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Supplier Selection Using the TAOV Method under T-Spherical Fuzzy Soft Environment with Aczel-Alsina Operators

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Abstract. The Aczel-Alsina (AA) aggregation operators (AOs) are highly effective in optimizing complex decision-making (DM) tasks involving extensive datasets, particularly in the context of T-spherical fuzzy (T-SF) environments. This research creates a refined decision framework that integrates the AA T-norm (TN) and T-conorm (T-CN) operations with T-spherical fuzzy soft sets (T-SFSs). Two novel aggregation operators, named the T-SFS AA Weighted Geometric (T-SFSAAWG) and T-SFS AA Weighted Averaging (T-SFSAAWA) operators, are put forward. Their mathematical behavior, properties, and specific instances are thoroughly investigated. The Technique for Alternative Ordering of Variables (TAOV) is incorporated into the model to reinforce the ranking process. With the TAOV method, the time and computational effort required for ranking can be diminished, all while preserving precision and stability. Our proposed methodology is applied to a supplier selection problem, demonstrating its superior optimization effectiveness compared to current T-SFS and AA-based approaches. The model has been shown to yield consistent and efficient results through comparative and sensitivity analyses, establishing it as a practical and trustworthy resource for making data-informed decisions.

2020 Mathematics Subject Classifications: 90B50, 03E72, 90C31

Key Words and Phrases: Multi-attribute decision-making, T-Spherical soft sets, Aczel-Alsina TN and TCN, T-Spherical Fuzzy Logic, Fuzzy Soft Aczel-Alsina Weighted Averaging, Geometric operators, Optimization

1. Introduction

We apply AA T-N and T-CN-based AOs for T-SFSs in the aggregation of attributes T-SFSAAWG and T-SFSAAWA to evaluate five companies based on five key characteristics, which are used as measures for suppliers in this study: production, product quality, service, risks, and lead time. Each of these attributes is important in determining the best supplier, and its relative importance represents a weighted DM process. AA staff enables the aggregation of these assessments and effectively deals with uncertainties and ambiguities in the DM process. The use of the T-SFSAAWG and T-SFSAAWA operators assures that every aspect of the provider's performance is considered, resulting in a more thorough and unbiased assessment of each applicant. Despite the competing values, DMs can effectively consider multiple factors and arrive at an optimal solution due to the flexibility of the proposed operator. The company can maintain a competitive supply chain performance by leveraging this combination of personnel to help make more informed decisions.

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1.1. Literature Review

By defining the membership degree (MD) from 0 to 1, Zadeh [1] introduced fuzzy sets (FSs), which improved the representation of information over traditional sets to reduce information loss. This was extended by Atanassov [2] who expanded this using intuitionistic fuzzy sets (IFSs) and added a non-membership degree (NMD) with the restriction $0 \leq \mathrm{MD} + \mathrm{NMD} \leq 1$. With Pythagorean FSs (PyFSs) and q-rung orthopair FSs (q-ROFSs), which employ squared or powered degrees to increase flexibility, Yager [3] further developed the idea. Even with these developments, there were still difficulties in capturing complex data. Cuong [4] was the first to introduce Picture Fuzzy Set (PFS), which included four degrees: MD, NMD, and AD (abstention degree). Then Mahmood et al. [5] extended PFSs to spherical Fuzzy Set (SFSs) and T-SFS, which enabled greater flexibility in representing complex cases.

Soft set (SS) theory was first proposed by Molodtsov [6] as a tool for dealing with ambiguous objects. Extending this, Ali et al. [7] analyzed new operational laws in SS theory. Combining fuzzy and soft cluster theory, Cagman et al. [8] developed a fuzzy soft AOs that improves DM processes. T-SF clusters are classified as soft clusters by Perveen et al. [9], with an emphasis on DM. The AOs were developed by Gunner et al. Ahmmad et al. [10] defined the concept of mixed SFS clusters. Further studies with different AOs on SFS aggregates were carried out by other authors [10, 11]. Rough sets were studied by Sarfraz et al. [12, 13] in the case of AA operators. With applications to triangular norms, AA [14] developed the characterization of certain classes of quasilinear functions. AA TN, and TCN have been used in recent studies [15] to develop SF information models. The study of AA TN and TCM AOs in SF contexts is still ongoing [16]. AA AOs are still a relatively new concept in fuzzy mathematics, despite substantial foundational work [17, 18]. To find the best AA TN, Farahbod and Eftekhari [19] evaluated a variety of TNs and TCNs in a complicated organizational context. By adding Frank norms and conorms, Sarfraz [20] improved the PFs on the Maclaurin symmetric mean aggregation operator. TN concept and the TCM aggregation operator based on SF information. There is presently ongoing research on the theory and request of AOs based on TNs and TCNs [21, 22].

Abdelhafeez et al. [23] defined the TOPSIS method for assessing blockchain. Akram et al. [24] articulated the CRITIC-EDAS method for PyFs using MCDM. Wang et al. [25] fostered the CoCoSo-D method based on sine trigonometric functions with PyFs. Lohrmann et al. [26] suggested entropy measures for supervised feature selection. Altin et al. [27] fostered WASPAS and VIKOR methods for different environments. Sharma et al. [28] developed the WASPAS method with the application of medical science. Yin et al. [29] put forward an analysis to enhance energy performance using the entropy method. Alsanousi et al. [30] described the TOPSIS method in an MCDM environment. Li et al. [31] devised the EDAS method for MCDM. Yang et al. [32] articulated the MULTI-MOORA method with the selection of electric vehicle power battery recycling. Aydin et al. [33] devised the TAOV method for the application of renewable energy. Liaqat et al. [34] developed a study on fuzzy using the trade flow application.

Wan et al. [35] used a supplier selection application with fuzzy hesitant measures. Rouyendegh et al. [36] utilized the supplier selection problem on the IF TOPSIS method. Liaqat et al. [34, 37] defined the study of fuzzy using the MADM environment. Modibbo et al. [38] introduced the supplier selection problem on the fuzzy TOPSIS method with MADM. Lu et al. [39] developed green supplier selection under the PF environment. Jain et al. [40] used supplier selection under the fuzzy inference system. Kilic et al. [41] introduced the green supplier selection with fuzzy goal programming. Hoseini et al. [42] developed supplier selection with a hybrid fuzzy environment.

1.2. Research Gap and Motivation

Our work focuses on creating new aggregation operators (AOs) for T-Spherical Fuzzy Sets (T-SFSs) using the T-conorm (TCN) and T-norm (TN) of Aczel–Alsina (AA). These operators address the shortcomings of earlier research and offer a more efficient framework for aggregating T-spherical fuzzy values (T-SFVs). Unlike previous AOs that failed to fully capture some nuances and complexities, the proposed AOs provide greater accuracy and flexibility. As far as we know, this particular AOs has not yet been investigated, adding more to the statistical uncertainty. The main part of this paper will demonstrate the efficiency and utility of our approach in practical decision situations: these are some of the contributions of my article.

(i) Using AA TN and TCN, new **T-SFSAAWG** and **T-SFSAAWA** AOs are proposed for efficiently handling T-SFS information.

(ii) To investigate the suitability and relevance of the suggested T-SFSAAWG and T-SFSAAWA operators in handling difficult decision-making (DM) issues, emphasizing their advantages over current approaches.

(iii) To carry out an exhaustive examination of the mathematical characteristics and features of the proposed AOs to ensure their consistency, dependability, and adaptability in real-world applications.

(iv) Using T-SFSAAWA and T-SFSAAWG operators in Multi-Attribute Decision-Making (MADM) scenarios to demonstrate their superior performance in managing ambiguity and uncertainty.

(v) To compare the results of the recommended operators with those of reputable aggregation operators to confirm the effectiveness of the T-SFS-based AOs in offering more accurate and comprehensive solutions to MADM problems.

(vi) To remedy the deficiencies of the prior AOs and bridge the gaps by presenting a new aggregation framework that provides DM with a more potent instrument for managing imprecise and inadequate data in fuzzy contexts.

In comparison to current fuzzy soft MADM methods, the proposed **TAOV−AA framework** offers three significant advantages:

(i) By combining the TAOV weighting mechanism with AA aggregation, it offers a more balanced and nonlinear information fusion.

(ii) It encompasses the T-SFS environment, enabling a flexible simultaneous modeling of hesitation, neutrality, and uncertainty.

(iii) The model guarantees parameter adaptability via the t parameter, which improves decision stability in comparison to traditional fuzzy, intuitionistic, and Pythagorean methods.

The structure of the paper is set out as follows: Section 1 includes the introduction, a synopsis of previous research, and relevant literature. Important ideas about T-SFSs, AOs, AA TN, and TCN are covered in Section 2. The properties of the suggested T-SFSAAWA and T-SFSAAWG operators are presented and examined in Section 3. The application of the T-SFSAAWA and T-SFSAAWG operators to an MADM algorithm is presented in Section 4. In Section 5, a thorough DM example is provided. A comparison of the suggested strategy and current approaches is provided in Section 6. Section 7 provides a summary of the findings and concluding remarks to bring the paper to a close.

2. Preliminaries

The key ideas covered in this section provide an overview of the suggested reading. These ideas will aid in our comprehension of this article. We give the meanings of AA, TN, TCN, and T-SFSs.

