EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2026, Vol. 19, Issue 1, Article Number 6855
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Variable-Order Fractional Delay Differential Equations with Integral Boundary Values: A Study on Existence, Uniqueness, and Stability

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Abstract. Variable-order fractional differential equations (VO-FDEs) significantly generalize classical fractional calculus by permitting the order of differentiation to vary with time or other system parameters. This flexibility offers a powerful framework for modeling complex phenomena characterized by evolving memory and dynamic heterogeneities, features that are beyond the reach of constant-order models. In this work, we conduct a rigorous analysis of a coupled system of VO-FDEs that incorporates multiple delays and nonlocal integral boundary conditions within the Caputo formalism. Our investigation first establishes sufficient criteria for the existence and uniqueness of solutions using Banach's and Schauder's fixed-point theorems. We then perform a comprehensive stability analysis, deriving explicit conditions for Ulam–Hyers and Ulam–Hyers–Rassias stability to guarantee the robustness of solutions against small perturbations. The practical applicability of our theoretical findings is demonstrated through detailed numerical examples, which serve to validate the efficacy of the proposed framework.

2020 Mathematics Subject Classifications: 34A07, 34A08, 60G22

Key Words and Phrases: Systems of delay differential equations, integral boundary values, existence, uniqueness and stability, Caputo variable order derivative.

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DOI: https://doi.org/10.29020/nybg.ejpam.v19i1.6855

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1. Introduction

Fractional calculus, a branch of mathematics concerned with derivatives and integrals of arbitrary real or complex order, has a history dating back to 1695, shortly after the inception of classical calculus. For centuries, it was largely regarded as a purely theoretical construct. However, this perception has shifted dramatically in recent decades, as Fractional calculus has proven to be exceptionally powerful for modeling the behavior of complex physical systems [1]. Many natural phenomena exhibit nonlocal properties, meaning their present state is influenced by their entire history, not just immediate past events.

Fractional operators provide a more natural and precise framework for capturing these hereditary effects than classical integer-order derivatives, for two primary reasons. First, the order of a fractional derivative is not restricted to integers, offering a continuous parameter to fit observed data. Second, these non-integer operators inherently incorporate memory, as their definitions depend on an integration over the past.

Various fractional operators have been extensively studied in the literature [2–4]. Among the most commonly used are the Caputo [5, 6], conformable [7, 8], Hadamard [9], Atangana–Baleanu [10], and Katugampola fractional operators [11].

Recent advancements have further generalized this concept to variable-order fractional derivatives (VO-FD), where the order itself can be a function of time, space, or other system parameters. This allows for the modeling of processes with dynamically evolving memory and heterogeneities. Consequently, VO-FD have emerged as a vital mathematical tool across diverse scientific and engineering domains, including anomalous diffusion, viscoelasticity, control theory, and petroleum engineering [12, 13].

Let $\mathrm{Q}(T)$ be the space of all integrable functions on T. In the Riemann–Liouville sense, the variable-order fractional integral of a function $x \in \mathrm{Q}(T)$ is defined as [14]:

$$\mathcal{I}_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(s)-1} x(s) ds, \tag{1}$$

where $\alpha(t) : T \to (0,1]$ is a continuous function.

The Caputo of VO-FD is defined as [14]:

$$D_t^{\alpha(t)} x(t) = \frac{1}{\Gamma(v-\alpha(t))} \int_0^t (t-s)^{v-\alpha(s)-1} x^{(v)}(s) ds, \ v \in N. \tag{2}$$

Let $x \in Q(T)$. For a fractional order $\alpha \in (0,1]$, the following relation holds [15]:

$$\mathcal{I}_t^{\alpha(t)} D_t^{\alpha(t)} x(t) = x(t) - x(0). \tag{3}$$

The conceptual foundation for VO-FD and integrals was first laid by Samko and Ross in 1993, who also established several of their fundamental properties [16]. This framework, which allows the fractional order to vary as a function of variable, provides a more powerful means of modeling processes with dynamically evolving memory effects across time or space. This pioneering work spurred significant research across various fields. Lorenzo and Hartley later applied VO-FD to describe complex diffusion processes

[17], while the solutions to Laplacian equations within this framework were investigated in [18]. Heydari contributed to the field by solving the nonlinear diffusion-wave equation using a VO-FD approach, developing a unique and convergent iterative series via the contraction mapping principle [19]. The critical questions of existence and uniqueness for solutions to VO-FD Laplacian equations were addressed by Chen et al. [20]. From a numerical perspective, the authors of [21] employed Legendre wavelet functions to solve a class of nonlinear VO-FDEs, and a similar approach using shifted Legendre operational matrices was applied to multi-term VO-FDEs in [22]. The utility of VO-FD also extends to applied physics and biology; for instance, Coimbra et al. utilized them to study viscoelasticity oscillators [23], and Sweilam and Al-Mekhlafi developed a novel multi-strain tuberculosis model based on the VO-FD concept [24].

The literature concerning the existence and uniqueness of solutions for VO-FDEs remains relatively limited. Establishing the existence and uniqueness of solutions is a critical prerequisite for the reliable application of VO-FD models, as it ensures their mathematical well-posedness and physical validity.

Beyond existence and uniqueness, stability analysis is essential for dynamic systems. While stability in the Lyapunov, Mittag-Leffler, and exponential senses has been thoroughly established for classical fractional models, recent research has increasingly focused on Hyers-Ulam stability and its variants.

Most of the current research on the existence, uniqueness, and stability of solutions focuses on constant-order fractional differential equations. This study seeks to fill that gap by starting a thorough investigation into the variable-order case. We first establish sufficient conditions for the existence and uniqueness of solutions by employing Banach's and Schauder's fixed-point theorems. Thereafter, we conduct a detailed stability analysis and establish clear criteria for Ulam–Hyers and Ulam–Hyers–Rassias stability. This ensures the stability of the solutions, even under minor modifications. Finally, to illustrate the practical relevance of our theoretical results, we present numerical examples that confirm the effectiveness and validity of the proposed framework.

We consider the following system of variable-order fractional delay differential equations with nonlocal integral boundary conditions.

$$\begin{cases} D_t^{\alpha(t)}x(t) = U\left(t, x(t-\phi), x(ht), y(ht), z(ht)\right), & h \in (0,1),\ \alpha(t) \in (0,1], \\ D_t^{\beta(t)}y(t) = U\left(t, y(t-\phi), y(ht), x(ht), z(ht)\right), & h \in (0,1),\ \beta(t) \in (0,1], \\ D_t^{\gamma(t)}z(t) = U\left(t, z(t-\phi), z(ht), x(ht), y(ht)\right), & h \in (0,1),\ \gamma(t) \in (0,1], \\ x(0) = \int_0^{\psi} \frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)} g\left(x(s), y(s), z(s)\right) ds + x_0, & \varkappa \in (0,1], \\ y(0) = \int_0^{\psi} \frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)} g\left(y(s), x(s), z(s)\right) ds + y_0, & \delta \in (0,1], \\ z(0) = \int_0^{\psi} \frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)} g\left(z(s), x(s), y(s)\right) ds + z_0, & \lambda \in (0,1], \end{cases} \tag{4}$$

where $T = [0, \psi]$ represents the time domain, $U : T \times R^4 \to R$ is the nonlinear function governing the system dynamics, $g : T \times T \times T \to R$ defines the nonlocal initial conditions, and $D_t^{\alpha(t)}, D_t^{\beta(t)}, D_t^{\gamma(t)}$ denote the Caputo-type variable-order fractional derivatives of orders $\alpha(t), \beta(t), \gamma(t)$, respectively.

The given system of variable-order fractional delay differential equations with nonlocal integral boundary conditions has wide applications in modeling complex real-world phenomena that exhibit memory, hereditary effects, and time-varying dynamics. It can be applied in physics and engineering to describe viscoelastic materials, electrical circuits with fractance, and control systems with time-dependent damping, as well as in biology and medicine to model population dynamics, epidemiological processes with incubation delays, and pharmacokinetics with cumulative drug effects. In addition, it is useful in environmental and physical sciences for studying anomalous diffusion, porous-media transport, and climate interactions. The variable fractional orders $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ represent evolving memory or diffusion properties, the delays account for finite response times or transport effects, and the nonlocal integral boundary conditions describe distributed or averaged initial data, making this framework highly effective for capturing the complex behavior of coupled and time-dependent dynamical systems.

This work establishes a comprehensive theoretical framework for the analyzed system. The existence and uniqueness of solutions are first rigorously proven using Banach's contraction mapping principle and Schauder's fixed-point theorem. An extensive stability analysis is then conducted, yielding sufficient conditions for various types of Ulam-Hyers stability, such as Hyers-Ulam, generalized Hyers-Ulam GHU, and Hyers-Ulam-Rassias. Finally, the validity and practical applicability of the theoretical findings are demonstrated through pertinent numerical examples and a detailed discussion.

