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Strong and Weak Dominating Sets of Graphs under Some Binary Operations

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Abstract. A set S of vertices of a graph G is a strong (resp. weak) dominating set of G if for every vertex v of G outside of S, there is a vertex u inside of S such that u and v are adjacent and $deg_G(v) \leq deg_G(u)$ (resp. $deg_G(v) \geq deg_G(u)$). The minimum cardinality of a strong (resp. weak) dominating set is called the strong (resp. weak) domination number of G, and is denoted by $\gamma_s(G)$ (resp. $\gamma_w(G)$). In this paper, we characterize the strong and weak dominating sets of graphs under some binary operations. As a result, we also determine the exact values of or sharp bounds for the corresponding strong and weak domination numbers.

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Key Words and Phrases: Strong dominating, weak dominating, shadow, join, corona, edge corona, lexicographic product

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1. Introduction

All throughout this paper, we consider only graphs which are simple, finite and undirected. Given a graph $G = (V(G), E(G))$, we call $V(G)$ the *vertex set* of G and $E(G)$ its *edge set*. The cardinality $|V(G)|$ of $V(G)$ is the *order* of G. All terminologies used here which are not being defined are adapted from [1].

Let G and H be disjoint graphs. By $G \cup H$, we mean the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *complementary prism* $G\overline{G}$ is formed from G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. If for each $v \in V(G)$, $\overline{v}$ is the vertex in $\overline{G}$ corresponding to v, then $G\overline{G}$ is formed by adding the edge $v\overline{v}$ for every $v \in V(G)$. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

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The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The *edge corona* of G and H is the graph $G \diamond H$ obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The *lexicographic product* of G and H is the graph $G[H]$ with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *degree* of v refers to the cardinality $|N_G(v)|$ of the open neighborhood of v, $\Delta(G)$ is the maximum degree of a vertex of G and $\delta(G)$ is the minimum degree of a vertex of G. If $|N_G(v)| = 1$, then v is an *endvertex*, and, in this case, if $u \in N_G(v)$, then u is the *support vertex* of v. The symbols $End(G)$ and $Supp(G)$ denote the set of all endvertices and the set of all support vertices of G, respectively.

The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. In case $N_G(S) = V(G)$, then S is a *total dominating set* of G. The minimum cardinality $\gamma(G)$ of a dominating set of G is the *domination number* of G, and the minimum cardinality $\gamma_t(G)$ of a total dominating set is the *total domination number of* G. A dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G. Similarly, a γ_t-*set* is a total dominating set of cardinality $\gamma_t(G)$. The reader is referred to [2–7] for the history, fundamental concepts and recent developments of domination in graphs as well as its various applications.

For two vertices $u, v \in V(G)$, v is said to *strongly dominate* u in G if $uv \in E(G)$ and $deg_G(v) \geq deg_G(u)$. In this case, we also say that u *weakly dominates* v. We write $v \succcurlyeq_G u$ or $u \preccurlyeq_G v$ to mean that v strongly dominates u or, equivalently, u weakly dominates v. A subset $S \subseteq V(G)$ is said to *strongly dominate* (resp. *weakly dominate*) $u \in V(G) \setminus S$ in G if there exists $v \in S$ for which $v \succcurlyeq_G u$ (resp. $u \succcurlyeq_G v$) in G. In this case we write $u \preccurlyeq_G S$ (resp. $u \succcurlyeq_G S$). For $S, D \subseteq V(G)$, S is said to *strongly dominate* (resp. *weakly dominate*) D if S strongly dominates (resp. weakly dominates) every vertex $v \in D \setminus S$. We say S is a *strong dominating set* (resp. *weak dominating set*) of G if S strongly dominates (resp. weakly dominates) $V(G)$, i.e., for each $u \in V(G) \setminus S$, $u \preccurlyeq_G S$ (resp. $u \succcurlyeq_G S$) in G. The minimum cardinality of a strong dominating set (resp. weak dominating set) of G is the *strong domination number* (resp. *weak domination number*) of G, which is denoted by $\gamma_s(G)$ (resp. $\gamma_w(G)$). Any strong dominating set (resp. weak dominating set) of G of cardinality $\gamma_s(G)$ (resp. $\gamma_w(G)$) is called a γ_s-*set* (resp. γ_w-*set*) of G.

The concepts of strong and weak domination were first introduced by E. Sampathkumar and L. Pushpa Latha [8] in 1996. Thereafter, several further studies have been done on these two concepts (see [9, 10], [11]- [12], [13, 14], [15]-[16]). In particular, properties and characteristics of strong and weak dominating sets are explored in [12, 17]. Bounds on

$\gamma_s(G)$ and $\gamma_w(G)$ are studied in [18–21], and investigation of strong and weak domination in families of graphs are done in [22–25].

In this present study, we continue the investigation of these two concepts, particularly on characterizing the strong and weak dominating sets in families of graphs involving the complementary prism of graphs, join, corona, edge corona and lexicographic product of graphs.

For $v \in V(G)$, write

$$N_G(v \succcurlyeq) = \{u \in V(G) : u \preccurlyeq v\},$$

$$N_G(v \preccurlyeq) = \{u \in V(G) : u \succcurlyeq v\},$$

$$N_G[v \succcurlyeq] = N_G(v \succcurlyeq) \cup \{v\}$$

and

$$N_G[v \preccurlyeq] = N_G(v \preccurlyeq) \cup \{v\}.$$

For $S \subseteq V(G)$, $N_G(S \succcurlyeq) = \cup_{v \in S} N_G(v \succcurlyeq)$ and $N_G(S \preccurlyeq) = \cup_{v \in S} N_G(v \preccurlyeq)$. We also write $N_G[S \succcurlyeq] = N_G(S \succcurlyeq) \cup S$ and $N_G[S \preccurlyeq] = N_G(S \preccurlyeq) \cup S$. Hence, S is a strong (resp. weak) dominating set of G if and only if $N_G[S \succcurlyeq] = V(G)$ (resp. $N_G[S \preccurlyeq] = V(G)$).

The symbol $\Gamma_s(G)$ (resp. $\Gamma_w(G)$) denotes the family of all strong (resp. weak) dominating sets of G. Thus, $\gamma_s(G) = \min\{|S| : S \in \Gamma_s(G)\}$ and $\gamma_w(G) = \min\{|S| : S \in \Gamma_w(G)\}$.

2. Preliminary results

It is worth noting that strong and weak dominating sets are necessarily dominating sets. Hence, for a graph of order n, $\gamma(G) \le \gamma_s(G) \le n - \Delta(G)$ and $\gamma(G) \le \gamma_w(G) \le n - \delta(G)$ [26].

The following are immediate observations.

Remark 1. *Let G be any graph of order n. Then*

(i) $\gamma_s(G) = 1$ *if and only if* $\gamma(G) = 1$*;*

(ii) $\gamma_w(G) = 1$ *if and only if* $G = K_n$*;*

(iii) $\gamma_s(G) = n$ *(resp.* $\gamma_w(G) = n$*) if and only if* $G = \overline{K_n}$*;*

(iv) $\gamma_s(G) = n - 1$ *if and only if* $G = K_2$ *or* $G = K_2 \cup \overline{K_{n-2}}$*; and*

(v) $\gamma_w(G) = n - 1$ *if and only if* $G = K_{1,n-1}$ *or* $G = K_{1,(n-k-1)} \cup \overline{K_k}$.

