subclass of type I as discussed in [30, 39].

Example 3. *[30, 39]*

(1) let $h(x, y) = (y + l)^x$, with $l > 1$.
(2) let $h(x, y) = (x + l)^y$, with $l > 0$.
(3) let $h(x, y) = x^n y^k$, such that $k > 0, n \in \mathbb{N} \cup \{0\}$.
(4) let $h(x, y) = \frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} y$.
(5) let $h(x, y) = (\frac{x^n + x^{n-1} + \ldots + x + 1}{n+1} + l)^y$, and $l > 1$.
(6) let $h(x, y) = \frac{mx+n}{m+n} y$, such that $m, n \in N$.

Remark 2. *From Definition 10, we observe that $y \leq h(1, y)$.*

Definition 11. *[30, 39] A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is said to be of subclass of type II, if $x, y \geq 1 \Longrightarrow h(1, 1, z) \leq h(x, y, z)$, for all $x, y, z \in \mathbb{R}^+$.*

Example 4. *[30, 39]*

The following functions are examples of a subclass of Type II.

(1) $h(x, y, z) = (z + l)^{xy}$, and $l > 1$.
(2) $h(x, y, z) = (xy + l)^z$, with $l > 0$.
(3) $h(x, y, z) = x^m y^n z^p$, such that $m, n \in \mathbb{N} \cup \{0\}, p > 0$.
(4) $h(x, y, z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1}) z$.
(5) $h(x, y, z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1} + l)^z$, and $l > 1$.
(6) $h(x, y, z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, with the assumption $m, n, p, q, k \in \mathbb{N}$.

Remark 3. *By Definition 11, we observe that $z \leq h(1, 1, z)$.*

Definition 12. *([30, 39]) Let $h, Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be functions. We say the pair (Q, h) is an upper class of type I, if h is a subclass of type I, and satisfies the following conditions:*

(i) $0 \le s \le 1 \Longrightarrow Q(s, t) \le Q(1, t)$,

(ii) $h(1, y) \le Q(s, t) \Longrightarrow y \le st$, *for all* $x, y, s, t \in \mathbb{R}^+$.

When we set $s = 1$ in (2), the pair (Q, h) is referred to as a special upper class of type I.

Example 5. *[30, 39]*

Examples of functions belonging to the upper class of Type I are given below.

(1)Let $h(x, y) = (y + l)^x$, *with* $l > 1$ *and,* $Q(s, t) = st + l$.
(2) Let $h(x, y) = (x + l)^y, l > 0$, *and* $Q(s, t) = (1 + l)^{st}$.
(3) Let $h(x, y) = x^n y^k$, *and* $Q(s, t) = s^m t^k, k > 0$.
(4) Let $h(x, y) = \frac{mx+n}{m+n} y$, *with* $m, n \in N$, *and* $Q(s, t) = st$.
(5) Let $h(x, y) = (\frac{x^n+x^{n-1}+...+x+1}{n+1} + l)^y, l > 1$, *and* $Q(s, t) = (1 + l)^{st}$.
(6) Let $h(x, y) = (y + l)^x, l > 1$ *and* $Q(s, t) = st + \frac{l}{k}, k \ge 1$.

Definition 13. *([30, 39]) Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be functions. We say the pair (Q, h) is an upper class of type II, if h is a subclass of type II, and satisfies the following conditions:*

(i) $0 \le s \le 1 \Longrightarrow Q(s, t) \le Q(1, t)$.

(ii) $h(1, 1, z) \le Q(s, t) \Longrightarrow z \le st$, *for all* $z, s, t \in \mathbb{R}^+$.

When we set $s = 1$ in (2), the pair (Q, h) is referred to as a special upper class of type II.

Example 6. *[30, 39]*

Examples of functions belonging to the upper class of Type II are given below.

(1) $h(x, y, z) = (z + l)^{xy}, l > 1$, *and* $Q(s, t) = st + l$.
(2) $h(x, y, z) = (xy + l)^z$, *with* $l > 0$, *and* $Q(s, t) = (1 + l)^{st}$.
(3) $h(x, y, z) = x^m y^n z^p, Q(s, t) = s^p t^p$, *such that* $m, n \in \mathbb{N} \cup \{0\}$ *and* $p > 0$.
(4) $h(x, y, z) = \frac{x^m+x^n y^p+y^q}{3} z^k$, *with* $m, n, p, q, k \in \mathbb{N}$ *and* $Q(s, t) = (st)^k$.
(5) $h(x, y, z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1} + l)^z, l > 1$, *and* $Q(s, t) = (1 + l)^{st}$.
(6) $h(x, y, z) = (\frac{\sum_{i=0}^{n} x^{n-i} y^i}{n+1}) z$, *and* $Q(s, t) = st$.
(7) $h(x, y, z) = (z + l)^{xy}, l > 1$, *and* $Q(s, t) = st + \frac{l}{k}, k \ge 1$.

Wardowski [23] introduced the notion of $\mathcal{F}-$contraction and developed new fixed point theorems applicable to complete metric spaces. The definition is provided below.

Definition 14. *[23] Let $\mathcal{F}$ represents the family of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the following conditions:*
$(W1)$ F is an strictly increasing function.
$(W2)$ Let $\{t_n\}$ be any sequence of positive real numbers, then

$$\lim_{n\to+\infty} t_n = 0 \textit{ iff } \lim_{n\to+\infty} F(t_n) = -\infty.$$

$(W3)$ $\lim_{t\to 0^+} t^k F(t) = 0$, for some $k \in (0, 1)$.
Then, $\mathcal{F}$ is referred to as an $\mathcal{F}$-contraction mapping.

Numerous researchers have adapted the $\mathcal{F}$-contraction mapping initially proposed by Wardowski [23] to fit various metric space contexts, as discussed in [26] and [29]. Additionally, Azmi [20] introduced a modified $\mathcal{F}$-contraction mapping specifically designed for S-metric type spaces.

Definition 15. *[20] Assume $(X, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. The mapping $T : X \to X$ is referred to as a modified $\mathcal{F}$-contraction mapping, if there exists a function $F \in \mathcal{F}$ and a constant $\tau > 0$, such that the following condition holds:*

$$\begin{aligned} \mathcal{S}(Tx, Ty, Tz) &> 0 \implies \tau + F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)), \\ \textit{for all} x, y, z &\in X. \end{aligned} \tag{3}$$

3. Main Results

In this section, we demonstrate the existence and uniqueness of fixed points within a complete $\mathcal{TC}$-S-$\mathcal{MTS}$ space by employing two different contraction mappings.