Definition 1: [2] Consider that F is a universe of dissertations, a T-SFS in F is an appearance φ that is provided through

$$E = \{(\xi, \Delta_E(\xi), \theta_E(\xi), \varphi_E(\xi) : \xi \in F)\}$$

Where the fundamentals Δ_E, θ_E and φ_E are MD, AD, and NMD function such that $\Delta_E : F \to [0,1]$, $\theta_E : F \to [0,1]$ and $\varphi_E : F \to [0,1]$ with $0 \leq \Delta_E^t(\xi) + \theta_E^t(\xi) + \varphi_E^t(\xi) \leq 1$, and $\pi_E(\xi) = 1 - \Delta_E^t(\xi) - \theta_E^t(\xi) - \varphi_E^t(\xi)$, $\forall \xi \in F$ is called the hesitancy degree (HD) of ξ to E. Further, $(\xi, \Delta_E(\xi), \theta_E(\xi), \varphi_E(\xi))$ is known as T-SFV.

Definition 2: [11] Consider that F is a universe of dissertations, a T-SFS in F is an appearance φ that is provided through. Consider E a parameter space and let F be a universal set. To represent the set of all T-SFs of F, let $T = SF(F)$. A T-SFS is a pair (L, A) where $A = E$ and L is a mapping defined as: $L : A \to T - SF(F)$. A T-SFS is defined as a parameterized family of SF subsets of F, not just a set. For any $h \in A$, $L(h)$ is a T-SFs of F, and can be expressed as follows:

$$E = \{(\xi, \Delta_E(\xi), \theta_E(\xi), \varphi_E(\xi) : \xi \in F)\}$$

Where the fundamentals Δ_E, θ_E and φ_E are MD, AD, and NMD function such that $\Delta_E : F \to [0,1]$, $\theta_E : F \to [0,1]$ and $\varphi_E : F \to [0,1]$ with $0 \leq \Delta_E^t(\xi) + \theta_E^t(\xi) + \varphi_E^t(\xi) \leq 1$, and $\pi_E(\xi) = 1 - \Delta_E^t(\xi) - \theta_E^t(\xi) - \varphi_E^t(\xi)$, $\forall \xi \in F$ is called the hesitancy degree (HD) of ξ to E. Further, $(\xi, \Delta_E(\xi), \theta_E(\xi), \varphi_E(\xi))$,it is known as T-SFSV.

Definition 3: [43] Consider $E_{11} = (\Delta_{E_{11}}, \theta_{E_{11}}, \varphi_{E_{11}})$ be a T-SFSV. Then

$$Sco(E_{11}) = \frac{(1 + \theta_{E_{11}}^t - Y_{E_{11}}^t - \varphi_{E_{11}}^t)}{2} \tag{1}$$

Be the score value of E_{11}.

Definition 4: [43] Consider $E_{11} = (\Delta_{E_{11}}, \theta_{E_{11}}, \varphi_{E_{11}})$ be a T-SFSV. Then

$$Acc(E_{11}) = \frac{(1 + \theta_{E_{11}}^t + \theta_{E_{11}}^t + \varphi_{E_{11}}^t)}{2} \tag{2}$$

Be the degrees of accuracy of E_{11}.

(i) If $Sco(E_{11}) < Sco(E_{22})$, then E_{11} has less partiality than E_{22}.

(ii) If $Sco(E_{11}) = Sco(E_{22})$, then E_{11} and E_{22} are same.

(iii) If $Acc(E_{11}) < Acc(E_{22})$, then E_{11} has less partiality than E_{22}.

(iv) If $Acc(E_{11}) = Acc(E_{22})$, then E_{11} and E_{22} are same.

Definition 5: [19]: The T-Ns $(L_A^\lambda)_{\lambda \in [0,\infty]}$ for Aczél–Alsina (AA) are distinct as

$$(\mathcal{L}_\lambda^\lambda)_{(\ell,\nu)} = \begin{cases} L_D(\ell,\nu) & \text{if } \lambda = 0 \\ \min(\ell,\nu) & \text{if } \lambda = \infty \\ e^{-((-\ln \ell)^\lambda + (-\ln \nu)^\lambda)^{\frac{1}{\lambda}}} & \text{otherwise} \end{cases} \tag{3}$$

The T-CNs $(S_\lambda^\lambda)_{\lambda \in [0,\infty]}$ for AA, are distinct as

$$(S^{\lambda}_{\lambda})_{(\ell,\nu)} = \begin{cases} S_D(\ell,\nu) & \text{if } \lambda = 0 \\ \max(\ell,\nu) & \text{if } \lambda = \infty \\ 1 - e^{-((-\ln(1-\ell))^{\lambda} + (-\ln(1-\nu))^{\lambda})^{\frac{1}{\lambda}}} & \text{otherwise} \end{cases} \tag{4}$$

Wherever $\lambda \in [0, \infty]$.

Definition 6: Consider $\Xi = (\Delta_{\Xi}, \Theta_{\Xi}, \phi_{\Xi})$, $\Xi_{11} = (\Delta_{\Xi_{11}}, \Theta_{\Xi_{11}}, \phi_{\Xi_{11}})$ and $\Xi_{22} = (\Delta_{\Xi_{22}}, \Theta_{\Xi_{22}}, \phi_{\Xi_{22}})$ be three T-SFSVs, $\lambda \geq 1$ and $\Phi \geq 0$. The definitions of the Aczél–Alsina T-norm (TN) and T-conorm (TCN) operations of T-SFSVs are as follows:

$$\Xi_{11} \oplus \Xi_{22} = \left(\sqrt[t]{1 - e^{-\left((-\ln(1-\Delta^t_{\Xi_{11}}))^{\lambda} + (-\ln(1-\Delta^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}}}, \right.$$
$$e^{-\left((-\ln(\Theta^t_{\Xi_{11}}))^{\lambda} + (-\ln(\Theta^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}},$$
$$\left. e^{-\left((-\ln(\phi^t_{\Xi_{11}}))^{\lambda} + (-\ln(\phi^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}} \right),$$

$$\Xi_{11} \otimes \Xi_{22} = \left(\sqrt[t]{e^{-\left((-\ln(\Delta^t_{\Xi_{11}}))^{\lambda} + (-\ln(\Delta^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}}}, \right.$$
$$1 - e^{-\left((-\ln(1-\Theta^t_{\Xi_{11}}))^{\lambda} + (-\ln(1-\Theta^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}},$$
$$\left. 1 - e^{-\left((-\ln(1-\phi^t_{\Xi_{11}}))^{\lambda} + (-\ln(1-\phi^t_{\Xi_{22}}))^{\lambda}\right)^{1/\lambda}} \right),$$

$$\Phi\Xi = \left(\sqrt[t]{1 - e^{-(\Phi(-\ln(1-\Delta^t_{\Xi}))^{\lambda})^{1/\lambda}}}, \right.$$
$$e^{-(\Phi(-\ln(\Theta^t_{\Xi}))^{\lambda})^{1/\lambda}},$$
$$\left. e^{-(\Phi(-\ln(\phi^t_{\Xi}))^{\lambda})^{1/\lambda}} \right),$$

$$\Xi^{\Phi} = \left(\sqrt[t]{e^{-(\Phi(-\ln(\Delta^t_{\Xi}))^{\lambda})^{1/\lambda}}}, \right.$$
$$1 - e^{-(\Phi(-\ln(1-\Theta^t_{\Xi}))^{\lambda})^{1/\lambda}},$$
$$\left. 1 - e^{-(\Phi(-\ln(1-\phi^t_{\Xi}))^{\lambda})^{1/\lambda}} \right).$$

3. Aczel-Alsina Weighted Averaging operators on T-SFS

Here we introduce some T-SFSAAWA and T-SFSAAWG operators using the AA operations. The indexing terms $(ij = 1, 2, \ldots, nm)$ will be used throughout the article.

Definition 7: Consider $\mathcal{E}_{ij} = \left(\Delta_{\mathcal{E}_{ij}}, \theta_{\mathcal{E}_{ij}}, \varphi_{\mathcal{E}_{ij}}\right)$ be a collection of T-SFSVs. Then T-SFSAAWA operator is a mapping $T - SFSAAWA : L^{*A} \to L^{*}$ and is defined as:

$$T - SFSAAWA(\mathcal{E}_{11}, \mathcal{E}_{22}, \ldots, \mathcal{E}_{nm}) = \bigoplus_{j=1}^{m} \zeta_j \left(\bigoplus_{i=1}^{n} \alpha_i \mathcal{E}_{ij} \right)$$

Using AA operations on T-SFSVs, we prove the following theorem.

Theorem 1: Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ as a set of T-SFSVs. Before the aggregated value of Ξ_{ij} by evolving the T-SFSAAWA operator is beyond a T-SFSV specified by:

$$\text{T-SFSAAWA}(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \left(\sqrt[t]{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \right.$$

$$e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},$$
$$\left.e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right) \tag{5}$$

Proof: Theorem 1 can be demonstrated in the following way using the mathematical induction method.