Remark 2. *Let G be a connected graph. Then* $\gamma_s(G) = 2$ *(resp.* $\gamma_w(G) = 2$*) if and only if* $\gamma(G) = 2$ *and G has a γ-set $S = \{u, v\}$ for which $x \preccurlyeq u$ (resp. $x \succcurlyeq u$) or $x \preccurlyeq v$ (resp. $x \succcurlyeq v$) for each $x \in V(G) \setminus S$.*

Lemma 1. *If G is connected of order $n \geq 3$, then for each $S \in \Gamma_s(G)$ there exists $S^* \in \Gamma_s(G)$ for which $S^* \cap End(G) = \varnothing$ and $|S^*| \leq |S|$. Consequently, G has a γ_s-set S for which $S \cap End(G) = \varnothing$.*

Proof. Let $S \in \Gamma_s(G)$. If $S \cap End(G) = \varnothing$, then let $S^* = S$. Suppose that $S \cap End(G) \neq \varnothing$. For each $v \in S \cap End(G)$, let x_v be the support vertex v. Then $|\{x_v \notin S : v \in S \cap End(G)\}| \leq |S \cap End(G)|$. Put $S^* = (S \setminus End(G)) \cup \{x_v \notin S : v \in S \cap End(G)\}$. Then $S^* \in \Gamma_s(G)$ and

$$|S^*| = |S \setminus End(G)| + |\{x_v \notin S : v \in S \cap End(G)\}| \leq |S|.$$

□

Lemma 2. *Let G be a connected graph. Then for each $S \in \Gamma_s(G)$ (resp. $S \in \Gamma_w(G)$), S contains a vertex v for which $deg_G(v) = \Delta(G)$ (resp. $deg_G(v) = \delta(G)$).*

Proof. Let $S \subseteq V(G)$ be a strong dominating set of G. Let $v \in V(G)$ for which $deg_G(v) = \Delta(G)$. If $v \in S$, then we are done. Suppose $v \notin S$. Since $S \in \Gamma_s(G)$, there exists $u \in S$ for which $v \preccurlyeq u$. Necessarily, $deg_G(u) = \Delta(G)$.

Parallel arguments will prove the case of the weak domination.

Remark 3. [27]

(i) *For a cycle C_n, $\gamma_s(C_n) = \gamma_w(C_n) = \lceil \frac{n}{3} \rceil$.*

(ii) *For a path P_n,*

$$\gamma_s(P_n) = \lceil \frac{n}{3} \rceil \ \text{and} \ \gamma_w(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil, & \text{if } n = 1(\text{mod } 3), \\ 1 + \lceil \frac{n}{3} \rceil, & \text{else.} \end{cases}$$

In what follows, for the purpose of emphasis, we write $x \preccurlyeq_G y$ to mean $x \preccurlyeq y$ in G.

3. In the complementary prism of graphs

Proposition 1. *Let G be any graph. Then*

(i) $\gamma_s(G\overline{G}) = 1$ *(resp. $\gamma_w(G\overline{G}) = 1$) if and only if $G = K_1$;*

(ii) $\gamma_s(G\overline{G}) = 2$ *if and only if exactly one of the following holds:*

  (a) *G has an isolated vertex x for which $\gamma(G - x) = 1$.*

  (b) *$\overline{G}$ has an isolated vertex x for which $\gamma(\overline{G} - x) = 1$.*

(iii) $\gamma_w(G\overline{G}) = 2$ *if and only if $G \in \{K_2, \overline{K_2}\}$.*

Proof. Statement (i) immediately follows from Observation 1(i). We prove (ii). Suppose that $\gamma_s(G\overline{G}) = 2$. By (i), $G \neq K_1$. By Observation 2, there exists a γ-set $S = \{u, v\}$ of $G\overline{G}$ for which $x \preccurlyeq_{G\overline{G}} u$ or $x \preccurlyeq_{G\overline{G}} v$ for each $x \in V(G\overline{G}) \setminus S$. If $u, v \in V(G)$ (resp. $u, v \in V(\overline{G})$), then $G = K_2$ (resp. $\overline{G} = K_2$) and (b) holds (resp. (a) holds). Assume WLOG that $u \in V(G)$ and $v \in V(\overline{G})$. Suppose further that $u\overline{v} \in E(G\overline{G})$ and $G \neq K_2$. Necessarily, $u \succcurlyeq_G V(G) \setminus \{u\}$ and $v \succcurlyeq_{\overline{G}} V(\overline{G}) \setminus \{\overline{u}, v\} \neq \varnothing$. The former implies that $N_G[u] = V(G)$ and consequently, $\overline{u}$ is a (unique) isolated vertex of $\overline{G}$. The latter implies that $N_{\overline{G}-\overline{u}}[v] = V(\overline{G}-\overline{u})$. This shows that (b) holds. Similarly, if $\overline{u}v \in E(\overline{G})$ and $\overline{G} \neq K_2$, then u is an isolated vertex of G and $\gamma(G - u) = 1$, showing that (a) holds.

Conversely, suppose that (a) holds for G. Let x be an isolated vertex of G and let $z \in V(G - x)$ such that $N_{G-x}[z] = V(G - x)$. Clearly, $G \neq K_1$ so that $\gamma_s(G\overline{G}) \geq 2$. Put $S = \{\overline{x}, z\}$. Since $N_{\overline{G}}[\overline{x}] = V(\overline{G})$, $\overline{x} \succcurlyeq_{G\overline{G}} V(\overline{G}) \cup \{x\} \setminus \{\overline{x}\}$. On the other hand, $z \succcurlyeq_G V(G) \setminus \{x, z\}$. By Observation 2, $S \in \Gamma_s(G\overline{G})$. Thus, $\gamma_s(G\overline{G}) \leq 2$. Similarly, if (b) holds, then $\gamma_s(G\overline{G}) = 2$. This proves (ii).

Now suppose that $\gamma_w(G\overline{G}) = 2$, and let $S = \{u, v\}$ be a γ-set of $G\overline{G}$ such that $u \preccurlyeq_{G\overline{G}} x$ or $v \preccurlyeq_{G\overline{G}} x$ for each $x \in V(G\overline{G}) \setminus S$. Suppose that $u \in V(G)$ and $v \in V(\overline{G})$. Assume $u\overline{v} \in E(G)$. Since $u \preccurlyeq_{G\overline{G}} \overline{u}$, there exists $w \in V(G)$ such that $\overline{u}\ \overline{w} \in E(\overline{G})$. Because $uw \notin E(G)$, $wv \in E(G\overline{G})$, which is impossible. Thus, $u, v \in V(G)$ or $u, v \in V(\overline{G})$. This implies that $G = K_2$ or $\overline{G} = K_2$. The converse is easy. This proves (iii).

Remark 4. *Let G be any graph. Then $S \subseteq V(G\overline{G})$ is a strong dominating set of $G\overline{G}$ if and only if $S = S_G \cup S_{\overline{G}}$ with $S_G \subseteq V(G)$ and $S_{\overline{G}}$ such that for each $x \in V(G) \setminus S_G$* (resp. $x \in V(\overline{G}) \setminus S_{\overline{G}}$), *$x \preccurlyeq_G S_G$* (resp. $x \preccurlyeq_{\overline{G}} S_{\overline{G}}$) *or $\overline{x} \in S_{\overline{G}}$* (resp. $\overline{x} \in S_G$) *and $x \preccurlyeq_{G\overline{G}} \overline{x}$.*

Let G be of order n. Put $S_G = \{x \in V(G) : \overline{x} \preccurlyeq_{G\overline{G}} x\}$ and $S_{\overline{G}} = \{\overline{x} : x \in V(G) \setminus S_G\}$. By Observation 4, $S = S_G \cup S_{\overline{G}} \in \Gamma_s(G\overline{G})$, showing $\gamma_s(G\overline{G}) \leq |S| = n$. If $\gamma(G) = 1$ or G has an isolated vertex, this bound coincides with the bound given in Equation (1) for $\gamma_s(G\overline{G})$. Replacing "$\preccurlyeq_{G\overline{G}}$" with "$\succcurlyeq_{G\overline{G}}$" in the definition of S_G will show that $\gamma_w(G\overline{G}) \leq n$. In particular, if $G \in \{K_n, \overline{K_n}\}$, then $\gamma_s(G\overline{G}) = n = \gamma_w(G\overline{G})$.

