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On Finite Groups with Transfer Maps and Weak Closure

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Abstract. Let G be a finite group, $P \in Syl_p(G)$ and $W \subseteq P$. We say that W is weakly closed in P with respect to G if $W^g \subseteq P$ for all $g \in G$. In this paper, we explore structural properties of finite groups using the transfer homomorphism and the notion of weak closure in Sylow subgroups. We establish that if a central subgroup $H \leq Z(G)$ has finite index $[G : H]$ coprime to $|H|$, then $G \cong H \times \ker(v)$, where $v : G \to H$ is the transfer. Furthermore, we characterize weakly closed subgroups $W \leq P \in Syl_p(G)$ as normal in both $N_G(P)$ and all Sylow p-subgroups containing W. Several consequences concerning conjugacy and normality are discussed.

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1. Introduction and Preliminaries

All groups are assumed to be finite. The internal structure of finite groups has long been investigated through the lens of transfer homomorphisms and Sylow theorem [1], tools which continue to provide profound insights into group decompositions and subgroup interactions [2]. The transfer map, introduced in classical group theory, remains a valuable technique for relating global properties of a group to its substructures, particularly when considering questions of direct product decompositions and coprime conditions [3]. For example, if H is a finite central subgroup of a group G, and the index $[G : H]$ is finite and relatively prime to $\mid H \mid$, then the transfer homomorphism $v : G \to H$ may be used to show that $G \cong H \times ker(v)$, effectively separating G into two commuting, intersecting-trivially components [4]. This classical result showcases the interplay between arithmetic conditions and the homomorphic structure of finite groups [5] and [6].

Beyond this, the concept of weak closure has emerged as a key feature in understanding conjugacy phenomena in Sylow subgroups. Given a Sylow p-subgroup P of G, a subgroup

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$W \subseteq P$ is said to be weakly closed in P with respect to G if no other G-conjugate of W lies within P. This property, though defined in terms of conjugation, has strong implications for normality: W is weakly closed if and only if it is normal in $N_G(P)$ and remains normal in every Sylow p-subgroup of G that contains it [7] and [8]. These conditions align well with results in fusion theory and control of transfer, where conjugacy within Sylow subgroups plays a critical role in understanding the larger ambient group [9].

This paper develops the underlying concepts in tandem, beginning with the identification of criteria that allow a finite group to be analyzed via the transfer method, and proceeding to a detailed examination of the relationship between weak closure and normality within Sylow subgroups. The central focus is on how the behavior of local subgroups both influences and reflects the overall structure of the group. The following Lemmas are well-known and basic concepts:

Lemma 1. *[9]A subgroup H of G having order a power of a prime is normal if and only if it is subnormal in G and weakly closed in G.*

Lemma 2. *[6] Let G be a group and let H be a weakly closed subgroup of G of prime power order. Assume $K \leq G$ and $N \trianglelefteq G$. Then:*

(i) If $H \leq K$, then H is weakly closed in K.

(ii) HN/N is weakly closed in G/N and HN is weakly closed in G.

(iii) K is weakly closed in G if and only if K/N is weakly closed in G.

Lemma 3. *[10] If P is a Sylow p-subgroup of a group G and $N \trianglelefteq G$ such that $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Definition 1. *[11] Let G be a finite group, $P \in Syl_p(G)$, and $W \subseteq P$. We say that W is weakly closed in P with respect to G if whenever $W^g \subseteq P$ for some $g \in G$, it follows that $W^g = W$.*

Definition 2. *[12] A subgroup of a group is called a characteristic subgroup if, for every automorphism of G, one has $\sigma(H) = H$.*

Lemma 4. *Let G be an abelian group and $H \subseteq G$ has index n. The transfer homomorphism from G to H is given by the map $g \to g^n$, i.e., for each $g \in G$, the transfer homomorphism maps g to g^n.*

Proof. We need to show that the transfer homomorphism $\tau : G \to H$ is the map $g \to g^n$. Since G is abelian, conjugation by any element of G leaves elements of H unchanged. Thus, for all $h \in H$, we have: $ghg^{-1} = h$. Now $g \to g^n$. The group G/H has index n, meaning there are n cosets of H in G. The $g \mapsto g^n$ in the quotient group G/H corresponds to raising each element to the power of n. For each $g \in G$, we have: $g^n \in H$. Thus, the transfer homomorphism maps each element $g \in G$ to g^n in H. Hence, we conclude that the transfer homomorphism from G to H is the map $g \to g^n$, as required.

