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Vertex-Edge Dominating Sets of Some Graphs under Binary Operations

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Abstract. Given a simple undirected graph $G = (V(G), E(G))$, a vertex $u \in V(G)$ vertex-edge dominates the edge $xy \in E(G)$ if one of the following holds: (1) $u = x$ or $u = y$, (2) $ux \in E(G)$ or $uy \in E(G)$. A subset $S \subseteq V(G)$ is a vertex-edge dominating set of G if for each $xy \in E(G)$, there exists $u \in S$ such that u vertex-edge dominates xy. A vertex-edge dominating set $S \subseteq V(G)$ is a total vertex-edge dominating set if for each $u \in S$, there exists $v \in S$ for which $uv \in E(G)$. The minimum cardinality of a vertex-edge (resp. total vertex-edge) dominating set of G is the vertex-edge domination number (resp. total vertex-edge domination number) of G.
This paper investigates the vertex-edge domination and total vertex-edge domination in the join, corona, lexicographic product, complementary prism and edge corona of graphs. It provides complete characterizations of both the vertex-edge dominating sets and total vertex-edge dominating sets in these families of graphs, and establishes sharp bounds, if not the exact values, for their respective vertex-edge domination and total vertex-edge domination numbers.

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1. Introduction

Vertex-edge domination in graphs was first introduced by Peters [1] in 1986, and is a graph protection strategy which basically came from the marriage of the two concepts, namely the *domination* (about static positioning of guards which protect the vertices) and *vertex covering* (a static positioning of guards which protect the edges) of graph. The importance of vertex-edge domination is best illustrated by the so called "searchlight problem" (see [2], [3], [4]) which, inspired by the famous art gallery problem, attempts to

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use searchlights to find an intruder in a graph. In this case, the guards, each of whom holds a searchlight, must shine a searchlight down some edge where they think there might be an intruder.

Vertex-edge domination as well as its variant, the total vertex-edge domination, is very-well studied in trees (see [5], [6], [7], [8], [9]), in some special graphs (see [8], [1]), in cubic graphs and grids (see [10], [7]), in connected C_5-free graphs and connected $K_{1,k}$ - free graphs (see [11]). Peters also dealt with the complexity problems of the parameter in [1].

In the present paper, we investigate the verter-edge and total vertex-edge domination in the join, corona, lexicographic product and edge corona of graphs.

All throughout this paper, we consider only graphs which are simple, finite and undirected.

Given a graph $G = (V(G), E(G))$, we call $V(G)$ the *vertex set* of G and $E(G)$ its *edge set*. The cardinality $|V(G)|$ of $V(G)$ is the **order** of G. If $E(G) = \varnothing$, then G is an *empty graph*. All terminologies used here which are not being defined are adapted from [12].

Given a graph G, $G - v$ refers to the resulting graph after removing vertex v and all incident edges from G. If the removal of vertex v from G increases the number of components, i.e., $G - v$ has more components than G, then v is called a *cut vertex* of G. For $S \subseteq V(G)$, $\langle S\rangle$ is the **induced subgraph** of G with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$.

Let G and H be disjoint graphs. The **join** of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The **corona** of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. In particular, we call $G \circ K_1$ the corona of G, and write $cor(G) = G \circ K_1$. The **edge corona** of G and H is the graph $G \diamond H$ obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The **composition** (or **lexicographic product**) of G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs. The **complementary prism** $G\overline{G}$ is formed from G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. If for each $v \in V(G)$, $\overline{v}$ is the vertex in $\overline{G}$ corresponding to v, then $G\overline{G}$ is formed by adding the edge $v\overline{v}$ for every $v \in V(G)$.

For vertices u and v of a graph G, a u-v **geodesic** is any shortest path in G joining u and v. The length of a u-v geodesic is the **distance** between u and v, and is denoted by $d_G(u, v)$. The **eccentricity** of v refers to the quantity $e(v) = \max\{d_G(u, v) : v \in V(G)\}$. Customarily, $diam(G) = \max\{e(v) : v \in V(G)\}$. In this paper, we write $e(G) = \min\{e(v) : v \in V(G)\}$.

Vertices u and v of a graph G are **neighbors** if $uv \in E(G)$. The **open neighborhood** of v refers to the set $N_G(v)$ consisting of all neighbors of v. If $N_G(v) = \varnothing$, then v is an **isolated vertex**. The **degree** of v refers to the cardinality $|N_G(v)|$ of the open neighborhood of v, and $\delta(G)$ is the minimum degree of a vertex of G. The **closed neighborhood** of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v\in S} N_G(v)$ and

$N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a **dominating set** of G if $N_G[S] = V(G)$. A dominating set S of G is a **total dominating set** if $S \subseteq N_G(S)$. The minimum cardinality $\gamma(G)$ of a dominating set of G is the **domination number** of G. The minimum cardinality $\gamma_t(G)$ of a total dominating set is the **total domination number** of G. A dominating (resp. total dominating) set of cardinality $\gamma(G)$ (resp. $\gamma_t(G)$) is called a γ**-set** (resp. γ_t**-set** of G. The reader is referred to [13], [14], [15], [16], [17], and [18] for the history, fundamental concepts and recent developments of domination in graphs as well as its various applications.

Vertex u of G is said to **vertex-edge dominate** (or ve**-dominate**) edge $xy \in E(G)$ if one of the following holds:

- $u = x$ or $u = y$;
- $ux \in E(G)$ or $uy \in E(G)$.

A subset $S \subseteq V(G)$ is a ve**-dominating set** of G if for every $xy \in E(G)$, there exists $u \in S$ for which u ve-dominates xy. A ve-dominating set is a **total** ve**-dominating set** if, in addition, $\langle S \rangle$ has no isolated vertex. The minimum cardinality of a ve-dominating set is the ve**-domination number** of G. Similarly, the minimum cardinality of a total ve-dominating set is the **total** ve**-domination number** of G. We use the symbols $\gamma_{ve}(G)$ and $\gamma_{ve}^t(G)$ to refer to the ve-domination number and total ve-domination number of G, respectively. We also use the terms γ_{ve}**-set** (resp. γ_{ve}^t**-set**) to refer to any ve-dominating (resp. total ve-dominating) set of cardinality $\gamma_{ve}(G)$ (resp. $\gamma_{ve}^t(G)$).

For every nontrivial connected graph G,

$$\gamma_{ve}(G) \leq \gamma_{ve}^t(G) \leq 2\gamma_{ve}(G)[11].$$

2. Preliminary results

The following formalizes the alternative definition given by W. Klostermeyer et al. in [10].

Proposition 1. *$S \subseteq V(G)$ is a ve-dominating set of G if and only if $V(G) \setminus N_G(S)$ is either empty or a nonempty independent subset of $V(G)$.*

Proof. First, let S be a ve-dominating set of G. If $V(G) \setminus N_G(S) = \varnothing$, then we are done. Suppose that $V(G) \setminus N_G(S) \neq \varnothing$ and let $x, y \in V(G) \setminus N_G(S)$. Suppose that $xy \in E(G)$. Then there exists $u \in S$ for which u ve-dominates xy. If $u = x$ (resp. $u = y$), then $y \in N_G(S)$ (resp. $x \in N_G(S)$), a contradiction. However, if $ux \in E(G)$ (resp. $uy \in E(G)$), then $x \in N_G(S)$ (resp. $y \in N_G(S)$), a contradiction. Since x and y are arbitrary, $V(G) \setminus N_G(S)$ is an independent set.

Conversely, first suppose that $V(G) \setminus N_G(S) = \varnothing$. Let $xy \in E(G)$. In particular, since $V(G) = N_G(S)$, there exists $u \in S$ for which $ux \in E(G)$. Observe that u, and thus S, ve-dominates xy. Next, suppose that $V(G) \setminus N_G(S)$ is a nonempty independent set, and let $xy \in E(G)$. Suppose that S does not ve-dominate xy. Then $x \notin N_G(S)$ and $y \notin N_G(S)$. Since $V(G) \setminus N_G(S)$ is an independent set, $xy \notin E(G)$, a contradiction. ■

If $G = \overline{K_n}$, then every $S \subseteq V(G)$ is a ve-dominating set of G. Hence, $\gamma_{ve}(G) = 0$ if and only if $G = \overline{K_n}$. If $G = K_n$ or G is the complete multipartite $K_{n_1,n_2,\ldots,n_k}$, then S is a ve-dominating set of G for every nonempty $S \subseteq V(G)$.

