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Efficient Estimation via Record Ranked Set Sampling for Copula-Linked Bivariate Exponentiated Inverted Weibull Distributions

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Abstract. This research examines the parameter estimation in the Morgenstern-type bivariate exponentiated inverted Weibull distribution utilizing an innovative sampling framework grounded in concomitant record-ranked set sampling (CRRSS). The main aim is to obtain the best linear unbiased estimator for the population mean using the CRRSS method and to assess its efficacy relative to the estimator derived using concomitant record values (CRV). Explicit formulations for the BLUE, its variance, and associated coefficients are derived using both methodologies. A detailed simulation study is performed to evaluate the influence of the correlation between the auxiliary and primary variables, along with the effect of the sample size. The findings indicate that the CRRSS-based BLUE constantly exceeds the CRV-based estimate in efficiency, especially under higher correlations and larger sample sizes. Graphical evaluations further support these findings, demonstrating enhanced stability and precision in the estimations derived from CRRSS. This study emphasizes the practical benefits of integrating auxiliary information and record-based sampling into the estimation process, providing a more efficient method for parameter inference in bivariate reliability and lifetime models.

2020 Mathematics Subject Classifications: 62D05, 62G30, 62H05, 62G20, 62F12

Key Words and Phrases: Copula-ranked methodology, concomitant record RSS, best linear unbiased estimators, computer simulation, data analysis

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1. Introduction

Ranked set sampling (RSS) was first proposed by [1] as a method to enhance the precision of the sample mean in approximating the population mean. This strategy is especially beneficial in situations when sampling units may be readily evaluated through judgment or visual evaluation. In McIntyre's original RSS scheme, n^2 units are randomly selected and divided into n sets, each containing n units. Within each set, the units are ranked using a judgment-based approach, and the ith-ranked unit is selected from each set for actual measurement. This method produces a more informative sample than simple random sampling, particularly when direct measurement of the variable of interest is expensive, time-intensive, or challenging. One limitation of the classical RSS approach is that it relies on perfect judgment-based ranking, which, if inaccurate, can lead to an increase in the mean squared error of the resulting estimators. To mitigate this issue, [2] introduced an extension to RSS that incorporates an auxiliary (concomitant) variable, X which is easier and less costly to measure, to assist in the ranking process. The ranks obtained from this auxiliary variable enable a probabilistic derivation of the distribution of the associated concomitant variables. As a result, the theory of concomitants of order statistics has become a powerful tool in developing inferential procedures within the RSS framework. A comprehensive account of the applications of this method is provided by [3]. More recent applications and developments of RSS based on concomitant variables can be found in the works of [4], [5, 6], [7], [8], [9], [10], [11], [12], [13–15].

The study of concomitants of record values (CRV) is a relatively recent yet important development in the field of ordered random variables. This area was initiated by [16] who first explored the structure of concomitants associated with record-breaking observations. This was followed by a significant theoretical contribution from [17], who formally developed the distribution theory of CRV and explored their probabilistic behavior under various settings. Building on these foundations, [18] derived the joint distribution of CRVs from the Morgenstern-type bivariate logistic distribution, and later extended this work to include best linear unbiased estimator (BLUE) procedures for parameter estimation [19, 20]. Their research demonstrated that under certain dependence structures, CRV could lead to more efficient estimators compared to those based on order statistics or random samples. Parallel work by [21, 22] focused on the Fisher information contained in record values and their concomitants. Further developments include studies on generalized k-record values and their concomitants and integration of information measures such as Shannon entropy, Fisher information, Kullback-Leibler divergence and extropy into the analysis of concomitants of upper and k-record values. For example, [23], [24], [25], [26], [27], [28]. [29] introduced the concept of record ranked set sampling (RRSS), in which an inexpensive ranking mechanism such as judgment ranking is used to construct a sequence of record values based on ranking scores. In a subsequent study, [30] investigated the point and interval estimation of stress-strength reliability using upper RRSS from a one-parameter exponential distribution. [31] explored the uncertainty and information content of RRSS samples by evaluating various information-theoretic measures, including Shannon entropy, Renyi entropy, and Kullback–Leibler divergence. More recently, [32] proposed methods for constructing prediction intervals for Rayleigh observations in the context of ordered extreme k record ranked set sampling. Continuing this line of research, [33] addressed parameter estimation in the generalized exponential distribution using ordered moving extremes under a lower k-record-ranked set sampling scheme. The RRSS approach is constrained by the inadequacy of its ranking system. To resolve this issue, [34] presented the concomitant record ranked set sampling (CRRSS) approach, which utilizes record-based ranking of an auxiliary variable X to pick informative units for assessing the primary variable of interest Y. The authors generated the BLUE for the location and scale parameters of Y utilizing CRRSS data from the Morgenstern family of bivariate distributions.

The growing interest in modifying ranked set and record-based methodologies to improve estimating efficiency is shown in the aforementioned contributions. They set the stage for the current investigation, which seeks to estimate parameters in the bivariate exponentiated inverted Weibull distribution (BEIWD) utilizing CRRSS. The exponentiated inverted Weibull distribution (EIWD) is very useful for modeling lifetime data with different survival rate patterns in one variable. A bivariate extension is commonly needed in real life when there are paired dependent variables. In this context, the Morgenstern-type BEIWD (MTBEIWD) satisfies this criterion by integrating a versatile EIWD marginal with a straightforward dependence structure. This makes the model easy to work with analytically and valuable in real life for investigations of dependability, survival, and the environment, especially when the variables are only weakly or moderately dependent on each other. Because the EIWD marginals are very flexible and can handle extreme value behavior and different survival rates, MTBEIWD is the best way to represent record values when you have paired record data. Additionally, MTBEIWD provides clear joint pdf and cdf, which makes it easier to find the distributions of record values and their related values. This is very important for making inferences under CRRSS. Section 3 shows the survival curves for both MTBEIWD and the upper record value concomitants, with varied parameter values and dependence structures. This makes it clear that this model may be used with record data sequences. Record values and CRRSS schemes focus on extreme observations, therefore survival plots provide a straightforward way to see how the tail behaves and how it depends on other data, which helps to see if the model accurately describes the distributional features of record data. The goal of this study is to find the BLUEs for the mean under the CRRSS scheme for the BEIWD model and to compare their performance to that of the BLUEs found under CRV. The efficiency improvements of CRRSS are analyzed by analytical derivation and simulation, considering different sample sizes and correlation values. CRRSS combines the natural occurrence of record values with the organized efficiency of ranked set sampling. This makes it a great tool for looking at bivariate lifespan data. If CRRSS gives far more accurate estimates than CRV, it makes a stronger case for employing MTBEIWD with a record-based data collection method.

2. Morgenstern-Type Bivariate Exponentiated Inverted Weibull Distribution

The Morgenstern family of distributions is a versatile framework for creating bivariate distribution families with predetermined marginals. A generalization of the Morgenstern approach was introduced by [35], known as the Farlie-Gumbel-Morgenstern (FGM) family of distributions. Reference [36] presented a BEIWD utilizing the FGM distribution system and examined the concomitants of order statistics within this framework. A generalized Morgenstern distribution system, known as the Bairamov-Kotz-Becki-Farlie-Gumbel-Morgenstern (BKB-FGM) family, exists, and recent research on generalized order statistics and their concomitants has been conducted by [37] and [38]. This study examines MTBEIWD with two shape parameters, β and θ, while assuming the scale parameter $\lambda = 1$. We posit that (X, Y) implies a bivariate population adhering to the MTBEIWD, with X signifying the auxiliary variable and Y indicating the variable of primary interest. A bivariate random variable (X, Y) is classified as following the MTBEIWD if its cumulative distribution function (cdf) is defined as:

$$F_{X,Y}(x,y) = e^{-\theta_1 x^{-\beta_1}} e^{-\theta_2 y^{-\beta_2}} \left[1 + \rho \left\{1 - e^{-\theta_1 x^{-\beta_1}}\right\} \left\{1 - e^{-\theta_2 y^{-\beta_2}}\right\}\right], \quad (1)$$

where $(X, Y) > 0$, $(\beta_1, \theta_1, \beta_2, \theta_2) > 0$, and $-1 \leq \rho \leq 1$. The corresponding probability density function (pdf) can be formulated as:

$$f_{X,Y}(x,y) = \beta_1\theta_1\beta_2\theta_2 x^{-(\beta_1+1)} e^{-\theta_1 x^{-\beta_1}} y^{-(\beta_2+1)} e^{-\theta_2 y^{-\beta_2}} \left[1 + \rho \left\{1 - 2e^{-\theta_1 x^{-\beta_1}}\right\} \left\{1 - 2e^{-\theta_2 y^{-\beta_2}}\right\}\right]. \quad (2)$$