2. Main results

We begin by outlining our main findings, starting with the existence and uniqueness of solutions for problem (4). Let $n \in \{1, 2, 3, \ldots\}$, and consider the partition of the interval T given by:

$$\{T_1 = [0, t_1],\ T_2 = (t_1, t_2],\ T_3 = (t_2, t_3],\ \ldots,\ T_n = (t_{n-1}, t_n]\}.$$

Let $\alpha, \beta, \gamma : T \to (0, 1]$ be piecewise constant functions defined as:

$$\alpha(t) = \sum_{j=1}^{n} \alpha_j(t)\mu_j(t) = \begin{cases} \alpha_1, & \text{if } t \in T_1, \\ \alpha_2, & \text{if } t \in T_2, \\ \vdots \\ \alpha_n, & \text{if } t \in T_n, \end{cases}$$

where $\mu_j(t)$ denotes the indicator function of the interval T_j. The functions $\beta(t)$ and $\gamma(t)$ are defined analogously.

$$\beta(t) = \sum_{j=1}^{n} \beta_j(t)\mu_j(t) = \begin{cases} \beta_1, & \text{if } t \in T_1, \\ \beta_2, & \text{if } t \in T_2, \\ \vdots \\ \beta_n, & \text{if } t \in T_n, \end{cases}$$

and

$$\gamma(t) = \sum_{j=1}^{n} \gamma_j(t)\mu_j(t) = \begin{cases} \gamma_1, & \text{if } t \in T_1, \\ \gamma_2, & \text{if } t \in T_2, \\ \vdots \\ \gamma_n, & \text{if } t \in T_n, \end{cases}$$

where $\alpha_j, \beta_j, \gamma_j \in (0, 1]$ and μ_j shows the indicator function of $T_j = (t_{j-1}, t]$ where $j = 1, 2, \dots, n$ as $t_0 = 0, t_n = \psi$, and

$$\mu_j(t) = \begin{cases} 1, & \text{for } t \in T_j, \\ 0, & \text{otherwise} . \end{cases}$$

Suppose that $V_j = \mathrm{C}\big[T_j, R\big]$ is a Banach space equipped with the norm

$$\|x\| = \max_{t \in T} |x(t)|.$$

Consequently, from Eq. (4) (for $n = 1$), we obtain the following:

$$\begin{cases} D_t^{\alpha(t)} x(t) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\alpha_j}}{\Gamma(1-\alpha_j)}\, x'(s)\, ds, \\ D_t^{\beta(t)} y(t) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)}\, y'(s)\, ds, \\ D_t^{\gamma(t)} z(t) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\gamma_j}}{\Gamma(1-\gamma_j)}\, z'(s)\, ds. \end{cases} \tag{5}$$

Taking into consideration Eq. (5), we can formulate our theoretical problem as follows:

$$\begin{cases} U\big(t, x(t-\phi), x(ht), y(ht), z(ht)\big) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\alpha_j}}{\Gamma(1-\alpha_j)}\, x'(s)\, ds, \\ U\big(t, y(t-\phi), y(ht), x(ht), z(ht)\big) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\beta_j}}{\Gamma(1-\beta_j)}\, y'(s)\, ds, \\ U\big(t, z(t-\phi), z(ht), x(ht), y(ht)\big) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t-s)^{-\gamma_j}}{\Gamma(1-\gamma_j)}\, z'(s)\, ds. \end{cases}$$

Now, assume that $x, y, z \in \mathrm{C}([0, \psi], R)$ are such that we need to deal with

$$\begin{cases} D^{\alpha_j}_{t_{j-1}} x(t) = U(t, x(t-\phi), x(ht), y(ht), z(ht)), t \in T_j, \\ D^{\beta_j}_{t_{j-1}} y(t) = U(t, y(t-\phi), y(ht), x(ht), z(ht)), t \in T_j, \\ D^{\gamma_j}_{t_{j-1}} z(t) = U(t, z(t-\phi), z(ht), x(ht), y(ht)), t \in T_j, \\ x\left(t_{j-1}\right) = \int_0^\psi \frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)} \mathrm{g}(x(s), y(s), z(s)) ds + x_0, \varkappa \in (0,1], \\ y\left(t_{j-1}\right) = \int_0^\psi \frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)} \mathrm{g}(y(s), x(s), z(s)) ds + y_0, \delta \in (0,1], \\ z\left(t_{j-1}\right) = \int_0^\psi \frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)} \mathrm{g}(z(s), x(s), y(s)) ds + z_0, \lambda \in (0,1]. \end{cases}$$

Lemma 1. *When $p, q, r \in Q(T_j)$, consider the following problem:*

$$\begin{cases} D_0^{\alpha_j} x(t) = p(t), \\ D_0^{\beta_j} y(t) = q(t), \\ D_0^{\gamma_j} z(t) = r(t), \\ x(t_{j-1}) = \displaystyle\int_0^\psi \frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)} \, g(s, x(s), y(s), z(s)) \, ds + x_0, \\ y(t_{j-1}) = \displaystyle\int_0^\psi \frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)} \, g(s, y(s), x(s), z(s)) \, ds + y_0, \\ z(t_{j-1}) = \displaystyle\int_0^\psi \frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)} \, g(s, z(s), x(s), y(s)) \, ds + z_0. \end{cases} \tag{6}$$

Then, the corresponding solution is given by

$$\begin{cases} x(t) = x_0 + \int_0^\psi \frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)} \, g(s, x(s), y(s), z(s)) \, ds + \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^t (t-s)^{\alpha_j - 1} p(s) \, ds, \\ y(t) = y_0 + \int_0^\psi \frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)} \, g(s, y(s), x(s), z(s)) \, ds + \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^t (t-s)^{\beta_j - 1} q(s) \, ds, \\ z(t) = z_0 + \int_0^\psi \frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)} \, g(s, z(s), x(s), y(s)) \, ds + \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^t (t-s)^{\gamma_j - 1} r(s) \, ds. \end{cases} \tag{7}$$

Proof. By applying the fractional integral operators $I^{\alpha_j}_{t_{j-1}}$, $I^{\beta_j}_{t_{j-1}}$, and $I^{\gamma_j}_{t_{j-1}}$ to Eq. (6), and subsequently using Eq. (3), we obtain

$$\begin{cases} x(t) = E + \dfrac{1}{\Gamma(\alpha_j)} \displaystyle\int_{t_{j-1}}^t (t-s)^{\alpha_j - 1} p(s) \, ds, \\ y(t) = F + \dfrac{1}{\Gamma(\beta_j)} \displaystyle\int_{t_{j-1}}^t (t-s)^{\beta_j - 1} q(s) \, ds, \\ z(t) = D + \dfrac{1}{\Gamma(\gamma_j)} \displaystyle\int_{t_{j-1}}^t (t-s)^{\gamma_j - 1} r(s) \, ds. \end{cases} \tag{8}$$

Letting $t \to 0$ in Eq. (8) and applying the corresponding initial conditions, we derive

$$\begin{cases} E = x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa-1}}{\Gamma(\varkappa)} g(s, x(s), y(s), z(s))\, ds, \\ F = y_0 + \int_0^{\psi} \frac{(\psi - s)^{\delta-1}}{\Gamma(\delta)} g(s, y(s), x(s), z(s))\, ds, \\ D = z_0 + \int_0^{\psi} \frac{(\psi - s)^{\lambda-1}}{\Gamma(\lambda)} g(s, z(s), x(s), y(s))\, ds. \end{cases} \tag{9}$$

Hence, we obtain that

$$\begin{cases} x(t) = x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa-1}}{\Gamma(\varkappa)} g(s, x(s), y(s), z(s))\, ds + \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} p(s)\, ds, \\ y(t) = y_0 + \int_0^{\psi} \frac{(\psi - s)^{\delta-1}}{\Gamma(\delta)} g(s, y(s), x(s), z(s))\, ds + \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t} (t-s)^{\beta_j - 1} q(s)\, ds, \\ z(t) = z_0 + \int_0^{\psi} \frac{(\psi - s)^{\lambda-1}}{\Gamma(\lambda)} g(s, z(s), x(s), y(s))\, ds + \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^{t} (t-s)^{\gamma_j - 1} r(s)\, ds. \end{cases} \tag{10}$$

Eq. (10) characterizes the coupled system of fractional integral equations associated with the original problem, wherein each component $x(t)$, $y(t)$, and $z(t)$ depends on its initial condition, the $g(\cdot)$, and the corresponding forcing terms $p(s)$, $q(s)$, and $r(s)$.

Corollary 1. *According to Lemma 1, the solution of the system under consideration, given in Eq. (4), can be expressed as*

$$\begin{cases} x(t) = x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa-1}}{\Gamma(\varkappa)} g(s, x(s), y(s), z(s))\, ds \\ \qquad + \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} U(s, x(s-\phi), x(hs), y(hs), z(hs))\, ds, \\ y(t) = y_0 + \int_0^{\psi} \frac{(\psi - s)^{\delta-1}}{\Gamma(\delta)} g(s, y(s), x(s), z(s))\, ds \\ \qquad + \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^{t} (t-s)^{\beta_j - 1} U(s, y(s-\phi), y(hs), x(hs), z(hs))\, ds, \\ z(t) = z_0 + \int_0^{\psi} \frac{(\psi - s)^{\lambda-1}}{\Gamma(\lambda)} g(s, z(s), x(s), y(s))\, ds \\ \qquad + \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^{t} (t-s)^{\gamma_j - 1} U(s, z(s-\phi), z(hs), x(hs), y(hs))\, ds. \end{cases} \tag{11}$$

Consequently, we establish the following hypotheses concerning the existence and uniqueness of the solution to Eq. (4):

($\mathbf{H}_1$) For $a, b, c, e, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{e} \in R$, there exists a constant $L_1 > 0$ such that

$$|U(t, a, b, c, e) - U(t, \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{e})| \leq L_1\big(|a - \widetilde{a}| + |b - \widetilde{b}| + |c - \widetilde{c}| + |e - \widetilde{e}|\big).$$

($\mathbf{H}_2$) For $b, c, e, \widetilde{b}, \widetilde{c}, \widetilde{e} \in R$, there exists a constant $L_2 > 0$ such that

$$|g(t, b, c, e) - g(t, \widetilde{b}, \widetilde{c}, \widetilde{e})| \leq L_2\big(|b - \widetilde{b}| + |c - \widetilde{c}| + |e - \widetilde{e}|\big).$$

Theorem 1. *The proposed problem* (4) *admits a unique solution provided that assumptions* ($\boldsymbol{H}_1$) *and* ($\boldsymbol{H}_2$) *hold, and the following conditions are satisfied:*

$$\frac{3L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)} + \frac{4L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)} < 1, \tag{12}$$

$$\frac{3L_2\psi^{\delta}}{\Gamma(\delta+1)} + \frac{4L_1\psi^{\beta_j}}{\Gamma(\beta_j+1)} < 1, \tag{13}$$

and

$$\frac{3L_2\psi^{\lambda}}{\Gamma(\lambda+1)} + \frac{4L_1\psi^{\gamma_j}}{\Gamma(\gamma_j+1)} < 1. \tag{14}$$