Proposition 2. [19] *For the complete graph K_n, the complete bipartite $K_{m,n}$, complete r-partite $K_{n_1,n_2,\ldots,n_r}$, path P_n and cycle C_n, we have:*

(i) $\gamma_{ve}(K_n) = \gamma_{ve}(K_{m,n}) = \gamma_{ve}(K_{n_1,n_2,\ldots,n_r}) = 1$;

(ii) $\gamma_{ve}(P_n) = \lfloor \frac{n+2}{4} \rfloor$;

(iii) $\gamma_{ve}(C_n) = \lfloor \frac{n+3}{4} \rfloor$.

Proposition 3. [1] *For any graph of order n, $\gamma_{ve}(G) \leq \frac{n}{2}$.*

Observation 1. *Let G be a disconnected graph with nontrivial components G_1, G_2, $\ldots$, G_n. Then $S \subseteq V(G)$ is a (total) ve-dominating set of G if and only if $S \cap V(G_k)$ is a (total) ve-dominating set of G_k for each $k = 1, 2, \ldots, n$. In particular,*

$$\gamma_{ve}(G) = \sum_{k=1}^{n} \gamma_{ve}(G_k) \text{ and } \gamma_{ve}^t(G) = \sum_{k=1}^{n} \gamma_{ve}^t(G_k).$$

Proposition 4. *Let G be any graph of order n. Then*

(i) $\gamma_{ve}(G) = n - 1$ *if and only if* $G = \{K_1, P_2\}$.

(ii) $\gamma_{ve}(G) = 1$ *if and only if G has exactly one nontrivial component G', where there exists a vertex $v \in V(G')$ such that $d_G(v,x) = 1$ or $d_G(v,y) = 1$ for every $xy \in E(G)$.*

Proof. The case where G is nonempty follows directly from Proposition 3. Suppose G is an empty graph. Then $\gamma_{ve}(G) = 0$. If $\gamma_{ve}(G) = n - 1$, then $n = 1$. Conversely, if $n = 1$, then $\gamma_{ve}(G) = n - 1$. Thus, (i) holds.

To prove (ii), first, suppose that $\gamma_{ve}(G) = 1$. In view of Observation 1, G has exactly one nontrivial component G' and $\gamma_{ve}(G') = \gamma_{ve}(G) = 1$. Let $S = \{v\}$, where $v \in V(G')$ for which v ve-dominates every $xy \in E(G)$. Let $xy \in E(G)$. Then $xy \in E(G')$. If $x = v$ (resp. $y = v$), then $d_G(v,y) = d_{G'}(v,y) = 1$ (resp. $d_G(v,x) = d_{G'}(v,x) = 1$). Suppose that $x \neq v$ and $y \neq v$. Since v ve-dominates xy, $d_G(v,x) = d_{G'}(v,x) = 1$ or $d_G(v,y) = d_{G'}(v,y) = 1$. Next, for the converse, let G have exactly one nontrivial component G', where there exists a vertex $v \in V(G')$ such that $d_G(v,x) = d_{G'}(v,x) = 1$ or $d_G(v,y) = d_{G'}(v,y) = 1$ for every $xy \in E(G)$. It is worth noting that $xy \in E(G)$ if and only if $xy \in E(G')$. Let $x, y \in V(G) \setminus N_G(v)$ with $x \neq y$. Suppose that $xy \in E(G)$. Then, by the assumption, $d_G(v,x) = d_{G'}(v,x) = 1$ or $d_G(v,y) = d_{G'}(v,y) = 1$. That is, $x \in N_G(v)$ or $y \in N_G(v)$, a contradiction. Therefore, $V(G) \setminus N_G(v)$ is an independent set. By Proposition 1, $S = \{v\}$ is a ve-dominating set of G. Consequently, $\gamma_{ve}(G) = |S| = 1$. ■

Observation 2. *A graph G admits a total ve-dominating set if and only if G is not an empty graph.*

Observation 3. *For the complete graph K_n $(n \geq 2)$, the complete bipartite $K_{m,n}$, complete r-partite $K_{n_1,n_2,\ldots,n_r}$, path P_n and cycle C_n, we have:*

(i) $\gamma^t_{ve}(K_n) = \gamma^t_{ve}(K_{m,n}) = \gamma^t_{ve}(K_{n_1,n_2,\ldots,n_r}) = 2$;

(ii) *For* $n \geq 2$, $$\gamma^t_{ve}(P_n) = \begin{cases} 2, & \text{if } n = 2, \\ 3 + 2\lfloor \frac{n-3}{5} \rfloor, & \text{if } n \neq 2 \text{ but } n \equiv 2 \mod 5, \\ 2 + 2\lfloor \frac{n-3}{5} \rfloor, & \text{if } n \equiv 0, 1, 3, 4 \mod 5. \end{cases}$$

(iii) *For* $n \geq 3$, $$\gamma^t_{ve}(C_n) = \begin{cases} 2, & \text{if } n \in \{3, 4, 5\}, \\ 3, & \text{if } n = 6, \\ 5 + 2\lfloor \frac{n-7}{5} \rfloor, & \text{if } n \geq 7 \text{ and } n \equiv 1 \mod 5, \\ 4 + 2\lfloor \frac{n-7}{5} \rfloor, & \text{if } n \geq 7 \text{ and } n \equiv 0, 2, 3, 4 \mod 5. \end{cases}$$

Proposition 5. *Let G be a nonempty graph of order n. Then*

(i) $\gamma^t_{ve}(G) = n$ *if and only if G is the (disjoint) union of copies of K_2 ;*

(ii) $\gamma^t_{ve}(G) = n - 1$ *if and only if G is one of the following: P_3, K_3, the (disjoint) union of K_1 and copies of K_2, the (disjoint) union of P_3 and copies of K_2, the (disjoint) union of K_3 and copies of K_2;*

(iii) $\gamma^t_{ve}(G) = 2$ *if and only if there exists $uv \in E(G)$ such that for every $xy \in E(G) \setminus \{uv\}$, $E(G) \cap \{ux, vx, uy, vy\} \neq \varnothing$.*

Proof. Suppose that $\gamma^t_{ve}(G) = n$, and let K be a nontrivial component of G. Suppose that $K \neq K_2$. Pick a non-cut vertex $x \in V(K)$. Since $S = V(K) \setminus \{x\}$ is a total ve-set of K, $\gamma^t_{ve}(K) \leq |V(K)| - 1$. By Observation 1, $\gamma^t_{ve}(G) \leq n - 1$, a contradiction. Thus, G is the union of copies of K_2. The converse is clear. Thus, (i) holds.

Suppose that $\gamma^t_{ve}(G) = n - 1$, and let $x \in V(G)$ such that $S = V(G) \setminus \{x\}$ is a γ^t_{ve}-set of G. Let C be the component of G for which $x \in V(C)$. Put $S_C = S \cap V(C) = V(C) \setminus \{x\}$. In view of Observation 1, S_C is a γ^t_{ve}-set of C. Suppose that $|V(C)| \geq 4$, and let $y \in V(C) \cap N_C(x)$. Since S_C is a total ve-dominating set, there exists $q \in S_C \cap N_G(y)$ $(hence, |N_C(y) \setminus \{x\}| \geq 1)$. Suppose $\langle S_C \setminus \{y\} \rangle$ has no isolated vertex. Let $w \in S_C \setminus \{y\}$ and let $p \in V(C) \cap N_C(w)$. Then w ve-dominates pw. Since w was arbitrarily chosen, it follows that $S_C \setminus \{y\}$ is a ve-dominating set of C. Moreover, since $\langle S_C \setminus \{y\} \rangle$ has no isolated vertex, $S_C \setminus \{y\}$ is a total ve-dominating set of C, a contradiction. Next, suppose $\langle S_C \setminus \{y\} \rangle$ has an isolated vertex, say p. Since $\langle S_C \rangle$ has no isolated vertex, it follows that $yp \in E(G)$. Note that $\langle S_C \setminus \{p\} \rangle$ has no isolated vertex because $\langle S_C \rangle$ has no isolated vertex and $p \in End(G)$. It is easy to verify (following n earlier argument) that $\langle S_C \setminus \{p\} \rangle$ is a ve-dominating set of C. Again, this gives a contradiction. Therefore, $|V(C)| \leq 3$.