The survival function of the MTBEIWD can be formulated as

$$S_{X,Y}(x,y) = \left\{1 - e^{-\theta_1 x^{-\beta_1}}\right\} \left\{1 - e^{-\theta_2 y^{-\beta_2}}\right\} \left[1 + \rho e^{-\theta_1 x^{-\beta_1}} e^{-\theta_2 y^{-\beta_2}}\right]. \quad (3)$$

The assumptions underlying the BEIWD are:

- The random variables X and Y follow the EIWD where X and Y are continuous and non-negative, i.e., $0 < (X, Y) < \infty$.
- The parameters involved in the distribution are positive, $(\beta_1, \theta_1, \beta_2, \theta_2) > 0$.
- The dependence parameter $\rho \in [-1, 1]$. For Morgenstern type distributions, the dependence parameter is weak.
- The existence of mean and variance depends on the values of β. For $\beta > 1$, the moments exist.

The graphs of the pdf, cdf, and survival curve of the MTBEIWD are presented in Figures 1 and 2, respectivey.

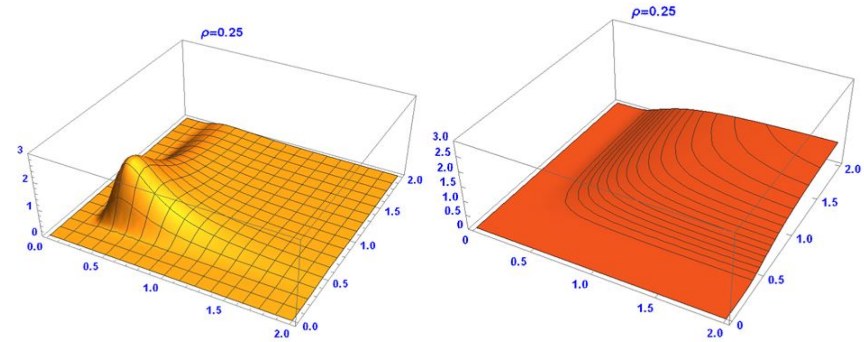

Figure 1: The pdf and cdf of the MTBEIWD for $\beta_1 = 2, \beta_2 = 2, \theta_1 = 1, \theta_2 = 2, \rho = 0.25$.

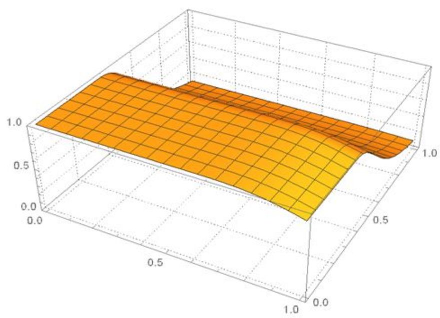

Figure 2: The survival function for $S_{(X,Y)}(x, y)$ with $\theta_1 = 2, \theta_2 = 2, \beta_1 = 3, \beta_2 = 4, \rho = 0.5, n = 10$.

Figure 2 illustrates that the survival curve of the MTBEIWD exhibits a monotonically decreasing trend with hefty tails contingent upon the form parameters. The precise form varies upward or downward according on the dependence parameter, rendering it appropriate for depicting joint survival with mild to moderate correlation. The marginal distributions X and Y are univariate EIWD with pdf defined as follows:

$$f_X(x) = \begin{cases} \theta_1 \beta_1 x^{-(\beta_1+1)} \left(e^{-x^{-\beta_1}}\right)^{\theta_1}, & x, \beta_1, \theta_1 > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

and

$$f_Y(y) = \begin{cases} \theta_2 \beta_2 y^{-(\beta_2+1)} \left(e^{-y^{-\beta_2}}\right)^{\theta_2}, & y, \beta_2, \theta_2 > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

The expected value and variation of the EIWD Y are as follows:

$$E(Y) = \mu_2 = \theta_2^{1/\beta_2} \Gamma\left(1 - \frac{1}{\beta_2}\right), \tag{6}$$

$$\mathrm{Var}(Y) = \theta_2^{2/\beta_2} \left[\Gamma\left(1 - \frac{2}{\beta_2}\right) - \left\{ \Gamma\left(1 - \frac{1}{\beta_2}\right) \right\}^2 \right]. \tag{7}$$

3. Concomitants of Record Values and Best Linear Unbiased Estimator

According to lower recorded values, [39] identified BLUEs and best linear invariant estimators (BLIEs) for the position and scale parameters of EIWD. [36] examined the distributional characteristics of CRV resulting from the MTBEIWD. It provides theoretical derivations and survival analyses corresponding to these recorded concomitants. In this study, we will consider the BLUE of the population mean μ_2 from (6) of MTBEIWD based on the upper record values that go along with it. Let (X_i, Y_i) be a sequence of independent and identically distributed random variables derived from (2). Define the series of upper record values associated with the marginal sequence X_i of observations as $\{R_n; n \geq 1\}$. The concomitant of the nth upper record value associated with R_n is denoted as $R_{[n]}$, and the pdf of the concomitant of the nth upper record value for the variable Y is:

$$f_{Y_{[n]}}(y) = \beta_2 \theta_2 y^{-(\beta_2+1)} e^{-\theta_2 y^{-\beta_2}} \left[1 + \rho \left\{ 1 - 2e^{-\theta_2 y^{-\beta_2}} \right\} (2^{1-n} - 1) \right]. \tag{8}$$

The survival function of the concomitant of the nth upper record value is given as:

$$S_{Y_{[n]}}(y) = 1 - e^{-\theta_2 y^{-\beta_2}} \left[1 + \rho \left\{ 1 - e^{-\theta_2 y^{-\beta_2}} \right\} (2^{1-n} - 1) \right]. \tag{9}$$

Figure 3 illustrates the survival plots of the concomitant of the n^{th} upper record value derived from (9).

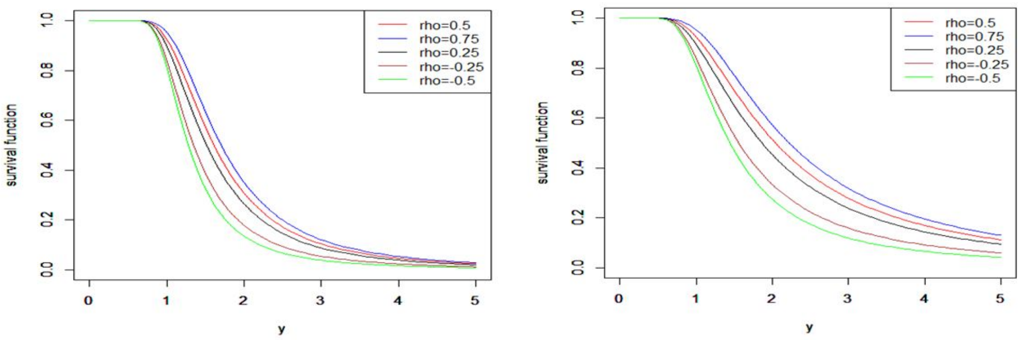

Figure 3: The survival function for $S_{Y_{[n]}}(y)$ with (a) $\theta_2 = 2, \beta_2 = 3, n = 10$ and (b)$\theta_2 = 2, \beta_2, n = 10$.