Proof. Define an operator $\Omega : V_j \times V_j \times V_j \to V_j$ by

$$\begin{aligned}\Omega(x, y, z)(t) =& x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} \mathrm{g}(s, x(s), y(s), z(s))ds \\ &+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t - s)^{\alpha_j - 1} U(s, x(s - \phi), x(hs), y(hs), z(hs))ds.\end{aligned} \tag{15}$$

It remains to show that the operator Ω is a condensing operator. To this end, let $x, y, \tilde{x}, \tilde{y}, z, \tilde{z} \in V_j$. Using the inequality $(\psi - t_{j-1})^{\alpha_j} \leq \psi^{\alpha}$, and applying the assumptions ($\mathbf{H}_1$) and ($\mathbf{H}_2$), we obtain the following:

$$\begin{aligned}&\|\Omega(x, y, z) - \Omega(\tilde{x}, \tilde{y}, \tilde{z})\| \\ &\leq \frac{L_2}{\Gamma(t)} \int_0^{\psi} (\psi - s)^{\varkappa - 1} ds \times (\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) \\ &+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t - s)^{\alpha_j - 1} ds \times (2\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) \\ &\leq \frac{L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}(\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) + \frac{L_1(\psi - t_{j-1})^{\alpha_j}}{\Gamma(\alpha_j + 1)} \\ &(2\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) \\ &\leq \frac{L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}(\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) + \frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j + 1)}(2\|x - \tilde{x}\| + \|y - \tilde{y}\| + \|z - \tilde{z}\|) \\ &= \left(\frac{L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)} + \frac{2L_1\psi^{\alpha_j}}{\Gamma(\alpha_j + 1)}\right)\|x - \tilde{x}\| + \left(\frac{L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)} + \frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j + 1)}\right)\|y - \tilde{y}\|\end{aligned}$$

$$+\left(\frac{L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{L_1\psi^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right)\|z-\tilde{z}\|$$
$$=\left(\frac{3L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right)\|(x,y,z)-(\tilde{x},\tilde{y},\tilde{z})\|. \tag{16}$$

Similarly, using $(\psi-t_{j-1})^{\beta_j}\leq\psi^{\beta}$ and $(\psi-t_{j-1})^{\gamma_j}\leq\psi^{\gamma}$, we have

$$\|\Omega(y,x,z)-\Omega(\tilde{y},\tilde{x},\tilde{z})\|\leq\left(\frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_j}}{\Gamma\left(\beta_j+1\right)}\right)\|(y,x,z)-(\tilde{y},\tilde{x},\tilde{z})\|. \tag{17}$$

and

$$\|\Omega(z,x,y)-\Omega(\tilde{z},\tilde{x},\tilde{y})\|\leq\left(\frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\gamma_j}}{\Gamma\left(\gamma_j+1\right)}\right)\|(z,x,y)-(\tilde{z},\tilde{x},\tilde{y})\|. \tag{18}$$

From inequalities (16), (17), and (18), it follows that the operator Ω satisfies a strict contraction-type property under the given assumptions. Consequently, Ω is a condensing operator, which implies that the problem admits a unique solution under the criteria specified in Theorem 1. This completes the proof.

We now prove that the considered problem (4) admits at least one solution on bounded sets. To this end, we propose the following assumptions.

$(\mathbf{H}_3)$ For $L_3>0$, we have

$$|U(t,a,b,c,e)|\leq L_3(|a|+|b|+|c|+|e|),\ \ for\ a,b,c,e\in R.$$

$(\mathbf{H}_4)$ If $L_4>0$, then

$$|\mathrm{g}(t,a,b,c)|\leq L_4(|a|+|b|+|c|),\ \ for\ a,b,c\in R.$$

Theorem 2. *The proposed problem Eq.* (4) *admits at least one solution in*

$$\mathrm{G}\times\mathrm{G}\times\mathrm{G}=\left\{(x,y,z)\in V_j\times V_j\times V_j:\|x\|\leq\ell,\ \|y\|\leq\ell,\ \|z\|\leq\ell\right\},$$

under the assumptions $(\mathbf{H}_1)$–$(\mathbf{H}_4)$*, where*

$$A_1=\left(\frac{3L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right),$$
$$A_2=\left(\frac{3L_2\psi^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_j}}{\Gamma(\beta_j+1)}\right),$$
$$A_3=\left(\frac{3L_2\psi^{\lambda}}{\Gamma(\lambda+1)}+\frac{4L_1\psi^{\gamma_j}}{\Gamma(\gamma_j+1)}\right).$$

Proof. The proof was divided into the subsequent steps:

Step 1. Prove that $\Omega:\mathrm{G}\times\mathrm{G}\times\mathrm{G}\to\mathrm{G}$ is bounded. Assuming that $(x,y,z)\in\mathrm{G}\times\mathrm{G}\times\mathrm{G}$, we have

$$\|\Omega(x,y,z)\|=\max_{t\in T}\left|x_0+\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}g(s,x(s),y(s),z(s))\,ds\right.$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} U(s, x(s-\phi), x(hs), y(hs), z(hs))\, ds \Bigg| \\
&\leq x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} |g(s, x(s), y(s), z(s))|\, ds \\
&+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} |U(s, x(s-\phi), x(hs), y(hs), z(hs))|\, ds \\
&\leq x_0 + \left(\frac{3L_2 \psi^{\varkappa}}{\Gamma(\varkappa + 1)} + \frac{4L_1 \psi^{\alpha_j}}{\Gamma(\alpha_j + 1)} \right) \ell \leq \ell.
\end{aligned}
$$

Similarly, we can write

$$
\|\Omega(y, x, z)\| \leq y_0 + \left(\frac{3L_2 \psi^{\delta}}{\Gamma(\delta + 1)} + \frac{4L_1 \psi^{\beta_j}}{\Gamma(\beta_j + 1)} \right) \ell \leq \ell,
$$

and

$$
\|\Omega(z, x, y)\| \leq z_0 + \left(\frac{3L_2 \psi^{\lambda}}{\Gamma(\lambda + 1)} + \frac{4L_1 \psi^{\gamma_j}}{\Gamma(\gamma_j + 1)} \right) \ell \leq \ell.
$$

Hence, $(\Omega(x, y, z), \Omega(y, x, z), \Omega(z, x, y)) \in \mathrm{G} \times \mathrm{G} \times \mathrm{G}$, so that Ω transforms a bounded set in V_j into another bounded set.

Step 2. Show that Ω is continuous. Let the sequences (x_n, y_n, z_n) converge to (x, y, z) in $\mathrm{G} \times \mathrm{G} \times \mathrm{G}$, for each $t \in T$, we get

$$
\begin{aligned}
&\|\Omega\left(x_n, y_n, z_n\right) - \Omega(x, y, z)\| \\
&= \max_{t \in T_j} \Bigg| \left\{ x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} \mathrm{g}\left(s, x_n(s), y_n(s), z_n(s)\right) ds \right. \\
&\left. + \frac{1}{\Gamma\left(\alpha_j\right)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} U\left(s, x_n(s-\phi), x_n(hs), y_n(hs), z_n(hs)\right) ds \right\} \\
&- \left\{ x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} \mathrm{g}(s, x(s), y(s), z(s)) ds \right. \\
&\left. + \frac{1}{\Gamma\left(\alpha_j\right)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} U(s, x(s-\phi), x(hs), y(hs), z(hs)) ds \right\} \Bigg| \\
&\leq \max_{t \in T_j} \left\{ \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} \left| \mathrm{g}\left(s, x_n(s), y_n(s), z_n(s)\right) - \mathrm{g}(s, x(s), y(s), z(s)) \right| ds \right. \\
&+ \frac{1}{\Gamma\left(\alpha_j\right)} \int_{t_{j-1}}^{t} (t-s)^{\alpha_j - 1} \mid U\left(s, x_n(s-\phi), x_n(hs), y_n(hs), z_n(hs)\right) \\
&\left. - U(s, x(s-\phi), x(hs), y(hs), z(hs)) \mid ds \right\},
\end{aligned}
$$

which implies that

$$
\|\Omega\left(x_n, y_n, z_n\right) - \Omega(x, y, z)\|
$$

$$\leq \max_{t\in T_j}\left\{L_4\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}ds\times(|x_n-x|+|y_n-y|+|z_n-z|)\right.$$
$$\left.+\frac{L_3}{\Gamma(\alpha_j)}\int_{t_{j-1}}^{t}(t-s)^{\alpha_j-1}ds\times(2|x_n-x|+|y_n-y|+|z_n-z|)\right\}$$
$$\leq \max_{t\in T_j}\left\{L_4\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}ds+\frac{2L_3}{\Gamma(\alpha_j)}(t-s)^{\alpha_j-1}ds\right\}(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|).$$

Similarly, we can write

$$\|\Omega(y_n,x_n,z_n)-\Omega(y,x,z)\|$$
$$\leq \max_{t\in T_j}\left\{L_4\int_0^{\psi}\frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)}ds+\frac{2L_3}{\Gamma(\beta_j)}(t-s)^{\beta_j-1}ds\right\}(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|),$$

and

$$\|\Omega(z_n,x_n,y_n)-\Omega(z,x,y)\|$$
$$\leq \max_{t\in T_j}\left\{L_4\int_0^{\psi}\frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)}ds+\frac{2L_3}{\Gamma(\gamma_j)}(t-s)^{\gamma_j-1}ds\right\}(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|).$$

Since $x_n\to x, y\to y_n, z\to z_n$ as $v\to\infty$, and Ω is bounded, we have $\|\Omega(x_n,y_n,z_n)-\Omega(x,y,z)\|\to 0$, $\|\Omega(y_n,x_n,z_n)-\Omega(y,x,z)\|\to 0$ and $\|\Omega(z_n,x_n,y_n)-\Omega(z,x,y)\|\to 0$ as $n\to 0$. Therefore, Ω is continuous.