Proposition 2. (i) *For all $n \geq 3$,*

$$\gamma_s(P_n\overline{P_n}) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil, & \text{if } 3 \leq n \leq 5; \\ 2 + \lceil \frac{n-2}{3} \rceil, & \text{if } n \geq 6, \end{cases}$$

and

$$\gamma_w(P_n\overline{P_n}) = \begin{cases} 3, & \text{if } n = 3, \\ 4, & \text{if } 4 \leq n \leq 6; \\ 2 + \lceil \frac{n}{3} \rceil, & \text{if } n \geq 7, \end{cases}$$

(ii) *For $n \geq 4$,*

$$\gamma_s(C_n\overline{C_n}) = \gamma_w(C_n\overline{C_n}) = \begin{cases} 3, & \text{if } n = 5; \\ 2 + \lceil \frac{n}{3} \rceil, & \text{else.} \end{cases}$$

Proof. For (i), put $G = P_n = [x_1, x_2, \ldots, x_n]$ and for any $S \subseteq V(G\overline{G})$, define $S_G = S \cap V(G)$ and $S_{\overline{G}} = S \cap V(\overline{G})$. The value of $\gamma_s(G\overline{G})$ can readily be checked when $3 \leq n \leq 5$. Let $n \geq 6$. Let $A \subseteq \{x_2, x_4, \ldots, x_{n-1}\}$ be a γ-set of the path $P = [x_2, x_4, \ldots, x_{n-1}]$. Then by Observation 4, $S = A \cup \{\overline{x_1}, \overline{x_n}\} \in \Gamma_s(G\overline{G})$. Hence, $\gamma_s(G\overline{G}) \leq |S| = 2 + \gamma(P_{n-2}) = 2 + \lceil \frac{n-2}{3} \rceil$. To get the other inequality, let $S \subseteq V(G\overline{G})$ be a γ_s-set of $G\overline{G}$. Since $deg_G(x) < deg_{\overline{G}}(\overline{x})$ for all $x \in V(G)$, Observation 4 implies that $S_{\overline{G}} \in \Gamma_s(\overline{G})$. In view of Lemma 2, we may assume $\overline{x_1} \in S_{\overline{G}}$. Since $V(\overline{G}) \setminus \{\overline{x_2}\} \preccurlyeq_{\overline{G}} \overline{x_1}$, $|S_{\overline{G}}| \geq 2$. In case $\overline{x_2} \in S_{\overline{G}}$, choose $x = x_2$. Otherwise, choose $x \in V(G)$ such that $\overline{x} \in S_{\overline{G}} \setminus \{\overline{x_1}\}$ and $\overline{x_2} \preccurlyeq \overline{x}$. If $x \notin S_G$, then $S^* = S \setminus \{\overline{x}, \overline{x_1}\}$ strongly dominates $V(G) \setminus \{x, x_1\}$. In this case, $|S| \geq 2 + \gamma_s(P_{n-2})$. If $x \in S_G$, then $S \setminus \{\overline{x}, \overline{x_1}\}$ strongly dominates $V(G) \setminus \{x_1\}$ so that $|S| \geq 2 + \gamma_s(P_{n-1}) \geq 2 + \gamma_s(P_{n-2})$. In any case, $\gamma_s(G\overline{G}) \geq 2 + \gamma_s(P_{n-2}) = 2 + \lceil \frac{n-2}{3} \rceil$.

For $\gamma_w(G\overline{G})$, the case where $3 \leq n \leq 6$ can be readily verified. Let $n \geq 7$. Put $S = \{x_1, \overline{x_2}, x_3, x_6, \ldots, x_{3k}\}$ whenever $n = 3k$; otherwise write $S = \{x_1, \overline{x_2}, x_3, x_6, \ldots, x_{3\lfloor \frac{n}{3} \rfloor}, x_n\}$. By Observation 4, S is a weak dominating set of $G\overline{G}$. Thus, $\gamma_w(G\overline{G}) \leq |S| = 2 + \lceil \frac{n}{3} \rceil$. Now, let $S \subseteq V(G\overline{G})$ be a γ_w-set of $G\overline{G}$. Because $deg_G(x) < deg_{\overline{G}}(\overline{x})$ for all $x \in V(G)$, $S_G \in \Gamma_w(G)$. Hence, $\gamma_w(G\overline{G}) = |S| \geq |S_{\overline{G}}| + \gamma_w(G)$. Moreover, since $n \geq 7$, $|S| \leq 2 + \lceil \frac{n}{3} \rceil \leq n - 2$, and consequently, $S_{\overline{G}} \neq \varnothing$. If $n = 1 (\text{mod } 3)$, then $|S_{\overline{G}}| \geq 2$. In view of Observation 3, in any case, $\gamma_w(G\overline{G}) = |S| \geq 2 + \lceil \frac{n}{3} \rceil$.

For (ii), the case where $4 \leq n \leq 5$ can be readily verified. Similar arguments used in the proof of statement (i) will prove the case where $n \geq 6$.

4. In the join of graphs

Remark 5. *Let G and H be connected graphs of orders m and n, respectively. Then*

(i) *For $u, v \in V(G)$, $u \preccurlyeq_{G+H} v$ if and only if $u \preccurlyeq_G v$.*

(ii) *For $v \in V(G)$ and $u \in V(H)$, $u \preccurlyeq_{G+H} v$ if and only if $deg_G(v) + n \geq deg_H(u) + m$.*

Theorem 1. *Let G, H be connected graphs of orders m and n, respectively. Let $S \subseteq V(G + H)$. Then $S \in \Gamma_s(G + H)$ if and only if one of the following holds:*

(i) *$S \subseteq V(G)$ such that $S \in \Gamma_s(G)$ and S contains a vertex v for which $deg_G(v) \geq \Delta(H) + m - n$.*

(ii) *$S \subseteq V(H)$ such that $S \in \Gamma_s(H)$ and S contains a vertex v for which $deg_H(v) \geq \Delta(G) + n - m$.*

(iii) *$S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$ and one of the following holds for each $u \in V(G + H) \setminus S$:*

(a) *$u \in V(G)$ and $u \preccurlyeq_G S_G$ or there exists $v \in S_H$ for which $deg_H(v) \geq deg_G(u) + n - m$;*

(b) *$u \in V(H)$ and $u \preccurlyeq_H S_H$ or there exists $v \in S_G$ for which $deg_G(v) \geq deg_H(u) + m - n$.*

Proof. Assume that $S \in \Gamma_s(G + H)$. Suppose that $S \subseteq V(G)$. By Observation 5(i), $S \in \Gamma_s(G)$. Pick $u \in V(H)$ for which $deg_H(u) = \Delta(H)$. Since $S \in \Gamma_s(G + H)$ there exists $v \in S$ for which $u \preccurlyeq_{G+H} v$. By Observation 5(ii), $deg_G(v) + n \geq \Delta(H) + m$ or, equivalently, $deg_G(v) \geq \Delta(H) + m - n$. Similarly, if $S \subseteq V(H)$, then (ii) holds. Next, suppose that S intersects both $V(G)$ and $V(H)$. Let $u \in V(G+H) \setminus S$, and suppose that $u \in V(G)$. Then there exists $v \in S$ for which $u \preccurlyeq_{G+H} v$. If $v \in V(G)$, then $u \preccurlyeq_G v$. This means that $u \preccurlyeq_G S_G$. If $v \in V(H)$, then by Observation 5(ii), $deg_H(v) \geq deg_G(u)+n-m$, and (a) holds. Similarly, if $u \in V(H)$, then (b) holds.