Figure 3 illustrates that the survival curve of the concomitants of the upper record under MTBEIWD has a monotonically decreasing trend, is more heavy-tailed, and moves downward with an increase in record numbers. The dependency parameter regulates the rate of curve decay, with $\rho > 0$ indicating a slower decay and $\rho < 0$ indicating a quicker decline. The data demonstrate that the model effectively represents the tail behavior and dependence structure; hence, the MTBEIWD is suitable for record-based inference. Let $R_{[m]}$ and $R_{[n]}$ denote the concomitants of the mth and nth record values in the sequence of observations X. The joint density of $R_{[m]}$ and $R_{[n]}$, where $m < n$, is expressed as follows:

$$\begin{aligned} f_{[m,n]}(y_1, y_2) =& \beta_2^2\theta_2^2 e^{-\theta_2[y_1^{-\beta_2}+y_2^{-\beta_2}]}(y_1y_2)^{-(\beta_2+1)} \\ & \left[1+\rho\left\{1-2e^{-\theta_2 y_1^{-\beta_2}}\right\}\left(\frac{1}{2^m}-1\right)+\rho\left\{1-2e^{-\theta_2 y_2^{-\beta_2}}\right\}\left(\frac{1}{2^n}-1\right)\right. \\ & \left.+\rho^2\left\{1-2e^{-\theta_2 y_1^{-\beta_2}}\right\}\left\{1-2e^{-\theta_2 y_2^{-\beta_2}}\right\}\left(1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\right)\right]. \end{aligned}$$

The kth moment of concomitance of $R_{[n]}$ $(n \geq 1)$ can be derived as:

$$E(R_{[n]}^k) = \theta_2^{k/\beta_2}\Gamma\left(1-\frac{k}{\beta_2}\right)\left[1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-k/\beta_2}}\right)\right]. \tag{10}$$

The product moment of $R_{[m]}$ and $R_{[n]}$ for $m < n$ can be listed as:

$$\begin{aligned} E[R_{[m]}R_{[n]}] = & \left[\theta_2^{1/\beta_2}\Gamma\left(1-\frac{1}{\beta_2}\right)\right]^2\left[1+\rho\left(\frac{1}{2^m}-1\right)\left(1-2^{1/\beta_2}\right)\right. \\ & +\rho\left(\frac{1}{2^n}-1\right)\left(1-2^{1/\beta_2}\right) \\ & \left.+\rho^2\left(1-2^{1/\beta_2}\right)\left(1-2^{1/\beta_2}\right)\left(1-\frac{1}{2^m}-\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}\right)\right]. \end{aligned} \tag{11}$$

Utilizing (6) and (10), we can derive the mean and variance of $R_{[n]}$ as outlined below:

$$E(R_{[n]}) = \mu_2\left[1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-1/\beta_2}}\right)\right], \tag{12}$$

$$V(R_{[n]}) = \mu_2^2\left[\frac{\Gamma\left(1-\frac{2}{\beta_2}\right)}{\left[\Gamma\left(1-\frac{1}{\beta_2}\right)\right]^2}\left\{1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-2/\beta_2}}\right)\right\}-\left\{1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-1/\beta_2}}\right)\right\}^2\right]. \tag{13}$$

Utilizing (6) and (11), the covariance between $R_{[m]}$ and $R_{[n]}$ for $m < n$ is expressed as:

$$\text{Cov}(R_{[m]}, R_{[n]}) = \mu_2^2\rho(2^{1/\beta_2}-1)\left[\frac{1}{2^m}+\frac{1}{2^n}+\rho(2^{1/\beta_2}-1)\left(\frac{1}{2^m}+\frac{1}{2^n}+\frac{1}{2^{n-m-2}3^{m+1}}-\frac{4}{2^m2^n}\right)\right]. \tag{14}$$

Assume:

$$\zeta_n = 1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-1/\beta_2}}\right), \quad n \geq 1,$$

$$\eta_{n,n} = \frac{\Gamma\left(1-\frac{2}{\beta_2}\right)}{\left[\Gamma\left(1-\frac{1}{\beta_2}\right)\right]^2}\left\{1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-2/\beta_2}}\right)\right\}-\left\{1+\rho(2^{1-n}-1)\left(1-\frac{2}{2^{1-1/\beta_2}}\right)\right\}^2.$$

For $m < n$:

$$\eta_{m,n} = \rho(2^{1/\beta_2} - 1)\left[\frac{1}{2^m} + \frac{1}{2^n} + \rho(2^{1/\beta_2} - 1)\left(\frac{1}{2^m} + \frac{1}{2^n} + \frac{1}{2^{n-m-2}3^{m+1}} - \frac{4}{2^m 2^n}\right)\right].$$

Subsequently, utilizing these identities, we can derive from (12), (13) and (14) as:

$$E(R_{[n]}) = \mu_2 \zeta_n, \tag{15}$$

$$V(R_{[n]}) = \mu_2^2 \eta_{n,n}. \tag{16}$$

For $m < n$:

$$\mathrm{cov}(R_{[m]}, R_{[n]}) = \mu_2^2 \eta_{m,n}. \tag{17}$$

Consider $R_{[n]} = (R_{[1]}, R_{[2]}, \ldots, R_{[n]})^T$ representing the vector of concomitants corresponding to the first n upper record values, derived from (2). Consequently, utilizing (15), (16), and (17), we obtain:

$$E(R_{[n]}) = \mu_2 \zeta, \tag{18}$$

$$D(R_{[n]}) = \mu_2^2 \Delta, \tag{19}$$

where $D(R_{[n]})$ is the dispersion matrix of $R_{[n]}$ and $\zeta = (\zeta_{[1]}, \zeta_{[2]}, \ldots, \zeta_{[n]})^T$ and $\Delta = (\eta_{m,n})$. If the dependence parameter ρ is supposed to be known, then (18) and (19) together defines a generalized Gauss Markov setup. Then, the BLUE of μ_2 is given by:

$$\hat{\mu}_2 = (\zeta^T \Delta^{-1} \zeta)^{-1} \zeta^T \Delta^{-1} R_{[n]}. \tag{20}$$

The variance of $\hat{\mu}_2$ is given by:

$$\mathrm{var}(\hat{\mu}_2) = (\zeta^T \Delta^{-1} \zeta)^{-1} \mu_2^2. \tag{21}$$

From (20), we see that $\hat{\mu}_2$ is a linear function of the concomitants of upper record values $R_{[r]}; r = 1, 2, \ldots n$. Then, $\hat{\mu}_2$ can be written as:

$$\hat{\mu}_2 = \sum_{r=1}^{n} a_r R_{[r]}, \tag{22}$$

where $a_r; r = 1, 2, \ldots, n$ are coefficients always sum to 1. These coefficients reflect the optimal weights assigned to each record value under the BLUE framework and adapt to both sample size and correlation strength. In order to compare the efficiency of the estimator $\hat{\mu}_2$, we introduce an unbiased estimator $\tilde{\mu}_2$ of μ_2 based on concomitants of upper records:

$$\tilde{\mu}_2 = \frac{R_{[n]}}{1 + \rho(2^{1-n} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)}, \tag{23}$$

and its variance is obtained as:

$$\mathrm{Var}(\tilde{\mu}_2) = \frac{\frac{\Gamma\left(1-\frac{2}{\beta_2}\right)}{\left[\Gamma\left(1-\frac{1}{\beta_2}\right)\right]^2}\left\{1 + \rho(2^{1-n} - 1)\left(1 - \frac{2}{2^{1-2/\beta_2}}\right)\right\} - \left\{1 + \rho(2^{1-n} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)\right\}^2}{\left[1 + \rho(2^{1-n} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)\right]^2} \mu_2^2. \tag{24}$$

4. Numerical Study

In this section, we evaluate the relative efficiency of the BLUE estimator $\hat{\mu}_2$ compared to an alternative estimator $\tilde{\mu}_2$ by computing the ratio:

$$\text{Efficiency} = \frac{\text{var}(\tilde{\mu}_2)}{\text{var}(\hat{\mu}_2)},$$

for various correlation values $\rho = \pm 0.25, \pm 0.50, \pm 0.75$ and sample sizes $n = 2(1)10$. The coefficients a_r associated with the record values $R_{[r]}; r = 1, 2, \ldots, n$ used in constructing the BLUE estimator $\hat{\mu}_2$ are also obtained and presented in Table 1.