(I) By applying the AA operations to T-SFSVs, we obtain for $nm = 2$,

$$\varrho_i\Xi_1 = \left(\sqrt[t]{1-e^{-\left(\varrho_i(-\ln(1-\Delta^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}}},\ e^{-\left(\varrho_i(-\ln(\Theta^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}},\ e^{-\left(\varrho_i(-\ln(\varphi^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$\bigoplus_{i=1}^{n}\varrho_i\Xi_1 = \left(\sqrt[t]{1-e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}}},\ e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}},\ e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_1}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$\bigoplus_{j=1}^{m}\zeta_j\left(\bigoplus_{i=1}^{n}\varrho_i\Xi_{1_{ij}}\right) = \left(\sqrt[t]{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},\right.$$
$$e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},$$
$$\left.e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)$$

$$\varphi_i\Xi_2 = \left(\sqrt[t]{1-e^{-\left(\varrho_i(-\ln(1-\Delta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}},\ e^{-\left(\varrho_i(-\ln(\Theta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}},\ e^{-\left(\varrho_i(-\ln(\varphi^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$\bigoplus_{i=1}^{n}\varrho_i\Xi_2 = \left(\sqrt[t]{1-e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}},\ e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}},\ e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$\bigoplus_{j=1}^{m}\zeta_j\left(\bigoplus_{i=1}^{n}\varrho_i\Xi_{2_{ij}}\right) = \left(\sqrt[t]{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},\right.$$
$$e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},$$
$$\left.t^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)$$

$$\text{T-SFSAAWA}(\Xi_{1_{ij}},\Xi_{2_{ij}}) = \left(\bigoplus_{j=1}^{m}\zeta_j\left(\bigoplus_{i=1}^{n}\varrho_i\Xi_{1_{ij}}\right)\right)\oplus\left(\bigoplus_{j=1}^{m}\zeta_j\left(\bigoplus_{i=1}^{n}\varrho_i\Xi_{2_{ij}}\right)\right)$$

$$= \begin{cases} \sqrt[t]{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \end{cases} \oplus \begin{cases} \sqrt[t]{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \end{cases}$$

$$= \begin{cases} \sqrt[t]{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^{\lambda}\right) + \sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{1_{ij}}}))^{\lambda}\right) + \sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{1_{ij}}}))^{\lambda}\right) + \sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{2_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \end{cases}$$

$$= \left(\sqrt[t]{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln(1-\Delta^t_{\Xi_{ij}})\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \right. \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln(1-\Theta^t_{\Xi_{ij}})\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ \left. e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln(1-\varphi^t_{\Xi_{ij}})\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \right)$$

Innovative, Eq.5 is accurate for $nm = 2$.

(II) Yield that Eq.3 is accurate for $nm = k$, previously

$$\text{T-SFSAAWA}(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{k_{ij}}) = \bigoplus_{ij=1}^{k} \left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i \Xi_{ij} \right) \right) \\ = \left(\sqrt[t]{1 - e^{-\left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \right. \\ e^{-\left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ \left. e^{-\left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \right)$$

Now for $nm = k + 1$, we advance

$$\text{T-SFSAAWA}(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{(k+1)_{ij}}) = \\ \bigoplus_{ij=1}^{k} \left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i \Xi_{ij} \right) \right) \oplus \left(\sum_{j=1}^{k} \zeta_j \left(\sum_{i=1}^{k} \varrho_i \Xi_{ij} \right) \right) \Xi_{(k+1)_{ij}}$$

$$=$$

$$\left(\sqrt[t]{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}},\right.$$
$$\left. e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$\oplus$$

$$\left(\sqrt[t]{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(1-\Delta^t_{\Xi_{(k+1)ij}}))^\lambda\right)^{\frac{1}{\lambda}}}},\right.$$
$$\left. e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\Theta^t_{\Xi_{(k+1)ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\sum_{ij=1}^{k}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\varphi^t_{\Xi_{(k+1)ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

$$=$$

$$\left(\sqrt[t]{1-e^{-\left(\sum_{ij=1}^{k+1}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}},\right.$$
$$\left. e^{-\left(\sum_{ij=1}^{k+1}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\sum_{ij=1}^{k+1}\zeta_j\sum_{ij=1}^{k+1}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)$$

Therefore, Eq. 5 is correct for $k+1$.

Forms (I) and (II) lead us to the conclusion that Eq. 5 is valid for any value of nm.

In the following, we scrutinize some cardinal features of the T-SFSWAA operator, such as idempotency, monotonicity, and boundedness properties is defined theorem 2, 3 and 4.

Theorem 2: (Idempotency) Consider all

$$\Xi_{ij} = (\Delta^t_{\Xi_{ij}}, \Theta^t_{\Xi_{ij}}, \varphi^t_{\Xi_{ij}}) = \Xi,$$

that is, $\Xi_{ij} = \Xi$ for all ij. Then

$$\text{T-SFAAWA}(\Xi_{11}, \Xi_{22}, \dots, \Xi_{nm}) = \Xi.$$

Proof: Consequently,

$$\Xi_{ij} = (\Delta^t_{\Xi_{ij}}, \Theta^t_{\Xi_{ij}}, \varphi^t_{\Xi_{ij}}) = \Xi = (\Delta^t_{\Xi_{ij}}, \Theta^t_{\Xi_{ij}}, \varphi^t_{\Xi_{ij}}).$$

Subsequently, by Eq. 3,

$$\dots$$

$$\text{T-SFSAAWA}(\Xi_{11}, \Xi_{22}, \dots, \Xi_{nm}) = \left(1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},\right.$$
$$e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},$$
$$\left. e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)$$

$$= \left(1 - e^{-\left((-\ln(1-\vartheta^t_{\Xi}))^{\lambda}\right)^{\frac{1}{\lambda}}},\right.$$
$$e^{-\left((-\ln(\Upsilon^t_{\Xi}))^{\lambda}\right)^{\frac{1}{\lambda}}},$$
$$\left. e^{-\left((-\ln(\varphi^t_{\Xi}))^{\lambda}\right)^{\frac{1}{\lambda}}}\right)$$
$$= \left(1 - e^{-\ln(1-\vartheta^t_{\Xi})}, \quad e^{\ln \Upsilon^t_{\Xi}}, \quad e^{\ln \varphi^t_{\Xi}}\right)$$
$$= \left(\vartheta^t_{\Xi},\, \Upsilon^t_{\Xi},\, \varphi^t_{\Xi}\right) = \Xi.$$

Therefore, T-SFSAAWA$(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \Xi$ embraces.

Theorem 3 (Monotonicity): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a set of T-SFSVs. Consider $\Xi^- = \min(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$ and $\Xi^+ = \max(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$. Then

$$\Xi^- \leq T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq \Xi^+.$$

Proof: Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a set of T-SFSVs.
Consider $\Xi^- = \min(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = (\Delta_{\Xi^-}, \Theta_{\Xi^-}, \varphi_{\Xi^-})$ and $\Xi^+ = \max(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = (\Delta_{\Xi^+}, \Theta_{\Xi^+}, \varphi_{\Xi^+})$. Therefore, the subsequent disparities occur:

$$\sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi^-}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}$$
$$\leq \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}$$
$$\leq \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi^+}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}$$

$$e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi^+}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$
$$\geq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$
$$\geq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi^-}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$

$$e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi^+}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$
$$\geq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$
$$\geq e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi^-}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}$$

Therefore $\Xi^- \leq T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq \Xi^+$.

Theorem 4 (Boundedness): Consider Ξ_{ij} and Ξ'_{ij} to be two sets of T-SFSVs. If $\Xi_{ij} \leq \Xi'_{ij}$ for all ij. Then

$$T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq T\text{-}SFSAAWA(\Xi'_{11}, \Xi'_{22}, \ldots, \Xi'_{nm}).$$

Proof: According to Theorems 2 and 3, we have

$$\overline{\Xi_{ij}^{+}} = T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \geq T\text{-}SFSAAWA(\Xi_{11}', \Xi_{22}', \ldots, \Xi_{nm}') = \overline{\Xi_{ij}^{-}}$$

and

$$\overline{\Xi_{ij}^{-}} = T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq T\text{-}SFSAAWA(\Xi_{11}', \Xi_{22}', \ldots, \Xi_{nm}') = \overline{\Xi_{ij}^{+}}.$$

Therefore,

$$\overline{\Xi_{ij}^{-}} \leq T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq \overline{\Xi_{ij}^{+}}.$$

Theorem 5 (Translation Invariance): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a set of T-SFSVs. If $\alpha = (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$ is a T-SFSV on k, then:

$$T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha, \Xi_{22} \oplus \alpha, \ldots, \Xi_{nm} \oplus \alpha) = T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus \alpha$$

Proof: First, we calculate $T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha, \Xi_{22} \oplus \alpha, \ldots, \Xi_{nm} \oplus \alpha)$.

$$T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta_{\Xi_{ij}}^t))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta_{\Xi_{ij}}^t))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi_{\Xi_{ij}}^t))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$\Xi_{ij} \oplus \alpha = \begin{pmatrix} \sqrt{1 - e^{-\left((-\ln(1-\Delta_{\Xi_{ij}}^t))^\lambda + (-\ln(1-\Delta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left((-\ln(\Theta_{\Xi_{ij}}^t))^\lambda + (-\ln(\Theta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left((-\ln(\varphi_{\Xi_{ij}}^t))^\lambda + (-\ln(\varphi_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha, \Xi_{22} \oplus \alpha, \ldots, \Xi_{nm} \oplus \alpha) =$$

$$\begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln\left(1 - \left(1 - e^{-\left((-\ln(1-\Delta_{\Xi_{ij}}^t))^\lambda + (-\ln(1-\Delta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln\left(e^{-\left((-\ln(\Theta_{\Xi_{ij}}^t))^\lambda + (-\ln(\Theta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i \left(-\ln\left(e^{-\left((-\ln(\varphi_{\Xi_{ij}}^t))^\lambda + (-\ln(\varphi_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha, \Xi_{22} \oplus \alpha, \ldots, \Xi_{nm} \oplus \alpha) =$$

$$\begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta_{\Xi_{ij}}^t))^\lambda\right) + (-\ln(1-\Delta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta_{\Xi_{ij}}^t))^\lambda\right) + (-\ln(\Theta_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi_{\Xi_{ij}}^t))^\lambda\right) + (-\ln(\varphi_\alpha^t))^\lambda\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11}\oplus\alpha,\Xi_{22}\oplus\alpha,\ldots,\Xi_{nm}\oplus\alpha)=$$
$$\begin{pmatrix}\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij\oplus\alpha}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij\oplus\alpha}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij\oplus\alpha}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\end{pmatrix}$$