Moreover, the definition of C implies that $C \neq K_2$. If $G = C$ (connected), then either $C = P_3$ or $C = K_3$. Otherwise, (i) implies that G is the disjoint union of copies of K_2 and exactly one of the following: K_1, P_3 and K_3. The converse of (ii) is clear.

Finally, suppose that $\gamma_{ve}^t(G) = 2$, and let $S = \{u, v\}$ be a γ_{ve}^t-set of G. Let $xy \in E(G) \setminus \{uv\}$ and put $T = \{ux, vx, uy, vy\}$. If uv and xy have a common vertex, then $T \cap E(G) \neq \varnothing$. Suppose otherwise. Assume WLOG that u ve-dominates xy. Then either $ux \in E(G)$ or $uy \in E(G)$. In any case, $T \cap E(G) \neq \varnothing$. The converse is straightforward. ■

Proposition 6. *For every positive integers a and b with $2 \leq a \leq b \leq 2a$, there exists a connected graph G for which $\gamma_{ve}(G) = a$ and $\gamma_{ve}^t(G) = b$.*

Proof. Suppose that $a = b$. For each $k \in \{1, 2, \ldots, a\}$, let $[x = x_1^k, x_2^k, x_3^k, x_4^k]$ denote the k^{th} copy of P_4. Take $G = G_1$, where G_1 is the graph provided in Figure 1 obtained by connecting these a copies of P_4 by having $x_1^1 = x_1^2 = \cdots = x_1^a$ and through the path $[x_3^1, x_3^2, \ldots, x_3^a]$.The set $S = \{x_3^1, x_3^2, \ldots, x_3^a\}$ is both a γ_{ve}-set and a γ_{ve}^t-set of G. Thus,

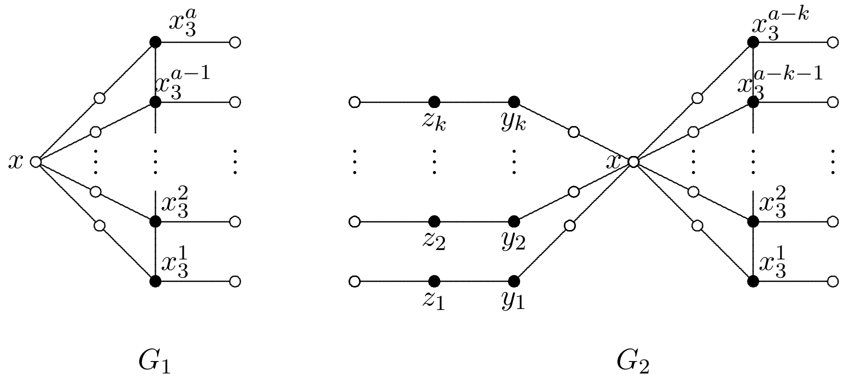

Figure 1: Graphs satisfying the conditions in Proposition 6

$\gamma_{ve}(G) = \gamma_{ve}^t(G) = a$. Now, let $b = a + k$, where $1 \leq k \leq a$. Construct a graph G_1 as above but using only $a - k$ copies of P_4. Obtain G as the graph G_2 in Figure 1 obtained by connecting to G_1 k copies of P_5 using vertex x. Then $S = \{x_3^1, x_3^2, \ldots, x_3^{a-k}\} \cup \{y_1, y_2, \ldots, y_k\}$ is a γ_{ve}-set of G. Also, $S \cup \{z_1, z_2, \ldots, z_k\}$ is a γ_{ve}^t-set of G. For this G, $\gamma_{ve}(G) = (a - k) + k = a$ and $\gamma_{ve}^t(G) = a + k = b$. ■

Corollary 1. *The difference $\gamma_{ve}^t(G) - \gamma_{ve}(G)$ can be made arbitrarily large.*

Proof. Let k be any positive integer. Choose an integer $a \geq k$ and put $b = a + k$. By Proposition 6, there exists a connected graph G for which $\gamma_{ve}^t(G) - \gamma_{ve}(G) = b - a = k$. Since k is arbitrary, the conclusion follows. ■

3. On families of graphs under binary operations

Proposition 7. *Let G and H be any graphs, and let $S \subseteq (G + H)$ with $S \neq \varnothing$. Then S is a (total) ve-dominating set of $G + H$ if and only if one of the following holds:*

(i) $S \subseteq V(G)$ *and* S *is a (total) ve-dominating set of* G*;*

(ii) $S \subseteq V(H)$ *and* S *is a (total) ve-dominating set of* H*;*

(iii) $S \cap V(G) \neq \varnothing$ *and* $S \cap V(H) \neq \varnothing$.

Proof. Assume S is a (total) ve-dominating set of $G+H$. Suppose that $S \subseteq V(G)$. If $S = V(G)$, then we are done. Suppose that $S \neq V(G)$. Since

$$V(G) \setminus N_G(S) = V(G+H) \setminus N_{G+H}(S), \tag{1}$$

$V(G) \setminus N_G(S)$ is an independent set. By Proposition 1, S is a (total) ve-dominating set of G, and (i) holds. Similarly, if $S \subseteq V(H)$, then (ii) holds. If both (i) and (ii) do not hold, then (iii) holds.

Conversely, if (i) holds, then Equation (1) implies that $V(G+H) \setminus N_{G+H}(S)$ is an independent set. Consequently, S is a (total) ve-dominating set of $G+H$. The same conclusion is attained if (ii) holds. Now, suppose that (iii) holds for S. Since $V(G+H) \setminus N_{G+H}(S) = \varnothing$, S is a (total) ve-dominating set of $G+H$ by Proposition 1. ■

Corollary 2. *Let* G *and* H *be any graphs. Then* $\gamma_{ve}^t(G+H) = 2$*, and*

$$\gamma_{ve}(G+H) = \begin{cases} 1, & if\ G\ or\ H\ is\ empty; \\ \min\{\gamma_{ve}(G), \gamma_{ve}(H), 2\}, & else. \end{cases}$$

In particular, $\gamma_{ve}(K_{m,n}) = 1$ *and* $\gamma_{ve}^t(K_{m,n}) = 2$ *for all* $m, n \geq 1$.

Proposition 8. *Let* G *be a connected graph and* H *be a nonempty graph. Then* $S \subseteq V(G \circ H)$ *is a ve-dominating set of* $G \circ H$ *if and only if*

$$S = A \cup \left(\cup_{v \in V(G)} S_v \right), \tag{2}$$

where $A \subseteq V(G)$ *and* $S_v \subseteq V(H^v)$ *for each* $v \in V(G)$ *such that* S_v *is a ve-dominating set of* H^v *for all* $v \in V(G) \setminus A$.

Proof. First, assume that S is a ve-dominating set of $G \circ H$. Put $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Then S satisfies Equation 1. Let $v \in V(G) \setminus A$, and $xy \in E(H^v)$. Since S is a ve-dominating set, there exists $u \in S$ such that u ve-dominates xy. Since $v \notin S$, $u \in S_v$. Accordingly, S_v is a ve-dominating set of H^v.

Conversely, suppose S is as given in Equation 1 such that S_v is a ve-dominating set of H^v for all $v \in V(G) \setminus A$. Let $xy \in E(G \circ H)$. We consider the following cases:

Case 1: $x \in V(G)$ or $y \in V(G)$

WLOG assume $x \in V(G)$. If $x \in A$, then S ve-dominates xy. Suppose that $x \notin A$. Since H is nonempty, H^x contains an edge ab. Because $x \notin S$, there exists $u \in S_x$ such that u ve-dominates ab. Since $ux \in E(G \circ H)$, u and therefore, S ve-dominates xy.

Case 2: $x \notin V(G)$ and $y \notin V(G)$

There exists $v \in V(G)$ for which $xy \in E(H^v)$. If $v \in A$, then v, and therefore S, ve-dominates xy. Suppose that $v \notin A$. Then S_v ve-dominates xy by the assumption. Thus, S ve-dominates xy.