Table 1: The coefficients $a_r; r = 1, 2, \ldots, n$; BLUE estimate $\widehat{\mu}_2$; $\text{Var}(\widehat{\mu}_2)$; $\text{Var}(\widetilde{\mu}_2)$ and efficiency $e = \frac{\text{Var}(\widetilde{\mu}_2)}{\text{Var}(\widehat{\mu}_2)}$.

n	ρ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	μ_2 (BLUE)	$\text{Var}(\widetilde{\mu}_2)$	$\text{Var}(\widehat{\mu}_2)$	e
2	0.25	0.3944	0.6056						1.5572	1.7264	0.7489	2.2050
	0.30	0.4001	0.5999						1.6302	1.8720	0.8344	2.2456
	0.75	0.4037	0.5963						1.7089	2.0139	0.9160	2.1986
	-0.25	0.3740	0.6260						1.3914	1.4233	0.5678	2.5070
	-0.50	0.3563	0.6437						1.3099	1.2655	0.4725	2.6787
	-0.75	0.3302	0.6698						1.2270	1.1034	0.3749	2.9435
3	0.25	0.2603	0.3997	0.3399					1.5586	0.6933	2.5219	0.2749
	0.50	0.2654	0.3980	0.3365					1.6454	0.7554	2.7297	0.2767
	0.75	0.2687	0.3988	0.3345					1.7319	0.8155	2.9424	0.2772
	-0.25	0.2417	0.4045	0.3539					1.3832	0.5635	2.1287	0.2647
	-0.50	0.2253	0.4069	0.3678					1.2938	0.4961	1.9553	0.2537
	-0.75	0.2007	0.4071	0.3922					1.2026	0.4275	1.8191	0.2350
4	0.25	0.1915	0.2940	0.2501	0.2644				1.5618	0.6099	1.6912	0.3617
	0.50	0.1962	0.2942	0.2487	0.2639				1.6542	0.6748	1.8504	0.3650
	0.75	0.1991	0.2941	0.2479	0.2589				1.7464	0.7378	2.0082	0.3670
	-0.25	0.1742	0.2916	0.2551	0.2751				1.3753	0.4746	1.3655	0.3480
	-0.50	0.1588	0.2869	0.2593	0.2949				1.2801	0.4042	1.1957	0.3380
	-0.75	0.1352	0.2742	0.2642	0.3265				1.1826	0.3321	1.0098	0.3290
5	0.25	0.1699	0.2548	0.2167	0.2291	0.1335			1.5695	0.4826	0.3995	1.2078
	0.50	0.1703	0.2554	0.2159	0.2265	0.1318			1.6647	0.5243	0.4518	1.1007
	0.75	0.1731	0.2557	0.2156	0.2251	0.1305			1.7598	0.5649	0.5014	1.1267
	-0.25	0.1498	0.2507	0.2193	0.2399	0.1402			1.3772	0.3954	0.2864	1.3807
	-0.50	0.1353	0.2444	0.2209	0.2512	0.1483			1.2791	0.3499	0.2242	1.5610
	-0.75	0.1124	0.2300	0.2197	0.2716	0.1682			1.1778	0.3035	0.1565	1.9391
6	0.25	0.1486	0.2282	0.1940	0.2052	0.1195	0.1045		1.5797	0.4644	0.2980	1.5384
	0.50	0.1527	0.2290	0.1936	0.2031	0.1182	0.1033		1.6774	0.3066	0.3369	1.5034
	0.75	0.1554	0.2295	0.1935	0.2020	0.1172	0.1025		1.7748	0.3477	0.3738	1.4631
	-0.25	0.1335	0.2234	0.1954	0.2137	0.1249	0.1091		1.3824	0.3760	0.2127	1.7077
	-0.50	0.1198	0.2164	0.1938	0.2224	0.1313	0.1146		1.2816	0.3298	0.1657	1.9899
	-0.75	0.0979	0.1987	0.1915	0.2366	0.1466	0.1286		1.1776	0.2819	0.1148	2.4351
	0.25	0.1119	0.1719	0.1462	0.1546	0.0901	0.0787	0.2464	1.5491	0.3679	0.2947	1.2482

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Table 1: The coefficients $a_r; r = 1, 2, \ldots, n$; BLUE estimate $\widehat{\mu}_2$; $\text{Var}(\widehat{\mu}_2)$; $\text{Var}(\widetilde{\mu}_2)$ and efficiency $e = \frac{\text{Var}(\widetilde{\mu}_2)}{\text{Var}(\widehat{\mu}_2)}$. (continued)

n	ρ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	μ_2 (BLUE)	$\text{Var}(\widetilde{\mu}_2)$	$\text{Var}(\widehat{\mu}_2)$	e
	0.50	0.1155	0.1733	0.1464	0.1536	0.0893	0.0782	0.2435	1.6488	0.3088	0.3334	1.1898
	0.75	0.1177	0.1739	0.1465	0.1530	0.0887	0.0776	0.2424	1.7485	0.4248	0.3704	1.1468
	-0.25	0.0988	0.1654	0.1447	0.1582	0.0924	0.0808	0.2536	1.3486	0.3075	0.2099	1.4651
	-0.50	0.0872	0.1576	0.1425	0.1620	0.0956	0.0835	0.2714	1.2471	0.2700	0.1624	1.6994
	-0.75	0.0700	0.1421	0.1369	0.1692	0.1048	0.0919	0.2848	1.1441	0.2438	0.1096	2.2345

Table 2: The coefficients $a_r; r = 1, 2, \ldots, n$; BLUE estimate $\widehat{\mu}_2$; $\text{Var}(\widehat{\mu}_2)$; $\text{Var}(\widetilde{\mu}_2)$ and efficiency $e = \frac{\text{Var}(\widetilde{\mu}_2)}{\text{Var}(\widehat{\mu}_2)}$. (continued Table 1)

n	ρ	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\mu_2(BLUE)$	$\text{Var}(\widetilde{\mu}_2)$	$\text{Var}(\widehat{\mu}_2)$	e
8	0.25	0.0974	0.1496	0.1272	0.1345	0.0783	0.0685	0.2144	0.1298			1.5542	0.3682	0.1725	2.1338
	0.50	0.1006	0.1509	0.1275	0.1338	0.0778	0.0680	0.2121	0.1290			1.6558	0.4047	0.1955	2.0707
	0.75	0.1026	0.1515	0.1277	0.1333	0.0773	0.0676	0.2112	0.1287			1.7575	0.4401	0.2177	2.0212
	-0.25	0.0856	0.1433	0.1253	0.1371	0.0801	0.0699	0.2249	0.1335			1.3501	0.2913	0.1232	2.3643
	-0.50	0.0752	0.1359	0.1228	0.1397	0.0825	0.0720	0.2341	0.1376			1.2469	0.2508	0.0956	2.6218
	-0.75	0.0599	0.1215	0.1170	0.1447	0.0896	0.0786	0.2435	0.1451			1.1426	0.2086	0.0646	3.2269
9	0.25	0.0759	0.1166	0.0991	0.1048	0.0611	0.0534	0.1671	0.1011	0.2206		1.5430	0.4018	0.1500	2.6781
	0.50	0.0786	0.1179	0.0996	0.1045	0.0608	0.0532	0.1657	0.1008	0.2185		1.6463	0.4464	0.1707	2.6136
	0.75	0.0801	0.1183	0.0998	0.1042	0.0604	0.0528	0.1651	0.1005	0.2184		1.7497	0.4898	0.1913	2.5601
	-0.25	0.0658	0.1102	0.0965	0.1055	0.0617	0.0538	0.1731	0.1027	0.2306		1.3359	0.3083	0.1055	2.9227
	-0.50	0.0574	0.1037	0.0938	0.1066	0.0629	0.0549	0.1786	0.1049	0.2369		1.2321	0.2590	0.0805	3.2165
	-0.75	0.0458	0.0928	0.0894	0.1105	0.0685	0.0601	0.1861	0.1108	0.2357		1.1295	0.2075	0.0529	3.9245
10	0.25	0.0687	0.1055	0.0897	0.0948	0.0553	0.0483	0.1513	0.0915	0.1997	0.0948	1.5580	0.1882	0.1902	0.9891
	0.50	0.0712	0.1068	0.0903	0.0947	0.5351	0.0482	0.1502	0.0913	0.1980	0.0940	1.6630	0.2057	0.2167	0.9493
	0.75	0.0726	0.1073	0.0904	0.0944	0.0547	0.0479	0.1496	0.0911	0.1979	0.0935	1.7683	0.2227	0.2428	0.9175
	-0.25	0.0593	0.0994	0.0869	0.0957	0.0555	0.0485	0.1560	0.0925	0.2078	0.0986	1.3468	0.1517	0.1338	1.1331
	-0.50	0.0515	0.0931	0.0841	0.0957	0.0565	0.0493	0.1603	0.0942	0.2126	0.1025	1.2404	0.1327	0.1021	1.2996
	-0.75	0.0408	0.0828	0.0798	0.0987	0.0611	0.0536	0.1661	0.0989	0.2104	0.1075	1.1347	0.1134	0.0663	1.7116