Conversely, suppose that (i) holds for S. Let $u \in V(G + H) \setminus S$. If $u \in V(G)$, then $u \preccurlyeq_G S$, and hence $u \preccurlyeq_{G+H} S$. Suppose that $u \in V(H)$. There exists $v \in S$ for which $deg_G(v) \geq \Delta(H) + m - n$. This means

$$deg_{G+H}(u) = deg_H(u) + m \leq \Delta(H) + m \leq deg_G(v) + n = deg_{G+H}(v).$$

Thus, $u \preccurlyeq_{G+H} v$, and consequently, $u \preccurlyeq_{G+H} S$. Accordingly, $S \in \Gamma_s(G+H)$. Similarly, if (ii) holds, then $S \in \Gamma_s(G+H)$. Finally, suppose that (iii) holds for S. Let $u \in V(G) \setminus S$. If $u \preccurlyeq_G S_G$, then $u \preccurlyeq_{G+H} S$. Suppose that S_G does not strongly dominate u. By condition (a), there exists $v \in S_H$ for which $deg_H(v)+m \geq deg_G(u)+n$. This means that $deg_{G+H}(v) \geq deg_{G+H}(u)$ and $u \preccurlyeq_{G+H} v$. Thus, $u \preccurlyeq_{G+H} S$. Similarly, if $u \in V(H) \setminus S$ and S_H does not strongly dominate u, then there exists $v \in S_G$ for which $u \preccurlyeq_{G+H} v$, and therefore $u \preccurlyeq_{G+H} S$. Therefore, $S \in \Gamma_s(G + H)$.

Corollary 1. *Let G and H be connected graphs of orders m and n, respectively.*

(i) *$\gamma_s(G + H) = 1$ if and only if $\gamma(G) = 1$ or $\gamma(H) = 1$.*

(ii) *Assume $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$.*

  (a) *If $\Delta(G) + n = \Delta(H) + m$, then $\gamma_s(G + H) = 2$.*

  (b) *If $\Delta(G) + n > \Delta(H) + m$, then*

$$\gamma_s(G + H) = \min\{\gamma_s(G), 1 + \gamma_s(K)\},$$

where $K = \langle V(G) \setminus N_{G+H}[u \succcurlyeq] \rangle$ and $u \in V(H)$ for which $deg_H(u) = \Delta(H)$.

Proof. Statement (i) follows immediately from Theorem 1(i). Assume that $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Suppose that $\Delta(G) + n = \Delta(H) + m$. Pick $u \in V(G)$ and $v \in V(H)$ such that $deg_G(u) = \Delta(G)$ and $deg_H(v) = \Delta(H)$. Since $S = \{u, v\}$ satisfies condition (iii) of Theorem 1, $S \in \Gamma_s(G+H)$. In this case, $\gamma_s(G+H) = |S| = 2$ and $(ii)(a)$ holds. To prove $(ii)(b)$, suppose that $\Delta(G) + n > \Delta(H) + m$. First, let $S \subseteq V(G)$ be a γ_s-set of G. By

Lemma 2, there exists $v \in S$ such that $deg_G(v) = \Delta(G)$. Then $deg_G(v) + n > \Delta(H) + m$. By Theorem 1, $S \in \Gamma_s(G + H)$. Thus,

$$\gamma_s(G + H) \leq |S| = \gamma_s(G).$$

Next, pick $u \in V(H)$ for which $deg_H(u) = \Delta(H)$. Since $\Delta(G) + n > \Delta(H) + m$, $V(G) \setminus N_{G+H}[u \not\succcurlyeq] \neq \varnothing$. Let $K = \langle V(G) \setminus N_{G+H}[u \not\succcurlyeq] \rangle$. Choose a γ_s-set S^* of K. Put $S = S^* \cup \{u\}$. Since S satisfies Theorem 1(iii), $S \in \Gamma_s(G + H)$. Thus,

$$\gamma_s(G + H) \leq |S| = 1 + \gamma_s(K).$$

Therefore, $\gamma_s(G + H) \leq \min\{\gamma_s(G), 1 + \gamma_s(K)\}$. Now, let $S \subseteq V(G + H)$ be a γ_s-set of $G + H$. By Lemma 2, $S \not\subseteq V(H)$. If $S \subseteq V(G)$, then by Theorem 1, $S \in \Gamma_s(G)$, showing $\gamma_s(G) \leq |S| = \gamma_s(G + H)$. Suppose that $S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$. Since $\Delta(G) + n > \Delta(H) + m$, $V(G) \setminus N_{G+H}[S_H \not\succcurlyeq] \neq \varnothing$. Put $K = \langle V(G) \setminus N_{G+H}[S_H \not\succcurlyeq] \rangle$. We claim that S_G is a strong dominating set of K. Let $u \in V(K) \setminus S_G$. Since $u \in V(G) \setminus S$, there exist $v \in S$ for which $uv \in E(G + K)$ and $u \preccurlyeq_{G+K} v$. Because u is not strongly dominated by S_H in $G + H$, $v \in S_G$. Since u is arbitrary, S_G is a strong dominating set of K. Thus, $\gamma_s(K) \leq |S_G|$. Since $|S_H| \geq 1$, $\gamma_s(G + H) = |S_G| + |S_H| \geq 1 + \gamma_s(K)$. The above results imply that $\gamma_s(G + H) = \min\{\gamma_s(G), 1 + \gamma_s(K)\}$.

Verify that $\gamma_s(P_4 + P_6) = \gamma_s(P_4) = 2$. On the other hand, if G is as in Figure 1, then $\gamma_s(G + P_8) = 1 + \gamma_s(K) = 2$, where $K = \langle V(G) \setminus N_{G+P_8}[u \not\succcurlyeq] \rangle = \langle \{v\} \rangle$ and $u \in V(P_8)$ with $deg_{P_8}(u) = 2$.

G :

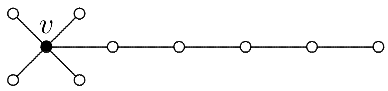

Figure 1: Graph G

The following versions for weak domination follow similar proofs.

Theorem 2. *Let G, H be connected graphs of orders m and n, respectively. Let $S \subseteq V(G + H)$. Then $S \in \Gamma_w(G + H)$ if and only if one of the following holds:*

(i) *$S \subseteq V(G)$ such that $S \in \Gamma_w(G)$ and S contains a vertex v for which $deg_G(v) \leq \delta(H) + m - n$.*

(ii) *$S \subseteq V(H)$ such that $S \in \Gamma_w(H)$ and S contains a vertex v for which $deg_H(v) \leq \delta(G) + n - m$.*

(iii) *$S_G = S \cap V(G) \neq \varnothing$ and $S_H = S \cap V(H) \neq \varnothing$ and one of the following holds for each $u \in V(G + H) \setminus S$:*

(a) *$u \in V(G)$ and $u \not\succcurlyeq_G S_G$ or there exists $v \in S_H$ for which $deg_H(v) \leq deg_G(u) + n - m$;*

(b) *$u \in V(H)$ and $u \succcurlyeq_H S_H$ or there exists $v \in S_G$ for which $deg_G(v) \leq deg_H(u) + m - n$.*

Corollary 2. *Let G and H be connected graphs of orders m and n, respectively. Assume $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Then the following holds:*

(a) *If $\delta(G) + n = \delta(H) + m$, then $\gamma_w(G + H) = 2$.*

(b) *If $\delta(G) + n < \delta(H) + m$, then*

$$\gamma_w(G + H) = \min\{\gamma_w(G), 1 + \gamma_w(K)\},$$

where $K = \langle V(G) \setminus N_{G+H}[u \preccurlyeq] \rangle$ and $u \in V(H)$ for which $deg_H(u) = \delta(H)$.