Since xy is arbitrary, S is a ve-dominating set of $G \circ H$. ■

Corollary 3. *Let G be a connected graph of order $n \geq 2$ and H any graph.*

(*i*) *If H is an empty graph, then $\gamma_{ve}(G \circ H) = \gamma(G)$ and $\gamma^t_{ve}(G \circ H) = \gamma_t(G)$.*

(*ii*) *If H is a nonempty graph, then $\gamma_{ve}(G \circ H) = \gamma^t_{ve}(G \circ H) = n$.*

Proof. Let H be an empty graph. Let $S \subseteq V(G)$ be a (total) dominating set of G. We claim that S is a (total) ve-dominating set of $G \circ H$. Let $xy \in E(G \circ H)$. Assume WLOG that $x \in V(G)$. If $x \in S$, then S ve-dominates xy. Suppose that $x \notin S$. Since S is a dominating set of G, there exists $u \in S$ for which $ux \in E(G)$. This means u, hence S, ve-dominates xy. Therefore, S is a (total) ve-dominating set of $G \circ H$. Since S is arbitrary,

$$\gamma_{ve}(G \circ H) \leq \gamma(G) \text{ and } \gamma^t_{ve}(G \circ H) \leq \gamma_t(G).$$

To get the other inequalities, first let $S \subseteq V(G \circ H)$ be a γ_{ve}-set of $G \circ H$. Put $A = S \cap V(G)$ and $B = \{v \in V(G) : S \cap V(H^v) \neq \varnothing\}$. Define $S^* = A \cup B$. Then $|S^*| \leq |S| = \gamma_{ve}(G \circ H)$. We claim that S^* is a dominating set of G. Let $v \in V(G) \setminus S^*$. Pick $u \in V(H^v)$. There exists $w \in S$ such that w ve-dominates uv. Since $S \cap V(H^v) = \varnothing$, $w \in A$. Thus, $w \in S^* \cap N_G(v)$. This means that S^* is a dominating set of G. Therefore,

$$\gamma(G) \leq \gamma_{ve}(G \circ H).$$

Next, suppose that S is a γ^t_{ve}-set of $G \circ H$. Let $A = S \cap V(G)$ and $B = \{v \in A : S \cap V(H^v) \neq \varnothing\}$. For each $v \in B$, choose $u_v \in N_G(v)$. Define $S^* = A \cup \{u_v : v \in B\}$. Let $v \in V(G) \setminus S^*$ and let $u \in V(H^v)$. There exists $w \in S$ such that w ve-dominates uv. Since S is a total ve-dominating set and $v \notin S$, $u \notin S$. Hence, $w \in A$ showing that S^* is a dominating set of G. Let $w \in S^*$. If $w \notin A$, then $w = u_v$ for some $v \in B \subseteq A$. Note here that $v \in S^*$ and $wv \in E(G)$. Suppose that $w \in A$. If $S \cap V(H^w) = \varnothing$, then since S is a total ve-dominating set, there exists $z \in A \cap N_G(w)$. If $S \cap V(H^w) \neq \varnothing$, then $w \in B$ and $u_w \in S^* \cap N_G(w)$. This completely shows that S^* is a total dominating set of G. Thus,

$$\gamma_t(G) \leq |S^*| \leq |S| = \gamma^t_{ve}(G \circ H).$$

This proves (*i*).

Now, we prove (*ii*). By Proposition 8, $V(G)$ is a ve-dominating set, and therefore a total ve-dominating set, of $G \circ H$. Thus,

$$\gamma_{ve}(G \circ H) \leq n \text{ and } \gamma^t_{ve}(G \circ H) \leq n.$$

Let $S \subseteq V(G \circ H)$ be a γ_{ve}-set of $G \circ H$. By Proposition 8, $S = A \cup \left(\cup_{v \in V(G)} S_v\right)$, where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ for each $v \in V(G)$ such that S_v is a ve-dominating set of H^v for all $v \in V(G) \setminus A$. Thus,

$$\gamma_{ve}^t(G \circ H) \geq \gamma_{ve}(G \circ H) = |S| \geq |A| + \sum_{v \in V(G) \setminus A} |S_v| \geq |V(G)| = n.$$

■

Proposition 9. *Let G be a graph of order n. Then*

(i) $\gamma_{ve}(G\overline{G}) = 1$ *if and only if* $G \in \{K_n, \overline{K_n}\}$;

(ii) $\gamma_{ve}^t(G\overline{G}) = 2$ *if and only if one of the following holds:*

- (a) *There exists* $v \in V(G)$ *for which* $\{v\}$ *and* $\{\overline{v}\}$ *are* ve*-dominating sets of* G *and* $\overline{G}$*, respectively.*
- (b) G *has a* γ_t*-set* $\{u, v\}$ *such that* $xy \in E(G)$ *for all* $x, y \in V(G) \setminus \{u, v\}$.
- (c) $\overline{G}$ *has a* γ_t*-set* $\{u, v\}$ *such that* $xy \in E(\overline{G})$ *for all* $x, y \in V(\overline{G}) \setminus \{u, v\}$;

(iii) $\gamma_{ve}^t(G\overline{G}) \neq 2n - 1$; *and*

(iv) $\gamma_{ve}^t(G\overline{G}) = 2n$ *if and only if* $G = K_1$.

Proof. If $G = K_n$ and $v \in V(G)$, then $d_{G\overline{G}}(x, v) = 1$ or $d_{G\overline{G}}(y, v) = 1$ for each $xy \in E(G\overline{G})$. By Proposition 4, $\gamma_{ve}(G\overline{G}) = 1$. In case $G = \overline{K_n}$, we use the same argument on $\overline{G} = K_n$. Conversely, suppose that $\gamma_{ve}(G\overline{G}) = 1$. Assume $G \notin \{K_n, \overline{K_n}\}$. Let $v \in V(G\overline{G})$ such that $\{v\}$ is a ve-dominating set of $G\overline{G}$. WLOG assume that $v \in V(G)$. We consider the following cases:

Case 1: Suppose that $d_G(u, v) = 1$ for all $u \in V(G) \setminus \{v\}$. Since $G \neq K_n$, there exist $x, y \in V(G)$ for which $d_G(x, y) = 2$. Observe that $\overline{x}\,\overline{y} \in E(G\overline{G})$ and v does not ve-dominate $\overline{x}\,\overline{y}$, a contradiction.

Case 2: Suppose that $d_G(u, v) = 2$ for some $u \in V(G)$. In this case, it is easy to see that v does not ve-dominate $u\overline{u} \in E(G\overline{G})$, a contradiction.

The above contradictions imply that $G \in \{K_n, \overline{K_n}\}$. This proves (i).

Suppose that $\gamma_{ve}^t(G\overline{G}) = 2$, and let $S = \{u, v\}$ be a γ_{ve}^t-set of $G\overline{G}$. Suppose that $u = \overline{v}$. Assume $v \in V(G)$. Let $xy \in E(G)$. If $x = v$ or $y = v$, then v ve-dominates xy. Suppose that $x \neq v$ and $y \neq v$. Since S ve-dominates xy, v ve-dominates xy. Similarly, if $xy \in E(\overline{G})$, then $\overline{v}$ ve-dominates xy. This proves $(ii)(a)$. Suppose that $S \subseteq V(G)$. First, we claim that S is a total dominating set of G. Let $x \in V(G) \setminus S$. Since u or v ve-dominates $x\overline{x}$, $ux \in E(G)$ or $vx \in E(G)$. Thus, S is a dominating set, hence a total dominating set, of G. Next, let $x, y \in V(G) \setminus S$. Suppose that $\overline{x}\,\overline{y} \in E(\overline{G})$. Then u or v ve-dominates $\overline{x}\,\overline{y}$. This is impossible since $\min\{d_{G\overline{G}}(u, \overline{x}), d_{G\overline{G}}(u, \overline{y}), d_{G\overline{G}}(v, \overline{x}), d_{G\overline{G}}(v, \overline{y})\} \geq 2$. Thus, $\overline{x}\,\overline{y} \notin E(\overline{G})$ and $(ii)(b)$ holds. Similarly, if $S \subseteq V(\overline{G})$, then $(ii)(c)$ holds.