Lemma 5. *Let G be a group and G'´is commutator subgroup. The transfer homomorphism $v : G \to G/G'$ is the same as the canonical homomorphism from G to G/G'. i.e., $v(g) = gG'$´for all $g \in G$.*

Proof. The canonical homomorphism from G to G/G' is defined by $\pi(g) = gG'$ for each $g \in G$. This is the natural projection map that sends each element $g \in G$ to its coset in the quotient group G/G'. The transfer homomorphism v from $G \to G/G'$ is defined as: $v(g) = \sum_{h \in G'} ghg^{-1}$. Since G' is the commutator subgroup of G, and G/G' is abelian (by definition of the commutator subgroup), the conjugation by any element of G leaves G' invariant. Therefore, the action of the transfer homomorphism v on any $g \in G$ results in the coset gG' in G/G'. Hence, we have: $v(g) = gG'$ for all $g \in G$.

Example 1. *Let $G = S_3$, the symmetric group on three elements, which has order 6. The commutator subgroup H of S_3 is the alternating group A_3, consisting of the even permutations. This subgroup has order 3 and is generated by the 3-cycles* $(123)(123)(123)$ *and* $(132)(132)(132)$.*The canonical homomorphism $\pi : G \to G/H$ is defined by $\pi(g) = gH$, which assigns to each element g the coset of g in the quotient group G/H. In this case, we have $G/H = S_3/A_3$, which is isomorphic to $Z/2Z$ since there are exactly two cosets corresponding to the parity of permutations (even or odd). The transfer homomorphism $v : G \to G/H$ can be described intuitively as measuring the effect of conjugation on H. Since $H = A_3$ is a normal subgroup and G/H is abelian, conjugation by any element of G leaves H invariant. In practice, for any $g \in G$, the transfer homomorphism gives the same result as the canonical projection. Now, if $g = (12)$, which is an odd permutation, the image under the transfer homomorphism is $v((12)) = (12)H$, the coset corresponding to odd permutations. This exactly matches $\pi(12)$.*

Therefore, in this example, the transfer homomorphism v coincides with the canonical homomorphism π.

Both the transfer homomorphism v and the canonical homomorphism π map (12) to the same coset, which is $(12)A_3$.

2. Main Results

in this section, we present the core theorems that connect the transfer homomorphism and the weak closure property within Sylow subgroups. These results provide explicit criteria for identifying weakly closed subgroups and reveal their structural role in finite group theory.

Lemma 6. *Let G be a finite group, $P \in Syl_p(G), P$ be a Sylow p-subgroup of G, and suppose $g \in G$ has order p, $g \in G'$(the commutator subgroup of G), and $g \notin P'$(the commutator subgroup of P). Then there exists an element $i \in G$ such that $i \in P'$and $i \notin P$.*

Proof. Let $u : G \to P/P'$ be the transfer homomorphism from G to P/P'. Since $g \in G'$ and $g \notin P'$ it follows that $g \in ker(u)$, meaning $u(g) = 0$ in P/P'. By the pretransfer map V, we then have $V(g) \in P'$. Since $g \notin P'$, it follows that$V(g) \notin P$. Therefore, there exists an element $i = V(g) \in G$ such that $i \in P'$ but $i \notin P$, completing the proof.

Example 2. *Consider the symmetric group S_5, the group of all permutations on 5 elements. Let $p = 2$, so we are interested in Sylow 2-subgroups of S_5. The order of a Sylow 2-subgroup of S_5 is the highest power of 2 dividing 120, which is 8. Hence, the Sylow 2-subgroups of S_5 have order 8. The commutator subgroup G' of S_5 is the alternating group A_5, consisting of all even permutations. The order of A_5 is 60.*

Consider the permutation $g = (12)(34)$, which swaps elements 1 with 2 and 3 with 4. This permutation has order 2, and since it is an even permutation, we know $g \in A_5 = G'$.

Suppose we consider the Sylow 2-subgroup $P = \langle (12)(34), (13)(24), (14)(23) \rangle$, a group of order 8. The commutator subgroup of P, denoted P', consists of the commutators of elements of P. However, $g = (12)(34)$ is not in P', because it cannot be written as a commutator of elements of P due to the structure of P.