We now consider an expression for $T\text{-}SFSAAWA(\Xi_{11},\Xi_{22},\ldots,\Xi_{nm})\oplus\alpha$:

$$T\text{-}SFSAAWA(\Xi_{11},\Xi_{22},\ldots,\Xi_{nm})=\begin{pmatrix}\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11},\Xi_{22},\ldots,\Xi_{nm})\oplus\alpha=$$
$$\begin{pmatrix}\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\end{pmatrix}\oplus(\Delta_\alpha,\Theta_\alpha,\varphi_\alpha)$$

$$T\text{-}SFSAAWA(\Xi_{11},\Xi_{22},\ldots,\Xi_{nm})\oplus\alpha=$$
$$\begin{pmatrix}\sqrt{1-e^{-\left((-\ln(1-(1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda+(-\ln(1-\Delta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}},\\ e^{-\left((-\ln(e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda+(-\ln(\Theta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}},\\ e^{-\left((-\ln(e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda+(-\ln(\varphi^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)+(-\ln(1-\Delta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)+(-\ln(\Theta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}},\\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)+(-\ln(\varphi^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}\end{pmatrix}$$

Then:

$$T\text{-}SFSAAWA(\Xi_{11}\oplus\alpha,\Xi_{22}\oplus\alpha,\ldots,\Xi_{nm}\oplus\alpha)=T\text{-}SFSAAWA(\Xi_{11},\Xi_{22},\ldots,\Xi_{nm})\oplus\alpha$$

Theorem 6 (Scalar Invariance): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a set of T-SFSVs, if $\Phi > 0$, then:

$$T\text{-}SFSAAWA(\Phi\Xi_{11}, \Phi\Xi_{22}, \ldots, \Phi\Xi_{nm}) = \Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$$

Proof: We have the following based on the operational laws listed in Section 2:

$$\Phi\Xi = \left(\sqrt{1 - e^{-\left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}}, e^{-\left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \right)$$

Theorem 1 conditions that we consider:

$$T\text{-}SFSAAWA(\Phi\Xi_{11}, \Phi\Xi_{22}, \ldots, \Phi\Xi_{nm}) =$$
$$\left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(-\ln\left(1-\left(1-e^{-\left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(-\ln\left(e^{-\left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(-\ln\left(e^{-\left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$\Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \Phi \left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - e^{-\left(\Phi\left(-\ln\left(1-\left(1-e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)\right)\right)^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\Phi\left(-\ln\left(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\Phi\left(-\ln\left(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}\right)\right)^\lambda\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

Therefore:

$$T\text{-}SFSAAWA(\Phi\Xi_{11}, \Phi\Xi_{22}, \ldots, \Phi\Xi_{nm}) = \Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$$

Theorem 7 (Linear Invariance): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a set of T-SFSVs, and $\alpha = (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$ be a T-SFSV on k.

$$T\text{-}SFSAAWA(\Phi\Xi_{11}\oplus\alpha, \Phi\Xi_{22}\oplus\alpha, \ldots, \Phi\Xi_{nm}\oplus\alpha) = \Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})\oplus\alpha$$

Proof:

$$T\text{-}SFSAAWA(\Phi\Xi_{11}\oplus\alpha, \Phi\Xi_{22}\oplus\alpha, \ldots, \Phi\Xi_{nm}\oplus\alpha) = \Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})\oplus\alpha$$

$$T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$\Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \Phi \left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$\Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \left(\begin{array}{c} \sqrt{1 - e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$\Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus \alpha = \left(\begin{array}{c} \sqrt{1 - e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}} \end{array} \right) \oplus (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$$

$$\Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus \alpha =$$
$$\left(\begin{array}{c} \sqrt{1 - e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right) + (-\ln(1-\Delta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\Phi\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right) + (-\ln(\Theta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\Phi\left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right) + (-\ln(\varphi^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

Now consider $T\text{-}SFSAAWA(\Phi\Xi_{11} \oplus \alpha, \Phi\Xi_{22} \oplus \alpha, \ldots, \Phi\Xi_{nm} \oplus \alpha)$:

$$\Phi\Xi_{ij} = \left(\sqrt{1 - e^{-\left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}}, e^{-\left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \right)$$

$$\Phi\Xi_{ij} \oplus \alpha =$$
$$\left(\sqrt{1 - e^{-\left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}}, e^{-\left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, e^{-\left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \right) \oplus (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$$

$$\Phi\Xi_{ij} \oplus \alpha =$$
$$\left(\begin{array}{c} \sqrt{1 - e^{-\left((-\ln(1-(1-e^{-\left(\Phi(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}))^\lambda + (-\ln(1-\Delta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left((-\ln(e^{-\left(\Phi(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}))^\lambda + (-\ln(\Theta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left((-\ln(e^{-\left(\Phi(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}))^\lambda + (-\ln(\varphi^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right) + (-\ln(1-\Delta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right) + (-\ln(\Theta^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right) + (-\ln(\varphi^t_\alpha))^\lambda\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

Therefore:

$$T\text{-}SFSAAWA(\Phi\Xi_{11} \oplus \alpha, \Phi\Xi_{22} \oplus \alpha, \ldots, \Phi\Xi_{nm} \oplus \alpha) = \Phi T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus \alpha$$

Theorem 8 (Additivity Property): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ and $\alpha_{ij} = (\Delta_{\alpha_{ij}}, \Theta_{\alpha_{ij}}, \varphi_{\alpha_{ij}})$ be two sets of T-SFSVs. Then

$$T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha_{11}, \Xi_{22} \oplus \alpha_{22}, \ldots, \Xi_{nm} \oplus \alpha_{nm}) = T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus T\text{-}SFSAAWA(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nm})$$

Proof: As per Theorem 1, we obtain:

$$\Xi_{ij} \oplus \alpha_{ij} = \begin{pmatrix} \sqrt{1 - e^{-\left((-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda + (-\ln(1-\Delta^t_{\alpha_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left((-\ln(\Theta^t_{\Xi_{ij}}))^\lambda + (-\ln(\Theta^t_{\alpha_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left((-\ln(\varphi^t_{\Xi_{ij}}))^\lambda + (-\ln(\varphi^t_{\alpha_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11} \oplus \alpha_{11}, \Xi_{16} \oplus \alpha_{16}, \ldots, \Xi_{nm} \oplus \alpha_{nm}) =$$
$$\begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left((-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda + (-\ln(1-\Delta^t_{\alpha_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left((-\ln(\Theta^t_{\Xi_{ij}}))^\lambda + (-\ln(\Theta^t_{\alpha_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i \left((-\ln(\varphi^t_{\Xi_{ij}}))^\lambda + (-\ln(\varphi^t_{\alpha_{ij}}))^\lambda\right)\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \oplus T\text{-}SFSAAWA(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nm}) =$$
$$\begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix} \oplus$$
$$\begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(1-\Delta^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\Theta^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\varphi^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1 - e^{-\left((-\ln(1-(1-e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda + \right.}} \\ (-\ln(1-(1-e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(1-\Delta^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda{}^{\frac{1}{\lambda}}, \\ e^{-\left((-\ln(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda + (-\ln(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\Theta^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda\right)^{\frac{1}{\lambda}}}, \\ e^{-\left((-\ln(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda + (-\ln(e^{-\left(\sum_{j=1}^m \zeta_j \left(\sum_{i=1}^n \varrho_i (-\ln(\varphi^t_{\alpha_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}))^\lambda\right)^{\frac{1}{\lambda}}} \end{pmatrix}$$

$$= \left(\begin{array}{c} \sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i\left((-\ln(1-\Delta^t_{\Xi_{ij}}))^{\lambda}+(-\ln(1-\Delta^t_{\alpha_{ij}}))^{\lambda}\right)\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i\left((-\ln(\Theta^t_{\Xi_{ij}}))^{\lambda}+(-\ln(\Theta^t_{\alpha_{ij}}))^{\lambda}\right)\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i\left((-\ln(\varphi^t_{\Xi_{ij}}))^{\lambda}+(-\ln(\varphi^t_{\alpha_{ij}}))^{\lambda}\right)\right)\right)^{\frac{1}{\lambda}}} \end{array} \right)$$

Definition 9: Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ is a set of T-SFSVs, and define $T\text{-}SFSAAWG : L^{*A} \to L^*$ as:

$$T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \bigotimes_{j=1}^{m} \zeta_j \left(\bigotimes_{i=1}^{n} \varrho_i \Xi_{ij} \right)$$

Theorem 9: Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ is a group of T-SFSVs. The T-SFSAAWG operator is then applied to the aggregated value to obtain T-SFSVs as well.

$$T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \left(\begin{array}{c} e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Theta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\varphi^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right) \quad (6)$$

Proof: Theorem 9 can be demonstrated as follows using the mathematical induction method:

Expending T-SFSVs' AA operations, we find for $nm = 2$,

$$\varrho_i \Xi_1 = \left(e^{-\left(\varrho_i(-\ln(1-\Delta^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}, \sqrt{1-e^{-\left(\varrho_i(-\ln(\Theta^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}}, \sqrt{1-e^{-\left(\varrho_i(-\ln(\varphi^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}} \right)$$

$$\bigotimes_{i=1}^{n} \varrho_i \Xi_1 = \left(\begin{array}{c} e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_1}))^{\lambda}\right)^{\frac{1}{\lambda}}}} \end{array} \right) \quad (1)$$

$$\bigotimes_{j=1}^{m} \zeta_j (\bigotimes_{i=1}^{n} \varrho_i \Xi_{1_{ij}}) = \left(\begin{array}{c} e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\Theta^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(\varphi^t_{\Xi_{1_{ij}}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

$$\varrho_i\Xi_2 = \left(e^{-\left(\varrho_i(-\ln(1-\Delta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}, \sqrt{1-e^{-\left(\varrho_i(-\ln(\Theta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}}, \sqrt{1-e^{-\left(\varrho_i(-\ln(\varphi^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}} \right)$$

$$\bigotimes_{i=1}^{n} \varrho_i\Xi_2 = \left(e^{-\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}, \sqrt{1-e^{-\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}}, \sqrt{1-e^{-\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_2}))^\lambda\right)^{\frac{1}{\lambda}}}} \right) \tag{2}$$

$$\bigotimes_{j=1}^{m} \zeta_j(\bigotimes_{i=1}^{n} \varrho_i\Xi_{2_{ij}}) = \left(\begin{array}{c} e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

$$T\text{-}SFSAAWG(\Xi_{1_{ij}}, \Xi_{2_{ij}}) = \left(\bigotimes_{j=1}^{m} \zeta_j(\bigotimes_{i=1}^{n} \varrho_i\Xi_{1_{ij}})\right) \otimes \left(\bigotimes_{j=1}^{m} \zeta_j(\bigotimes_{i=1}^{n} \varrho_i\Xi_{2_{ij}})\right)$$

$$= \left(\begin{array}{c} e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{1_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right) \otimes \left(\begin{array}{c} e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

$$= \left(\begin{array}{c} e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{1_{ij}}}))^\lambda\right)+\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(1-\Delta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{1_{ij}}}))^\lambda\right)+\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\Theta^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{1_{ij}}}))^\lambda\right)+\sum_{j=1}^m \zeta_j\left(\sum_{i=1}^n \varrho_i(-\ln(\varphi^t_{\Xi_{2_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

$$= \left(e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \right.$$
$$\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},$$
$$\left.\sqrt{1-e^{-\left(\sum_{j=1}^{m}\zeta_j\left(\sum_{i=1}^{n}\varrho_i(-\ln(1-\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}\right)$$

Therefore, Eq.6 is accurate for $nm = 2$.

Assume that Eq.6 is accurate for $nm = k$, then:

$$T\text{-}SFSAAWG(\Xi_{11},\Xi_{22},\ldots,\Xi_{k_{ij}}) = \bigotimes_{ij=1}^{k}\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(\Xi_{ij})\right)\right)$$
$$= \left(e^{-\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \right.$$
$$\sqrt{1-e^{-\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}},$$
$$\left.\sqrt{1-e^{-\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}\right)$$

Now for $nm = k + 1$, we progress:

$$T\text{-}SFSAAWG(\Xi_{11},\Xi_{22},\ldots,\Xi_{(k+1)_{ij}}) =$$
$$\bigotimes_{ij=1}^{k}\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(\Xi_{ij})\right)\right)\otimes\left(\sum_{j=1}^{k}\zeta_j\left(\sum_{i=1}^{k}\varrho_i(\Xi_{ij})\right)\right)(\Xi_{(k+1)_{ij}})$$

$$= \begin{pmatrix} e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{pmatrix} \otimes$$
$$\begin{pmatrix} e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(1-\Delta^t_{\Xi_{(k+1)_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(\Theta^t_{\Xi_{(k+1)_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k}\zeta_j\left(\sum_{ij=1}^{k+1}\varrho_i(-\ln(\varphi^t_{\Xi_{(k+1)_{ij}}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{pmatrix}$$

$$= \left(\begin{array}{c} e^{-\left(\sum_{ij=1}^{k+1} \zeta_j \left(\sum_{ij=1}^{k+1} \varrho_i(-\ln(1-\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k+1} \zeta_j \left(\sum_{ij=1}^{k+1} \varrho_i(-\ln(\Theta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{ij=1}^{k+1} \zeta_j \left(\sum_{ij=1}^{k+1} \varrho_i(-\ln(\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

Therefore, Eq.6 is suitable for $nm = k+1$.

Forms (I) and (II) lead us to the conclusion that Eq.6 is valid for any value of nm. The following properties, given in Theorems 10, 11, and 12, are probably satisfied by the T-SFSAAWG operator.

Theorem 10 (Idempotency): Consider that all $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ where T-SFSVs are collected. If for each ij, $\Xi_{ij} = \Xi$. Then $T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \Xi$.

Proof: Then $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})\Xi = (\vartheta_\Xi, \Theta_\Xi, \varphi_\Xi)$. Next, we have Equation (6) as follows:

$$T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \left(\begin{array}{c} e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i(-\ln(\Delta^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i(-\ln(1-\Upsilon^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1-e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i(-\ln(1-\varphi^t_{\Xi_{ij}}))^\lambda\right)\right)^{\frac{1}{\lambda}}}} \end{array} \right)$$

$$= \left(e^{-\left((-\ln(\vartheta^t_\Xi))^\lambda\right)^{\frac{1}{\lambda}}}, \sqrt{1-e^{-\left((-\ln(1-\Upsilon^t_\Xi))^\lambda\right)^{\frac{1}{\lambda}}}}, \sqrt{1-e^{-\left((-\ln(1-\varphi^t_\Xi))^\lambda\right)^{\frac{1}{\lambda}}}} \right)$$
$$= \left(1-e^{-\ln(1-\vartheta^t_\Xi)}, e^{\ln \Upsilon^t_\Xi}, e^{\ln \varphi^t_\Xi}\right) = \left(\vartheta^t_\Xi, \Upsilon^t_\Xi, \varphi^t_\Xi\right) = \Xi$$

Therefore $T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = \Xi$ holds.

Theorem 11 (Monotonicity): Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ is a set of T-SFSVs. Consider $\Xi^- = \min(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$ and $\Xi^+ = \max(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})$. Therefore $\Xi^- \leq T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq \Xi^+$.

Proof: Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ characterizes a set of T-SFSVs. Consider that $\Xi^+ = \max(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = (\vartheta^{t+}_\Xi, \Theta^{t+}_{\Xi_{ij}}, \varphi^{t+}_{\Xi_{ij}})$ and $\Xi^- = \min(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) = (\vartheta^{t-}_\Xi, \Theta^{t-}_{\Xi_{ij}}, \varphi^{t-}_{\Xi_{ij}})$. Therefore, the subsequent differences occur:

$$1-e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(1-\vartheta^{t-}_\Xi))^\lambda\right)^{\frac{1}{\lambda}}} \geq 1-e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(1-\vartheta^t_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \geq 1-e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(1-\Theta^{t-}_\Xi))^\lambda\right)^{\frac{1}{\lambda}}}$$
$$e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\Theta^{t+}_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \leq e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\Theta^t_\Xi))^\lambda\right)^{\frac{1}{\lambda}}} \leq e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\Theta^{t-}_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}$$
$$e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\varphi^{t+}_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}} \leq e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\varphi^t_\Xi))^\lambda\right)^{\frac{1}{\lambda}}} \leq e^{-\left(\sum_{i=1}^{n} \varrho_i(-\ln(\varphi^{t-}_{\Xi_{ij}}))^\lambda\right)^{\frac{1}{\lambda}}}$$

Hence, $\Xi^- \leq T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \leq \Xi^+$.

Theorem 12 (Boundedness): Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ characterizes a set of T-SFSVs. If α is T-SFSV on k and $\alpha = (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$.

$$T\text{-}SFSAAWG(\Xi_{11} \otimes \alpha, \Xi_{22} \otimes \alpha, \ldots, \Xi_{nm} \otimes \alpha) = T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \otimes \alpha$$

Proof: Theorem 12 can also be proved using the same method as Theorem 5.

Theorem 13 (Scalar Invariance): Consider $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ be a collection of T-SFSVs. If $r > 0$, then:

$$T\text{-}SFSAAWG(\Xi_{11}^r, \Xi_{22}^r, \ldots, \Xi_{nm}^r) = T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})^r$$

Proof: Theorem 13 can also be proved using the same method as Theorem 6.

Theorem 14 (Linear Invariance): Consider that $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$, symbolize a set of T-SFSVs. If $\alpha = (\Delta_\alpha, \Theta_\alpha, \varphi_\alpha)$ is T-SFSV on k and $r > 0$, then:

$$T\text{-}SFSAAWG(\Xi_{11}^r \otimes \alpha, \Xi_{22}^r \otimes \alpha, \ldots, \Xi_{nm}^r \otimes \alpha) = T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm})^r \otimes \alpha$$

Proof: Theorem 14 can also be proved using the same method as Theorem 7.