Assume that $(ii)(a)$ holds for G. Let $xy \in E(G\overline{G}) \setminus \{v\overline{v}\}$. If $x \in \{v, \overline{v}\}$ or $y \in \{v, \overline{v}\}$, then $E(G\overline{G}) \cap \{xv, x\overline{v}, yv, y\overline{v}\} \neq \varnothing$. Suppose that $\{x, y\} \cap \{v, \overline{v}\} = \varnothing$. If $xy \in E(G)$, then v ve-dominates xy. Thus, $E(G\overline{G}) \cap \{vx, vy\} \neq \varnothing$. Similarly, if $xy \in E(\overline{G})$, then $E(G\overline{G}) \cap \{x\overline{v}, y\overline{v}\} \neq \varnothing$. Now, suppose that $y = \overline{x}$ and assume $x \in V(G)$. Then either $vx \in E(G)$ or $\overline{v}\,\overline{x} \in E(\overline{G})$. Thus, $E(G\overline{G}) \cap \{xv, y\overline{v}\} \neq \varnothing$. By Proposition 5, $\gamma^t_{ve}(G\overline{G}) = 2$. Next, assume that $(ii)(b)$ holds. Let $xy \in E(G\overline{G}) \setminus \{uv\}$. If $x \in \{u, v\}$ or $y \in \{u, v\}$, then $E(G\overline{G}) \cap \{ux, uy, vx, vy\} \neq \varnothing$. Suppose that $\{x, y\} \cap \{u, v\} = \varnothing$. If $x \in V(G)$, then since $\{u, v\}$ is a dominating set of G, $E(G\overline{G}) \cap \{ux, vx\} \neq \varnothing$. Suppose that $x, y \in V(\overline{G})$. Then (exactly) one of the following is true: $x = \overline{v}$, $y = \overline{v}$, $x = \overline{u}$, $y = \overline{u}$. In any case, $E(G\overline{G}) \cap \{ux, uy, vx, vy\} \neq \varnothing$. By Proposition 5, $\gamma^t_{ve}(G\overline{G}) = 2$. Similarly, if $(ii)(c)$ holds, then $\gamma^t_{ve}(G\overline{G}) = 2$.

Statements (iii) and (iv) immediately follow from Proposition 5(ii) and Proposition 5(i), respectively. ■

Graphs G in Proposition 9(ii) need not have to be complete. Graphs $G_1 = K_{1,3}$ and $G_2 = C_4$ in Figure 2 are examples for statements $(ii)(a)$ and $(ii)(b)$, respectively. In each case, $\gamma^t_{ve}(G\overline{G}) = 2$.

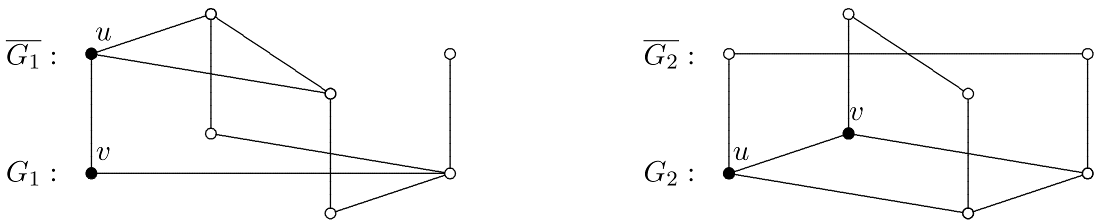

Figure 2: Examples of a noncomplete graph G for which $\gamma^t_{ve}(G\overline{G}) = 2$.

Proposition 10. *For any graph G,*

(i) $\gamma_{ve}(G\overline{G}) \leq \min\{\gamma(G) + \gamma_{ve}(\overline{G}), \gamma(\overline{G}) + \gamma_{ve}(G)\}$;

(ii) *provided G and $\overline{G}$ have no isolated vertices,*

$$\gamma^t_{ve}(G\overline{G}) \leq \min\{\gamma_t(\overline{G}) + \gamma^t_{ve}(G), \gamma_t(G) + \gamma^t_{ve}(\overline{G})\}.$$

Proof. Let $S \subseteq V(G)$ be a γ-set of G and $S^* \subseteq V(\overline{G})$ a γ_{ve}-set of $\overline{G}$. Then $S \cup S^*$ ve-dominates $V(G) \cup V(\overline{G})$. Let $u \in V(G)$. There exists $v \in S$ such that $u \in N_G[u]$. If $u = v$, then v ve-dominates $u\overline{u}$. If $v \neq u$, then $uv \in E(G)$ and v ve-dominates $u\overline{u}$. In any case, $S \cup S^*$ ve-dominates $u\overline{u}$. Since u is arbitrary, $S \cup S^*$ is a ve-dominating set of $G\overline{G}$. Hence, $\gamma_{ve}(G\overline{G}) \leq |S| + |S^*| = \gamma(G) + \gamma_{ve}(\overline{G})$. Similarly, $\gamma_{ve}(G\overline{G}) \leq \gamma(\overline{G}) + \gamma_{ve}(G)$. This proves (i).

The inequality in (ii) is proved similarly. ■

Proposition 11. *Let G be a graph with one isolated vertex v. Then*

$$\gamma_{ve}(G\overline{G}) \leq 1 + \gamma_{ve}(G - v) \tag{3}$$

and

$$\gamma^t_{ve}(G\overline{G}) \leq 2 + \gamma^t_{ve}(G - v). \tag{4}$$

Equality in Equation 3 is attained if $G\overline{G}$ has a γ_{ve}-set which contains $\overline{v}$. Under the same condition

$$1 + \gamma_{ve}(G - v) \leq \gamma^t_{ve}(G\overline{G}), \tag{5}$$

and this lower bound is sharp.

Proof. Equation 3 follows from Proposition 10. Equation 4 follows from the fact that if $S \subseteq V(G - v)$ is a γ^t_{ve}-set of $G - v$, then $S \cup \{v, \overline{v}\}$ is a total ve-dominating set of $G\overline{G}$. Let $S \subseteq V(G\overline{G})$ be a γ_{ve}-set of $G\overline{G}$ with $\overline{v} \in S$. Let $S^* = \{w \in V(G - v) : \overline{w} \in S\}$, and put $T = S^* \cup (S \cap V(G - v))$. Let $xy \in E(G - v)$. There exists $w \in S$ for which w ve-dominates xy. If $w \in V(G - v)$, then T ve-dominates xy. Suppose that $w \in V(\overline{G})$. Then $\overline{w} \in S^*$ and either $\overline{w} = x$ or $\overline{w} = y$. In either case, T ve-dominates xy. Since xy is arbitrary, T is a ve-dominating set of $G - v$. Hence,

$$\gamma_{ve}(G\overline{G}) = |S| \geq 1 + |T| \geq 1 + \gamma_{ve}(G - v).$$

If S were taken as a γ^t_{ve}-set of $G\overline{G}$, then the above argument implies that

$$\gamma^t_{ve}(G\overline{G}) = |S| \geq 1 + |T| \geq 1 + \gamma_{ve}(G - v).$$

If, in particular, G is the graph $G_1 = K_1 \cup P_3$ in Figure 2(a), then

$$\gamma^t_{ve}(G\overline{G}) = 2 = 1 + \gamma_{ve}(G),$$

showing that the lower bound in Equation 5 is sharp. ■

Strict inequality can be attained in Proposition 11. Consider the graph $G_2 = K_1 \cup (K_2 \cup K_2)$. Then $\overline{G_2}$ is the wheel $K_1 + C_4$, and $G_2\overline{G_2}$ is the graph in Figure 3(b).Observe that $\gamma_{ve}(G\overline{G}) = 2$ and is determined by the ve-dominating set $S = \{x, y\}$ of $G\overline{G}$. Note, on the other hand, that $1 + \gamma_{ve}(G - v) = 3$, where v is the isolated vertex of G.