Example 1. *For $m, n \geq 4$,*

(1) $\gamma_s(P_m + P_n) = \begin{cases} 2, & \text{if } m = n; \\ \lceil \frac{m}{3} \rceil, & \text{if } m < n, \end{cases}$

(2) $\gamma_w(P_m + P_n) = \begin{cases} 2, & \text{if } m = n; \\ 3, & \text{if } m = n + 1; \\ \lceil \frac{m}{3} \rceil, & \text{if } m \geq n + 2; m \equiv 1 \pmod 3; \\ \lceil \frac{m}{3} \rceil + 1, & \text{if } m \geq n + 2; m \equiv 0, 2 \pmod 3, \end{cases}$

(3) $\gamma_s(C_m + C_n) = \begin{cases} 2, & \text{if } m = n; \\ \lceil \frac{m}{3} \rceil, & \text{if } m < n, \end{cases}$ $\quad \gamma_w(C_m + C_n) = \begin{cases} 2, & \text{if } m = n; \\ \lceil \frac{n}{3} \rceil, & \text{if } m < n; \end{cases}$

5. In the corona of graphs

Proposition 3. *Let G be a nontrivial connected graph and H any graph, and let $S \subseteq V(G \circ H)$. Then $S \in \Gamma_s(G \circ H)$ if and only if*

$$S = A \cup \left(\cup_{v \in V(G)} S_v\right), \tag{1}$$

where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ such that the following hold:

(i) *$A \in \Gamma_s(G)$; and*

(ii) *For each $v \in V(G) \setminus A$, $S_v \in \Gamma_s(H^v)$.*

Proof. Assume that $S \in \Gamma_s(G \circ H)$. Put $A = S \cap V(G)$ and $S_v = A \cap V(H^v)$ for all $v \in V(G)$. Then Equation 1 holds. To prove (i), let $v \in V(G) \setminus A$. There exists $u \in S$ for which $v \preccurlyeq_{G \circ H} u$. Since $deg_{G \circ H}(v) > deg_{G \circ H}(w)$ for all $w \in V(H^v)$, $u \notin S_v$. Hence, $u \in A$. This means that $v \preccurlyeq_G A$. Thus, $A \in \Gamma_s(G)$, and (i) holds. To prove (ii), let $v \in V(G) \setminus A$

and let $u \in V(H^v) \setminus S_v$. There exists $w \in S$ for which $u \preccurlyeq_{G\circ H} w$. Since $w \neq v$, $w \in S_v$. Thus, $u \preccurlyeq_{H^v} S_v$. Therefore, $S_v \in \Gamma_s(H^v)$, and (ii) holds.

Conversely, suppose that Equation 1 holds for S together with conditions (i) and (ii). Let $w \in V(G \circ H) \setminus S$, and let $v \in V(G)$ for which $w \in V(H^v + v)$. If $w \in V(H^v)$ and $v \in A$, then $w \preccurlyeq_{G\circ H} S$. On the other hand, if $w \in V(H^v)$ and $v \notin A$, then $w \preccurlyeq_{H^v} S_v$ by (ii) so that $w \preccurlyeq_{G\circ H} S$. Suppose that $w = v$. By (i), $w \preccurlyeq_G A$ and, consequently, $w \preccurlyeq_{G\circ H} A$. Thus, $w \preccurlyeq_{G\circ H} S$. Accordingly, $S \in \Gamma_s(G \circ H)$.

Proposition 4. *Let G be a nontrivial connected graph and H any graph, and let $S \subseteq V(G \circ H)$. Then $S \in \Gamma_w(G \circ H)$ if and only if*

$$S = A \cup \left(\cup_{v\in V(G)} S_v\right),$$

where $A \subseteq V(G)$ and $S_v \in \Gamma_w(H^v)$ for each $v \in V(G)$.

Proof. Suppose that $S \in \Gamma_w(G \circ H)$. Then $S = A \cup \left(\cup_{v\in V(G)} S_v\right)$, where $A = S \cap V(G)$ and $S_v = S \cap V(H^v)$ for each $v \in V(G)$. Let $v \in V(G)$, and let $x \in V(H^v) \setminus S_v$. There exists $y \in S$ for which $x \succcurlyeq_{G\circ H} y$. Since $deg_{G\circ H}(x) < deg_{G\circ H}(v)$, $y \in S_v$ and $x \succcurlyeq_{H^v} y$. Thus, $S_v \in \Gamma_w(H^v)$.

Conversely, suppose that $S = A \cup \left(\cup_{v\in V(G)} S_v\right)$, where $A \subseteq V(G)$ and $S_v \in \Gamma_w(H^v)$ for each $v \in V(G)$. Let $x \in V(G \circ H) \setminus S$, and let $v \in V(G)$ such that $x \in V(H^v + v)$. Note that $S_v \neq \varnothing$. If $x = v$, then pick any $w \in S_v$. Then $x \succcurlyeq_{G\circ H} w$. If $x \neq v$, then $x \in V(H^v) \setminus S_v$, and there exists $y \in S_v$ such that $x \succcurlyeq_{H^v} y$. This means $x \succcurlyeq_{G\circ H} y$. Therefore, $S \in \Gamma_w(G \circ H)$.

Corollary 3. *If G is a connected graph of order $n \geq 2$. Then*

- (i) [25] $\gamma_s(G \circ H) = n$ *for any graph H.*
- (ii) $\gamma_w(G \circ H) = n\gamma_w(H)$ *for any graph H.*

Proof. By Proposition 3, $V(G)$ is a strong dominating set of $G \circ H$. Thus, $\gamma_s(G\circ H) \leq n$. Now, let $S \subseteq V(G \circ H)$ be a strong dominating set of $G \circ H$, and let $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ be as provided in Proposition 3 such that $S = A \cup \left(\cup_{v\in V(G)} S_v\right)$. By Proposition 3$(ii)$, $|S_v| \geq 1$ for all $v \in V(G) \setminus A$. Thus, $|S| \geq |A| + |V(G) \setminus A| = n$. Since S is arbitrary, $\gamma_s(G \circ H) \geq n$.

Statement (ii) is easy.

5.1. In the edge corona of graphs

Given graphs G and H, we write H^{uv} to denote that copy of H that is being joined with the endvertices of the edge $uv \in E(G)$ in the edge corona $G \diamond H$. For $uv \in E(G)$, we write $H^{uv} + uv = H^{uv} + \langle\{u, v\}\rangle$. If $x \in V(H)$, then we write x^{uv} to denote the corresponding vertex in H^{uv}.