Theorem 15 (Additivity Property): Consider there are two assemblies of T-SFSVs: $\Xi_{ij} = (\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ and $\alpha_{ij} = (\Delta_{\alpha_{ij}}, \Theta_{\alpha_{ij}}, \varphi_{\alpha_{ij}})$. Then

$$\begin{aligned} &T\text{-}SFSAAWG(\Xi_{11} \otimes \alpha_{11}, \Xi_{22} \otimes \alpha_{22}, \ldots, \Xi_{nm} \otimes \alpha_{nm}) \\ &\quad = T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \otimes T\text{-}SFSAAWG(\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nm}) \end{aligned}$$

Proof: Theorem 15 can also be proved using the same method as Theorem 8.

4. Proposed MADM-TAOV Method

We compared the new MADM methodology with the TAOV method in this section.

4.1. Weight Determination Method

Given that one of the most important aspects of the MADM process is the criteria weight. This section outlines a technique for calculating such weights using suggested distance measures in terms of optimistic and pessimistic utility values. The steps listed in this method are as follows.

Step 1: Compute criteria weights. Arrange the expert ratings towards each alternative in terms of DM as given in Eq. (7).

$$E_{ij} = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{m1} & U_{m2} & \cdots & U_{mn} \end{bmatrix} \tag{7}$$

Step 2: For each criteria N_m the optimistic and pessimistic values are taken as:

$$\text{Optimistic values:} \quad V^+ = (S_1^+, S_2^+, \ldots, S_n^+) \tag{8}$$

$$\text{Pessimistic values:} \quad V^- = (S_1^-, S_2^-, \ldots, S_n^-) \tag{9}$$

Where

$$S_j^+ = \begin{cases} \max_{1\leq i\leq n} & \text{for benefit criteria} \\ \min_{1\leq i\leq n} & \text{for cost criteria} \end{cases} \tag{10}$$

$$S_j^- = \begin{cases} \min_{1\leq i\leq n} & \text{for benefit criteria} \\ \max_{1\leq i\leq n} & \text{for cost criteria} \end{cases} \tag{11}$$

Step 3: Compute the distance measures between S_{ij} and S_j^+, S_j^- as:

$$d_j^+ = \sqrt{\sum_{i=1}^{m}(D_a(S_{ij}, S_j^+))^2} \tag{13}$$

$$= \sqrt{\sum_{i=1}^{m}\left(\frac{|M(\Lambda_{ij}) - M(\Lambda_{ij}^+)| + |M(\Lambda_{ij}) - M(\beta_{ij}^+)| + |M(\Lambda_{ij}) - M(\pi_{ij}^+)|}{4\sqrt{n}}\right)} \tag{14}$$

$$d_j^- = \sqrt{\sum_{i=1}^{m}(D_a(S_{ij}, S_j^-))^2} \tag{15}$$

$$= \sqrt{\sum_{i=1}^{m}\left(\frac{|M(\Lambda_{ij}) - M(\Lambda_{ij}^-)| + |M(\Lambda_{ij}) - M(\beta_{ij}^-)| + |M(\Lambda_{ij}) - M(\pi_{ij}^-)|}{4\sqrt{n}}\right)} \tag{16}$$

Each value is divided by the sum of the values in its column

$$\sigma_j = \frac{d_j^+}{d_j^+ + d_j^-} \quad \text{for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n \tag{17}$$

Step 4: Calculate the weights of attributes

$$w_j = \frac{\sigma_j}{\sum_{j=1}^{n}\sigma_j} \tag{18}$$

4.2. T-SFSV-Based Examined Operators in MADM Methods

To ensure the method's efficacy and dependability, a MADM technique is developed based on the suggested AOs under the T-SFSs framework. A set of alternatives, represented as $\wp = \{\wp^{(1)}, \wp^{(2)}, \wp^{(3)}, \wp^{(4)}\}$, is typically dealt with in an SS environment. Each alternative is linked to several attributes. A single alternative can be associated with multiple values for a given attribute in the SS context, which is significant because it captures the hesitancy and uncertainty of DM. And a set of attributes $I = \{I_1, I_2, \ldots I_{ij}\}$. Each option's evaluation $\wp$ about attribute I_{ij}, as determined by DM, is represented by a T-SFSV:

In this case, $(\Delta_{\Xi_{ij}}, \Theta_{\Xi_{ij}}, \varphi_{\Xi_{ij}})$ stand for the levels of MD, AD, and NMD, respectively. These numbers have to meet the requirement: $0 \leq (\Delta_{\iota ij}^t + \Theta_{\iota ij}^t + \varphi_{\iota ij}^t) \leq 1$, $\iota = 1, 2, \ldots m$, $ij = 1, 2, \ldots n$.

Normalization of attribute values is not required if all attributes I_{ij} $(ij = 1, 2, \ldots, t)$ are of the same type (benefit-type or cost-type). The attribute values do not require normalization if every attribute I_{ij} $(ij = 1, 2, \ldots, t)$ is of the same type. If not, we normalize $K^t = (K_{\iota ij}^t)_{m\times t}$ into $R^t = (r_{\iota ij}^t)_{m\times t}$ the DM matrix, where:

$$r_{\iota ij}^t = \begin{cases} k_{\iota ij}^t, & \text{for benefit attribute } I_{ij} \\ k_{\iota ij}^t, & \text{for cost attribute } I_{ij} \end{cases}$$

Next, we create a MADM strategy in a T-SFSVs environment using the T-SFSAAWA operator. The main steps that are involved are as follows:

Step 1: If needed to normalize the Data.

Step 2: Utilize the T-SFSAAWA operator supplied by:

$$\begin{aligned} r_{\iota ij} &= T\text{-}SFSAAWA(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \\ &= \begin{pmatrix} \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Theta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\varphi^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}} \end{pmatrix} \end{aligned}$$

Or the T-SFSAAWG operator:

$$\begin{aligned} r_{\iota ij} &= T\text{-}SFSAAWG(\Xi_{11}, \Xi_{22}, \ldots, \Xi_{nm}) \\ &= \begin{pmatrix} e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(\Delta^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}, \\ \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\Upsilon^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}}, \\ \sqrt{1 - e^{-\left(\sum_{j=1}^{m} \zeta_j \left(\sum_{i=1}^{n} \varrho_i (-\ln(1-\varphi^t_{\Xi_{ij}}))^{\lambda}\right)\right)^{\frac{1}{\lambda}}}} \end{pmatrix} \end{aligned}$$

Step 3: Using the score function outlined in Section 2, order each option as follows:

$$Sco(\Xi_\iota) = \frac{(1 + \vartheta^t_{\Xi_{ij}} - \Theta^t_{\Xi} - \varphi^t_{\Xi})}{2} \qquad \iota = 1, 2, \ldots m$$

Step 4: Rank the best applicants.

Step 5: Finish.

The procedural flow of the proposed MADM scheme is portrayed in Figure 1.

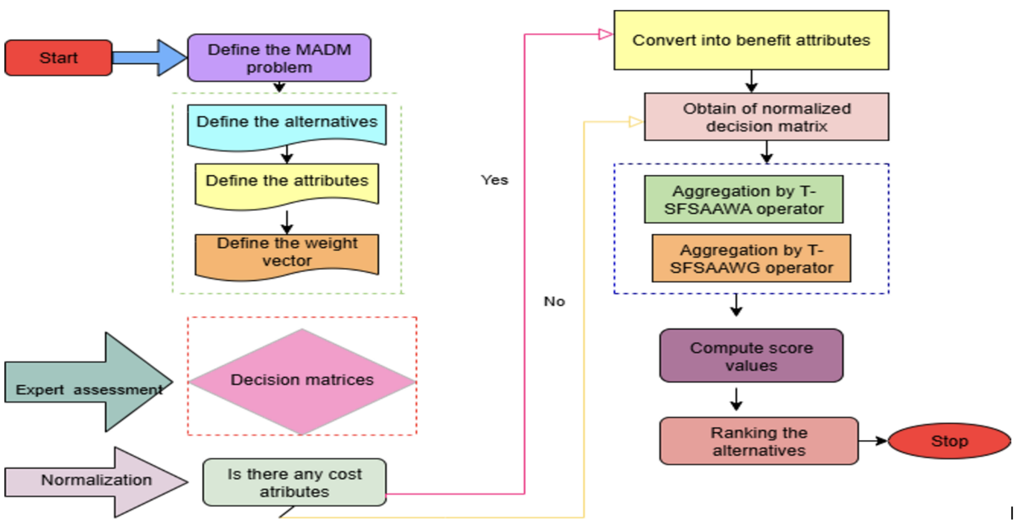

Figure 1: Procedural flow of the proposed MADM scheme.

4.3. Practical Example

In this section, the DM process aims to determine the optimal traffic management plan for an urban city center that fits the budget, safety requirements, and sustainability goals of the city. The city wants to lessen its negative effects on the environment, improve overall road safety, and reduce congestion. The city also gives top priority to plans that further its overarching goal of encouraging effective public transportation and sustainable urban development.

To do this, the city evaluates traffic systems according to priorities such as cost, environmental impact, safety, feasibility of use, and driving technology. Ultimately, this approach will affect the city's ability to provide reliable and safe transportation for citizens while maintaining long-term sustainability and fiscal responsibility.

Let's assume that a city government wants to implement the best peak traffic policy possible for its downtown. This example illustrates how companies can choose the best logistics provider when presented with different options and needs. By applying important criteria such as price, delivery time, reliability, customer service, and sustainability, DM can guarantee a comprehensive evaluation process for their corporate objectives. This approach has the advantage of prioritization and consideration of expert opinion, bringing informed and unbiased DM, helping companies choose sustainable suppliers with consistent presence to maximize operational efficiency.