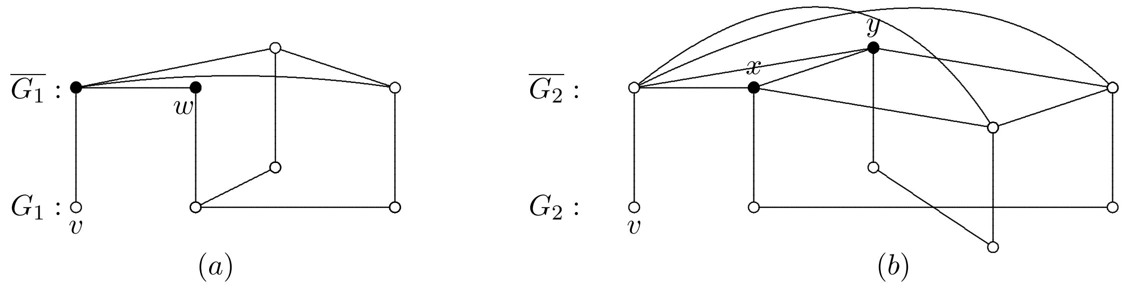

Figure 3: Examples of graphs illustrating Proposition 11.

Proposition 12. *Let G and H be connected graphs with H nontrivial, and $C \subseteq V(G[H])$. Then C is a ve-dominating set of $G[H]$ if and only if*

$$C = \cup_{x \in S} \left(\{x\} \times T_x\right), \tag{6}$$

where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ such that each of the following holds:

(*i*) *S is a dominating set of G;*

(*ii*) *T_x is a ve-dominating set of H for each $x \in S \setminus N_G(S)$.*

Proof. First, assume that C is a *ve*-dominating set of $G[H]$. Let S be the G-projection C_G of C, i.e., $S = C_G = \{x \in V(G) : (x, y) \in C \text{ for some } y \in V(H)\}$. Then $C = \cup_{x \in S} (\{x\} \times T_x)$, where $T_x = \{y \in V(H) : (x, y) \in C\}$ for each $x \in S$. We claim that S is a dominating set of G. Suppose not, and let $x \in V(G) \setminus N_G[S]$. Let $wz \in E(H)$. Then $(x, w)(x, z) \in E(G[H])$ and there exists $(u, v) \in C$ such that (u, v) *ve*-dominates $(x, w)(x, z)$. Since $x \notin S$, $u \neq x$. Thus, $ux \in E(G)$, a contradiction. This means that $V(G) = N_G[S]$, and the claim is done. Therefore, (i) holds. Now, to prove (ii), let $x \in S \setminus N_G(S)$. Let $wz \in E(H)$. Then $(x, w)(x, z) \in E(G[H])$ and there exists $(u, v) \in C$ for which (u, v) *ve*-dominates $(x, w)(x, z)$ in $G[H]$. If $(u, v) = (x, w)$, then $u = x$ so that $v = w \in T_x$ and T_x *ve*-dominates wz. Similarly, if $(u, v) = (x, z)$, then T_x *ve*-dominates wz. Suppose that $(u, v)(x, w) \in E(G[H])$. Since $x \notin N_G(S)$, $u = x$ and $vw \in E(H)$. This means that v, and hence T_x, *ve*-dominates wz. Similarly, if $(u, v)(x, z) \in E(G[H])$, then T_x *ve*-dominates wz. Accordingly, T_x is a *ve*-dominating set of H.

Next, assume that C satisfies Equation 6 where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ satisfying conditions (i) and (ii), respectively. Note first that since S is a dominating set, S is a *ve*-dominating set of G. Let $(x, y)(w, z) \in E(G[H])$. We consider the following cases:

Case 1: Suppose that $xw \in E(G)$. By a *ve*-dominating set, there exists $u \in S$ such that u *ve*-dominates xw. Pick $v \in T_u$. If $u = x$ (resp. $u = w$), then $(u, v)(w, z) \in E(G[H])$ (resp. $(u, v)(x, y) \in E(G[H])$). This means that (u, v), and hence C, *ve*-dominates $(x, y)(w, z)$. On the other hand, if $ux \in E(G)$ (resp. $uw \in E(G)$), then $(u, v)(x, y) \in E(G[H])$ (resp. $(u, v)(w, z) \in E(G[H])$). Thus, (u, v), and hence, C *ve*-dominates $(x, y)(w, z)$.

Case 2: Now suppose that $x = w$ and $yz \in E(H)$. If $x \in N_G(S)$ and $u \in S \cap N_G(x)$, pick $v \in T_u$. Then $(u, v) \in C$ and $(u, v)(x, y) \in E(G[H])$. This means that (u, v), and hence C, ve-dominates $(x, y)(w, z)$. Suppose, on the other hand, that $x \notin N_G(S)$. Since S is a dominating set, $x \in S \setminus N_G(S)$. By condition (ii), there exists $v \in T_x$ such that v ve-dominates yz. This means that $(x, v) \in C$ and (x, v), and hence C, ve-dominates $(x, y)(w, z)$. Accordingly, C is a ve-dominating set of $G[H]$. ■

Lemma 1. [13, 14] *If S is a dominating set of a graph G without isolated vertices, then*

$$\gamma_t(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

Consequently, $\gamma_t(G) \leq 2\gamma(G)$.

Corollary 4. *Let G and H be connected graphs with H nontrivial.*

- (i) *If $\gamma_{ve}(H) = 1$, then $\gamma_{ve}(G[H]) = \gamma(G)$.*
- (ii) *If $\gamma_{ve}(H) \geq 2$, then $\gamma_{ve}(G[H]) = \gamma_t(G)$.*

Proof. Let $S \subseteq V(G)$ be a γ-set and $v \in V(H)$ for which $\{v\}$ is a ve-dominating set of H. Put $T_x = \{v\}$ for all $x \in S$. By Proposition 12, $S \times \{v\} = \cup_{x\in S} (\{x\} \times T_x)$ is a ve-dominating set of $G[H]$ yielding

$$\gamma_{ve}(G[H]) \leq |S| = \gamma(G).$$

For the other inequality, note from Proposition 12 that every ve-dominating set of $G[H]$ is of the form $C = \cup_{x\in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ is a dominating set of G and $T_x \subseteq V(H)$ is a ve-dominating set of H for each $x \in S \setminus N_G(S)$. For any such C, we have

$$|C| = \sum_{x\in S} |T_x| \geq |S| = \gamma(G).$$

Therefore, $\gamma_{ve}(G[H]) \geq \gamma(G)$. This proves (i).

Now we prove (ii) using Proposition 12. For all total dominating sets S of G and $v \in V(H)$, since $S \setminus N_G(S) = \varnothing$, $C = S \times \{v\} = \cup_{x\in S} (\{x\} \times \{v\})$ is a ve-dominating set of $G[H]$ so that

$$\gamma_{ve}(G[H]) \leq |S| = \gamma_t(G).$$

Now, let $C = \cup_{x\in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$, be a ve-dominating set of $G[H]$. Necessarily, S is a dominating set of G and T_x is a ve-dominating set of H for each $x \in S \setminus N_G(S)$. By Lemma 1,

$$|C| = \sum_{x\in S} |T_x| \geq 2|S| \geq 2\gamma(G) \geq \gamma_t(G).$$

Since C is arbitrary, $\gamma_{ve}(G[H]) \geq \gamma_t(G)$. ■

The following immediately follows from Proposition 12

Proposition 13. *Let G and H be connected nontrivial graphs, and let $C \subseteq V(G[H])$. Then C is a total ve-dominating set of $G[H]$ if and only if*

$$C = \cup_{x\in S}\left(\{x\} \times T_x\right), \tag{7}$$

where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ such that one of the following holds:

(*i*) *S is a total dominating set of G;*

(*ii*) *Each of the following holds:*

- (*a*) *S is a dominating set of G; and*
- (*b*) *T_x is a total ve-dominating set of H for each $x \in S \setminus N_G(S)$.*

Corollary 5. *If G and H are connected nontrivial graphs, then*

$$\gamma^t_{ve}(G[H]) = \gamma_t(G).$$

Proof. By Proposition 13, $S \times \{y\}$ is a total ve-dominating set of $G[H]$ for all total dominating sets $S \subseteq V(G)$ of G and all $y \in V(H)$. Therefore,

$$\gamma^t_{ve}(G[H]) \leq \gamma_t(G).$$

Now, let $C = \cup_{x\in S}\left(\{x\} \times T_x\right) \subseteq V(G[H])$ be a γ^t_{ve}-set of $G[H]$. If S is a total dominating set of G, then

$$|C| \geq |S| \geq \gamma_t(G).$$

Otherwise, Lemma 1 and Proposition 13 imply that

$$\begin{aligned} |C| &= \sum_{x\in S\cap N_G(S)} |T_x| + \sum_{x\in S\setminus N_G(S)} |T_x| \\ &\geq |S \cap N_G(S)| + 2|S \setminus N_G(S)| \\ &\geq \gamma_t(G). \end{aligned}$$

In any case, $\gamma^t_{ve}(G[H]) \geq \gamma_t(G)$. ■

For each $uv \in E(G)$, H^{uv} denotes that copy of H being joined to G through edge uv. We also write $H^{uv} + uv$ to denote that subgraph of $G \diamond H$ induced by $V(H^{uv}) \cup \{u, v\}$.