Definition 16. *[20] Assume $(X, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. A mapping $T : X \to X$ is said to be an $(\alpha_s$-$\mathcal{F})$-contraction mapping, if there exists a function $\alpha_s : X^3 \to [0, +\infty)$, $F \in \mathcal{F}$, and a constant $\tau > 0$ such that the following inequality holds:*

$$\tau + \alpha_s(x, y, z) F(\mathcal{S}(Tx, Ty, Tz)) \leq F(\mathcal{S}(x, y, z)),$$

for $x, y, z \in X$, with $\mathcal{S}(Tx, Ty, Tz) > 0$.

We now introduce our new $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})-$contraction mapping on $\mathcal{TC}$-S-$\mathcal{MTS}$, detailed as follows.

Definition 17. *Assume that* $(X, \mathcal{S})$ *is a* $\mathcal{TC}$*-S-*$\mathcal{MTS}$*, where* $X \neq \emptyset$*, and let the pair* (Q, h) *be an upper class of type I. A mapping* $T : X \to X$ *is said to be an* $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})-$*contraction mapping, if there exist functions* $\alpha_s : X^3 \to [0, +\infty)$ *and* $\nu_s : X^3 \to [0, +\infty)$*, as in Definitions 8 and 9, with* $F \in \mathcal{F}$*, and some constant* $\tau > 0$*, such that subsequent inequality holds:*

$$h\Big(\alpha_s(x, y, z), \tau + F(\mathcal{S}(Tx, Ty, Tz)\Big) \leq Q\Big(\nu_s(x, y, z), F(\mathcal{S}(x, y, z))\Big), \tag{4}$$

for $x, y, z \in X$*, with* $\mathcal{S}(Tx, Ty, Tz) > 0$ *.*

We present our first main result.

Theorem 1. *Let* $(X, \mathcal{S})$ *be a complete* $\mathcal{TC}$*-S-*$\mathcal{MTS}$*, where* $X \neq \emptyset$*. Let the pair* (Q, h) *be an upper class of type I and let* $T : X \to X$ *be* $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})-$*contraction mapping. Assume the following conditions hold:*

(1) T *is* α_s*-admissible and* ν_s*-subadmissible mapping.*

(2) There is $x_0 \in X$*, such that* $\alpha_s(x_0, x_0, Tx_0) \geq 1$*,* $\nu_s(x_0, x_0, Tx_0) \leq 1$*.*

(3) For $x_0 \in X$*, the sequence* $\{x_n\}$*, is defined by* $x_n = T^n x_0$*, and the following inequality holds:*

$$\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1. \tag{5}$$

For every $x \in X$*, the following limits exist and are finite;*

$$\lim_{n \to +\infty} \beta(x, x_n) \ , \ \lim_{n \to +\infty} \mu(x_n, x) \text{ and } \lim_{n \to +\infty} \gamma(x_n, x) . \tag{6}$$

Then, T *has a fixed point. For the uniqueness of the fixed point, assume both* u*, and* v *are fixed points such that* $\alpha_s(u, u, v) \geq 1$*, and* $\nu_s(u, u, v) \leq 1$*, then* T *has a unique fixed point in* X*.*

Proof.

Select $x_0 \in X$ such that, $\alpha_s(x_0, x_0, Tx_0) \geq 1$, and $\nu_s(x_0, x_0, Tx_0) \leq 1$. A sequence $\{x_n\}$ is formed by $Tx_0 = x_1, T^2x_0 = Tx_1 = x_2$. Therefore, for any $n \in \mathbb{N}$, we have

$$T^n x_0 = T^{n-1} x_1 = \cdots = T x_{n-1} = x_n.$$

In addition, $T^n x_0 \neq T^{n+1} x_0$ holds for all $n \geq 0$.

Utilizing the fact that T is an $(\alpha_s, \mu_s, (Q, h) - \mathcal{F})-$contraction mapping and referring to Definition 12, we can derive:

$$\begin{aligned} h\Big(1, \tau + F(\mathcal{S}(x_n, x_n, x_{n+1}))\Big) &= h\Big(1, \tau + F(\mathcal{S}(T x_{n-1}, T x_{n-1}, T x_n))\Big). \\ &\leq h\Big(\alpha_s(x_{n-1}, x_{n-1}, x_n), \tau + F(\mathcal{S}(T x_{n-1}, T x_{n-1}, T x_n))\Big). \end{aligned}$$

$$\leq Q\Big(\nu_s(x_{n-1},\ x_{n-1},\ x_n), F(\mathcal{S}(\ x_{n-1},\ x_{n-1},\ x_n))\Big).$$
$$\leq Q\Big(1, F(\mathcal{S}(\ x_{n-1},\ x_{n-1},\ x_n))\Big).$$

As the pair (Q,h) is an upper class of type I, hence we have

$$\tau + F(\mathcal{S}(\ x_n,\ x_n,\ x_{n+1})) \leq F(\mathcal{S}(\ x_{n-1},\ x_{n-1},\ x_n)),$$

which implies

$$\begin{aligned} F(\mathcal{S}(x_n,x_n,x_{n+1})) &\leq F(\mathcal{S}(x_{n-1},x_{n-1},x_n)) - \tau. \\ &\leq F(\mathcal{S}(x_{n-2},x_{n-2},x_{n-1})) - 2\tau. \\ &\leq \cdots \leq F(\mathcal{S}(x_0,x_0,x_1)) - n\tau. \end{aligned} \tag{7}$$

Letting $n \to +\infty$ in equation 7, and with $\tau > 0$, we have

$$\lim_{n\to+\infty} F(\mathcal{S}(x_n,x_n,x_{n+1})) = -\infty. \tag{8}$$

As $F \in \mathcal{F}$, utilizing (W2) of Definition 14, we deduce that $\lim_{n\to+\infty} \mathcal{S}(x_n,x_n,x_{n+1}) = 0$, and by (W3), there exists $k \in (0,1)$ such that

$$\lim_{n\to+\infty} (\mathcal{S}(x_n,x_n,x_{n+1}))^k F((\mathcal{S}(x_n,x_n,x_{n+1})) = 0. \tag{9}$$

From equation 7, we obtain

$$F(\mathcal{S}(\ x_n,\ x_n,\ x_{n+1})) - F(\mathcal{S}(x_0,\ x_0,\ x_1)) \leq -n\tau.$$