Out of five options, a retail company wants to choose the best delivery option. Let

$$I_{ij} = (I_1, I_2, I_3, I_4, I_5)$$

be the set of alternatives. The business assesses the suppliers based on five key criteria

$$M_{ij} = (M_1, M_2, M_3, M_4, M_5)$$

defined as follows:

- **Cost Efficiency** M_1: Total cost-effectiveness of the service offered by the provider.
- **Delivery Time** M_2: The rate at which the logistics supplier can deliver merchandise to clients.
- **Service Reliability** M_3: Dependability and precision of the supplier in terms of on-time and error-free shipment delivery.
- **Customer Service** M_4: The caliber and promptness of assistance provided to customers when problems occur.
- **Sustainability Practices** M_5: Supplier's dedication to environmentally friendly methods, reducing waste and carbon emissions.

4.4. Attributes Weight Vector

The attributes' weight vector in terms of MD, AD, and NMD is shown below:

Alternative	MD	AD	NMD
I_1	0.34	0.22	0.45
I_2	0.16	0.28	0.33
I_3	0.45	0.31	0.24
I_4	0.23	0.16	0.21
I_5	0.25	0.33	0.21

The attributes' weight vector is

$$(0.2157, 0.1943, 0.2156, 0.1777, 0.1967),$$

indicating the relative importance of Each attribute to the business. The ξth attribute's weight can then be determined by Eq. (19):

$$\varphi_{\xi} = \frac{\Delta_{\Xi} + \pi_{\Xi}\left(\frac{\Delta_{\Xi}}{\Delta_{\Xi}+\Theta_{\Xi}+\varphi_{\Xi}}\right)}{\sum_{\xi=1}^{n}\left(\Delta_{\Xi} + \pi_{\Xi}\left(\frac{\Delta_{\Xi}}{\Delta_{\Xi}+\Theta_{\Xi}+\varphi_{\Xi}}\right)\right)} \tag{19}$$

4.5. Alternative Weight Vector

To assess the logistics providers, five industry experts are consulted. Their judgments are weighted according to

$$(0.1888, 0.2190, 0.1888, 0.1820, 0.2214),$$

and these weights are used in Eq. (18). For each alternative $I = (1, 2, 3, 4, 5)$, the providers are ranked to choose the best option based on these factors. This illustration shows how businesses can assess logistics providers impartially, balancing sustainability, cost, and efficiency.

4.6. Decision-Making Steps Using T-SFSAAWA/T-SFSAAWG Operators

Step 1: All criteria are benefit types and do not need normalization.

Table 1: Linguistic Terms for Importance Degrees

Linguistic Terms	T-SFSs
Very Very Important (VVI)	$(0.86, 0.22, 0.79)$
Very Important (VI)	$(0.82, 0.35, 0.81)$
Important (I)	$(0.73, 0.44, 0.85)$
Medium (M)	$(0.51, 0.55, 0.91)$

Table 2: Evaluations of Decision Maker X_1

Criteria	t_1	t_2	t_3	t_4	t_5
X_1	VI	VVI	I	VVI	I
X_2	M	I	I	VI	M
X_3	M	I	M	M	VVI
X_4	VI	M	M	I	M

Step 2: Derive the criteria weights as described in Section 4.

Step 3: Compute V^+ and V^- by Eqs. (8-9):

$$V^{+} = \begin{bmatrix} 0.8200 & 0.3500 & 0.3200 \\ 0.8600 & 0.2200 & 0.1800 \\ 0.7300 & 0.4400 & 0.4100 \\ 0.8600 & 0.2200 & 0.1800 \\ 0.8600 & 0.2200 & 0.1800 \end{bmatrix}, \quad V^{-} = \begin{bmatrix} 0.5100 & 0.5500 & 0.8111 \\ 0.5100 & 0.5500 & 0.6614 \\ 0.5100 & 0.5500 & 0.6614 \\ 0.5100 & 0.5500 & 0.6614 \\ 0.5100 & 0.5500 & 0.6614 \end{bmatrix} \tag{20}$$

Step 4: Compute the values of dispersion:

$$\sigma_j = (0.5000, 0.5801, 0.5000, 0.4821, 0.5866) \tag{21}$$

Step 5: Using Eq. (21), the criteria weights are computed as:

$$w_j = (0.1888, 0.2190, 0.1888, 0.1820, 0.2214) \tag{22}$$

which will be used to rank the alternatives and select the best logistics provider.

4.7. Solving the MADM Problem Based on TAOV Method

Step 1: Based on the T-SFSVs for each parameter, each expert rates each applicant as shown in Tables 3,4,5 and 6

Table 3: Ratings for applicant $\mathcal{P}^{(3)}$

	M_1			M_2			M_3			M_4			M_5		
I	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD
I_1	0.54	0.34	0.56	0.45	0.45	0.56	0.45	0.56	0.67	0.54	0.78	0.45	0.54	0.36	0.47
I_2	0.44	0.33	0.43	0.47	0.44	0.52	0.54	0.54	0.55	0.45	0.44	0.45	0.45	0.44	0.43
I_3	0.43	0.36	0.34	0.46	0.39	0.51	0.44	0.44	0.56	0.45	0.56	0.49	0.39	0.37	0.42
I_4	0.42	0.39	0.54	0.53	0.45	0.47	0.39	0.36	0.44	0.67	0.47	0.52	0.56	0.56	0.45
I_5	0.46	0.42	0.45	0.35	0.47	0.38	0.49	0.46	0.56	0.56	0.45	0.45	0.54	0.35	0.47

Table 4: Ratings for applicant $\mathcal{P}^{(4)}$

MADM	M_1			M_2			M_3			M_4			M_5		
	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD
I_1	0.44	0.44	0.45	0.44	0.45	0.45	0.56	0.44	0.43	0.43	0.33	0.52	0.33	0.45	0.46
I_2	0.37	0.48	0.56	0.45	0.39	0.46	0.44	0.47	0.44	0.45	0.38	0.51	0.34	0.47	0.53
I_3	0.39	0.45	0.34	0.56	0.33	0.49	0.47	0.45	0.45	0.45	0.37	0.51	0.45	0.45	0.54
I_4	0.36	0.56	0.78	0.45	0.35	0.49	0.45	0.44	0.51	0.47	0.39	0.56	0.42	0.46	0.45
I_5	0.42	0.34	0.34	0.34	0.34	0.45	0.39	0.39	0.45	0.39	0.33	0.52	0.34	0.45	0.44

Table 5: Ratings for applicant $\mathcal{P}^{(5)}$

	M_1			M_2			M_3			M_4			M_5		
I	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD
I_1	0.36	0.34	0.52	0.43	0.34	0.52	0.37	0.45	0.38	0.37	0.56	0.45	0.45	0.47	0.56
I_2	0.34	0.33	0.44	0.46	0.38	0.54	0.38	0.52	0.36	0.43	0.52	0.47	0.46	0.39	0.45
I_3	0.44	0.38	0.47	0.47	0.36	0.53	0.44	0.51	0.35	0.44	0.44	0.42	0.52	0.38	0.67
I_4	0.42	0.39	0.54	0.45	0.35	0.44	0.47	0.56	0.34	0.39	0.43	0.45	0.49	0.44	0.48
I_5	0.45	0.37	0.41	0.56	0.34	0.45	0.45	0.44	0.45	0.45	0.37	0.42	0.42	0.42	0.47

Table 6: Ratings for applicant $\mathcal{P}^{(6)}$

	M_1			M_2			M_3			M_4			M_5		
I	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD	MD	AD	NMD
I_1	0.45	0.37	0.52	0.43	0.36	0.52	0.45	0.42	0.43	0.47	0.37	0.34	0.44	0.46	0.52
I_2	0.44	0.39	0.51	0.38	0.45	0.45	0.47	0.43	0.45	0.48	0.45	0.45	0.45	0.34	0.56
I_3	0.46	0.44	0.53	0.34	0.44	0.56	0.45	0.45	0.45	0.45	0.45	0.47	0.52	0.37	0.52
I_4	0.43	0.42	0.45	0.47	0.47	0.43	0.47	0.47	0.53	0.44	0.49	0.53	0.51	0.49	0.49
I_5	0.45	0.41	0.44	0.44	0.45	0.53	0.45	0.47	0.54	0.48	0.44	0.56	0.54	0.46	0.46

Step 2: The combined values of various applicants obtained by applying the T-SFSAAWA and T-SFSAAWG operators are shown in Table 7.

Table 7: Ratings for applicant $\mathcal{P}^{(7)}$

Applicant	T-SFSAAWA			T-SFSAAWG		
	MD	AD	NMD	MD	AD	NMD
$\mathcal{P}^{(1)}$	0.1178	0.0834	0.1109	0.0834	0.9166	0.8891
$\mathcal{P}^{(2)}$	0.0848	0.0699	0.1083	0.9152	0.9301	0.8917
$\mathcal{P}^{(3)}$	0.0879	0.0697	0.0987	0.9121	0.9303	0.9013
$\mathcal{P}^{(4)}$	0.0960	0.0777	0.1160	0.9040	0.9223	0.8840

Step 3: Using the score function

$$Sco(E_{11}) = \frac{(1 + \theta^t_{E_{11}} - Y^t_{E_{11}} - \varphi^t_{E_{11}})}{2} \tag{23}$$

The score values for each applicant using T-SFSAAWA:

$$S(N^{(1)}) = 0.4998, \quad S(N^{(2)}) = 0.9990, \quad S(N^{(3)}) = 0.4997, \quad S(N^{(4)}) = 0.4994$$

Step 4: According to the T-SFSAAWA operator, the applicants are ranked as:

$$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$$

Hence, $\mathcal{P}^{(2)}$ is the most suitable applicant.