According lemma 6, there must exist an element $i \in G_i$ such that $i \in P'_i$ in $P'_i \in P'$ but $i \notin P_i$ notin $P_i \notin P$. In this case, the element i that satisfies this is $i = (12)(34)$, which lies in $G' = A_5$ but not in P, and $i \in P'_i$, since $g \in G'$.

Proposition 1. *Let G be a group, and let G'denote its commutator subgroup. The transfer homomorphism v: $G \to G/G'$is identical to the canonical homomorphism from $G \to G/G'$That is, for all $g \in G$, we have: $v(g) = gG'$ where $v(g)$ is the transfer homomorphism and gG' is the coset of g in the quotient group G/G'.*

Proof. The canonical homomorphism from $G \to G/G'$ is defined by $\pi(g) = gG'$ for each $g \in G$, which is the natural projection map sending g to its coset in the quotient group G/G'.

The transfer homomorphism v is defined as:

$v(g) = \Sigma_{h \in G} ghg^{-1}$. Since G' is the commutator subgroup of G, and G/G' is abelian, the conjugation by any element of G leaves G' invariant. Therefore, the action of the transfer homomorphism v on any $g \in G$ results in the coset gG' in G/G'.

Hence, $v(g) = gG'$ for all $g \in G$, establishing that the transfer homomorphism is the same as the canonical homomorphism.

Theorem 1. *Let G be a simple group, and let P be an abelian Sylow 2-subgroup of G of order 25. Then P is elementary abelian.*

Proof. Now, P is abelian and of order 25, it must be isomorphic to either $Z/25Z$ or $Z/5Z \times Z/5Z$ However, G is simple, and if $P \cong Z/25Z$, then the cyclic group generated by an element of order 25 would be a nontrivial normal subgroup of G, contradicting the simplicity of G. Therefore, P must be isomorphic to $Z/5Z \times Z/5Z$.

The group $Z/5Z \times Z/5Z$ is elementary abelian because it can be viewed as a 2-dimensional vector space over $Z/2Z$, where every non-identity element has order 2. Hence, P is elementary abelian. Thus, P is elementary abelian.

Theorem 2. *Let $P \in Syl_p(G)$, and suppose H and K are normal subgroups of P that are conjugate in G. Then, H and K are conjugate in the normalizer $N_G(P)$of P. If H is characteristic in P, then $H = K$.*

Proof. Let H and K be conjugate in G, meaning there exists $g \in G$ such that $K = gHg^{-1}$. Since both H and K are normal subgroups of P, we have $H \trianglelefteq P$ and $K \trianglelefteq P$, so for every $p \in P$, we have $pHp^{-1} = H$ and $pKp^{-1} = K$.

To show that H and K are conjugate in $N_G(P)$, note that $K = gHg^{-1}$. To ensure K is conjugate in $N_G(P)$, we require that $g \in N_G(P)$, and since H and K lie within P, g must normalize P. Therefore, $g \in N_G(P)$, and thus H and K are conjugate in $N_G(P)$.

Next, suppose H is characteristic in P . By definition 2, since H and$K$ are conjugate in$N_G(P)$, there exists some $n \in N_G(P)$ such that $K = nHn^{-1}$. However, since H is characteristic in P, K must be identical to H.

Theorem 3. *Let G be a finite group, and suppose that $W \subseteq P \subseteq G$, where $P \in Syl_p(G)$. We say that W is weakly closed in P with respect to G if for every $g \in G^g W^g \subseteq P \Longrightarrow W^g = W$. Then W is weakly closed in P with respect to G if and only if $W \trianglelefteq N_G(P)$ and $W \trianglelefteq Q$ for every $Q \in Syl_p(G)$ such that $W \subseteq Q$.*

Proof. $\Rightarrow$) Assume W is weakly closed in P with respect to G.

Since $N_G(P) \leq G$, then for all $n \in N_G(P)$, we have $W^n \leq P$.