Thus for any n, we have

$$\begin{aligned} (\mathcal{S}(\ x_n,\ x_n,\ x_{n+1}))^k F(\mathcal{S}(x_n,\ x_n,\ x_{n+1}) - (\mathcal{S}(x_n,\ x_n,\ x_{n+1})^k F(\mathcal{S}(x_0,x_0,x_1)) \\ \leq -n\tau(\mathcal{S}(x_n,x_n,x_{n+1}))^k \leq 0. \end{aligned} \tag{10}$$

As n approaches infinity in equation 10, we find that

$$\lim_{n\to+\infty} n(\mathcal{S}(x_n,x_n,x_{n+1}))^k = 0. \tag{11}$$

This gives, $\lim_{n\to+\infty} n^{1/k}(\mathcal{S}(x_n,x_n,x_{n+1})) = 0$, hence there exists some $n_0 \in \mathbb{N}$, such that

$$\mathcal{S}(x_n,x_n,x_{n+1}) \leq \frac{1}{n^{1/k}},\ \text{for all}\, n \geq n_0. \tag{12}$$

To show that the sequence $\{x_n\}$ is a Cauchy sequence, we consider any natural numbers m and n such that $n < m$. We find that:

$$\mathcal{S}(x_n,\ x_n,\ x_m) \leq \beta(x_n,\ x_{n+1})\mathcal{S}(\ x_n, x_n,\ x_{n+1}) + \mu(x_n,x_{n+1})\mathcal{S}(x_n,\ x_n,x_{n+1})$$

$$
\begin{aligned}
&+\gamma(x_{n+1},x_m)\mathcal{S}(x_{n+1},x_{n+1},x_m).\\
&\le \beta(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})+\mu(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})\\
&+\gamma(x_{n+1},x_m)[\beta(x_{n+1},x_{n+2})\mathcal{S}(x_{n+1},x_{n+1},x_{n+2})+\mu(x_{n+1},x_{n+2})\mathcal{S}(x_{n+1},x_{n+1},x_{n+2})\\
&+\gamma(x_{n+2},x_m)\mathcal{S}(x_{n+2},x_{n+2},x_m)].\\
&\le \beta(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})+\mu(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})\\
&+\gamma(x_{n+1},x_m)\beta(x_{n+1},x_{n+2})\mathcal{S}(x_{n+1},x_{n+1},x_{n+2})\\
&+\gamma(x_{n+1},x_m)\mu(x_{n+1},x_{n+2})\mathcal{S}(x_{n+1},x_{n+1},x_{n+2})\\
&+\gamma(x_{n+1},x_m)\gamma(x_{n+2},x_m)[\beta(x_{n+2},x_{n+3})\mathcal{S}(x_{n+2},x_{n+2},x_{n+3})\\
&+\mu(x_{n+2},x_{n+3})\mathcal{S}(x_{n+2},x_{n+2},x_{n+3})+\gamma(x_{n+3},x_m)\mathcal{S}(x_{n+3},x_{n+3},x_m)].\\
&\vdots
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\mathcal{S}(x_n,x_n,x_m)\le{}& \beta(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})+\mu(x_n,x_{n+1})\mathcal{S}(x_n,x_n,x_{n+1})\\
&+\sum_{i=n+1}^{m-2}[\beta(x_i,x_{i+1})+\mu(x_i,x_{i+1})]\mathcal{S}(x_i,x_i,x_{i+1})\left(\prod_{j=n+1}^{i}\gamma(x_j,x_m)\right)\\
&+\prod_{i=n+1}^{m-1}\gamma(x_i,x_m)\mathcal{S}(x_{m-1},x_{m-1},x_m).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{S}(x_n,x_n,x_m)\le{}& [\beta(x_n,x_{n+1})+\mu(x_n,x_{n+1})]\mathcal{S}(x_n,x_n,x_{n+1})\\
&+\sum_{i=n+1}^{m-1}[\beta(x_i,x_{i+1})+\mu(x_i,x_{i+1})]\mathcal{S}(x_i,x_i,x_{i+1})\left(\prod_{j=n+1}^{i}\gamma(x_j,x_m)\right)
\end{aligned}
\tag{13}
$$

Applying equation 12 into inequality 13, it becomes

$$
\begin{aligned}
\mathcal{S}(x_n,x_n,x_m)\le{}& [\beta(x_n,x_{n+1})+\mu(x_n,x_{n+1})](\frac{1}{n^{1/k}})\\
&+\sum_{i=n+1}^{m-1}[\beta(x_i,x_{i+1})+\mu(x_i,x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=n+1}^{i}\gamma(x_j,x_m)\right).\\
&\le[\beta(x_n,x_{n+1})+\mu(x_n,x_{n+1})](\frac{1}{n^{1/k}})\\
&+\sum_{i=1}^{m-1}[\beta(x_i,x_{i+1})+\mu(x_i,x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\gamma(x_j,x_m)\right).
\end{aligned}
$$

(14)

Let $L_p = \sum_{i=1}^{p-1}[\beta(x_i, x_{i+1}) + \mu(x_i, x_{i+1})](\frac{1}{i^{1/k}})\Big(\prod_{j=1}^{p}\gamma(x_j, x_m)\Big)$.

Therefore, inequality 14 can be expressed as:

$$\mathcal{S}(x_n, x_n, x_m) \leq [\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})](\frac{1}{n^{1/k}}) + (L_{m-1} - L_n). \tag{15}$$

The application of the ratio test to inequality 15, followed by taking the limit as both n and m approach infinity while utilizing equations 5, leads us to the conclusion that $\lim_{n,m\to+\infty}[L_{m-1}-L_n] = 0$. Furthermore, the use of 6 indicates that $\lim_{n\to+\infty}[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})](\frac{1}{n^{1/k}}) = 0$.