Step 3. If $t_1,t_2\in T, t_1<t_2$, we get

$$\begin{aligned}
&|\Omega(x,y,z)(t_1)-\Omega(x,y,z)(t_2)|\\
&=\Big|x_0+\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}\mathrm{g}(s,x(s),y(s),z(s))\,ds\\
&\quad+\frac{1}{\Gamma(\alpha_j)}\int_{t_{j-1}}^{t_1}(t_1-s)^{\alpha_j-1}U(s,x(s-\phi),x(hs),y(hs),z(hs))\,ds\\
&\quad-x_0-\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}\mathrm{g}(s,x(s),y(s),z(s))\,ds\\
&\quad-\frac{1}{\Gamma(\alpha_j)}\int_{t_{j-1}}^{t_2}(t_2-s)^{\alpha_j-1}U(s,x(s-\phi),x(hs),y(hs),z(hs))\,ds\Big|\\
&\leq\frac{2L_3\ell}{\Gamma(\alpha_j)}\left[\int_{t_{j-1}}^{t_1}(t_1-s)^{\alpha_j-1}\,ds-\int_{t_{j-1}}^{t_2}(t_2-s)^{\alpha_j-1}\,ds\right].
\end{aligned}$$

By the same method, we have

$$|\Omega(y,x,z)(t_1)-\Omega(y,x,z)(t_2)|\leq\frac{1}{\Gamma(\beta_j)}\left\{\int_{t_{j-1}}^{t_1}(t_1-s)^{\beta_j-1}\,ds-\int_{t_{j-1}}^{t_2}(t_2-s)^{\beta_j-1}\,ds\right\}2L_3\ell.$$

and

$$|\Omega(z,x,y)\,(t_1)-\Omega(z,x,y)\,(t_2)| \leq \frac{1}{\Gamma(\gamma_j)}\left\{\int_{t_{j-1}}^{t_1}(t_1-s)^{\gamma_j-1}\,ds-\int_{t_{j-1}}^{t_2}(t_2-s)^{\gamma_j-1}\,ds\right\}2L_3\ell.$$

Since $t_1 \to t_2, |\Omega(x,y,z)\,(t_1)-\Omega(x,y,z)\,(t_2)| \to 0$, $|\Omega(y,x,z)\,(t_1)-\Omega(y,x,z)\,(t_2)| \to 0$ and $|\Omega(z,x,y)\,(t_1)-\Omega(z,x,y)\,(t_2)| \to 0$. From Steps (1) and (2), Ω is bounded and continuous.
Therefore,

$$\begin{aligned}&\|\Omega(x,y,z)\,(t_1)-\Omega(x,y,z)\,(t_2)| \to 0, |\Omega(y,x,z)\,(t_1)-\Omega(y,x,z)\,(t_2)| \to 0\\&and\,|\Omega(z,x,y)\,(t_1)-\Omega(z,x,y)\,(t_2)\| \to 0.\end{aligned}$$

Hence, Ω is entirely continuous in V_j.
Step 4. We prove that

$$W=\left\{(x,y,z)\in V_j\times V_j\times V_j: \begin{array}{c}x=\sigma\Omega(x,y,z),\\ y=\sigma^*\Omega(y,x,z),\\ z=\tilde{\sigma}\Omega(z,x,y)\end{array}\quad \text{for some } \sigma,\sigma^*,\tilde{\sigma}\in[0,1]\right.$$

are a priori constraints. For all $x,y,z \in W$, we get

$$\begin{aligned}\|x\| &= \max_{t_j\in T_j}|\sigma\Omega(x,y,z)|\\ &\leq \max_{t\in T}\left|(x_0,y_0,z_0)+\int_0^{\psi}\frac{(\psi-s)^{\varkappa-1}}{\Gamma(\varkappa)}g(s,x(s),y(s),z(s))\,ds\right.\\ &\quad\left.+\frac{1}{\Gamma(\alpha_j)}\int_{t_{j-1}}^{t}(t-s)^{\alpha_j-1}U(s,x(s-\phi),x(hs),y(hs),z(hs))\,ds\right|\\ &\leq (x_0,y_0,z_0)+\left\{\frac{3L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right\}\ell \leq \ell.\end{aligned} \tag{19}$$

Similarly, one has

$$\|y\| \leq (y_0,x_0,z_0)+\left\{\frac{3L_2\psi^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_j}}{\Gamma(\beta_j+1)}\right\}\ell \leq \ell, \tag{20}$$

and

$$\|z\| \leq (z_0,x_0,y_0)+\left\{\frac{3L_2\psi^{\lambda}}{\Gamma(\lambda+1)}+\frac{4L_1\psi^{\gamma_j}}{\Gamma(\gamma_j+1)}\right\}\ell \leq \ell. \tag{21}$$

From Eqs. (19), (20) and (21), we see that

$$\ell \geq \frac{(x_0,y_0,z_0)}{1-A_1}, \quad \ell \geq \frac{(y_0,x_0,z_0)}{1-A_2}, \quad \ell \geq \frac{(z_0,x_0,y_0)}{1-A_3}.$$

Correspondingly, this results in $\|x\| \leq \ell$, $\|y\| \leq \ell$ and $\|z\| \leq \ell$. As a result, Theorem 2 states that the suggested problem Eq. (4) has at least one solution.

3. Stability Analysis

The stability analysis of problem Eq.(4) begins with some fundamental concepts. We first introduce the following definition:

Definition 1. *For every $\Delta \geq 0$, there exists a constant $W_U > 0$ such that for each solution $(x(t), y(t), z(t)) \in V_j \times V_j \times V_j$ satisfying:*

$$|\mathcal{D}^{\alpha_j}_{t_{j-1}} x(t) - U(t, x(t-\phi), x(ht), y(ht), z(ht))| \leq \Delta,$$
$$|\mathcal{D}^{\beta_j}_{t_{j-1}} y(t) - U(t, y(t-\phi), y(ht), x(ht), z(ht))| \leq \Delta,$$
$$|\mathcal{D}^{\gamma_j}_{t_{j-1}} z(t) - U(t, z(t-\phi), z(ht), x(ht), y(ht))| \leq \Delta,$$

for all $t \in T_j$, and for a unique solution $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \in V_j \times V_j \times V_j$ of Eq. (4), one has

$$\|(x, \tilde{x}) - (y, \tilde{y}) - (z, \tilde{z})\| \leq \Delta W_U.$$

The system Eq. (4) is then called Ulam-Hyers stable. If there exists a function $\xi : (0, \infty) \to (0, \infty)$ with $\xi(0) = 0$ such that

$$\|(x, \tilde{x}) - (y, \tilde{y}) - (z, \tilde{z})\| \leq W_U \xi(\Delta),$$

the solution is called generalized Ulam-Hyers stable.

Remark 1. *We say that the pair $(x(t), y(t), z(t)) \in V_j \times V_j \times V_j$ is a solution to problem (4) if and only if there exist functions $X, Y, Z \in C(T_j)$, for every $t \in T_j$, such that:*

(i) $|X(t)| \leq \Delta, \quad |Y(t)| \leq \Delta, \quad$ *and* $\quad |Z(t)| \leq \Delta$,

(ii) $D^{\alpha_j}_{t_{j-1}} x(t) - U\big(t, x(t-\phi), x(ht), y(ht), z(ht)\big) - X(t) = 0$,

(iii) $D^{\beta_j}_{t_{j-1}} y(t) - U\big(t, y(t-\phi), y(ht), x(ht), z(ht)\big) - Y(t) = 0$,

(iv) $D^{\gamma_j}_{t_{j-1}} z(t) - U\big(t, z(t-\phi), z(ht), x(ht), y(ht)\big) - Z(t) = 0$.

Definition 2. *The solution $(x(t), y(t), z(t))$ of the proposed problem Eq. (4) is Ulam-Hyers-Rassias stable for the continuous function $I \in V_j$, if there exists a constant $W_U > 0$ such that for all $t \in T_j$:*

$$|D^{\alpha_j}_{t_{j-1}} x(t) - U(t, x(t-\phi), x(ht), y(ht), z(ht))| \leq I(t)\Delta,$$
$$|D^{\beta_j}_{t_{j-1}} y(t) - U(t, y(t-\phi), y(ht), x(ht), z(ht))| \leq I(t)\Delta,$$
$$|D^{\gamma_j}_{t_{j-1}} z(t) - U(t, z(t-\phi), z(ht), x(ht), y(ht))| \leq I(t)\Delta,$$

and for a unique solution $(\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)) \in V_j \times V_j \times V_j$ of Eq. (4), one has

$$\|(x, \tilde{x}) - (y, \tilde{y}) - (z, \tilde{z})\| \leq \Delta W_U I(t).$$

Moreover, if there exists a function $\xi : (0, \infty) \to (0, \infty)$ with $\xi(0) = 0$ such that

$$\|(x, \tilde{x}) - (y, \tilde{y}) - (z, \tilde{z})\| \leq W_U I(t) \xi(\Delta),$$

the solution is then referred to as generalized Ulam-Hyers-Rassias stable.