Proposition 14. *Let G be a nontrivial connected graph and H be any nonempty graph. Then $S \subseteq V(G \diamond H)$ is a ve-dominating set of $G \diamond H$ if and only if*

$$S = A \cup \left(\cup_{uv\in E(G)} S_{uv}\right), \tag{8}$$

where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ for each $uv \in E(G)$ such that S_{uv} is a ve-dominating set of H^{uv} whenever $\{u, v\} \cap A = \varnothing$.

Proof. Let $S \subseteq V(G \diamond H)$ be a ve-dominating set of $G \diamond H$. Put $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then S satisfies Equation 8. Let $uv \in E(G)$ such that $\{u, v\} \cap A = \varnothing$, and let $xy \in E(H^{uv})$. Since S is a ve-dominating set of $G \diamond H$, there exists $w \in S$ such that w ve-dominates xy. Since $\{u, v\} \cap S = \varnothing$, $w \in S_{uv}$. Thus, S_{uv} is a ve-dominating set of H^{uv}.

Conversely, suppose that the set S in Equation 8 has the property that S_{uv} is a ve-dominating set of H^{uv} for each $uv \in E(G)$ with $\{u, v\} \cap A = \varnothing$. Let $xy \in E(G \diamond H)$ and let $uv \in E(G)$ such that $xy \in E(H^{uv} + uv)$. First, suppose that $\{u, v\} \cap A = \varnothing$. Then S_{uv} is a ve-dominating set of H^{uv}. If $\{x, y\} \cap \{u, v\} \neq \varnothing$, then since $\{zu, zv\} \subseteq E(G \diamond H)$, z ve-dominates xy for all $z \in S_{uv}$. If, on the other hand, $\{x, y\} \cap \{u, v\} = \varnothing$, then $xy \in E(H^{uv})$ and there exists $z \in S_{uv}$ such that z ve-dominates xy. In this case, S ve-dominates xy. Next, suppose that $\{u, v\} \cap A \neq \varnothing$. WLOG, assume that $u \in A$. If $x = u$, then u, and hence S, ve-dominates xy. If $x \neq u$, then $ux \in E(G \diamond H)$ and so u, and hence S, ve-dominates xy. Accordingly, S is a ve-dominating set of $G \diamond H$. ■

Corollary 6. *Let G be a nontrivial connected graph.*

(i) If H is an empty graph, then $\gamma_{ve}(G \diamond H) = \gamma(G)$.

(ii) If H is a nonempty graph, then

$$\gamma_{ve}(G \diamond H) = \min\{|S| : S \subseteq V(G) \text{ and } N_G(V(G) \setminus S) \subseteq S\}.$$

Proof. Suppose that H is an empty graph. Let $S \subseteq V(G)$ be a γ-set of G. Let $xy \in E(G \diamond H)$, and let $uv \in E(G)$ such that $xy \in E(H^{uv} + uv)$. If $u \in S$ (resp. $v \in S$), then u (resp. v) ve-dominates xy. Suppose that $u \notin S$ and $v \notin S$. Since S is a dominating set, there exist $w, z \in S$ for which $uw \in E(G)$ and $yz \in E(G)$. In this case, w ve-dominates xy or z ve-dominates xy. Thus, S is a ve-dominating set of $G \diamond H$. Consequently,

$$\gamma_{ve}(G \diamond H) \leq |S| = \gamma(G).$$

To get the other inequality, let $S \subseteq V(G \diamond H)$ be a ve-dominating set of $G \diamond H$. Put $S_1 = S \cap V(G)$. For each $uv \in E(G)$, put $S_{uv} = S \cap V(H^{uv})$. For each $uv \in E(G)$ for which $S_{uv} \neq \varnothing$, put either $x_{uv} = u$ or $x_{uv} = v$. Define $S_2 = \{x_{uv} : uv \in E(G) \text{ with } S_{uv} \neq \varnothing\}$ and define $S^* = S_1 \cup S_2$. We claim that S^* is a dominating set of G. Let $z \in V(G) \setminus S^*$ and choose $w \in V(G)$ such that $zw \in E(G)$. If $S_{zw} \neq \varnothing$, then since $z \notin S_2$, $w \in S_2$. In this case, $z \in N_G(S^*)$. Suppose that $S_{zw} = \varnothing$. Pick $t \in V(H^{zw})$. There exists $u \in S$ for which u ve-dominates zt. If $u \in V(G)$, then $u \in S_1$ and $uz \in E(G)$ so that $z \in N_G(S^*)$. If $u \notin V(G)$, then $u \in S_{vz}$ for some $v \in N_G(z)$. Then $x_{vz} = v \in S_2$ so that $z \in N_G(S^*)$. Since z is arbitrary, S^* is a dominating set of G. Hence, $\gamma(G) \leq |S^*| \leq |S|$. Since S is arbitrary, $\gamma(G) \leq \gamma_{ve}(G \diamond H)$. This completely proves (i).

Now we prove (ii). Note first that if $S = V(G) \setminus \{x\}$, where $x \in V(G)$, then $N_G(V(G) \setminus S) = N_G(x) \subseteq S$. Put $\alpha = \min\{|S| : S \subseteq V(G) \text{ and } N_G(V(G) \setminus S) \subseteq S\}$. Let $S \subseteq V(G)$

such that $N_G(x) \subseteq S$ for each $x \in V(G) \setminus S$. Write $S = A \cup \left(\cup_{uv\in E(G)} S_{uv}\right)$, where $A = S$ and $S_{uv} = \varnothing$ for all $uv \in E(G)$. Note that by the definition of S, $\{u, v\} \cap A \neq \varnothing$ for all $uv \in E(G)$. By Proposition 14, S is a ve-dominating set of $G \diamond H$. Consequently, $\gamma_{ve}(G \diamond H) \leq |S|$. Since S is arbitrary,

$$\gamma_{ve}(G \diamond H) \leq \alpha.$$

For the other inequality, let $S \subseteq V(G \diamond H)$ be a γ_{ve}-set of $G \diamond H$. By Proposition 14, $S = A \cup \left(\cup_{uv\in E(G)} S_{uv}\right)$, where $A = S \cap V(G)$ and $S_{uv} \subseteq V(H^{uv})$ with S_{uv} a ve-dominating set of H^{uv}, hence nonempty, whenever $\{u, v\}\cap A = \varnothing$. For each $uv \in E(G)$ with $\{u, v\}\cap A = \varnothing$, put either $x_{uv} = u$ or $x_{uv} = v$. Define $B = \{x_{uv} : uv \in E(G) \text{ with } \{u, v\} \cap A = \varnothing\}$ and put $S^* = A \cup B$. It is worth noting that $A \cap B = \varnothing$ and

$$|S^*| = |A| + |B| \leq |A| + \sum_{uv\in E(G)} |S_{uv}| = |S| = \gamma_{ve}(G \diamond H).$$

Let $v \in V(G) \setminus S^*$, and let $u \in N_G(v)$. If $u \notin S^*$, then $\{u, v\} \cap A = \varnothing$. This means that $x_{uv} = u$ or $x_{uv} = v$, which is impossible since $x_{uv} \in B \subseteq S^*$. Therefore, $u \in S^*$. Since u is arbitrary, $N_G(v) \subseteq S^*$. Hence, $\alpha \leq |S^*| \leq \gamma_{ve}(G \diamond H)$. ■

It is clear from Corollary 6(ii) that if G is a connected nontrivial graph and H is a nonempty graph, then

$$\gamma(G) \leq \gamma_{ve}(G \diamond H) \leq |V(G)| - 1.$$

In view of Proposition 14, the following is clear.