Thus, we have demonstrated that:

$$\lim_{n,m\to+\infty}\mathcal{S}(x_n, x_n, x_m) = 0.$$

We deduce that the sequence $\{x_n\}$ is a Cauchy sequence. Given the completeness of the space $(X, \mathcal{S})$, it follows that this sequence must converge to a limit $u \in X$, i.e.

$$\lim_{n\to+\infty}\mathcal{S}(x_n, x_n, u) = 0. \tag{16}$$

Assume that $\mathcal{S}(T x_n, T x_n, Tu) > 0$for all n. By Definition 15, we have

$$\begin{aligned} h\Big(1, \tau + F(\mathcal{S}(Tx_n, Tx_n, Tu))\Big) &\leq h\Big(\alpha_s(x_n, x_n, u), \tau + F(\mathcal{S}(Tx_n, Tx_n, Tu))\Big) \\ &\leq Q\Big(\nu_s(x_n, x_n, u), F(\mathcal{S}(x_n, x_n, u))\Big) \\ &\leq Q(1, F(\mathcal{S}(x_n, x_n, u))). \end{aligned}$$

As the pair (Q, h) is an upper class of type I, hence the following inequality holds:

$$\tau + F(\mathcal{S}(Tx_n, Tx_n, Tu)) \leq F(\mathcal{S}(x_n, x_n, u)). \tag{17}$$

Taking the limit as n approaches infinity in equation 17, and applying equation 16 along with (W2) from Definition 14, we find that $\lim_{n\to+\infty} F(\mathcal{S}(Tx_n, Tx_n, Tu)) = -\infty$. According to Definition 14, this means that $\lim_{n\to+\infty}\mathcal{S}(Tx_n, Tx_n, Tu) = 0$.

Taking the limit as n tends to infinity in equation 17, and using equation 16, and (W2) from Definition 14, we obtain $\lim_{n\to+\infty} F(\mathcal{S}(Tx_n, Tx_n, Tu)) = -\infty$. Again, by Definition 14 this implies $\lim_{n\to+\infty}\mathcal{S}(Tx_n, Tx_n, Tu) = 0$.

To show that u is a fixed point, note that;

$$\begin{aligned}\mathcal{S}(Tu, Tu, u) = \mathcal{S}(u, u, Tu) &\leq \beta(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) + \mu(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) \\ &+ \gamma(Tu, x_{n+1})\mathcal{S}(Tu, Tu, Tx_n). \\ &\leq \beta(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) + \mu(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) \\ &+ \gamma(Tu, x_{n+1})\mathcal{S}(Tx_n, Tx_n, Tu).\end{aligned}$$

As n tends to $+\infty$ in the preceding inequality, we conclude that $\mathcal{S}(Tu, Tu, u) = 0$, implying $Tu = u$. Now, we proceed to establish the uniqueness of the fixed point. Suppose there exist two fixed points, u and v, with $u \neq v$ such that $\alpha_s(u, u, v) \geq 1$, and $\mu_s(u, u, v) \leq 1$. Since $Tu = u \neq v = Tv$, it implies that $\mathcal{S}(Tu, Tu, Tv) > 0$. Given that T is an $(\alpha_s, \mu_s, (Q, h) - \mathcal{F})$-contraction mapping, utilizing equation 4, we obtain

$$\begin{aligned}h\Big(1, \tau + F(\mathcal{S}(Tu, Tu, Tv))\Big) &\leq h\Big(\alpha_s(u, u, v), \tau + F(\mathcal{S}(Tu, Tu, Tv))\Big) \\ &\leq Q\Big(\nu_s(u, u, v), F(\mathcal{S}(u, u, v))\Big) \\ &\leq Q\Big(1, F(\mathcal{S}(u, u, v))\Big).\end{aligned}$$

Since the pair(Q, h) is an upper class of type I, we get

$$\begin{aligned}\tau + F(\mathcal{S}(Tu, Tu, Tv)) &\leq \tau + \alpha_s(u, u, v)F(\mathcal{S}(Tu, Tu, Tv)) \\ &\leq F(\mathcal{S}(u, u, v)) = F(\mathcal{S}(Tu, Tu, Tv)).\end{aligned}$$

This implies $\tau \leq 0$, leading to a contradiction. Therefore, $u = v$, implying the uniqueness of the fixed point.

We will now provide an example that supports Theorem 1, based on the work of Azmi [9].

Example 7. *Let* $X = [0, +\infty)$*, and consider the mapping* $\mathcal{S} : X^3 \to [0, +\infty)$*, defined by* $\mathcal{S}(x, y, z) = |x - y| + |y - z|$*. Then* $(X, \mathcal{S})$ *is a complete* $\mathcal{TC}$*-S-*$\mathcal{MTS}$*, where* $\beta, \mu, \gamma : X^2 \to [1, +\infty)$ *are defined by*

$\beta(x, y) = \max\{x, y\} + 1, \quad \mu(x, y) = \max\{x, y\} + 2,$ *and*

$$\gamma(x, y) = \begin{cases} x + y & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases}$$

The mapping $T : X \to X$*, is defined by*

$$T(x) = \begin{cases} \frac{x}{3} & \text{if } x \in [0, 1], \\ 2x - \frac{5}{3} & \text{if } x > 1. \end{cases}$$

Let $\alpha_s, \nu_s : X^3 \to (-\infty, +\infty)$, *and* $F : (0, +\infty) \to (-\infty, +\infty)$ *be defined by,*

$$\alpha_s(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_s(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [0, 1], \\ 2 & \text{otherwise.} \end{cases}$$

and $F(t) = ln(t)$.

To show T*is* $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})$*-contraction mapping; we only need to consider the case when* $x, y, z \in [0, 1]$. *Note that*

$$\mathcal{S}(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| = \frac{1}{3}|x - y| + \frac{1}{3}|y - z| = \frac{1}{3}\mathcal{S}(x, y, z) < \frac{2}{3}\mathcal{S}(x, y, z).$$

Hence, the following inequality holds:

$$\alpha_s(x, y, z)(ln(\frac{3}{2}) + ln(\mathcal{S}(Tx, Ty, Tz))) \leq ln(\frac{3}{2}) + ln(\mathcal{S}(Tx, Ty, Tz)) \leq \nu_s(x, y, z) ln(\mathcal{S}(x, y, z)). \tag{18}$$

By taking $\tau = ln(\frac{3}{2}) > 0$, *in 18, we find that*

$$\alpha_s(x, y, z)(\tau + F(\mathcal{S}(Tx, Ty, Tz))) \leq \nu_s(x, y, z) F(\mathcal{S}(x, y, z)).$$

Therefore, we have

$$h\Big(\alpha_s(x, y, z), \tau + F(\mathcal{S}(Tx, Ty, Tz)\Big) \leq Q\Big(\nu_s(x, y, z), F(\mathcal{S}(x, y, z))\Big),$$

which implies that T *is* $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})$*-contraction mapping. Let* $x_0 = 1$, *then* $\alpha_s(x_0, x_0, Tx_0) \geq 1, \nu_s(x_0, x_0, Tx_0) \leq 1$. *We form a sequence by* $x_1 = T(x_0) = T(1) = \frac{1}{3}$, *hence* $x_n = T^n(x_0) = T^n(1) = \frac{1}{3^n}$ *for all* $n \geq 1$.