Step 5: The score values using T-SFSAAWG:

$$S(N^{(1)}) = -0.2362, \quad S(N^{(2)}) = 0.1107, \quad S(N^{(3)}) = 0.0632, \quad S(N^{(4)}) = 0.1318$$

$$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$$

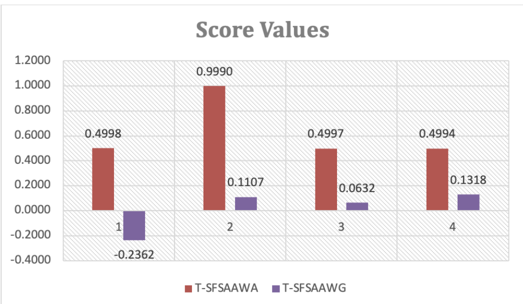

Figure 2: Result of T-SFSAAWA and T-SFSAAWG operators.

The ranking is both T-SFSAAWA and T-SFSAAWG operators identify $\mathcal{P}^{(2)}$ as the best applicant. The difference lies in the aggregation method: T-SFSAAWA uses arithmetic averaging while T-SFSAAWG uses geometric averaging.

5. Sensitivity Analysis

The AA TN and TCN models include a control parameter λ, which is crucial for determining the strictness or flexibility of the aggregation. This parameter λ provides an adjustable mechanism that enhances the model's ability to manage different levels of uncertainty.

A thorough sensitivity analysis is conducted to confirm the robustness and reliability of the proposed TAOV-based framework, with λ adjusted within the range $[1, 100]$. This assessment investigates the effect of variations in λ on the overall score values and ranking positions of alternatives in the supplier selection problem.

Tables 8-9 summarize the computed results for various values of λ, while Figure 3 illustrates the ranking variations graphically.

Table 8: Aggregated T-SFSAAWA and T-SFSAAWG values for applicants $\mathcal{P}^{(8)}$

λ	$\mathcal{P}^{(1)}$	$\mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$	$\mathcal{P}^{(4)}$	Ranking
1	0.1947	0.4340	0.3267	0.3045	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
5	0.3128	0.4562	0.4176	0.3452	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
10	0.2678	0.4332	0.3644	0.3422	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
20	0.3122	0.4213	0.3852	0.3462	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
40	0.2875	0.4623	0.3652	0.3126	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
60	0.2265	0.4032	0.3854	0.3064	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
80	0.2954	0.4743	0.3975	0.3124	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
100	0.2855	0.4323	0.3523	0.3372	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$

Table 9: Results attained by the T-SFSAAWG operator under different inputs $\mathcal{P}^{(9)}$

λ	$\mathcal{P}^{(1)}$	$\mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$	$\mathcal{P}^{(4)}$	Ranking
1	0.2365	0.4532	0.3733	0.3127	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
5	0.2563	0.4630	0.4875	0.3475	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
10	0.2654	0.4633	0.3458	0.3754	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
20	0.2173	0.4376	0.3782	0.3563	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
40	0.2543	0.4465	0.3672	0.3256	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
60	0.2563	0.4432	0.3844	0.3354	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
80	0.2375	0.4563	0.3932	0.3264	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$
100	0.2456	0.4322	0.3673	0.3453	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)} \succ \mathcal{P}^{(1)}$

The results suggest that the TAOV-based aggregation process yields consistent outcomes across different λ values, confirming the method's stability and robustness in uncertain DM environments. Even with changes in λ, the top-ranked alternatives remain largely unchanged.

Figure 3(a): Impact of λ under the T-SFSAAWA operator

Figure 3(b): Influence of λ through the T-SFSAAWG operator

Figure 3: Sensitivity of λ

Figure 3: Sensitivity of λ on score values and rankings of alternatives.

6. Comparative Studies

This section presents a detailed comparison between the MADM method using the proposed operators and several other aggregation-based approaches employing various AOs. Arora [44] introduced the concept of intuitionistic fuzzy soft weighted Einstein averaging. Sarfraz et al. [11] defined the concept of mixed SFS clusters. Ali et al. [45] developed the concept of AA AOs with IFS. The results generated by the current methods are consistent with the targeted strategy, as demonstrated in Table 10.

The two aggregation operators in this paper, T-SFSAAWA and T-SFSAAWG, are used to combine evaluation data, and the candidates are then ranked using the score function.

Table 10: Comparative studies of different aggregation operators $\mathcal{P}^{(10)}$

Approach	$\mathcal{P}^{(1)}$	$\mathcal{P}^{(2)}$	$\mathcal{P}^{(3)}$	$\mathcal{P}^{(4)}$	Ranking
T-SFSAAWA	0.4998	0.9990	0.4997	0.4994	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
T-SFSAAWG	-0.2362	0.1107	0.0632	0.1318	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
IFSWEA [44]	0.4432	0.8561	0.4213	0.4123	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
IFSWEG [44]	0.2432	0.6643	0.2232	0.1123	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
SFSWA [11]	0.4321	0.7632	0.3421	0.3221	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
SFSOWA [11]	0.3421	0.5632	0.2342	0.1985	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
IFSAAWA [45]	0.4326	0.7845	0.3452	0.2353	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$
IFSAAWG [45]	0.2341	0.6654	0.2231	0.1986	$\mathcal{P}^{(2)} \succ \mathcal{P}^{(1)} \succ \mathcal{P}^{(3)} \succ \mathcal{P}^{(4)}$

Table 10 shows that the results from different operators are consistent, indicating that the applicant $\mathcal{P}^{(2)}$ is the best alternative. Delivery time is the most important attribute in our study. The ranking results are identical across all methods, highlighting the significance of the recommended operators.

The proposed operators are based on flexible operational laws, namely AATRM and AATCRM, and thus yield more adaptable results compared to existing operators. The geometric representation of the comparative study is illustrated in Figure 4.

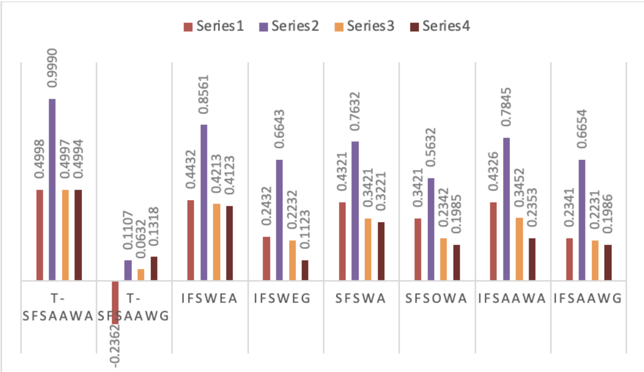

Figure 4: Geometrical representation of the comparison between different operators.

The proposed TAOV AA AOs within the T-SFS environment demonstrate consistent and reliable outcomes across various aggregation methods. The results from weighted averaging and geometric operators align with conventional fuzzy aggregation methods, confirming the framework's robustness. By considering contrast intensity and correlation among criteria, the TAOV method effectively determines attribute weights, improving fairness and precision.

The AA operations guarantee smooth aggregation and closure within $[0, 1]$, while capturing uncertainty, hesitation, and bipolarity that traditional MADM approaches overlook. This integration reduces information loss and improves accuracy and decision stability in uncertain environments. The proposed operators are highly suitable for practical MADM applications, such as supplier selection, energy assessment, and system management. Overall, the method provides a balanced, consistent, and realistic decision-making tool, improving upon conventional fuzzy and intuitionistic aggregation models.

7. Conclusions

This paper examined the use of AA aggregation operators (AOs) for T-SFSVs to solve supplier selection problems. We introduced and analyzed the T-SFSAAWA and T-SFSAAWG operators, highlighting their characteristics and practical applications. The study addressed the MADM challenges inherent in supplier selection, where different attributes require varying levels of importance, by incorporating the AA TN and TCN. The proposed methodology effectively demonstrated its value in managing the complexities of supplier evaluation. Comparisons with existing AOs revealed that the proposed operators yield consistent and reliable results.

(a) By integrating subjective judgments and uncertain information into a single framework, the proposed TAOV AA model assists managers in choosing the most suitable supplier. Unlike classical methods such as TOPSIS or AHP, which assume crisp or single-valued preferences, the T-SFS environment accommodates hesitation and flexibility in expert opinions. This allows procurement managers to evaluate suppliers more realistically, especially when data are incomplete or uncertain.

(b) The TAOV component ensures equitable weighting of attributes, and the AA operator maintains consistency and balance in the presence of conflicting attributes. This enables decision-makers to prioritize suppliers that offer cost-effective solutions while aligning with quality, delivery reliability, and sustainability objectives.

Future Research Directions

- Incorporating alternative uncertainty modeling frameworks such as T-spherical fuzzy rough sets, neutrosophic sets, or other T-spherical fuzzy settings to broaden applicability.
- Integrating T-SF SS-based operators with optimization or machine learning algorithms to enable automated decision-support systems in dynamic, data-intensive environments.
- Adding consensus-building and conflict-resolution techniques for multiple decision-makers in uncertain scenarios to enhance group decision-making.

Limitations

- The approach assumes attribute weights are fixed and known, which may not reflect real-world situations where preferences evolve over time.
- Interdependencies between criteria, which can influence decision outcomes in practice, are not considered in the current model.

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