Tables 1 and 2 demonstrate the comparative efficacy of the BLUE estimator $\hat{\mu}_2$ with an unbiased estimate $\tilde{\mu}_2$ of μ_2, obtained from concomitants of higher records. It has been observed that with a constant ρ, as n rises, the BLUE estimator $\hat{\mu}_2$ converges and stabilizes. Increased ρ, whether positive or negative, generally results in reduced variances, indicating enhanced ranking accuracy due to higher correlation. Efficiency escalates with n, particularly at elevated absolute ρ levels. This indicates that CRV demonstrates superior performance compared to baseline approaches, particularly when there is an abundance of data and a higher correlation among variables.

5. Concomitant Record Ranked Set Sampling

Reference [34] formalized the notion of CRRSS, utilizing record values of an auxiliary variable instead of mere order statistics in the RSS methodology. The method captures the associated concurrent observations by integrating the timing and sequencing of record occurrences, so it is designated as CRRSS. In CRRSS, an auxiliary variable X, which is cost-effective and readily measurable, is chosen to be jointly distributed with the primary variable of interest Y. Measurements are initially conducted on X, and based on the ordering of recorded values (higher or lower records) on X, the appropriate units are chosen for measurement on Y.

5.1. Upper Concomitant Record Ranked Set Sample

Assume that measurements of the auxiliary variable X are acquired in accordance with the sequence of upper record values. From the ith upper record value in this sequence, the associated unit is chosen, and a measurement is conducted on the principal variable of interest Y. The observed values are represented as $Y_{U[1]1}, Y_{U[2]2}, \ldots, Y_{U[n]n}$, indicating the concomitant of the ith upper record value, where $i = 1, 2, \ldots, n$. The collection $Y_{U[1]1}, Y_{U[2]2}, \ldots, Y_{U[n]n}$ forms the upper CRRSS. Let $f_{X,Y}(x, y)$ represent the joint probability density function of the bivariate random vector (X, Y), with marginal probability density functions $f_X(x)$, $f_Y(y)$, and conditional probability density function $f_{Y|X}(y|x)$. The probability density function of the concomitant of the ith upper record value, denoted as $Y_{U[i]i}$, is expressed as follows:

$$f_{U[i]i}(y) = \frac{1}{(i-1)!} \int f_{Y|X}(y|x) \left\{-\ln[1-F(x)]\right\}^{i-1} f_{U(i)i}(x)dx, \tag{25}$$

where $f_{U(i)i}(x)$ is the pdf of the ith upper record value of X.

5.2. Lower Concomitant Record Ranked Set Sample

Similarly, if ranking is conducted based on the lower record values of X, the resultant sample is designated as the lower CRRSS. Let $Y_{L[i]i}$ be the concomitant of the ith lower record value. Consequently, its PDF may be enumerated as:

$$f_{L[i]i}(y) = \frac{1}{(i-1)!} \int f_{Y|X}(y|x) \left\{-\ln[F(x)]\right\}^{i-1} f_{U(i)i}(x)dx.$$

5.3. Concomitant Record Ranked Set Sampling Design

Let X and Y denote a pair of continuous random variables that adhere to a Multivariate Truncated Bivariate Exponential Inverse Weibull Distribution, with X serving as the auxiliary variable and Y as the variable of interest. Our objective is to estimate the location parameter of Y utilizing an efficient sampling design predicated on higher CRRSS. Stepwise CRRSS Protocol:

(i) Form m sets, each of size k from the observations drawn randomly from the population.

(ii) In each set:

- Measure the auxiliary variable X for all k units.
- Identify the unit corresponding to the upper record value of X in that set.
- Select the concomitant Y value from this unit for measurement.

(iii) Repeat this process for r cycles, producing a total sample size $n = mr$.

This approach combines the ranking efficiency of RSS and the temporal advantage of record values, improving estimator performance over SRS and standard RSS.

5.4. Best Linear Unbiased Estimator of MTBEIWD under CRRSS

Let $Y_{U[1]1}, Y_{U[2]2}, \ldots, Y_{U[n]n}$ denote the observed values of Y associated with the upper record values $X_{U(1)1}, X_{U(2)2}, \ldots, X_{U(n)n}$ of X over n CRRSS selections. Utilizing the pdf of the upper record value $X_{U(i)}$ concerning the auxiliary variable X, along with the conditional pdf $f_{Y|X}(y|x)$ of MTBEIWD, we derive the pdf of the concomitant $Y_{U[i]i}$ under upper CRRSS as specified in (25):

$$f_{U[i]i}(y) = \theta_2\beta_2 y^{-(\beta_2+1)} e^{-\theta_2 y^{-\beta_2}} \left[1 + \rho\left\{1 - 2e^{-\theta_2 y^{-\beta_2}}\right\}(2^{1-i} - 1)\right]. \tag{26}$$

The mean and variance are defined for $i \geq 1$ as follows:

$$E(Y_{U[i]i}) = \mu_2 \left[1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)\right], \tag{27}$$

$$V(Y_{U[i]i}) = \mu_2^2 \left[\frac{\Gamma\left(1 - \frac{2}{\beta_2}\right)}{\left[\Gamma\left(1 - \frac{1}{\beta_2}\right)\right]^2}\left\{1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-2/\beta_2}}\right)\right\} - \left\{1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)\right\}^2\right]. \tag{28}$$

Let us assume,

$$\xi_i = 1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right),$$

$$\phi_i = \frac{\Gamma\left(1 - \frac{2}{\beta_2}\right)}{\left[\Gamma\left(1 - \frac{1}{\beta_2}\right)\right]^2}\left\{1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-2/\beta_2}}\right)\right\} - \left\{1 + \rho(2^{1-i} - 1)\left(1 - \frac{2}{2^{1-1/\beta_2}}\right)\right\}^2.$$

The mean and variance are then obtained from (27) and (28) as:

$$E(Y_{U[i]i}) = \mu_2 \xi_i, \tag{29}$$

$$V(Y_{U[i]i}) = \mu_2^2 \phi_i. \tag{30}$$

In contrast to spontaneous record values, observations under CRRSS arise from a controlled sampling mechanism involving ranked sets and replication. This structure introduces:

- Between cycle independence (assuming independent sets).
- Within cycle dependence (due to ranking),

Hence, when the model assumes independent sets, (when i, j are from different cycles)

$$\text{Cov}[Y_{U[i]i}, Y_{U[j]j}] = 0; \quad 1 \leq i \leq j \leq n. \tag{31}$$

When the model is within a cycle or replicate sets, the dependence is as follows:

$$\begin{aligned}\text{Cov}[Y_{U[i]i}, Y_{U[j]j}] =& \mu_2^2 \rho (2^{1/\beta_2} - 1)\left[\frac{1}{2^i} + \frac{1}{2^j}\right. \\ &\left. + \rho(2^{1/\beta_2} - 1)\left(\frac{1}{2^i} + \frac{1}{2^j} + \frac{1}{2^{i-j-2}3^{i+1}} - \frac{4}{2^i 2^j}\right)\right].\end{aligned} \tag{32}$$

Assuming (32) as:

$$\text{Cov}[Y_{U[i]i}, Y_{U[j]j}] = \mu_2^2 \rho \phi_{i,j}, \tag{33}$$

where $\phi_{i,j}$ signifies the dependence established by record ranking inside each set.