Finally, to show that the equation 5 holds, consider that

$$\begin{aligned} &\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} \\ &= \sup_{m \geq 1} \lim_{n \to +\infty} (\frac{([\max\{\frac{1}{3^{n+1}}, \frac{1}{3^{n+2}}\} + 1) + \max\{\frac{1}{3^{n+1}}, \frac{1}{3^{n+2}}\} + 2)]}{[(\max\{\frac{1}{3^n}, \frac{1}{3^{n+1}}\} + 1) + (\max\{\frac{1}{3^n}, \frac{1}{3^{n+1}}\} + 2])})(\frac{1}{3^{n+1}} + \frac{1}{3^m}) < 1. \end{aligned}$$

Moreover, for any $x \in X$, *all the following limits* $\lim_{n \to +\infty} \beta(x, x_n)$, $\lim_{n \to +\infty} \mu(x_n, x)$, *and* $\lim_{n \to +\infty} \gamma(x_n, x)$ *exists and are finite. Therefore,* T *fulfills all the hypotheses of Theorem 1. Consequently,* T *has a fixed point, which is* $x = 0$.

Remark 4. *In Example 7, if the mapping T is not regarded as α_s-admissible and ν_s-subadmissible, it follows that T cannot be classified as a contraction mapping. It is important to observe that for any $x, y, z \in (1, +\infty)$, the following expression holds:*

$$\mathcal{S}(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| = 2(|x - y| + |y - z|) = 2\mathcal{S}(x, y, z) > \mathcal{S}(x, y, z),$$

which implies that T is not a contraction.

Remark 5. *Let $(X, \mathcal{S})$ be a $\mathcal{TC}$-S-$\mathcal{MTS}$, as stated in Theorem 1. If $\beta = \mu = \gamma$, then $(X, \mathcal{S})$ becomes a controlled S-metric type space according to definition 4. Therefore, we arrive at our next result.*

Theorem 2. *Assume that $(X, \mathcal{S})$ is a complete controlled S-metric type space, where X is a nonempty set. Let $T : X \to X$ be an $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})-$contraction mapping, such that the following conditions hold:*

(1) T is α_s-admissible and ν_s-subadmissible mapping.

(2) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, and $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(3) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, moreover, assume these hold

$$\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\beta(x_{n+1}, x_m)\beta(x_{n+1}, x_{n+2})}{\beta(x_n, x_{n+1})} < 1. \tag{19}$$

And suppose that,

$$\lim_{n \to +\infty} \beta(x, x_n) \text{ , exists and finite.} \tag{20}$$

Then, T has a fixed point. Furthermore, if there are two fixed points of T in the space X, labeled as u and v, such that $\alpha_s(u, u, v) \geq 1$ and $\nu_s(u, u, v) \leq 1$, it follows that T has a unique fixed point within X.

Proof.
The proof is completed by following a similar approach to Theorem 1, using $\beta = \mu = \gamma$.

Definition 18. *Assume that $(X, \mathcal{S})$ is a $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. Let the pair (Q, h) represent an upper class of type II. A mapping $T : X \to X$ is defined as an $(\alpha_s, \eta_s, \nu_s, (Q, h) - \mathcal{F})$-contraction mapping if there are functions $\alpha_s, \eta_s, \nu_s : X^3 \to [0, +\infty)$, a function $F \in \mathcal{F}$, and a constant $\tau > 0$ such that the following condition holds:*

$$h\Big(\alpha_s(x, y, z), \eta_s(x, y, z), \tau + F(\mathcal{S}(Tx, Ty, Tz))\Big) \leq Q\Big(\nu_s(x, y, z), F(\mathcal{S}(x, y, z))\Big),$$

for all $x, y, z \in X$, with $\mathcal{S}(Tx, Ty, Tz) > 0$.

Theorem 3. *Let $(X, \mathcal{S})$ be a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. Let $T : X \to X$ be an $(\alpha_s, \eta_s, \nu_s, (Q, h) - \mathcal{F})-$contraction mapping. Assume that the following conditions hold:*

(1) T is α_s-admissible, η_s-admissible and ν_s-subadmissible mapping.

(2) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1, \eta_s(x_0, x_0, Tx_0) \geq 1$, and $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(3) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, and the following inequalityholds:

$$\sup_{m\geq 1} \lim_{n\to+\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1. \tag{21}$$

In addition, for every $x \in X$, the following limits exist and are finite:

$$\lim_{n\to+\infty} \beta(x, x_n) \ , \ \lim_{n\to+\infty} \mu(x_n, x) \text{ and } \lim_{n\to+\infty} \gamma(x_n, x) \ . \tag{22}$$

Then, T has a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(u, u, v) \geq 1, \eta_s(u, u, v) \geq 1$, and $\nu_s(u, u, v) \leq 1$, then T has a unique fixed point in X.

Proof. Select $x_0 \in X$ so $\alpha_s(x_0, x_0, Tx_0) \geq 1, \eta_s(x_0, x_0, Tx_0) \geq 1, \nu_s(x_0, x_0, Tx_0) \leq 1$. A sequence $\{x_n\}$ is formed by $Tx_0 = x_1, T^2x_0 = Tx_1 = x_2$. Therefore, for any $n \in \mathbb{N}$, we have

$$T^n x_0 = T^{n-1} x_1 = \cdots = T x_{n-1} = x_n.$$

In addition, $T^n x_0 \neq T^{n+1} x_0$ holds for all $n \geq 0$.

As T is an α_s-admissible mapping and ν_s-subadmissible mapping, this implies that $\alpha_s(x_n, x_n, x_{n+1}) \geq 1, \nu_s(x_n, x_n, x_{n+1}) \leq 1$, for all $n \in \mathbb{N}$.