Proposition 5. *Let G and H be a nontrivial connected graphs with $\delta(G) \geq 2$ or $\gamma(H) \geq 2$, and let $S \subseteq V(G \diamond H)$. Then $S \in \Gamma_s(G \diamond H)$ if and only if*

$$S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right), \tag{2}$$

where $A \subseteq V(G)$ and $S_{uv} \subseteq V(H^{uv}$ satisfying the following:

(i) $A \in \Gamma_s(G)$; and

(ii) For each $uv \in E(G)$ for which $\{u, v\} \cap A = \varnothing$, $S_{uv} \in \Gamma_s(H^{uv})$.

Proof. Assume that S is strong dominating set of $G \circ H$. Put $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then Equation 2 holds. Let $v \in V(G) \setminus A$. There exists $u \in S$ for which $v \preccurlyeq_{G \diamond H} u$. If $\delta(G) \geq 2$ or $\gamma(H) \geq 2$, then $deg_{G \diamond H}(w) < deg_{G \diamond H}(v)$ for all $w \in V(H^{xv})$, for all $x \in N_G(v)$. Thus, $u \in A$. Hence, $u \preccurlyeq_G A$ and $A \in \Gamma_s(G)$. This proves (i). Now, let $uv \in E(G)$ with $u \notin A$ and $v \notin A$, and let $w \in V(H^{uv}) \setminus S_{uv}$. There exists $z \in S$ for which $w \preccurlyeq_{G \diamond H} z$. Clearly, $z \in S_{uv}$ so that $w \preccurlyeq S_{uv}$. Therefore, S_{uv} is a strong dominating set of H^{uv}.

Conversely, assume that (i) and (ii) hold for S. Let $v \in V(G \diamond H) \setminus S$. There exists $xy \in E(G)$ such that $v \in V(H^{xy} + xy)$. First, suppose that $v \in V(H^{xy})$. If $x \in A$, then $v \preccurlyeq x$. Similarly, if $y \in A$, then $v \preccurlyeq y$. In any case, $v \preccurlyeq S$. Suppose that $x, y \notin A$. By (ii), $v \preccurlyeq S_{xy}$. Thus $v \preccurlyeq S$. Next, suppose that $v = x$ or $v = y$. By (i), there exists $w \in A$ for which $v \preccurlyeq w$. Therefore, $v \preccurlyeq S$. Accordingly, S is a strong dominating set of $G \diamond H$.

The set A in Proposition 5 need not be a strong dominating set of G whenever $\delta(G) = 1 = \gamma(H)$. Consider the edge corona $P_5 \diamond P_3$ in Figure 2. The set S of darkened vertices is a strong dominating set of $P_5 \diamond P_3$. However $A = S \cap V(P_5)$ is not a strong dominating set of P_5.

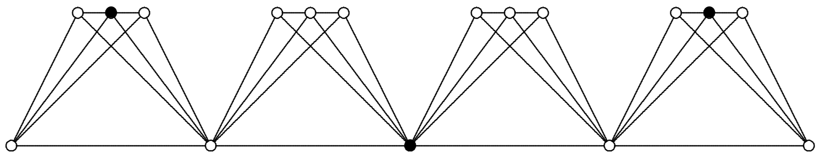

Figure 2: Graph $P_5 \diamond P_3$

Proposition 6. *Let G and H be a nontrivial connected graphs where $\gamma(H) \neq 1$, and let $S \subseteq V(G \diamond H)$. Then $S \in \Gamma_w(G \diamond H)$ if and only if*

$$S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right),$$

where $A \subseteq V(G)$ and $S_{uv} \in \Gamma_w(H^{uv})$ for each $uv \in E(G)$.

Proof. Assume that $S \in \Gamma_w(G \diamond H)$. Let $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for each $uv \in E(G)$. Then $S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right)$. Let $uv \in E(G)$ and let $x \in V(H^{uv}) \setminus S_{uv}$.

There exists $y \in S$ such that $x \succcurlyeq_{G\diamond H} y$. Since $\min\{deg_{G\diamond H}(u), deg_{G\diamond H}(v)\} > deg_{G\diamond H}(x)$, $y \notin \{u, v\}$. Thus, $y \in S_{uv}$ and $y \succcurlyeq_{H^{uv}} S_{uv}$. This shows that $S_{uv} \in \Gamma_w(H^{uv})$.

Conversely, Let $x \in V(G \diamond H) \setminus S$. Let $uv \in E(G)$ such that $x \in V(H^{uv} + uv)$. Since $S_{uv} \in \Gamma_w(H^{uv})$, $S_{uv} \neq \varnothing$. If $x \in \{u, v\}$, then $x \succcurlyeq_{G\diamond H} w$ for each $w \in S_{uv}$. Also, if $x \in V(H^{uv}) \setminus S_{uv}$, then there exists $w \in S_{uv}$ for which $x \succcurlyeq_{H^{uv}} w$. This means $x \succcurlyeq_{G\diamond H} w$. Therefore, $S \in \Gamma_w(G \diamond H)$.

For a nonempty $A \subseteq V(G)$, define

$$A^e = \{uv \in E(G) : u \notin A \text{ and } v \notin A\}.$$

Corollary 4. *Let G and H be connected graphs where G is nontrivial. Then*

(i) $\gamma_s(G \diamond H) = \min\{|A| + |A^e| : A \in \Gamma_s(G)\}$.

(ii) $\gamma_w(G \diamond H) = |E(G)|\gamma_w(H)$.

Proof. To prove (i), put $\alpha = \min\{|A| + |A^e| : A \in \Gamma_s(G)\}$. Let $n = |V(G)|$. If $n = 2$, then $\gamma_s(G \diamond H) = 1 = \alpha$. Assume that $n \geq 3$. We consider the following cases:

Case 1: Suppose that $\delta(G) \geq 2$ or $\gamma(H) \geq 2$.

Let $A \in \Gamma_s(G)$. For each $uv \in A^e$, denote by w_{uv} exactly one of u and v. Then w_{uv} strongly dominates $V(H^{uv})$. Define $A^* = A \cup \{w_{uv} : uv \in A^e\}$. Then $A^* \in \Gamma_s(G)$ with $(A^*)^e = \varnothing$ and $|\{w_{uv} : uv \in A^e\}| \leq |A^e|$. By Proposition 5, $A^* \in \Gamma_s(G \diamond H)$ so that $\gamma_s(G \diamond H) \leq |A^*| \leq |A| + |A^e|$. Since A is arbitrary, $\gamma_s(G \diamond H) \leq \alpha$.

To get the other inequality, let $S \subseteq V(G \diamond H)$ be a γ_s-set of $G \diamond H$. By Proposition 5, there exists $A \in \Gamma_s(G)$ such that $S = A \cup \left(\cup_{uv \in E(G)} S_{uv}\right)$, where $S_{uv} \in \Gamma_s(H^{uv})$ for each $uv \in A^e$. We have

$$\begin{aligned}\gamma_s(G \diamond H) = |S| &\geq |A| + \sum_{uv \in A^e} |S_{uv}| \\ &\geq |A| + |A^e|\gamma_s(H) \\ &\geq \alpha.\end{aligned}$$

Case 2: Suppose that $\delta(G) = 1 = \gamma(H)$.