Let $R_{U[n]} = (Y_{U[1]1}, Y_{U[2]2}, \ldots, Y_{U[n]n})^T$ denote the vector of concomitants of upper records derived from CRRSS. Utilizing equations (29), (30), and (33), we derive the mean vector and dispersion matrix of $R_{U[n]}$ as follows:

$$E(R_{U[n]}) = \mu_2 \xi, \tag{34}$$

$$D(R_{U[n]}) = \mu_2^2 \Phi, \tag{35}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ an $n \times 1$ vector and $\Phi = (\phi_{i,j})_{n \times n}$ for $i, j = 1, 2, \ldots, n$ are as defined in the assumed identities. If the covariance structure is known, say ρ is given, then the Gauss-Markov theorem justifies BLUE as the minimum variance linear unbiased estimator. Hence, from (34) and (35) BLUE of μ_2 can be obtained as:

$$\hat{\mu}_2^* = (\xi^T \Phi^{-1} \xi)^{-1} \xi^T \Phi^{-1} R_{U[n]}, \tag{36}$$

and variance is given by:

$$\text{Var}(\hat{\mu}_2^*) = (\xi^T \Phi^{-1} \xi)^{-1} \mu_2^2. \tag{37}$$

The coefficients of the BLUE estimator under CRRSS are derived from the general formula of BLUE for correlated observations. These coefficients determine the weights assigned to each record value in estimating the population mean. From (36), it is seen that $\hat{\mu}_2^*$ is linear function of record observations, so, it can be written as:

$$\hat{\mu}_2^* = \sum_{i=1}^{n} b_i R_{U[i]i} = b^T R,$$

where the BLUE coefficient vector $b = (b_1, b_2, \ldots, b_n)^T$ is given by:

$$b = (\xi^T \Phi^{-1} \xi)^{-1} \Phi^{-1} \xi.$$

6. Application and Pictorial Representation

In this section, we compute the BLUE estimates $(\hat{\mu}_2)_{\text{CRV}}$ and $(\hat{\mu}_2)_{\text{CRRSS}}$ for the mean of the variable of interest under the CRV and CRRSS sampling schemes, respectively, across various combinations of set sizes m, cycle sizes r and correlation values ρ. Additionally, we calculate the corresponding variances, $\text{Var}(\hat{\mu}_2)_{\text{CRV}}$ and $\text{Var}(\hat{\mu}_2)_{\text{CRRSS}}$ for each estimator. The relative efficiency of the CRRSS-based BLUE estimator with respect to the CRV-based estimator is then obtained as: $e = \frac{\text{Var}(\hat{\mu}_2)_{\text{CRV}}}{\text{Var}(\hat{\mu}_2)_{\text{CRRSS}}}$. A comparative summary of these estimates, variances, and efficiencies is presented in Table 3. This numerical study and the graphical presentation are done using the R software applying the packages `MASS`, `stats` and `ggplot2`. Table 3 indicates that as r (number of cycles) and n (sample sizes) rise, the efficiency of CRRSS continually improves. For instance, with $\rho = -0.75$, efficiency increases from 3.75 to 8.85. For a constant sample size, efficiency escalates with ρ. For example, at $n = 10$, the efficiency at $\rho = -0.75$ is 3.75, whereas at $\rho = 0.75$, it is 3.92. At $n = 25$, efficiency decreases from 8.85 to 6.59, indicating a non-monotonic trend attributable to simulation randomness or estimator variability. BLUE estimations under CRV and CRRSS are relatively similar, suggesting that unbiasedness is maintained. For instance, for $\rho = 0.25$ and $n = 25$, the CRV yields $\hat{\mu}_2 = 1.98$, while the CRRSS results in $\hat{\mu}_2 = 1.9706$. The variance under CRRSS is consistently inferior to that under CRV. Subsequently, we ascertain the coefficients b_i $(i = 1, 2, \ldots, n)$ of the record values $R_{U[n]}$ in the Best Linear Unbiased Estimator $\hat{\mu}_2^*$ inside the CRRSS framework. The coefficients that delineate the ideal linear combination of the observed record values are displayed in Table 3.

Table 3: The BLUE estimates $(\hat{\mu}_2)_{\mathrm{CRV}}$ and $(\hat{\mu}_2)_{\mathrm{CRRSS}}$, corresponding variances $\mathrm{Var}(\hat{\mu}_2)_{\mathrm{CRV}}$ and $\mathrm{Var}(\hat{\mu}_2)_{\mathrm{CRRSS}}$ with efficiencies for various sizes of m, r and ρ.

n	ρ	$(\hat{\mu}_2)_{\mathrm{CRV}}$	$\mathrm{Var}(\hat{\mu}_2)_{\mathrm{CRV}}$	$m;r$	$(\hat{\mu}_2)_{\mathrm{CRRSS}}$	$\mathrm{Var}(\hat{\mu}_2)_{\mathrm{CRRSS}}$	e
10	-0.75	1.4531	0.8801	5;2	1.4723	0.2341	3.7502
	-0.50	1.4764	0.9812		1.5280	0.2593	3.7980
	-0.25	1.8422	0.9671		1.5831	0.2839	3.4271
	0.25	1.8760	1.1121		1.9670	0.3301	3.3549
	0.50	2.1090	1.0901		2.3080	0.3532	3.0980
	0.75	2.5561	1.2560		2.2821	0.3768	3.9230
15	-0.75	1.3122	0.9600	5;3	1.4908	0.1512	6.3712
	-0.50	1.4683	0.8967		1.5761	0.1698	5.3010
	-0.25	2.0323	1.0536		1.6071	0.1872	5.6012
	0.25	2.0210	1.0323		1.9320	0.2221	4.6760
	0.50	2.0132	1.0815		2.1132	0.2392	4.5231
	0.75	2.3956	1.1349		2.5010	0.2555	4.4405
20	-0.75	1.4831	0.8130	5;4	1.4356	0.1101	7.3760
	-0.50	1.4268	0.8523		1.5231	0.1256	6.9230
	-0.25	1.7569	0.8709		1.7098	0.1392	6.2439
	0.25	1.7830	0.9170		2.0051	0.1675	5.4980
	0.50	2.2457	1.0250		1.9807	0.1807	5.6871
	0.75	2.3385	1.0297		2.6073	0.1932	5.2870
25	-0.75	1.3761	0.7680	5;5	1.4421	0.0868	8.8547
	-0.50	1.5940	0.8242		1.5360	0.0991	8.3190
	-0.25	1.4321	0.8641		1.5811	0.1110	7.7731
	0.25	1.9820	0.9141		1.9706	0.1342	6.8365
	0.50	2.0805	1.0801		2.1046	0.1453	7.4880
	0.75	2.8156	1.0321		2.5534	0.1567	6.5991

Table 4: The coefficients b_i $(i = 1, 2, \ldots, n)$ in the BLUE μ_2^* under CRRSS scheme.

n	$m;r$	p	Coefficients											
			b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
2	2,1	-0.25	0.5	0.5										
		-0.50	0.5	0.5										
		-0.75	0.5	0.5										
		0.25	0.5	0.5										
		0.50	0.5	0.5										
		0.75	0.5	0.5										
3	3,1	-0.25	0.334	0.3336	0.3314									
		-0.50	0.3364	0.3339	0.3296									
		-0.75	0.3379	0.3343	0.3278									
		0.25	0.3317	0.3330	0.3353									
		0.50	0.3299	0.3328	0.3373									
		0.75	0.3281	0.3325	0.3394									
4	4,1	-0.25	0.2527	0.2511	0.2440	0.2473								
		-0.50	0.2553	0.2521	0.2479	0.2446								
		-0.75	0.2577	0.2532	0.2470	0.2420								
		0.25	0.2472	0.2489	0.2511	0.2528								
		0.50	0.2413	0.2479	0.2522	0.2536								
		0.75	0.2413	0.2483	0.2534	0.2558								
5	5,1	-0.25	0.2034	0.2017	0.1997	0.1982	0.1970							
		-0.50	0.2066	0.2034	0.1995	0.1964	0.1940							
		-0.75	0.2098	0.2051	0.1993	0.1947	0.1911							
		0.25	0.1965	0.1983	0.2003	0.2019	0.2031							
		0.50	0.1929	0.1966	0.2006	0.2037	0.2061							
		0.75	0.1892	0.1949	0.2010	0.2057	0.2093							
6	3,2	-0.25	0.1705	0.1689	0.1670	0.1656	0.1645	0.1636						
		-0.50	0.1742	0.1711	0.1674	0.1645	0.1623	0.1605						
		-0.75	0.1778	0.1733	0.1678	0.1635	0.1601	0.1574						
		0.25	0.0628	0.1645	0.1663	0.1678	0.1689	0.1698						