Remark 2. *For functions $X, Y, Z \in \mathrm{C}(T_j)$, and for every $t \in T_j$, we have*

(i) $|X(t)| \leq \Delta, \quad |Y(t)| \leq \Delta, \quad$ *and* $\quad |Z(t)| \leq \Delta,$

(ii)
$$D^{\alpha_j}_{t_{j-1}} x(t) - U\big(t, x(t-\phi), x(ht), y(ht), z(ht)\big) - X(t) = 0,$$

(iii)
$$D^{\beta_j}_{t_{j-1}} y(t) - U\big(t, y(t-\phi), y(ht), x(ht), z(ht)\big) - Y(t) = 0,$$

(iv)
$$D^{\gamma_j}_{t_{j-1}} z(t) - U\big(t, z(t-\phi), z(ht), x(ht), y(ht)\big) - Z(t) = 0.$$

Lemma 2. *Thanks to Remark 1 and Lemma 1, the solution of the perturbed system*

$$\begin{cases} D^{\alpha_j}_{t_{j-1}} x(t) = U\big(t, x(t-\phi), x(ht), y(ht), z(ht)\big) + X(t), & \alpha_j \in (0,1], \\ D^{\beta_j}_{t_{j-1}} y(t) = U\big(t, y(t-\phi), y(ht), x(ht), z(ht)\big) + Y(t), & \beta_j \in (0,1], \\ D^{\gamma_j}_{t_{j-1}} z(t) = U\big(t, z(t-\phi), z(ht), x(ht), y(ht)\big) + Z(t), & \gamma_j \in (0,1], \\ x(0) = \displaystyle\int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)}\, g\big(x(s), y(s), z(s)\big)\, ds + x_0, & \varkappa \in (0,1], \\ y(0) = \displaystyle\int_0^{\psi} \frac{(\psi - s)^{\delta - 1}}{\Gamma(\delta)}\, g\big(y(s), x(s), z(s)\big)\, ds + y_0, & \delta \in (0,1], \\ z(0) = \displaystyle\int_0^{\psi} \frac{(\psi - s)^{\lambda - 1}}{\Gamma(\lambda)}\, g\big(z(s), x(s), y(s)\big)\, ds + z_0, & \lambda \in (0,1], \end{cases} \tag{22}$$

satisfies the following inequalities for all $t \in T$:

$$|x(t) - \Omega(x, y, z)(t)| \leq \frac{\Delta \psi^{\alpha_j}}{\Gamma(\alpha_j + 1)}, \tag{23}$$

$$|y(t) - \Omega(y, x, z)(t)| \leq \frac{\Delta \psi^{\beta_j}}{\Gamma(\beta_j + 1)}, \tag{24}$$

$$|z(t) - \Omega(z, x, y)(t)| \leq \frac{\Delta \psi^{\gamma_j}}{\Gamma(\gamma_j + 1)}, \tag{25}$$

with the operator Ω defined by

$$\Omega(x, y, z)(t) = (x_0, y_0, z_0) + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)}\, g(s, x(s), y(s), z(s))\, ds$$
$$+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^{t} (t - s)^{\alpha_j - 1} p(s)\, ds.$$

Proof. *Using Lemma 1, problem Eq.* (22) *implies that*

$$x(t) = x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa - 1}}{\Gamma(\varkappa)} g(s, x(s), y(s), z(s))\, ds$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha_j)} \int_0^t (t-s)^{\alpha_j-1} X(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^t (t-s)^{\alpha_j-1} X(s)\, ds,
\end{aligned}
$$

$$
\begin{aligned}
y(t) = y_0 &+ \int_0^{\psi} \frac{(\psi-s)^{\delta-1}}{\Gamma(\delta)} g(s, y(s), x(s), z(s))\, ds \\
&+ \frac{1}{\Gamma(\beta_j)} \int_0^t (t-s)^{\beta_j-1} Y(s)\, ds \\
&+ \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^t (t-s)^{\beta_j-1} Y(s)\, ds,
\end{aligned}
$$

and

$$
\begin{aligned}
z(t) = z_0 &+ \int_0^{\psi} \frac{(\psi-s)^{\lambda-1}}{\Gamma(\lambda)} g(s, z(s), x(s), y(s))\, ds \\
&+ \frac{1}{\Gamma(\gamma_j)} \int_0^t (t-s)^{\gamma_j-1} Z(s)\, ds \\
&+ \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^t (t-s)^{\gamma_j-1} Z(s)\, ds,
\end{aligned}
$$

which implies that

$$
|x(t) - \Omega(x,y,z)(t)| \leq \frac{\Delta \psi^{\alpha_j}}{\Gamma(\alpha_j+1)}.
$$

$$
|y(t) - \Omega(y,x,z)(t)| \leq \frac{\Delta \psi^{\beta_j}}{\Gamma(\beta_j+1)},
$$

and

$$
|z(t) - \Omega(z,x,y)(t)| \leq \frac{\Delta \psi^{\gamma_j}}{\Gamma(\gamma_j+1)}.
$$

Theorem 3. *Under assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$*, the solution to problem* (4) *is both Ulam-Hyers and generalized Ulam-Hyers stable, given that*

$$
1 \neq \frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)}.
$$

Proof. Using Lemma 1, if x^*, y^*, and z^* are solutions of Eq. (4), then for any t:

$$
\begin{aligned}
|x(t) - x^*(t)| &= |x(t) - \Omega(x^*, y^*, z^*)(t)| \\
&= |x(t) - \Omega(x,y,z)(t) + \Omega(x,y,z)(t) - \Omega(x^*, y^*, z^*)(t)| \\
&\leq |x(t) - \Omega(x,y,z)(t)| + |\Omega(x,y,z)(t) - \Omega(x^*, y^*, z^*)(t)| \\
&\leq \frac{\Delta \psi^{\alpha_j}}{\Gamma(\alpha_j+1)} + \left(\frac{2L_2 \psi^{\varkappa}}{\Gamma(\varkappa+1)} + \frac{3L_1 \psi^{\alpha_j}}{\Gamma(\alpha_j+1)} \right) \|x - x^*\|
\end{aligned}
$$

$$+\left(\frac{L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\|y-y^*\|$$
$$+\left(\frac{L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\|z-z^*\|.$$

which implies that

$$\begin{aligned}&\left(1-\left(\frac{2L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{3L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\right)\|x-x^*\|\\ &\leq\frac{\Delta\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}+\left(\frac{L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\|y-y^*\|\\ &+\left(\frac{L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\|z-z^*\|.\end{aligned}$$

Hence, we derive the following system of inequalities:

$$\|x-x^*\|-\frac{\theta}{1-\theta}\left(\|y-y^*\|+\|z-z^*\|\right)\leq\frac{\Delta\psi^{\alpha_j}}{\Gamma(\alpha_j+1)(1-\theta)}=\Delta\tau, \tag{26}$$
$$\|y-y^*\|-\frac{\theta^*}{1-\theta^*}\left(\|x-x^*\|+\|z-z^*\|\right)\leq\frac{\Delta\psi^{\beta_j}}{\Gamma(\beta_j+1)(1-\theta^*)}=\Delta\tau^*, \tag{27}$$
$$\|z-z^*\|-\frac{\Theta}{1-\Theta}\left(\|x-x^*\|+\|y-y^*\|\right)\leq\frac{\Delta\psi^{\gamma_j}}{\Gamma(\gamma_j+1)(1-\Theta)}=\Delta\tilde{\tau}, \tag{28}$$

where the parameters are defined as:

$$\begin{aligned}\theta&=\frac{2L_2\psi^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{3L_1\psi^{\alpha_j}}{\Gamma(\alpha_j+1)},\\ \theta^*&=\frac{3L_2\psi^{\delta}}{\Gamma(\delta+1)}+\frac{2L_1\psi^{\beta_j}}{\Gamma(\beta_j+1)},\\ \Theta&=\frac{3L_2\psi^{\lambda}}{\Gamma(\lambda+1)}+\frac{2L_1\psi^{\gamma_j}}{\Gamma(\gamma_j+1)},\\ \tau&=\frac{\psi^{\alpha_j}}{\Gamma(\alpha_j+1)(1-\theta)},\\ \tau^*&=\frac{\psi^{\beta_j}}{\Gamma(\beta_j+1)(1-\theta^*)},\\ \tilde{\tau}&=\frac{\psi^{\gamma_j}}{\Gamma(\gamma_j+1)(1-\Theta)}.\end{aligned}$$

The inequalities (26)–(28) can be written in matrix form as:

$$\begin{bmatrix}1 & -\frac{\theta}{1-\theta} & -\frac{\theta}{1-\theta}\\ -\frac{\theta^*}{1-\theta^*} & 1 & -\frac{\theta^*}{1-\theta^*}\\ -\frac{\Theta}{1-\Theta} & -\frac{\Theta}{1-\Theta} & 1\end{bmatrix}\begin{bmatrix}\|x-x^*\|\\ \|y-y^*\|\\ \|z-z^*\|\end{bmatrix}\leq\begin{bmatrix}\Delta\tau\\ \Delta\tau^*\\ \Delta\tilde{\tau}\end{bmatrix}. \tag{29}$$

Solving the system (29) yields:

$$\begin{bmatrix} \|x-x^*\| \\ \|y-y^*\| \\ \|z-z^*\| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{G} & \frac{\theta}{1-\theta}\frac{1}{G} & \frac{\theta}{1-\theta}\frac{1}{G} \\ \frac{\theta^*}{1-\theta^*}\frac{1}{G} & \frac{1}{G} & \frac{\theta^*}{1-\theta^*}\frac{1}{G} \\ \frac{\Theta}{1-\Theta}\frac{1}{G} & \frac{\Theta}{1-\Theta}\frac{1}{G} & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \Delta\tau \\ \Delta\tau^* \\ \Delta\tilde{\tau} \end{bmatrix}, \tag{30}$$

where $G = 1 - \frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)}$.

From the system (30), we obtain the component-wise bounds:

$$\begin{aligned} \|x-x^*\| &\leq \frac{\Delta\tau}{G} + \frac{\Delta\tau^*\theta}{(1-\theta)G} + \frac{\Delta\tilde{\tau}\theta}{(1-\theta)G}, \\ \|y-y^*\| &\leq \frac{\Delta\tau\theta^*}{(1-\theta^*)G} + \frac{\Delta\tau^*}{G} + \frac{\Delta\tilde{\tau}\theta^*}{(1-\theta^*)G}, \\ \|z-z^*\| &\leq \frac{\Delta\tau\Theta}{(1-\Theta)G} + \frac{\Delta\tau^*\Theta}{(1-\Theta)G} + \frac{\Delta\tilde{\tau}}{G}. \end{aligned}$$

Summing these inequalities gives the total bound:

$$\begin{aligned} &\|x-x^*\| + \|y-y^*\| + \|z-z^*\| \\ &\quad \leq \left(\frac{\tau+\tau^*+\tilde{\tau}}{G} + \frac{\tau^*\theta+\tilde{\tau}\theta}{(1-\theta)G} + \frac{\tau\theta^*+\tilde{\tau}\theta^*}{(1-\theta^*)G} + \frac{\tau\Theta+\tau^*\Theta}{(1-\Theta)G} \right) \Delta. \end{aligned}$$

Defining the stability constant:

$$W_U = \frac{\tau+\tau^*+\tilde{\tau}}{G} + \frac{\tau^*\theta+\tilde{\tau}\theta}{(1-\theta)G} + \frac{\tau\theta^*+\tilde{\tau}\theta^*}{(1-\theta^*)G} + \frac{\tau\Theta+\tau^*\Theta}{(1-\Theta)G},$$

we conclude that:

$$\|(x,y,z) - (x^*,y^*,z^*)\| \leq W_U\Delta.$$

Therefore, the solution to problem (4) is Ulam-Hyers stable. Moreover, taking $\xi(\Delta) = \Delta$ with $\xi(0) = 0$, we obtain:

$$\|(x,y,z) - (x^*,y^*,z^*)\| \leq W_U\xi(\Delta),$$

which guarantees the generalized Ulam-Hyers stability of Eq. (4).