Since T is $(\alpha_s, \eta_s, \nu_s, (Q, h) - \mathcal{F})-$contraction mapping, we obtain

$$\begin{aligned} h\Big(1, 1, \tau + F(\mathcal{S}(x_n, x_n, x_{n+1}))\Big) &= h\Big(1, 1, \tau + F(\mathcal{S}(T x_{n-1}, T x_{n-1}, T x_n))\Big) \\ &\leq h\Big(\alpha_s(x_{n-1}, x_{n-1}, x_n), \eta_s(x_{n-1}, x_{n-1}, x_n), \\ &\quad \tau + F(\mathcal{S}(T x_{n-1}, T x_{n-1}, T x_n))\Big) \\ &\leq Q\Big(\nu_s(x_{n-1}, x_{n-1}, x_n), F(\mathcal{S}(x_{n-1}, x_{n-1}, x_n))\Big) \\ &\leq Q\Big(1, F(\mathcal{S}(x_{n-1}, x_{n-1}, x_n))\Big). \end{aligned}$$

As (Q, h) is an upper class pair of typeII, the following inequality holds:

$$\tau + F(\mathcal{S}(x_n, x_n, x_{n+1})) \leq F(\mathcal{S}(x_{n-1}, x_{n-1}, x_n)),$$

which implies that

$$\begin{aligned} F(\mathcal{S}(x_n, x_n, x_{n+1})) &\leq F(\mathcal{S}(x_{n-1}, x_{n-1}, x_n)) - \tau. \\ &\leq F(\mathcal{S}(x_{n-2}, x_{n-2}, x_{n-1})) - 2\tau. \\ &\leq \cdots \leq F(\mathcal{S}(x_0, x_0, x_1)) - n\tau. \end{aligned} \tag{23}$$

Letting $n \to +\infty$ in equation 23, and with $\tau > 0$, we have

$$\lim_{n \to +\infty} F(\mathcal{S}(x_n, x_n, x_{n+1})) = -\infty. \tag{24}$$

As $F \in \mathcal{F}$, utilizing (W2) of Definition 14, we deduce that $\lim_{n \to +\infty} \mathcal{S}(x_n, x_n, x_{n+1}) = 0$, and by (W3), there exists $k \in (0, 1)$ such that

$$\lim_{n \to +\infty} (\mathcal{S}(x_n, x_n, x_{n+1}))^k F((\mathcal{S}(x_n, x_n, x_{n+1})) = 0. \tag{25}$$

From equation 23, we obtain

$$F(\mathcal{S}(x_n, x_n, x_{n+1})) - F(\mathcal{S}(x_0, x_0, x_1)) \leq -n\tau.$$

Thus for any n, we have

$$\begin{aligned} (\mathcal{S}(x_n, x_n, x_{n+1}))^k F(\mathcal{S}(x_n, x_n, x_{n+1}) - (\mathcal{S}(x_n, x_n, x_{n+1})^k F(\mathcal{S}(x_0, x_0, x_1)) \\ \leq -n\tau(\mathcal{S}(x_n, x_n, x_{n+1}))^k \leq 0. \end{aligned} \tag{26}$$

As n approaches infinity in equation 26, we find that

$$\lim_{n \to +\infty} n(\mathcal{S}(x_n, x_n, x_{n+1}))^k = 0. \tag{27}$$

This leads to the conclusion that $\lim_{n \to +\infty} n^{1/k}(\mathcal{S}(x_n, x_n, x_{n+1})) = 0$. Consequently, there exists a natural number n_0 such that the following inequality holds:

$$\mathcal{S}(x_n, x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}, \text{ for all } n \geq n_0. \tag{28}$$

To show that $\{x_n\}$ is a Cauchy sequence, we will use the same approach as in the proof of Theorem 1. This leads us to the conclusion that

$$\lim_{n,m \to +\infty} \mathcal{S}(x_n, x_n, x_m) = 0.$$

Consequently, $\{x_n\}$ is a Cauchy sequence. From the completeness of $(X, \mathcal{S})$, therefore the sequence converges to a point $u \in X$. This means that

$$\lim_{n \to +\infty} \mathcal{S}(x_n, x_n, u) = 0. \tag{29}$$

In the next paragraph, we will demonstrate that u is a fixed point of the mapping T, meaning that $Tu = u$. First, we will prove that $\lim_{n\to+\infty} \mathcal{S}(Tx_n, Tx_n, Tu) = 0$.

Assume that $\mathcal{S}(T x_n, T x_n, Tu) > 0$, for all naturan number n. By Definition 15, we have

$$\begin{aligned}
&h\Big(1, 1, \tau + F(\mathcal{S}(Tx_n, Tx_n, Tu))\Big)\\
&\leq h\Big(\alpha_s(x_n, x_n, u), \eta_s(x_n, x_n, u), \tau + F(\mathcal{S}(Tx_n, Tx_n, Tu))\Big)\\
&\leq Q\Big(\nu_s(x_n, x_n, u), F(\mathcal{S}(x_n, x_n, u))\Big)\\
&\quad \leq Q\Big(1, F(\mathcal{S}(x_n, x_n, u))\Big).
\end{aligned}$$

As the pair(Q, h) is an upper class of typeII, hence we find that

$$\tau + F(\mathcal{S}(Tx_n, Tx_n, Tu)) \leq F(\mathcal{S}(x_n, x_n, u)). \tag{30}$$

Taking the limit as n approaches infinity in equation 30, and applying equation 29 along with (W2) from Definition 14, we find that $\lim_{n\to+\infty} F(\mathcal{S}(Tx_n, Tx_n, Tu)) = -\infty$. Furthermore, according to Definition 14, this means that $\lim_{n\to+\infty} \mathcal{S}(Tx_n, Tx_n, Tu) = 0$.

To prove that u is a fixed point, note the following;

$$\begin{aligned}
\mathcal{S}(Tu, Tu, u) = \mathcal{S}(u, u, Tu) &\leq \beta(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) + \mu(u, x_{n+1})\mathcal{S}(u, u, x_{n+1})\\
&+ \gamma(Tu, x_{n+1})\mathcal{S}(Tu, Tu, Tx_n).\\
&\leq \beta(u, x_{n+1})\mathcal{S}(u, u, x_{n+1}) + \mu(u, x_{n+1})\mathcal{S}(u, u, x_{n+1})\\
&+ \gamma(Tu, x_{n+1})\mathcal{S}(Tx_n, Tx_n, Tu).
\end{aligned}$$

As n approaches infinity in the previous inequality, we find that $\mathcal{S}(Tu, Tu, u) = 0$, which means $Tu = u$. Next, we will show that the fixed point is unique. Assume there are two fixed points, u and v, where $u \neq v$ and

$$\alpha_s(u, u, v) \geq 1, \eta_s(u, u, v) \geq 1, \nu_s(u, u, v) \leq 1.$$

Since $Tu = u$ and $v = Tv$, it follows that $\mathcal{S}(Tu, Tu, Tv) > 0$. Given that T is an $(\alpha_s, \eta_s, \nu_s, (Q, h) - \mathcal{F})-$contraction mapping, we obtain

$$\begin{aligned}
&h\Big(1, 1, \tau + F(\mathcal{S}(Tu, Tu, Tv))\Big)\\
&\leq h\Big(\alpha_s(u, u, v), \eta_s(u, u, v), \tau + F(\mathcal{S}(Tu, Tu, Tv))\Big)\\
&\leq Q\Big(\nu_s(u, u, v), F(\mathcal{S}(u, u, v))\Big)
\end{aligned}$$

$$\leq Q\Big(1, F(\mathcal{S}(u,\, u, v))\Big).$$

By the property of the pair(Q, h) being an upper class of typeII, allows to say that

$$\tau + F(\mathcal{S}(\,Tu,\, Tu,\, Tv)) \leq F(\mathcal{S}(u, u, v)) = F(\mathcal{S}(\,Tu,\, Tu,\, Tv)).$$

This implies $\tau \leq 0$, leading to a contradiction. Consequently, it follows that $u = v$, which indicates the uniqueness of the fixed point.