By weak closure, $W^n = W$. Hence, W is invariant under conjugation by all elements of $N_G(P)$, i.e., $W \trianglelefteq N_G(P)$. Let $Q \in Syl_p(G)$ with $W \leq Q$. By Sylow's Theorem, $Q = P^g$ for some $g \in G$, so $W \leq P_g \Rightarrow Wg^{-1} \leq P$. By weak closure, $Wg^{-1} = W \Rightarrow W^g = W \subseteq P^g = Q$.For any $x \in Q^x$, write $x = gpg^{-1}$ for some $p \in P$. Then $W^x = W^{gpg^{-1}} = (W^g)^{pg^{-1}} = W^{pg^{-1}}$. Since $W^g \leq Q$, and $p \in P$, $W^{pg^{-1}} \leq Q$. But this is again a G-conjugate of W lying in P, so again by weak closure, $W^{pg^{-1}} = W \Rightarrow W^x = W$. Thus, $W \trianglelefteq Q$.

$\Longleftarrow$) Now assume: $W \trianglelefteq N_G(P)$, For every $Q \in Syl_p(G)$ with $W \leq QW$, we have $W \trianglelefteq QW$.

Let $g \in G$ such that $W^g \leq P$. Then P^g is another Sylow p-subgroup of G, so there exists $x \in G$ such that $P^g = P^x$, or equivalently $gx^{-1} \in N_G(P)$. Define $h = gx^{-1} \in N_G(P)$ $\Longrightarrow g = hxh$. Then: $W^g = Wh^x = (W^x)h$.

Since $W^x \leq P^x = P^g = P$, and $W \trianglelefteq N_G(P) \Longrightarrow (W^x)h = W^x$. So $W^g = W^x$, but $W^x \subseteq P$, so it is a G-conjugate of W lying in P.

Now, since $W \trianglelefteq Q$ for any Sylow$p$-subgroup Q containing it, and $W^x \subseteq P$, and $W^x \trianglelefteq P$, it must be that $W^x = W$.

Hence $W^g = W$, so W is weakly closed in P with respect to G.

Corollary 1. *Let G be a finite group and p be a prime, and let $P \in Syl_p\ (G)$.Suppose $W \leq P$ is weakly closed in P with respect to G, i.e..$\forall g \in G : W^g \leq P \Longrightarrow W^g = W$.Then every G-conjugate of W that is contained in P is (trivially) P-conjugate to W.*

3. Conclusion

This study investigated the internal structure of finite groups via the transfer homomorphism $v : G \to H$ and the weakclosure property in a Sylow subgroup $P \in Syl_p(G)$, characterized by the conditions $W \trianglelefteq N_G\ (P)$ and $W \trianglelefteq Q$ for every $Q \in Syl_p(G)$ with

$W \leq Q$. The results unify classical subgroup properties with modern insights into conjugacy control and normality within p-local settings. Future work could explore generalizing these characterizations to other subgroup invariance concepts in algebra, aiming to identify broader structural patterns in finite group theory.

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References

[1] K. M. Aljamal, A. T. Ab Ghani, and R. M. Saleh. On preimages of technology. In *Proceedings of the International Conference on Information Technology (ICIT)*, pages 340–343. IEEE, 2021.

[2] A. M. Alotaibi and K. M. Aljamal. Exploring the associated groups of quasi-free groups. *European Journal of Pure and Applied Mathematics*, 17(3):2329–2335, 2024.

[3] A. Alotaibi. Sign-symmetry and frustration index in signed graphs. Master's thesis, Mississippi State University, 2023.

[4] A. M. Alotaibi, K. Al-Tahat, and K. M. Aljamal. Notes on finite groups with nearly s-permutable and nearly s-permutable-transitive subgroups. *European Journal of Pure and Applied Mathematics*, 18(3):6033–6033, 2025.

[5] W. Burnside. *Theory of Groups of Finite Order.* Cambridge University Press, 2 edition, 1911.

[6] I. M. Isaacs. *Finite Group Theory*, volume 92 of *Graduate Studies in Mathematics.* American Mathematical Society, 2008.

[7] D. Gorenstein. *Finite Groups.* Harper & Row, 1968.

[8] J. L. Alperin. *Local Representation Theory.* Cambridge University Press, 1986.

[9] M. Aschbacher, R. Kessar, and B. Oliver. *Fusion systems in algebra and topology.* Cambridge University Press, New York, NY, 2011.

[10] J. Tate. Nilpotent quotient groups. *Topology*, 3:109–111, 1964.

[11] L. Puig. Frobenius categories versus fusion systems. *Groups, Geometry, and Dynamics*, 3(1):59–120, 2009.

[12] D. S. Dummit and R. M. Foote. *Abstract algebra.* John Wiley & Sons, Hoboken, NJ, 3 edition, 2004.