Lemma 3. *For the solution of Eq.* (22)*, the conditions in Remark 2 is true:*

$$\begin{aligned} |x(t) - \Omega(x,y,z)(t)| &\leq \frac{\Delta\psi^{\alpha_j}}{\Gamma(\alpha_j+1)} I(t), \\ |y(t) - \Omega(y,x,z)(t)| &\leq \frac{\Delta t^{\beta_j}}{\Gamma(\beta_j+1)}, \\ |z(t) - \Omega(z,x,y)(t)| &\leq \frac{\Delta t^{\gamma_j}}{\Gamma(\gamma_j+1)}. \end{aligned}$$

Proof. Utilizing Remark 2, system Eq. (22) yields

$$x(t) = x_0 + \int_0^{\psi} \frac{(\psi - s)^{\varkappa-1}}{\Gamma(\varkappa)} \mathrm{g}(s, x(s), y(s), z(s))ds + \frac{1}{\Gamma(\alpha_j)} \int_0^t (t-s)^{\alpha_j - 1} X(s)ds + \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^t (t-s)^{\alpha_j-1} X(s)ds,$$

$$y(t) = y_0 + \int_0^{\psi} \frac{(\psi - s)^{\delta-1}}{\Gamma(\delta)} \mathrm{g}(s, y(s), x(s), z(s))ds + \frac{1}{\Gamma(\beta_j)} \int_0^{\psi} (\psi-s)^{\beta_j - 1} Y(s)ds + \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^t (t-s)^{\beta_j-1} Y(s)ds,$$

and

$$z(t) = z_0 + \int_0^{\psi} \frac{(\psi - s)^{\lambda-1}}{\Gamma(\lambda)} \mathrm{g}(s, z(s), x(s), y(s))ds + \frac{1}{\Gamma(\gamma_j)} \int_0^{\psi} (\psi-s)^{\gamma_j - 1} Z(s)ds + \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^t (t-s)^{\gamma_j-1} Z(s)ds,$$

which both leads to

$$\begin{aligned}|x(t) - \Omega(x,y,z)(t)| &\le \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^t (t-s)^{\alpha_j-1} |X(s)|ds \\ &\le \frac{1}{\Gamma(\alpha_j)} \int_{t_{j-1}}^t (t-s)^{\alpha_j-1} \Delta I(t)ds \le \frac{\Delta\psi^{\alpha_j}}{\Gamma(\alpha_j+1)} I(t),\end{aligned}$$

$$|y(t) - \Omega(y,x,z)(t)| \le \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^t (t-s)^{\beta_j-1} |Y(s)|ds \le \frac{1}{\Gamma(\beta_j)} \int_{t_{j-1}}^t (t-s)^{\beta_j-1} \Delta Y(t)ds \le \frac{\Delta\psi^{\beta_j}}{\Gamma(\beta_j+1)} Y(t),$$

and

$$|z(t) - \Omega(z,x,y)(t)| \le \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^t (t-s)^{\gamma_j-1} |Z(s)|ds \le \frac{1}{\Gamma(\gamma_j)} \int_{t_{j-1}}^t (t-s)^{\gamma_j-1} \Delta Z(t)ds \le \frac{\Delta\psi^{\gamma_j}}{\Gamma(\gamma_j+1)} Z(t).$$

Hence the proof is complete.

Theorem 4. *Under assumptions* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$*, the solution of problem* (4) *is both Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable if*

$$\frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)} \ne 1.$$

Proof. Using Lemma 3 and arguments analogous to those in Theorem 3, we establish the desired result.

4. Examples

In this section, we demonstrate our findings through numerical examples.

Example 4.1 Consider the following system:

$$\begin{cases} D_t^{\alpha(t)} x(t) = e^{-t}\left(\frac{|x(t-0.35)|}{15+|x(t-0.35)|} + \frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|} + \frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|} + \frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \ t \in [0,2], \\ D_t^{\beta(t)} y(t) = e^{-t}\left(\frac{|y(t-0.35)|}{15+|y(t-0.35)|} + \frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|} + \frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|} + \frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \\ D_t^{\gamma(t)} z(t) = e^{-t}\left(\frac{|z(t-0.35)|}{15+|z(t-0.35)|} + \frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|} + \frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|} + \frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}\right), \\ x(0) = \frac{1}{\Gamma(0.6)}\int_0^1 (1-s)^{-0.2}\left(\frac{|x(s)-y(s)-z(s)|}{12+|x(s)-y(s)-z(s)|}\right) ds + 0.031, \\ y(0) = \frac{1}{\Gamma(0.7)}\int_0^1 (1-s)^{-0.3}\left(\frac{|y(s)-x(s)-z(s)|}{12+|y(s)-x(s)-z(s)|}\right) ds + 0.037, \\ z(0) = \frac{1}{\Gamma(0.8)}\int_0^1 (1-s)^{-0.4}\left(\frac{|z(s)-x(s)-y(s)|}{12+|z(s)-x(s)-y(s)|}\right) ds + 0.043. \end{cases}$$

In the presented system, the state variables $x(t)$, $y(t)$, and $z(t)$ represent the concentrations or flow potentials of three interacting fluid components, such as water, oil, and a chemical surfactant. The nonlinear function U governs the coupled dynamics and inter-species interactions, encapsulating the complex physical processes like capillary forces and relative permeability effects. The nonlocal initial conditions are defined by the function g, which models the initial pre-saturation history of the reservoir, reflecting how the system's starting state is dependent on its configuration over the interval $[0,\psi]$. The operators $D_t^{\alpha(t)}$, $D_t^{\beta(t)}$, and $D_t^{\gamma(t)}$ denote the Caputo derivatives of variable orders $\alpha(t)$, $\beta(t)$, and $\gamma(t)$, respectively, which are crucial for capturing the time-evolving memory and heterogeneities in the fluid's rheological behavior. The entire dynamics are analyzed over the compact time domain $T = [0,\psi]$.

It is clear that for $\psi = 2, \varkappa = 0.6, \delta = 0.7, \lambda = 0.8$, and $h = 0.25$, we have

$$\alpha(t) = \begin{cases} 0.65, & t \in [0,1], \\ 0.4, & t \in (1,2], \end{cases}, \beta(t) = \begin{cases} 0.75, & t \in [0,1], \\ 0.5, & t \in (1,2], \end{cases} \text{ and } \gamma(t) = \begin{cases} 0.85, & t \in [0,1], \\ 0.6, & t \in (1,2]. \end{cases}$$

Obviously, for $j = 1, 2, x_0 = 0.031$, $y_0 = 0.037$, $z_0 = 0.043$

$$\begin{cases} U(t, x(t-\phi), x(ht), y(ht), z(ht)) \\ = e^{-t}\left(\frac{|x(t-0.35)|}{15+|x(t-0.35)|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \\ U(t, y(t-\phi), y(ht), x(ht), z(ht)) \\ = e^{-t}\left(\frac{|y(t-0.35)|}{15+|y(t-0.35)|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \\ U(t, z(t-\phi), z(ht), x(ht), y(ht)) \\ = e^{-t}\left(\frac{|z(t-0.35)|}{15+|z(t-0.35)|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}\right), \\ \mathrm{g}(x(s), y(s), z(s)) = \left(\frac{|x(s)-y(s)-z(s)|}{12+|x(s)-y(s)-z(s)|}\right), \mathrm{g}(y(s), x(s), z(s)) \\ = \left(\frac{|y(s)-x(s)-z(s)|}{12+|y(s)-x(s)-z(s)|}\right), \mathrm{g}(z(s), x(s), y(s)) = \left(\frac{|z(s)-x(s)-y(s)|}{12+|z(s)-x(s)-y(s)|}\right). \end{cases}$$

Considering $x, x^* \in V_j, j = 1, 2$, we have

$$\begin{aligned} &|U(t, x(t-\phi), x(ht), y(ht), z(ht)) - U\left(t, x^*(t-\phi), x^*(ht), y^*(ht), z^*(ht)\right)| \\ &\leq \left|e^{-t}\right|\left|\frac{x(t-0.35)}{15+|x(t-0.35)|}-\frac{x^*(t-0.35)}{15+|x^*(t-0.35)|}\right| \\ &+\left|\frac{x\left(\frac{t}{4}\right)}{15+\left|x\left(\frac{t}{4}\right)\right|}-\frac{x^*\left(\frac{t}{4}\right)}{1+\left|x^*\left(\frac{t}{4}\right)\right|}\right|+\left|\frac{y\left(\frac{t}{4}\right)}{15+\left|y\left(\frac{t}{4}\right)\right|}-\frac{y^*\left(\frac{t}{4}\right)}{1+\left|y^*\left(\frac{t}{4}\right)\right|}\right|+\left|\frac{z\left(\frac{t}{4}\right)}{15+\left|z\left(\frac{t}{4}\right)\right|}-\frac{z^*\left(\frac{t}{4}\right)}{1+\left|z^*\left(\frac{t}{4}\right)\right|}\right| \\ &\leq \frac{1}{15}\left|x(t-0.35)-x^*(t-0.35)\right| + \\ &\left|x\left(\frac{t}{4}\right)-x^*\left(\frac{t}{4}\right)\right|+\left|y\left(\frac{t}{4}\right)-y^*\left(\frac{t}{4}\right)\right|+\left|z\left(\frac{t}{4}\right)-z^*\left(\frac{t}{4}\right)\right|, \end{aligned}$$

and

$$\begin{aligned} &|\mathrm{g}(x(s), y(s), z(s)) - \mathrm{g}\left(x^*(s), y^*(s), z^*(s)\right)| \\ &= \left|\frac{|x(s)-y(s)-z(s)|}{12+|x(s)-y(s)-z(s)|}-\frac{|x^*(s)-y^*(s)-z^*(s)|}{12+|x^*(s)-y^*(s)-z^*(s)|}\right| \\ &\leq \frac{1}{12}\left|(x(s)-y(s)-z(s))-(x^*(s)-y^*(s)-z^*(s))\right| \\ &\leq \frac{1}{12}\left(|x(s)-x^*(s)|+|y(s)-y^*(s)|+|z(s)-z^*(s)|\right). \end{aligned}$$