Table 4: The coefficients b_i $(i = 1, 2, \ldots, n)$ in the BLUE μ_2^* under CRRSS scheme (continued).

n	$m;r$	p	Coefficients											
			b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
		0.50	0.1587	0.1623	0.1660	0.1689	0.1711	0.1729						
		0.75	0.1546	0.1601	0.1658	0.1701	0.1734	0.1760						
8	4,2	-0.25	0.1293	0.1278	0.1262	0.1250	0.1240	0.1232	0.1225	0.1220				
		-0.50	0.1335	0.1306	0.1275	0.1250	0.1230	0.1214	0.1201	0.1190				
		-0.75	0.1376	0.1335	0.1288	0.1250	0.1220	0.1196	0.1176	0.1106				
		0.25	0.1207	0.1222	0.1238	0.1251	0.1260	0.1268	0.1274	0.1280				
		0.50	0.1163	0.1195	0.1227	0.1251	0.1271	0.1286	0.1299	0.1309				
		0.75	0.1118	0.1167	0.1215	0.1252	0.1281	0.1304	0.1323	0.1339				
9	3,3	-0.25	0.1155	0.1141	0.1126	0.1115	0.1129	0.1098	0.1092	0.1087	0.1082			
		-0.50	0.1198	0.1171	0.1142	0.1118	0.1100	0.1084	0.1072	0.1062	0.1053			
		-0.75	0.1241	0.1202	0.1158	0.1122	0.1094	0.1071	0.1052	0.1037	0.1024			
		0.25	0.1067	0.1082	0.1096	0.1108	0.1117	0.1124	0.1130	0.1135	0.1140			
		0.50	0.1022	0.1052	0.1082	0.1105	0.1123	0.1138	0.1150	0.1160	0.1168			
		0.75	0.0977	0.1024	0.1068	0.1115	0.1105	0.1151	0.1169	0.1184	0.1196			
10	5,2	-0.25	0.1044	0.1031	0.1017	0.1006	0.0998	0.0991	0.0985	0.0980	0.0970	0.0972		
		-0.50	0.1088	0.1063	0.1035	0.1013	0.0995	0.0981	0.0969	0.0960	0.0951	0.0944		
		-0.75	0.1132	0.1095	0.1053	0.1020	0.0993	0.0972	0.0954	0.0939	0.0926	0.0915		
		0.25	0.0956	0.0969	0.0983	0.0994	0.1002	0.1009	0.1015	0.1020	0.1024	0.1027		
		0.50	0.0911	0.0939	0.0967	0.0988	0.1005	0.1019	0.1030	0.1040	0.1048	0.1055		
		0.75	0.0866	0.0909	0.0950	0.0983	0.1008	0.1028	0.1045	0.1059	0.1071	0.1081		
12	4,3	-0.25	0.0878	0.0866	0.0854	0.0844	0.0836	0.0830	0.0825	0.0820	0.0816	0.0813	0.0810	0.0808
		-0.50	0.0922	0.0900	0.0875	0.0855	0.0839	0.0826	0.0816	0.0807	0.0799	0.0792	0.0787	0.0781
		-0.75	0.0967	0.0934	0.0897	0.0867	0.0843	0.0823	0.0807	0.0793	0.0781	0.0771	0.0762	0.0755
		0.25	0.0789	0.0801	0.0813	0.0823	0.0831	0.0837	0.0842	0.0846	0.0850	0.0853	0.0856	0.0859
		0.50	0.0745	0.0770	0.0794	0.0813	0.0828	0.0840	0.0850	0.0859	0.0866	0.0873	0.0878	0.0883
		0.75	0.0701	0.0739	0.0775	0.0803	0.0826	0.0844	0.0859	0.0872	0.0822	0.0892	0.0900	0.0907

Table 4 indicates that for small sample sizes (e.g., $n = 2$), the weights are uniform. As the sample size increases (e.g., $n = 4, 5, \ldots, 12$), coefficients generally stabilize around the median ranks while exhibiting some asymmetry if ranking performance is influenced by correlation. As ρ grows, indicating a greater positive association, the weights adjust marginally. Heavier weights are typically positioned more centrally. A lower ρ (e.g., -0.75) results in a more pronounced coefficient dispersion, indicating a less effective ranking system.

6.1. Application Using Simulated Data

To illustrate the performance of the proposed BLUE estimator under CRRSS relative to CRV, we conducted a simulation using data generated from a BEIWD with known marginal parameters. An algorithm is given for the simulation study:

(i) Fix the distributional set up of MTBEIWD with chosen shape parameters and correlation coefficient ρ. Fix set size m, cycle size r, total sample size $n = mr$.

(ii) Simulate n pairs (X_i, Y_i) from MTBEIWD.

(iii) Identify record values from primary variable X and extract concomitants from Y. Then obtain CRV sample.

(iv) Obtain a sample of size m^2 partitioned into m sets each of size m. Within each set, rank primary variable X based on upper records. Select concomitants from Y corresponding to record positions and obtain the CRRSS sample.

(v) Compute the BLUE of population mean μ_2 using CRV sample as well as CRRSS sample. Also, obtain corresponding variances.

(vi) Compute the relative efficiency ($\text{Var}^{\text{CRV}}/\text{Var}^{\text{CRRSS}}$).

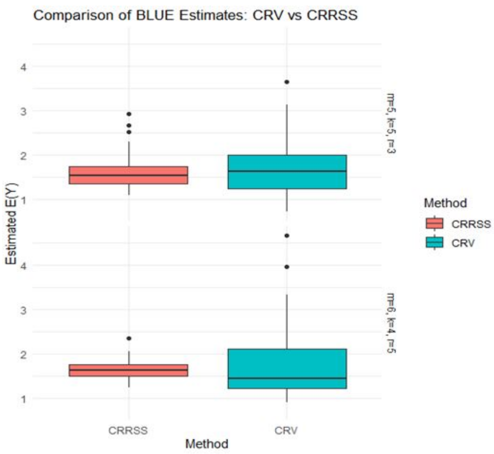
(vii) Loop steps 3-6 for multiple replications. Summarize mean estimates, variances and efficiencies.

The simulation was performed with a set size $m = 5$, cycle size $r = 3$; $r = 6$ and a fixed correlation coefficient $\rho = 0.25$ yielding a total sample size of $n = 15, 30$ for both CRRSS and CRV methods. For each scheme, the BLUE estimator $\hat{\mu}_2$ of the population mean was computed along with its corresponding variance. This illustrates that, with moderate positive correlation, CRRSS offers a significantly more

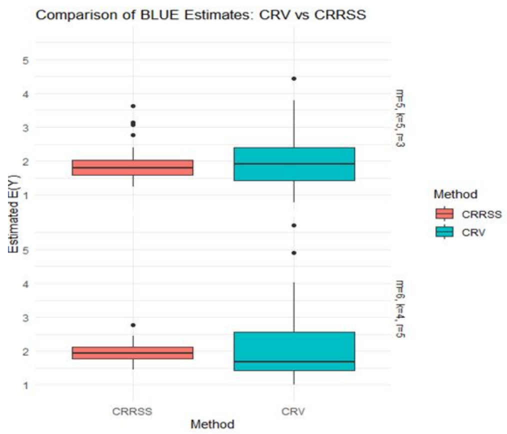
Table 5: The BLUE, variance, and its efficiency under CRV and CRRSS.