4. Corollaries

This section presents several corollaries derived from Theorem 1. As introduced in Definition 17, we have established an $(\alpha_s, \mu_s, (Q, h) - \mathcal{F})$ contraction mapping denoted as T in the framework of the space $\mathcal{TC}$-S-$\mathcal{MTS}$ $(X, \mathcal{S})$, that satisfies the condition 4. Here, (Q, h) represents a pair from the upper class of type I. We will now provide specific examples of such pairs (Q, h) belonging to this upper class.

By applying a similar approach with a pair (Q, h) from the upper class of type II, one can derive corollaries for Theorem 3.

By selecting the pair (Q, h) of upper class of type I, where $h(x, y) = x^n y^k$, $Q(s, t) = s^m t^k$, $m, n \in \mathbb{N} \cup \{0\}, k > 0$, we drive the following Corollary from Theorem 1.

Corollary 1. *Let $(\,X,\, \mathcal{S})$ be a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. Let*

$$\alpha_s(x, y, z)^n(\tau + F(\mathcal{S}(Tx, Ty, Tz)))^k \leq \nu_s(x, y, z)^m F(\mathcal{S}(x, y, z))^k,$$

where, $m, n \in \mathbb{N} \cup \{0\}, k > 0$. Assume the following conditions hold:

(i) T is α_s-admissible and μ_s-subadmissible mapping.

(ii) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(iii) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, and the following inequality holds:

$$\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\gamma(\,x_{n+1},\, x_m)[\beta(\,x_{n+1},\, x_{n+2}) + \mu(\,x_{n+1},\, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} \quad < \quad 1.$$

In addition, for every $x \in X$, the following limits exist and are finite:

$$\lim_{n \to +\infty} \beta(x, x_n)\ ,\ \lim_{n \to +\infty} \mu(x_n, x) \text{ and } \lim_{n \to +\infty} \gamma(x_n, x).$$

Then, T has a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(\,u,\, u,\, v) \geq 1$, and $\nu_s(u,\, u,\, v) \leq 1$, then T has a unique fixed point in X.

It is important to observe that by selecting $n = m = k = 1$ in Corollary 1, we obtain the subsequent corollary, as illustrated in Example 7.

Corollary 2. *Let $(X, \mathcal{S})$ be a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. Let*

$$\alpha_s(x, y, z)(\tau + F(\mathcal{S}(Tx, Ty, Tz))) \leq \nu_s(x, y, z)F(\mathcal{S}(x, y, z)).$$

Assume the following conditions hold:

(i) T is α_s-admissible and ν_s-subadmissible mapping.

(ii) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(iii) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, and the following inequality holds:

$$\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1.$$

In addition, For every x in X, the following limits exist and are finite:

$$\lim_{n \to +\infty} \beta(x, x_n) \ , \ \lim_{n \to +\infty} \mu(x_n, x) \text{ and } \lim_{n \to +\infty} \gamma(x_n, x).$$

Then, T has a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(u, u, v) \geq 1$, and $\nu_s(u, u, v) \leq 1$, then T has a unique fixed point in X.

By selecting the pair (Q, h) of upper class of type I, where, $h(x, y) = (y + l)^x, l > 1$, and $Q(s, t) = st + l$, we drive the following Corollary from Theorem 1.

Corollary 3. *Let $(X, \mathcal{S})$ be a complete $\mathcal{TC}$-S-$\mathcal{MTS}$, where $X \neq \emptyset$. Let*

$$(\tau + F(\mathcal{S}(Tx, Ty, Tz)) + l)^{\alpha_s(x,y,z)} \leq \nu_s(x, y, z)F(\mathcal{S}(x, y, z)) + l,$$

where, $l > 1$. Assume the following conditions hold:

(i) T is α_s-admissible and ν_s-subadmissible mapping.

(ii) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(iii) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, and the following inequality holds:

$$\sup_{m \geq 1} \lim_{n \to +\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1.$$

In addition, for every $x \in X$, the following limits exist and are finite:

$$\lim_{n\to+\infty} \beta(x, x_n)\ ,\ \lim_{n\to+\infty} \mu(x_n, x) \text{ and } \lim_{n\to+\infty} \gamma(x_n, x)\ .$$

Then, T has a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(u, u, v) \geq 1$, and $\nu_s(u, u, v) \leq 1$, then T has a unique fixed point in X.

By selecting the pair (Q, h) of upper class of type I, where $h(x, y) = (x + l)^y, l > 0$, and $Q(s, t) = (1 + l)^{st}$, we drive the following Corollary from Theorem 1.

Corollary 4. *Let $(X, \mathcal{S})$ be a complete $\mathcal{TC}$-$\mathcal{S}$-$\mathcal{MTS}$, where, $X \neq \emptyset$. Let*

$$(\tau + F(\mathcal{S}(Tx, Ty, Tz)) + l)^{\alpha_s(x,y,z)} \leq (1 + l)^{\nu_s(x,y,z)F(\mathcal{S}(x,y,z))},$$

where, $l > 0$. Assume the following conditions hold:

(i) T is α_s-admissible and ν_s-subadmissible mapping.

(ii) There is $x_0 \in X$, such that $\alpha_s(x_0, x_0, Tx_0) \geq 1$, $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(iii) For $x_0 \in X$, the sequence $\{x_n\}$, is defined by $x_n = T^n x_0$, and the following inequality holds:

$$\sup_{m\geq1} \lim_{n\to+\infty} \frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1.$$

In addition,For every $x \in X$, the following limits exist and are finite:

$$\lim_{n\to+\infty} \beta(x, x_n)\ ,\ \lim_{n\to+\infty} \mu(x_n, x) \text{ and } \lim_{n\to+\infty} \gamma(x_n, x).$$

Then, T has a fixed point. For the uniqueness of the fixed point, assume both u, and v are fixed points such that $\alpha_s(u, u, v) \geq 1$, and $\nu_s(u, u, v) \leq 1$, then T has a unique fixed point in X.