Case I: $(j=1,\psi=1)$, we have

$$\begin{cases} D_t^{\alpha_1(t)}x(t)=e^{-t}\left(\frac{|x(t-0.35)|}{15+|x(t-0.35)|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \quad t\in[0,1],\\ D_t^{\beta_1(t)}y(t)=e^{-t}\left(\frac{|y(t-0.35)|}{15+|y(t-0.35)|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right),\\ D_t^{\gamma_1(t)}z(t)=e^{-t}\left(\frac{|z(t-0.35)|}{15+|z(t-0.35)|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}\right),\\ x(0)=\frac{1}{\Gamma(0.6)}\int_0^1(1-s)^{-0.2}\left(\frac{|x(s)-y(s)-z(s)|}{12+|x(s)-y(s)-z(s)|}\right)ds+0.031,\\ y(0)=\frac{1}{\Gamma(0.7)}\int_0^1(1-s)^{-0.3}\left(\frac{|y(s)-x(s)-z(s)|}{12+|y(s)-x(s)-z(s)|}\right)ds+0.037,\\ z(0)=\frac{1}{\Gamma(0.8)}\int_0^1(1-s)^{-0.4}\left(\frac{|z(s)-x(s)-y(s)|}{12+|z(s)-x(s)-y(s)|}\right)ds+0.043. \end{cases} \tag{31}$$

Here, $L_1=\frac{1}{15}$ and $L_2=\frac{1}{12}$. Hence, clearly the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold. Also, we examine

$$\frac{3L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_1}}{\Gamma(\alpha_1+1)}\approx 0.40104<1\ ,\ \frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_1}}{\Gamma(\beta_1+1)}\approx 0.39045<1\ ,$$
$$\frac{3L_2(\psi)^{\lambda}}{\Gamma(\lambda+1)}+\frac{4L_1\psi^{\gamma_1}}{\Gamma(\gamma_1+1)}\approx 0.37986<1. \tag{32}$$

As a result, Theorem 1 provides a unique solution for Eq. (31). Additionally, one has

$$\frac{\theta\theta^*\Theta}{(1-\theta)\,(1-\theta^*)(1-\Theta))}\approx\frac{(0.22413)(0.23679)(0.24945)}{(1-0.22413)(1-0.23679)(1-0.24945)}\approx 0.029789\neq 1.$$

Therefore, the solution of Eq. (31) is both generalized Ulam–Hyers stable and Ulam–Hyers stable. Furthermore, Eq. (31) is also generalized Ulam–Hyers–Rassias stable and Ulam–Hyers–Rassias stable when the function $\xi(t)=\frac{t}{3}$ for $t\in[0,1]$.

Case II: $(j=2,\psi=2)$, we get

$$\begin{cases} D_t^{\alpha_2(t)}x(t)=e^{-t}\left(\frac{|x(t-0.35)|}{15+|x(t-0.35)|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right), \quad t\in[1,2],\\ D_t^{\beta_2(t)}y(t)=e^{-t}\left(\frac{|y(t-0.35)|}{15+|y(t-0.35)|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}\right),\\ D_t^{\gamma_2(t)}z(t)=e^{-t}\left(\frac{|z(t-0.35)|}{15+|z(t-0.35)|}+\frac{\left|z\left(\frac{t}{4}\right)\right|}{15+\left|z\left(\frac{t}{4}\right)\right|}+\frac{\left|x\left(\frac{t}{4}\right)\right|}{15+\left|x\left(\frac{t}{4}\right)\right|}+\frac{\left|y\left(\frac{t}{4}\right)\right|}{15+\left|y\left(\frac{t}{4}\right)\right|}\right),\\ x(0)=\frac{1}{\Gamma(0.6)}\int_0^1(1-s)^{-0.2}\left(\frac{|x(s)-y(s)-z(s)|}{12+|x(s)-y(s)-z(s)|}\right)ds+0.031,\\ y(0)=\frac{1}{\Gamma(0.7)}\int_0^1(1-s)^{-0.3}\left(\frac{|y(s)-x(s)-z(s)|}{12+|y(s)-x(s)-z(s)|}\right)ds+0.037,\\ z(0)=\frac{1}{\Gamma(0.8)}\int_0^1(1-s)^{-0.4}\left(\frac{|z(s)-x(s)-y(s)|}{12+|z(s)-x(s)-y(s)|}\right)ds+0.043. \end{cases} \tag{33}$$

With the same procedure as in Eq. (33), we may show that for $\psi = 2$

$$\frac{3L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)} + \frac{4L_1\psi^{\alpha_2}}{\Gamma(\alpha_2+1)} \approx 0.61713 < 1 \ , \ \frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)} + \frac{4L_1\psi^{\beta_2}}{\Gamma(\beta_2+1)} \approx 0.65083 < 1 \ ,$$
$$\frac{3L_2(\psi)^{\lambda}}{\Gamma(\lambda+1)} + \frac{4L_1\psi^{\gamma_2}}{\Gamma(\gamma_2+1)} \approx 0.68453 < 1. \tag{34}$$

Theorem 1 provides a unique solution for the system Eq. (33). Furthermore,

$$\frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)} \approx \frac{(0.29887)(0.36176)(0.42465)}{(1-0.29887)(1-0.36176)(1-0.42465)} \approx 0.178328 \neq 1.$$

As a result, the solution of Eq. (33) is both generalized Ulam–Hyers stable and Ulam–Hyers stable. Furthermore, Eq. (33) is also generalized Ulam–Hyers–Rassias stable and Ulam–Hyers–Rassias stable when the function $\xi(t) = \frac{t}{3}$ is defined for $t \in (1, 2]$.

Example 4.2 We now present the following system for further analysis:

$$\begin{cases} D_t^{\alpha(t)}x(t) = \sin\left(e^{-t}\right)\left(\frac{|x(t-0.25)|}{150+|x(t-0.25)|} + \frac{|x(\frac{t}{3})|}{150+|x(\frac{t}{3})|} + \frac{|y(\frac{t}{3})|}{150+|y(\frac{t}{3})|} + \frac{|z(\frac{t}{3})|}{150+|z(\frac{t}{3})|}\right), \quad t \in [0,3], \\ D_t^{\beta(t)}y(t) = \sin\left(e^{-t}\right)\left(\frac{|y(t-0.25)|}{150+|y(t-0.25)|} + \frac{|y(\frac{t}{3})|}{150+|y(\frac{t}{3})|} + \frac{|x(\frac{t}{3})|}{150+|x(\frac{t}{3})|} + \frac{|z(\frac{t}{3})|}{150+|z(\frac{t}{3})|}\right), \\ D_t^{\gamma(t)}z(t) = \sin\left(e^{-t}\right)\left(\frac{|z(t-0.25)|}{150+|z(t-0.25)|} + \frac{|z(\frac{t}{3})|}{150+|z(\frac{t}{3})|} + \frac{|x(\frac{t}{3})|}{150+|x(\frac{t}{3})|} + \frac{|y(\frac{t}{3})|}{150+|y(\frac{t}{3})|}\right), \\ x(0) = \frac{1}{\Gamma(0.6)}\int_0^1 (1-s)^{-0.4}\left(\frac{|x(s)-y(s)-z(s)|}{30+|x(s)-y(s)-z(s)|}\right) ds + 0.02, \\ y(0) = \frac{1}{\Gamma(0.5)}\int_0^1 (1-s)^{-0.5}\left(\frac{|y(s)-x(s)-z(s)|}{30+|y(s)-x(s)-z(s)|}\right) ds + 0.03, \\ z(0) = \frac{1}{\Gamma(0.4)}\int_0^1 (1-s)^{-0.6}\left(\frac{|z(s)-x(s)-y(s)|}{30+|z(s)-x(s)-y(s)|}\right) ds + 0.04. \end{cases}$$

For $\psi = 3, \varkappa = 0.6, \delta = 0.5, \lambda = 0.4, h = \frac{1}{3}$, one has

$$\alpha(t) = \begin{cases} 0.8, & t \in [0,1], \\ 0.875, & t \in (1,3], \end{cases} \quad \beta(t) = \begin{cases} 0.7, & t \in [0,1], \\ 0.785, & t \in (1,3], \end{cases} \text{ and } \gamma(t) = \begin{cases} 0.6, & t \in [0,1], \\ 0.695, & t \in (1,3]. \end{cases}$$