For each $uv \in E(G)$, let $x^{uv} \in V(H^{uv})$ such that $N_{H^{uv}}[x^{uv}] = V(H^{uv})$. It is worth noting that if $u \in End(G)$ or $v \in End(G)$, say $u \in End(G)$, then $deg_{G\diamond H}(x^{uv}) = deg_{G\diamond H}(u)$. Also, $deg_{G\diamond H}(x^{uv}) < deg_{G\diamond H}(y)$ for all $y \in \{u, v\} \setminus End(G)$ and that $\{x^{uv}\} \in \Gamma_s(H^{uv})$. Choose $A \in \Gamma_s(G)$ such that $|A| + |A^e| = \alpha$. Construct an $A^* \in \Gamma_s(G)$ as in the proof of Lemma 1 such that $|A^*| \leq |A|$ and $A^* \cap End(G) = \varnothing$. More precisely, $A^* = (A \setminus End(G)) \cup \{x_v \notin A : v \in A \cap End(G)\}$. Define $S^* = A^* \cup (\cup_{uv \in A^e}\{x^{uv}\})$. Then $S^* \in \Gamma_s(G \diamond H)$. Thus, $\gamma_s(G \diamond H) \leq |S^*| \leq |A| + |A^e| = \alpha$.

To get the other inequality, let $S \subseteq V(G \diamond H)$ be a γ_s-set of $G \diamond H$. Put $A = S \cap V(G)$ and $S_{uv} = S \cap V(H^{uv})$ for all $uv \in E(G)$. Let $A^e = \{uv \in E(G) : u, v \notin A\}$. If $uv \in A^e$, then S_{uv} is a γ_s-set of H^{uv}. Hence, $S_{uv} = \{x^{uv}\}$ where $x^{uv} \in V(H^{uv})$ such that $N_{H^{uv}}[x^{uv}] = V(H^{uv})$. Thus,

$$\begin{aligned} \gamma_s(G \diamond H) = |S| &= |A| + \sum_{uv \in A^e} |S_{uv}| + \sum_{uv \in E(G) \setminus A^e} |S_{uv}| \\ &= |A| + |A^e| + \sum_{uv \in E(G) \setminus A^e} |S_{uv}| \\ &\geq |A| + |A^e| = \alpha. \end{aligned}$$

To prove (ii), note that if $\gamma(H) \neq 1$, then it follows from Proposition 6 that $S \subseteq V(G \diamond H)$ is a γ_w-set of $G \diamond H$ if and only if $S = \cup_{uv \in E(G} S_{uv}$, where S_{uv} is a γ_w-set of H^{uv} for each $uv \in E(G)$. In this case, $\gamma_w(G \diamond H) = |E(G)|\gamma_w(H)$. Now suppose that $\gamma(H) = 1$. If H is complete, then $\gamma_w(G \diamond H) = |E(G)| = |E(G)|\gamma_w(H)$. Assume H is not complete. In view of Lemma 2, S contains x^{uv} for which $\delta(H^{uv}) = deg_{H^{uv}}(x^{uv})$ for each $uv \in E(G)$. Since S is a γ_w-set and $u \not\succcurlyeq_{G \diamond H} x^{uv}$ and $v \not\succcurlyeq_{G \diamond H} x^{uv}$, $\{u, v\} \cap S_{uv} = \varnothing$ for all $uv \in E(G)$. Consequently, $S = \cup_{uv \in E(G)} S_{uv}$ where S_{uv} is a γ_s-set of H^{uv}. Therefore, $\gamma_s(G \diamond H) = |S| = |E(G)|\gamma_w(H)$.

The value of $\gamma_s(G \diamond H)$ in Corollary 4 is not necessarily determined by a γ_s-set A of G. Observe that $\gamma_s(C_6 \diamond H) = 3$ for any connected graph H, and is not determined by any γ_s-set of C_6.

Example 2. *Let G be any graph. For positive integers $n \geq 2$ and $m \geq 3$,*

(i) [25] $\gamma_s(P_n \diamond G) = \lfloor \frac{n}{2} \rfloor$ *and* $\gamma_s(C_m \diamond G) = \lceil \frac{m}{2} \rceil$;

(ii) $\gamma_w(P_n \diamond G) = (n+1)\gamma_s(G)$ *and* $\gamma_w(C_m \diamond G) = m\gamma_w(G)$.

6. In the lexicographic product of graphs

Here we note that for $(u, v) \in V(G[H])$,

$$deg_{G[H]}((u, v)) = |V(H)|deg_G(u) + deg_H(v).$$

For $S \subseteq V(G[H])$, the *projection of S with respect to G* refers to the set $S_G = \{x \in V(G) : \exists y \in V(H) \text{ for which } (x, y) \in S\}$.

If $S_1 \in \Gamma_s(G)$ (resp. $S_1 \in \Gamma_w(G)$) and $S_2 \in \Gamma_s(H)$ (resp. $S_2 \in \Gamma_w(H)$), then $S_1 \times S_2 \in \Gamma_s(G[H])$ (resp. $S_1 \times S_2 \in \Gamma_w(G[H])$). Consequently,

$$\gamma_s(G[H]) \leq \gamma_s(G)\gamma_s(H)$$

$$(\text{resp. } \gamma_w(G[H]) \geq \gamma_w(G)\gamma_w(H)).$$

Proposition 7. *Let G and H be nontrivial connected graphs, and let $S \in \Gamma_s(G[H])$. Then $S = \cup_{x\in A} (\{x\} \times T_x)$, where $A \subseteq V(G)$ and $T_x \subseteq V(H)$ satisfying the following.*

(*i*) *$A \in \Gamma_s(G$; and*

(*ii*) *For each $x \in A \setminus N_G(A \succcurlyeq)$, $T_x \in \Gamma_s(H)$.*

Proof. Put $A = S_G$, the projection of S under G. For each $x \in A$, define $T_x = \{y \in V(H) : (x,y) \in S\}$. Then $S = \cup_{x\in A} (\{x\} \times T_x)$. Let $x \in V(G) \setminus A$, and pick $z \in V(H)$ such that $deg_H(z) = \Delta(H)$. Since $(x,z) \notin S$, there exists $(u,v) \in S$ for which $(x,z) \preccurlyeq_{G[H]} (u,v)$. Since $x \neq u$, $u \in A \cap N_G(x)$. Moreover, since $(x,z) \preccurlyeq_{G[H]} (u,v)$ and $deg_H(z) \geq deg_H(v)$, $deg_G(x) \leq deg_G(u)$, i.e., $x \preccurlyeq_G u$. Thus, $A \in \Gamma_s(G)$, and (*i*) holds. To show (*ii*), put $n = |V(H)|$ and let $x \in A \setminus N_G(A \succcurlyeq)$. We claim that $T_x \in \Gamma_s(H)$. To this end, let $y \in V(H) \setminus T_x$. Since $(x,y) \notin S$, there exists $(u,v) \in S$ for which $(x,y) \preccurlyeq_{G[H]} (u,v)$. If $x \neq u$, then since $x \notin N_G(A \succcurlyeq)$, $deg_G(u) < deg_G(x)$. Thus, $n \leq n[deg_G(x) - deg_G(u)] \leq deg_H(v) - deg_H(y)$, which is impossible. Thus, $x = u$ so that $v \in T_x \cap N_H(y)$, and necessarily, $y \preccurlyeq_H v$. Accordingly, $T_x \in \Gamma_s(H)$.

Proposition 8. *Let G and H be connected nontrivial graphs, and $S = \cup_{x\in A} (\{x\} \times T_x)$, where $A \subseteq V(G)$ and $T_x = \{y \in V(H) : (x,y) \in S\}$. Suppose that each of the following holds for A:*

(*i*) *$A \in \Gamma_s(G)$;*

(*ii*) *For each $x \in A \cap N_G(A \succcurlyeq)$, there exists $y \in T_x$ such that $deg_H(y) = \Delta(H)$; and*

(*iii*) *For each $x \in A \setminus N_G(A \succcurlyeq)$, $T_x \in \Gamma_s(H)$.*

Then $S \in \Gamma_s(G[H])$.