Proposition 15. *Let G be a nontrivial connected graph and H be any nonempty graph. Then $S \subseteq V(G \diamond H)$ is a total ve-dominating set of $G \diamond H$ if and only if*

$$S = A \cup \left(\cup_{uv\in E(G)} S_{uv}\right), \tag{9}$$

where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv})$ for each $uv \in E(G)$ satisfying the following:

(i) S_{uv} is a total ve-dominating set of H^{uv} for each $uv \in E(G)$ for which $\{u, v\} \cap A = \varnothing$.

(ii) $A \cap N_G(u) \neq \varnothing$ for all $u \in A$ for which $S_{uv} = \varnothing$ for all $v \in N_G(u)$.

Corollary 7. *Let G be a nontrivial connected graph.*

(i) If H is an empty graph, then $\gamma^t_{ve}(G \diamond H) = \gamma_t(G)$.

(ii) If H is a nonempty graph, then

$$\gamma^t_{ve}(G \diamond H) = \min\{|S| : S \subseteq V(G) \text{ with } S \subseteq N_G(S) \text{ and } N_G(V(G) \setminus S) \subseteq S\}.$$

Proof. Assume H is an empty graph. Let $S \subseteq V(G)$ be a γ_t-set of G. Since S is a dominating set of G, following the necessity proof of Corollary 6(i) will show that S is a ve-dominating set of $G \diamond H$. Consequently, S is a total ve-dominating set of $G \diamond H$. Thus, $\gamma_{ve}^t(G \diamond H) \leq |S| = \gamma_t(G)$. To get the other inequality, let $S \subseteq V(G \diamond H)$ be a γ_{ve}^t-set of $G \diamond H$. Define $S_1 = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Note that since S is a total ve-dominating set, if $S_{uv} \neq \varnothing$, then $u \in S_1$ or $v \in S_1$. For each $uv \in E(G)$ for which $S_{uv} \neq \varnothing$, write $u = x_{uv}$ whenever $u \notin S_1$ and write $v = x_{uv}$ whenever $v \notin S_1$. Put $S_2 = \{x_{uv} : uv \in E(G) \text{ with } S_{uv} \neq \varnothing\}$. Let $S^* = S_1 \cup S_2$. Then $u, v \in S^*$ for all $uv \in E(G)$ for which $S_{uv} \neq \varnothing$. First, let $z \in V(G) \setminus S^*$. Choose $w \in V(G)$ such that $zw \in E(G)$. If $w \in S^*$, then $z \in N_G(S^*)$. Suppose that $w \notin S^*$. Since S is a ve-dominating set, there exists $x \in S$ such that x ve-dominates zw. Since $z \notin S^*$ and $w \notin S^*$, $x \in S_1$. This means that $z \in N_G(S^*)$. Next, let $z \in S^*$. If $z \in S_2$, then $z \in N_G(S^*)$. Suppose that $z \in S_1$. Since S is a total ve-dominating set, there exists $x \in S \cap N_{G \diamond H}(z)$. If $x \in S_1$, then $z \in N_G(S^*)$. If, on the other hand, $x \notin S_1$ and $w \in V(G)$ for which $x \in S_{zw}$, then $w = x_{zw} \in S^*$. Hence $z \in N_G(S^*)$. The above implies that S^* is a total dominating set of G. Hence,

$$\gamma_{ve}^t(G \diamond H) = |S| \geq |S^*| \geq \gamma_t(G).$$

This proves (i).

Now we prove (ii). Put $\alpha = \min\{|S| : S \subseteq V(G) \text{ with } S \subseteq N_G(S) \text{ and } N_G(V(G) \setminus S) \subseteq S\}$. First, note that if $S \subseteq V(G)$ such that $S \subseteq N_G(S)$ and $N_G(V(G) \setminus S) \subseteq S$, then following the argument in the necessity proof of Corollary (ii) will show that S is a total ve-dominating set of $G \diamond H$. Hence,

$$\gamma_{ve}^t(G \diamond H) \leq \alpha.$$

Next, let $S \subseteq V(G \diamond H)$ be a γ_{ve}^t-set of $G \diamond H$. By Proposition 15, $S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right)$, where $A = S \cap V(G)$ and $S_{uv} \subseteq V(H^{uv})$ with S_{uv} a total ve-dominating set of H^{uv}, hence $|S_{uv}| \geq 2$, whenever $\{u, v\} \cap A = \varnothing$. Moreover, $A \cap N_G(u) \neq \varnothing$ for all $u \in A$ for which $S_{uv} = \varnothing$ for all $v \in N_G(u)$. For each $u \in A$, we say u has the property P_A and write $u \in P_A$ if there exists $v \in N_G(u)$ such that $S_{uv} \neq \varnothing$. For each $u \in P_A$, we pick one such v and write $v = v_u$. Put $C = \{v_u : u \in P_A\}$ and let $B = \{u \in V(G) \setminus A : \exists v \in V(G) \setminus A \text{ for which } uv \in E(G)\}$. Define $S^* = A \cup B \cup C$, and let $X = \{uv \in E(G) : \{u, v\} \cap A = \varnothing\}$ and $Y = \{uv \in E(G) : \{u, v\} \cap A \neq \varnothing\}$. Then

$$\begin{aligned} \gamma_{ve}^t(G \diamond H) = |S| &= |A| + \sum_{uv \in E(G)} |S_{uv}| \\ &= |A| + \sum_{uv \in Y} |S_{uv}| + \sum_{uv \in X} |S_{uv}| \\ &\geq |A \cup C| + |B| \\ &= |S^*|. \end{aligned}$$

Let $u \in B$. Then there exists $v \in B$ such that $uv \in E(G)$ and $\{u, v\} \cap A = \varnothing$. Consequently, $v \in S^*$ so that $u \in N_G(S^*)$. Let $u \in A$. If $u \in P_A$, then there exists $v_u \in C \cap N_G(u)$.

Thus, $u \in N_G(S^*)$. On the other hand, if $u \notin P_A$, then $A \cap N_G(u) \neq \varnothing$. This means $u \in N_G(S^*)$. Now, let $u \in C$. Then $u = u_x$ for some $x \in P_A$ and so $u \in N_G(S^*)$. All these imply that $S^* \subseteq N_G(S^*)$. Finally, let $v \in V(G) \setminus S^*$, and let $u \in N_G(v)$. If $u \notin S^*$, then $\{u, v\} \cap A = \varnothing$. This means that $u, v \in B$, which is impossible. Therefore, $u \in S^*$. Since u is arbitrary, $N_G(v) \subseteq S^*$. Hence, $\alpha \leq |S^*| \leq \gamma_{ve}^t(G \diamond H)$. ■

It also follows from Corollary7(ii) that if G is a connected graph of order $n \geq 3$ and H is a nonempty graph, then

$$\gamma_t(G) \leq \gamma_{ve}^t(G \diamond H) \leq n - 1.$$

4. Conclusion

This study has established several characterizations concerning the vertex-edge domination and total vertex-edge domination in graphs under various binary operations, namely the join, corona, complementary prism, lexicographic product, and edge corona. Each result provided necessary and sufficient conditions for a subset of vertices to be a (total) vertex-edge dominating set within the resulting graph. The results demonstrate that the (total) vertex-edge domination behavior of these resulting graphs is fundamentally determined by the vertex-edge domination properties of their component graphs. In the join of graphs, vertex-edge domination depends on the subset relationships among the component vertex sets; in the corona and edge corona, the vertex-edge domination properties are inherited through the copies of the secondary graph attached to each vertex or edge of the primary graph. Moreover, in the complementary prism, specific bounds for the vertex-edge domination and total vertex-edge domination numbers were established, while in the lexicographic product, the (total) vertex-edge dominating sets were expressed through unions of product sets derived from the component graphs.

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