For $j = 1, 2, x_0 = 0.02$, $y_0 = 0.03$, and $z_0 = 0.04$, it is clear that

$$\begin{aligned} &U(t, x(t-\phi), x(ht), y(ht), z(ht)) \\ &= \sin\left(e^{-t}\right)\left(\frac{|x(t-0.25)|}{150+|x(t-0.25)|} + \frac{|x\left(\frac{t}{3}\right)|}{150+|x\left(\frac{t}{3}\right)|} + \frac{|y\left(\frac{t}{3}\right)|}{150+|y\left(\frac{t}{3}\right)|} + \frac{|z\left(\frac{t}{3}\right)|}{150+|z\left(\frac{t}{3}\right)|}\right), \\ &U(t, y(t-\phi), y(ht), x(ht), z(ht)) \\ &= \sin\left(e^{-t}\right)\left(\frac{|y(t-0.25)|}{150+|y(t-0.25)|} + \frac{|y\left(\frac{t}{3}\right)|}{150+|y\left(\frac{t}{3}\right)|} + \frac{|x\left(\frac{t}{3}\right)|}{150+|x\left(\frac{t}{3}\right)|} + \frac{|z\left(\frac{t}{3}\right)|}{150+|z\left(\frac{t}{3}\right)|}\right), \\ &U(t, z(t-\phi), z(ht), x(ht), y(ht)) \end{aligned}$$

$$
\begin{aligned}
&= \sin\left(e^{-t}\right)\left(\frac{|z(t-0.25)|}{150+|z(t-0.25)|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}\right),\\
&\mathrm{g}(x(s),y(s),z(s)) = \left(\frac{|x(s)-y(s)-z(s)|}{30+|x(s)-y(s)-z(s)|}\right),\\
&\mathrm{g}(y(s),x(s),z(s)) = \left(\frac{|y(s)-x(s)-z(s)|}{30+|y(s)-x(s)-z(s)|}\right),\\
&\mathrm{g}(z(s),x(s),y(s)) = \left(\frac{|z(s)-x(s)-y(s)|}{30+|z(s)-x(s)-y(s)|}\right).
\end{aligned}
$$

Let $x, x^* \in V_j, j = 1, 2$, one has

$$
\begin{aligned}
&|U(t, x(t-\phi), x(ht), y(ht), z(ht)) - U\left(t, x^*(t-\phi), x^*(ht), y^*(ht), z^*(ht)\right)|\\
&\leq \left|\sin\left(e^{-t}\right)\right|\left|\frac{x(t-0.25)}{150+|x(t-0.25)|}-\frac{x^*(t-0.25)}{150+|x^*(t-0.25)|}\right|\\
&+\left|\frac{x\left(\frac{t}{3}\right)}{150+\left|x\left(\frac{t}{3}\right)\right|}-\frac{x^*\left(\frac{t}{3}\right)}{1+\left|x^*\left(\frac{t}{3}\right)\right|}\right|+\left|\frac{y\left(\frac{t}{3}\right)}{150+\left|y\left(\frac{t}{3}\right)\right|}-\frac{y^*\left(\frac{t}{3}\right)}{150+\left|y^*\left(\frac{t}{3}\right)\right|}\right|\\
&+\left|\frac{z\left(\frac{t}{3}\right)}{150+\left|z\left(\frac{t}{3}\right)\right|}-\frac{z^*\left(\frac{t}{3}\right)}{150+\left|z^*\left(\frac{t}{3}\right)\right|}\right|\\
&\leq \frac{1}{150}\left|x(t-0.25)-x^*(t-0.25)\right|+\left|x\left(\frac{t}{3}\right)-x^*\left(\frac{t}{3}\right)\right|+\left|y\left(\frac{t}{3}\right)-y^*\left(\frac{t}{3}\right)\right|\\
&+\left|z\left(\frac{t}{3}\right)-z^*\left(\frac{t}{3}\right)\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathrm{g}(x(s),y(s),z(s)) - \mathrm{g}\left(x^*(s),y^*(s),z^*(s)\right)|\\
&= \left|\frac{|x(s)-y(s)-z(s)|}{40+|x(s)-y(s)-z(s)|}-\frac{|x^*(s)-y^*(s)-z^*(s)|}{40+|x^*(s)-y^*(s)-z^*(s)|}\right|\\
&\leq \frac{1}{30}\left|(x(s)-y(s)-z(s))-(x^*(s)-y^*(s)-z^*(s))\right|\\
&\leq \frac{1}{30}\left(|x(s)-x^*(s)|+|y(s)-y^*(s)|+|z(s)-z^*(s)|\right).
\end{aligned}
$$

Case I: $(j=1,\psi=1)$, we have

$$\begin{cases} D_t^{\alpha_1(t)}x(t)=\sin\left(e^{-t}\right)\left(\frac{|x(t-0.25)|}{150+|x(t-0.25)|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}\right), \quad t\in[0,1],\\ D_t^{\beta_1(t)}y(t)=\sin\left(e^{-t}\right)\left(\frac{|y(t-0.25)|}{150+|y(t-0.25)|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}\right),\\ D_t^{\gamma_1(t)}z(t)=\sin\left(e^{-t}\right)\left(\frac{|z(t-0.25)|}{150+|z(t-0.25)|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}\right),\\ x(0)=\frac{1}{\Gamma(0.6)}\int_0^1(1-s)^{-0.4}\left(\frac{|x(s)-y(s)-z(s)|}{30+|x(s)-y(s)-z(s)|}\right)ds+0.02,\\ y(0)=\frac{1}{\Gamma(0.5)}\int_0^1(1-s)^{-0.5}\left(\frac{|y(s)-x(s)-z(s)|}{30+|y(s)-x(s)-z(s)|}\right)ds+0.03,\\ z(0)=\frac{1}{\Gamma(0.4)}\int_0^1(1-s)^{-0.6}\left(\frac{|z(s)-x(s)-y(s)|}{30+|z(s)-x(s)-y(s)|}\right)ds+0.04. \end{cases} \tag{35}$$

Clearly, $L_1=\frac{1}{150}$ and $L_2=\frac{1}{30}$. Hence, the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold. Also, we have

$$\frac{3L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_1}}{\Gamma(\alpha_1+1)}\approx 0.0782<1\ ,\ \frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_1}}{\Gamma(\beta_1+1)}\approx 0.0972<1\ ,$$
$$\frac{3L_2(\psi)^{\lambda}}{\Gamma(\lambda+1)}+\frac{4L_1\psi^{\gamma_1}}{\Gamma(\gamma_1+1)}\approx 0.0972<1. \tag{36}$$

The Eq. (35) has a unique solution using Theorem 1. Additionally, we get

$$\frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)}\approx\frac{(0.05878)(0.08256)(0.10634)}{(1-0.05878)(1-0.08256)(1-0.10634)}\approx 0.0006688\neq 1.$$

The solution of Eq. (35) is both Ulam–Hyers stable and generalized Ulam–Hyers stable. Furthermore, if we assume that $t\in[0,1]$ and $\xi(t)=\frac{t}{3}$, then Eq. (35) is stable in the sense of both Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability.
Case II: $(j=2,\psi=3)$, we have

$$\begin{cases} D_t^{\alpha_2(t)}x(t)=\sin\left(e^{-t}\right)\left(\frac{|x(t-0.25)|}{150+|x(t-0.25)|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}\right),\ t\in[1,3],\\ D_t^{\beta_2(t)}y(t)=\sin\left(e^{-t}\right)\left(\frac{|y(t-0.25)|}{150+|y(t-0.25)|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}\right),\\ D_t^{\gamma_2(t)}z(t)=\sin\left(e^{-t}\right)\left(\frac{|z(t-0.25)|}{150+|z(t-0.25)|}+\frac{\left|z\left(\frac{t}{3}\right)\right|}{150+\left|z\left(\frac{t}{3}\right)\right|}+\frac{\left|x\left(\frac{t}{3}\right)\right|}{150+\left|x\left(\frac{t}{3}\right)\right|}+\frac{\left|y\left(\frac{t}{3}\right)\right|}{150+\left|y\left(\frac{t}{3}\right)\right|}\right),\\ x(0)=\frac{1}{\Gamma(0.6)}\int_0^1(1-s)^{-0.4}\left(\frac{|x(s)-y(s)-z(s)|}{30+|x(s)-y(s)-z(s)|}\right)ds+0.02,\\ y(0)=\frac{1}{\Gamma(0.5)}\int_0^1(1-s)^{-0.5}\left(\frac{|y(s)-x(s)-z(s)|}{30+|y(s)-x(s)-z(s)|}\right)ds+0.03,\\ z(0)=\frac{1}{\Gamma(0.4)}\int_0^1(1-s)^{-0.6}\left(\frac{|z(s)-x(s)-y(s)|}{30+|z(s)-x(s)-y(s)|}\right)ds+0.04. \end{cases} \tag{37}$$

Using the same procedure, we obtain the following.

$$\frac{3L_2(\psi)^{\varkappa}}{\Gamma(\varkappa+1)}+\frac{4L_1\psi^{\alpha_2}}{\Gamma\left(\alpha_2+1\right)}<1\ ,\ \frac{3L_2(\psi)^{\delta}}{\Gamma(\delta+1)}+\frac{4L_1\psi^{\beta_2}}{\Gamma\left(\beta_2+1\right)}<1\ ,\ \frac{3L_2(\psi)^{\lambda}}{\Gamma(\lambda+1)}+\frac{4L_1\psi^{\gamma_2}}{\Gamma\left(\gamma_2+1\right)}<1.$$

Theorem 1 provides a unique solution for Eq. (37). Furthermore, we acquire

$$\frac{\theta\theta^*\Theta}{(1-\theta)(1-\theta^*)(1-\Theta)}\neq 1.$$

The solution to Eq. (37) satisfies both the generalized Ulam–Hyers stability and the Ulam–Hyers stability. Furthermore, for $t\in(1,3]$, we can infer that Eq. (37) is both Ulam–Hyers–Rassias stable and generalized Ulam–Hyers–Rassias stable if we take $\xi(t)=\frac{t}{3}$.

5. Conclusions

In this study, we investigated a class of fractional differential equations with mixed-type delays and variable-order fractional derivatives subject to integral boundary conditions. By employing fixed point theory, we established results concerning the existence and uniqueness of solutions. It was observed that proving such results for variable-order fractional derivatives presents significant challenges due to the complexity of the system.

Furthermore, we derived important results related to various stability concepts, including Ulam–Hyers stability, generalized Ulam-Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. These analyses are particularly challenging when dealing with systems involving mixed delays and variable-order fractional derivatives.

Finally, numerical examples were provided to illustrate and validate the theoretical results obtained in this work.

6. Future Directions

Future research will focus on establishing theorems regarding existence, uniqueness, and stability for variable-order fractional stochastic differential equations, thereby extending the current framework.

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