Proof. Let $(x,y) \in V(G[H]) \setminus S$. We consider the following cases:

Case 1: $x \notin A$

If $x \notin A$, then by (*i*), there exists $u \in A$ such that $x \preccurlyeq_G u$. If $u \notin N_G(A \succcurlyeq)$, then by (*iii*), $T_u \in \Gamma_s(H)$ so that, by Lemma 2, T_u contains a vertex w for which $deg_H(w) = \Delta(H)$. Here we have $(u,w) \in S \cap N_{G[H]}((x,y))$ and $(x,y) \preccurlyeq_{G[H]} (u,w)$. Suppose that $u \in N_G(A \succcurlyeq)$. Then by (*ii*), there exists $v \in T_u$ for which $deg_H(v) = \Delta(H)$. It means $(u,v) \in S \cap N_{G[H]}((x,y))$ and $(x,y) \preccurlyeq_{G[H]} (u,v)$.

Case 2: $x \in A \cap N_G(A \succcurlyeq)$

If $x \in A \cap N_G(A \succcurlyeq)$ and $u \in A$ such that $x \preccurlyeq_G u$, then by (*ii*), there exists $v \in T_u$ such that $deg_H(v) = \Delta(H)$. Thus, $(u,v) \in S \cap N_{G[H]}((x,y))$ and $(x,y) \preccurlyeq_{G[H]} (u,v)$.

Case 3: $x \in A \setminus N_G(A \succcurlyeq)$

Suppose that $x \in A \setminus N_G(A \succcurlyeq)$. Then $T_x \in \Gamma_s(H)$. Thus, there exists $w \in T_x$ for which $y \preccurlyeq_H w$. Here we have $(x,w) \in S \cap N_{G[H]}((x,y))$ and $(x,y) \preccurlyeq_{G[H]} (x,w)$.

All 3 cases above imply that $S \in \Gamma_s(G[H])$.

Similar arguments will also prove the following two propositions for weak domination:

Proposition 9. *Let G and H be nontrivial connected graphs, and let $S \in \Gamma_w(G[H])$). Then $S = \cup_{x\in A}(\{x\} \times T_x)$, where $A \subseteq V(G)$ and $T_x \subseteq V(H)$ satisfying the following.*

(i) $A \in \Gamma_w(G)$*; and*

(ii) *For each* $x \in A \setminus N_G(A \preccurlyeq)$, $T_x \in \Gamma_w(H)$.

Proposition 10. *Let G and H be connected nontrivial graphs, and $S = \cup_{x\in A}(\{x\} \times T_x)$, where $A \subseteq V(G)$ and $T_x = \{y \in V(H) : (x, y) \in S\}$. Suppose that each of the following holds for A:*

(i) $A \in \Gamma_w(G)$*;*

(ii) *For each* $x \in A \cap N_G(A \preccurlyeq)$*, there exists* $y \in T_x$ *such that* $deg_H(y) = \delta(H)$*; and*

(iii) *For each* $x \in A \setminus N_G(A \preccurlyeq)$, $T_x \in \Gamma_w(H)$.

Then $S \in \Gamma_w(G[H])$.

Proposition 11. *Let G and H be nontrivial connected graphs. Then for each $S^* \in \Gamma_s(G[H])$, there exists $S = \cup_{x\in A}(\{x\} \times T_x) \in \Gamma_s(G[H])$ satisfying the following:*

(i) $|S| = |S^*|$*;*

(ii) *For each* $x \in A \cap N_G(A \succcurlyeq)$*, there exists* $y \in T_x$ *for which* $deg_H(y) = \Delta(H)$.

Proof. Let $v \in V(H)$ such that $deg_H(v) = \Delta(H)$. Write $S^* = \cup_{x\in A^*}(\{x\} \times T_x^*)$, where $A^* \in \Gamma_s(G)$ and $T_x^* \in \Gamma_s(H)$ for each $x \in A^* \setminus N_G(A^* \succcurlyeq)$. For each $x \in A^* \cap N_G(A^* \succcurlyeq)$, let $y_x \in T_x$. Define the following:

- $A = A^*$;
- $T_x = T_x^*$ for each $x \in A^* \setminus N_G(A^* \succcurlyeq)$; and
- $T_x = (T_x^* \setminus \{y_x\}) \cup \{v\}$ for each $x \in A^* \cap N_G(A^* \succcurlyeq)$.

By Proposition 8, $S = \cup_{x\in A}(\{x\} \times T_x) \in \Gamma_s(G[H])$. Moreover, by the construction of S, $|S| = |S^*|$.

Corollary 5. *Let G and H be nontrivial connected graphs. Then*

$$\gamma_s(G[H]) = \min\{|A \cap N_G(A \succcurlyeq)| + \gamma_s(H)|A \setminus N_G(A \succcurlyeq)| : A \in \Gamma_s(G)\},$$

and

$$\gamma_w(G[H]) = \min\{|A \cap N_G(A \preccurlyeq)| + \gamma_w(H)|A \setminus N_G(A \preccurlyeq)| : A \in \Gamma_w(G)\}.$$

Proof. Put $\alpha = \min\{|A \cap N_G(A \succcurlyeq)| + \gamma_s(H)|A \setminus N_G(A \succcurlyeq)| : A \in \Gamma_s(G)\}$. Let $v \in V(H)$ such that $deg_H(v) = \Delta(H)$, and let $A \in \Gamma_s(G)$ such that $\alpha = |A| + \gamma_s(H)|A \setminus N_G(A \succcurlyeq)|$. For each $x \in A \cap N_G(A \succcurlyeq)$, let $T_x = \{v\}$ and for each $x \in A \setminus N_G(A \succcurlyeq)$, let $T_x \subseteq V(H)$ be a γ_s-set of H. By Proposition 8, $S = \cup_{x \in A} (\{x\} \times T_x) \in \Gamma_s(G[H])$. Thus, $\gamma_s(G[H]) \leq |S| = |A \cap N_G(A \succcurlyeq)| + \gamma_s(H)|A \setminus N_G(A \succcurlyeq) = \alpha$.

Let $S = \cup_{x \in A} (\{x\} \times T_x)$ be a γ_s-set of $G[H]$. In view of Proposition 11, $S = \cup_{x \in A} (\{x\} \times T_x)$ with $A \subseteq V(G)$ and $T_x \subseteq V(H)$, where $A \in \Gamma_s(G)$, $T_x \neq \varnothing$ for each $x \in A \cap N_G(A \succcurlyeq)$, and $T_x \in \Gamma_s(H)$ for each $x \in A \setminus N_G(A \succcurlyeq)$. Thus,

$$\gamma_s(G[H]) = |S| \geq |A \cap N_G(A \succcurlyeq)| + \gamma_s(H)|A \setminus N_G(A \succcurlyeq)| \geq \alpha.$$

Proof for the weak domination case is similar.

Corollary 6. *The following hold for nontrivial connected graphs G and H:*

(i) *If $\gamma(H) = 1$, then $\gamma_s(G[H]) = \gamma_s(G)$.*

(ii) *If H is a regular graph, then*

$$\gamma_s(G[H]) = \min\{|A \cap N_G(A \succcurlyeq)| + \gamma(H)|A \setminus N_G(A \succcurlyeq)| : A \in \Gamma_s(G)\}$$

and

$$\gamma_w(G[H]) = \min\{|A \cap N_G(A \preccurlyeq)| + \gamma(H)|A \setminus N_G(A \preccurlyeq)| : A \in \Gamma_w(G)\}.$$

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