Scheme	Sample size	BLUE $\hat{\mu}_2$	Variance	Efficiency (Var^{CRV}/Var^{CRRSS})
CRV	15	2.5267	1.0651	—
CRRSS	$(m = 5; r = 3)$	2.0429	0.2217	4.8033
CRV	30	2.3966	0.8066	—
CRRSS	$(m = 5; r = 6)$	2.0562	0.1118	7.2108

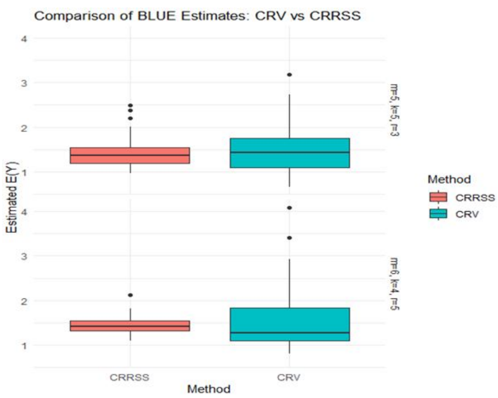
efficient estimation of the population mean than the conventional CRV method. Figure 4 presents a comparison of the BLUE estimates of the population mean derived from the CRV and those acquired from the CRRSS technique. Figure 5 illustrates the fluctuation in average efficiency of the estimators across various correlation values ρ and sample sizes n. Figure 4 indicates that the BLUE estimator maintains approximate unbiasedness over the spectrum of correlation values under both CRV and CRRSS methods, exhibiting enhanced stability at elevated absolute values of ρ. Regardless of whether ρ is positive or negative, the mean estimates stay centralized and symmetrical, reinforcing the distribution's framework and the efficacy of the ranking. Figure 5 illustrates the mean efficiency of the BLUE estimator under CRRSS in relation to CRV, displayed against the correlation coefficient ρ for different sample sizes n. The figure demonstrates that efficiency rises with both absolute correlation and sample size, confirming the benefit of CRRSS in utilizing auxiliary information for enhanced estimation accuracy. It is evident that, across all sample sizes, efficiency enhances as the absolute value of ρ grows. This pattern indicates that larger correlations boost the reliability of rankings, hence improving the efficacy of CRRSS. As n grows, the efficiency curves ascend, indicating that CRRSS derives greater benefits from larger record sets. For every ρ and n, the efficiency consistently exceeds 1, substantiating that CRRSS uniformly surpasses CRV regarding estimator variance.



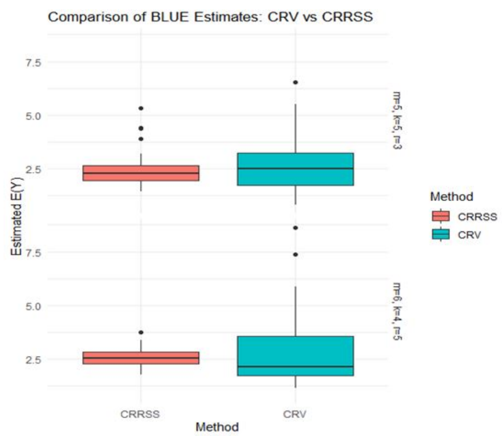
(a) $\rho = -0.25$



(b) $\rho = 0.25$

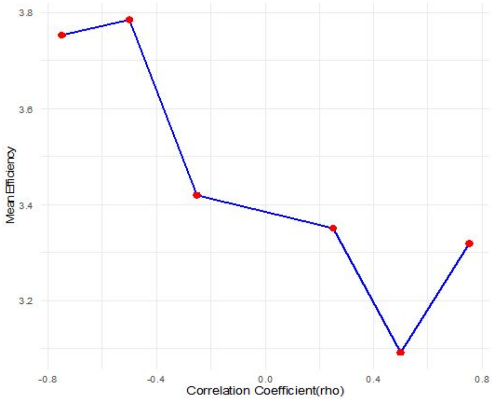


(c) $\rho = -0.75$

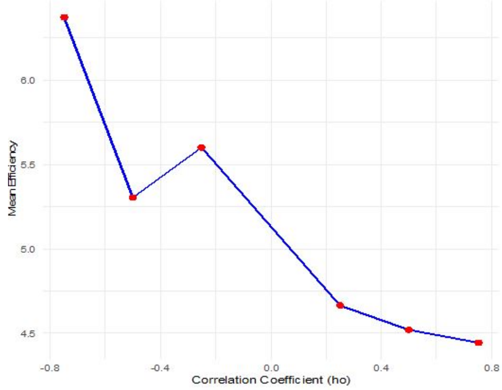


(d) $\rho = 0.75$

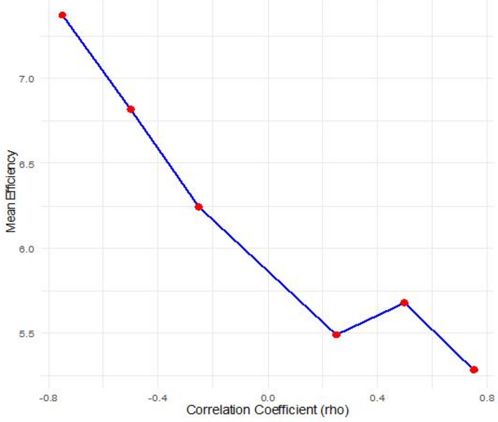
Figure 4: Comparision of estimated mean E(Y) under the method of CRV and CRRSS when (a) $\rho = -0.25$ (b) $\rho = 0.25$ (c) $\rho = 0.75$ (d) $\rho = 0.75$.



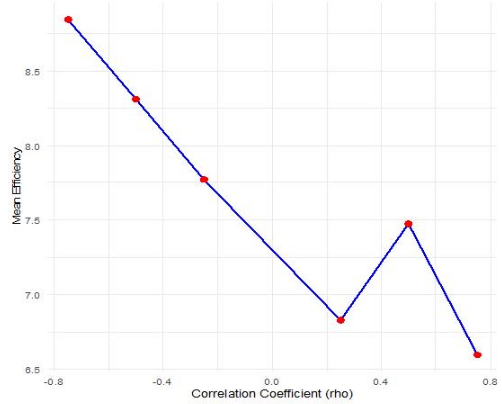
(a) n=10;m=5,r=2



(b) n=15;m=5,r=3



(c) n=20;m=5,r=4



(d) n=25;m=5,r=5

Figure 5: Mean efficiency of the estimators with respect to correlation coefficient ρ for (a) $n = 10$ (b) $n = 15$ (c) $n = 20$ and (d) $n = 25$.

7. Conclusion

This study focused on estimating the parameters of the MTBEIWD using the CRRSS methodology. The optimum BLUE for the population mean was derived from the framework of concomitants of upper record values under CRRSS. Analytical expressions for the mean, variance, and covariance structure were established, facilitating the creation of an efficient estimator. A thorough numerical analysis comparing the BLUEs under CRRSS and CRV across different sample sizes and correlation levels revealed the superior efficiency of CRRSS. Notably, as the absolute value of the correlation coefficient increased and the sample size expanded, the relative efficiency of the CRRSS-based BLUE significantly above that of the CRV-based alternative. The simulation findings validated that CRRSS preserves the unbiased characteristics of the BLUE estimator and that the resultant BLUE coefficients embody adaptive weighting procedures, hence ensuring minimal variation while retaining unbiasedness. Graphical analyses confirmed that CRRSS consistently produces more stable and efficient estimates than CRV across various correlation patterns and sample sizes. In conclusion, the CRRSS approach serves as a feasible alternative to traditional sampling strategies. Its integration into the estimating framework for MTBEIWD improves accuracy and efficiency, rendering it a desirable instrument for real applications defined by ranking or record data structures. The proposed methodology demonstrates clear benefits; nonetheless, it assumes the presence of a highly informative auxiliary variable and flawless ranking, which may not consistently occur in real-world scenarios. Furthermore, the model fitting presupposes that the marginal distribution adheres to the exponentiated inverted Weibull form, necessitating diagnostic testing on actual datasets. Subsequent investigations may refine this methodology by incorporating imperfect or probabilistic ranking, exploring Bayesian estimators inside the CRRSS framework, or generalizing the model to include additional bivariate distributions, such as the Marshall-Olkin or copula-based families. Research opportunities are available in applications including censored data, stress-strength models, and reliability systems with common components.

Use of Artificial Intelligence (AI) Tools Declaration

The authors declare that they have not used AI tools in the creation of this article.

Data Availability Statement

The datasets used and analyzed in this study are available within the paper.

Author Contributions

The author confirms sole responsibility for the conception of the study, design of methodology, data curation, formal analysis, software coding, presentation of results, and preparation of the manuscript.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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