By selecting the pair (Q, h) of upper class of type I, where $h(x, y) = \frac{mx+n}{m+n} y$, with $m, n \in N$, and $Q(s, t) = st$, we drive the following Corollary from Theorem 1.

Corollary 5. *Let* $(X, \mathcal{S})$ *be a complete* $\mathcal{TC}$*-S-*$\mathcal{MTS}$*, where* $X \neq \emptyset$*. Let*

$$\frac{m\alpha_s(x,y,z)+n}{m+n}(\tau + F(\mathcal{S}(Tx,Ty,Tz)) \leq \nu_s(x,y,z)F(\mathcal{S}(x,y,z).$$

Where $m \in N, n \in N \cup \{0\}$*Assume the following conditions hold:*

(i) T *is* α_s*-admissible and* ν_s*-subadmissible mapping.*

(ii) There is $x_0 \in X$*, such that* $\alpha_s(x_0, x_0, Tx_0) \geq 1$, $\nu_s(x_0, x_0, Tx_0) \leq 1$.

(iii) For $x_0 \in X$*, the sequence* $\{x_n\}$*, is defined by* $x_n = T^n x_0$*, and the following inequality holds:*

$$\sup_{m\geq 1}\lim_{n\to+\infty}\frac{\gamma(x_{n+1}, x_m)[\beta(x_{n+1}, x_{n+2}) + \mu(x_{n+1}, x_{n+2})]}{[\beta(x_n, x_{n+1}) + \mu(x_n, x_{n+1})]} < 1.$$

In addition, for every $x \in X$*, the following limits exist and are finite:*

$$\lim_{n\to+\infty}\beta(x, x_n)\ ,\ \lim_{n\to+\infty}\mu(x_n, x) \text{ and } \lim_{n\to+\infty}\gamma(x_n, x).$$

Then, T *has a fixed point. For the uniqueness of the fixed point, assume both* u*, and* v *are fixed points such that* $\alpha_s(u, u, v) \geq 1$*, and* $\nu_s(u, u, v) \leq 1$*, then* T *has a unique fixed point in* X*.*

Remark 6. *It is important to observe that by selecting* $n = 0$ *in Corollary 5, we obtain Corollary 2.*

5. Application

This section demonstrates the application of the main theorem presented earlier, with a particular focus on Theorem 1, which plays a crucial role in establishing the existence of a unique real solution for an mth-degree polynomial. While various approaches exist for solving root-finding problems—especially numerical techniques—the use of fixed point theory, as outlined below, provides a clear and effective alternative. We begin with the following theorem. The application of fixed point results offers a direct and elegant method for proving Theorem 1. Since the proof closely follows the arguments of Theorem 4.1 in [20], we omit it here.

Theorem 4. *[20] For any natural number* $m \geq 3$*, the following equation*

$$x^m - (m^4 - 1)x^{m+1} - m^4 x + 1 = 0, \tag{31}$$

has exactly one solution in the interval $[-1, 1]$*.*

Theorem 5. *For any natural number $m \geq 3$, the following equation*

$$\sin^m x - (m^4 - 1)\sin^{m+1} x - m^4 \sin x + 1 = 0, \tag{32}$$

has a unique solution in the interval $[-1, 1]$.

Proof. Let $y = \sin x$. Thus, $|y| \leq 1$. Now apply Theorem 4.

Definition 19. *[40] The functions* ${}_p\tan_{i\ j}, {}_p\tanh_{i\ j} : \mathbb{R} \to \mathbb{R},\ i, j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N}, i \neq j$ *are defined as follows:*

$$\begin{aligned} {}_p\tan_{i\ j}(t) &= \frac{T_{p\ i}(t)}{T_{p\ j}(t)}, \quad {}_p\tanh_{i\ j}(t) = \frac{H_{p\ i}(t)}{H_{p\ j}(t)}. \\ &\Longrightarrow \\ {}_p\tan_{1\ 0}(t) &= \frac{T_{p\ 1}(t)}{T_{p\ 0}(t)}, \quad {}_p\tanh_{1\ 0}(t) = \frac{H_{p\ 1}(t)}{H_{p\ 0}(t)}. \end{aligned}$$

Where $T_{p\ j}, H_{p\ j} : \mathbb{R} \to \mathbb{R},\ j = 0, 1, 2, \cdots, p-1,\ p \in \mathbb{N}$, *are nested functions defined by:*

$$T_{p\ j}(t) = \sum_{n=0}^{+\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!}, \quad H_{p\ j}(t) = \sum_{n=0}^{+\infty} \frac{t^{pn+j}}{(pn+j)!}.$$

Consult [41] and [42] for more details on the nested functions.

Theorem 6. *For $m \geq 3$ any natural number m, the following equation*

$$\frac{{}_p\tanh_{i0}(x)}{x^i}^m - (m^4 - 1)\frac{{}_p\tanh_{i0}(x)}{x^i}^{m+1} - m^4 \frac{{}_p\tanh_{i0}(x)}{x^i} + 1 = 0, \tag{33}$$

has a unique solution in the interval $[-1, 1]$.

Proof. Let $y = \frac{{}_p\tanh_{i0}(x)}{x^i}$; thus, $|y| \leq 1$. Now, apply Theorem 4.

6. Conclusion

This paper presents two new types of contraction mappings: $(\alpha_s, \nu_s, (Q, h) - \mathcal{F})$-contraction and $(\alpha_s, \eta_s, \nu_s, (Q, h) - \mathcal{F})$-contraction, within the framework of triple-controlled S-metric spaces. These mappings are based on α_s and η_s-admissible mappings, ν_s-subadmissible mappings, upper-class functions (Q, h), and Wardowski's $\mathcal{F}$-contraction. By expanding the $(\alpha_s - \mathcal{F})$-contraction, we have shown that fixed points exist uniquely in complete triple-controlled S-metric spaces. Furthermore, we provided several corollaries of Theorem 1 using specific examples of pairs (Q, h) from the upper class of type I. The findings of this paper enhance the understanding of fixed point theory in generalized metric spaces and lay the groundwork for future studies in this field.

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Conflict of interest

The authors declare no conflict of interest.

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