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Two Numerical Approaches to Solving Fractional Differential Equations with a Generalized Mittag–Leffler Kernel Using Bernstein Polynomials

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Abstract. This paper presents a solution to fractional differential equations containing three parameters, utilizing Bernstein polynomials through two efficient computational approaches. In the first approach, the solution is expressed as a linear combination of Bernstein polynomials. In contrast, in the second, the fractional derivative itself is represented in terms of Bernstein polynomials. The key properties of both algorithms are derived and analyzed. The generalized Atangana Baleanu Caputo definition of fractional derivative that uses the Mittag-Leffler function as the kernel of the integration form of the fractional derivative, characterized by three tunable parameters, is adopted throughout this study. Those parameters can adjust the existence and the behavior of the solution for the fractional derivative equations. A set of initial value problems, including both linear and nonlinear fractional differential equations, are solved using the suggested approaches. The solution profiles illustrate the performance of the numerical solutions and the impact of the Atangana Baleanu Caputo definition on the obtained findings, demonstrating that Bernstein polynomials provide improved accuracy and efficiency in extracting solutions for the considered fractional models. The computational simulation of this comparative analysis reveals that the second approach yields higher accuracy with smaller absolute errors and additionally provides insight into the existence of solutions, as illustrated through the studied fractional models.

2020 Mathematics Subject Classifications: 26A33, 41A30, 65N12, 33C45, 33E12, 65N22

Key Words and Phrases: Generalized ABC fractional derivative, generalized Mittag-Leffler kernel, Bernstein polynomials, Riemann-Liouville fractional derivative, fractional calculus, AB fractional integral

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1. Introduction

Differential equations are one of the most used for modeling real-life phenomena that are based on calculating the changes of the systems based on dimensions or time. Replacing the standard derivatives with fractional ones will enrich the problem with several parameters that can be fitted to real data [1, 2]. Most of those fractional derivatives(FDs) are close to the standard one in the case of an integer derivative.

Fractional order calculus has garnered significant interest in both theoretical and applied sciences over the past twenty years. Diverse kinds of fractional operators have been introduced, such as Grunwald-Letnikov, Riemann-Liouville (R-L), Riesz, Caputo, and Atangana Baleanu Caputo (ABC) FDs. [3–5], and have been thoroughly studied as worthy tools for depicting genetic features, memory influences, and material convey procedures in a variety of applied mathematics, engineering [6], and physics disciplines, including dynamics, elasticity, control theory, and mechanics. The calculus of variations is a further discipline in which Fractional order calculus has shown great utility (see [7, 8]). Here, FDs are used to model functionals rather than a path's first-order derivative. This is often the case in many mathematical or engineering challenges, providing a more accurate quantification of physical processes. A great deal of work has been done on the fractional order calculus of variations (e.g., Sun et al. [9], Agrawal [10], Atanackovic et al. [11] Araz and Çetin [12], Arik and Araz [13], Jahan et al. [14] and Almeida and Torres [15]). Although the fractional order calculus of variations literature is currently extensive, further investigation is still necessary. FDs can be constructed using various techniques that do not always yield the same result. Some of these are defined using a fractional integral. Due to definition incompatibility, it is usually necessary to be explicit about which definition is utilized, even for smooth functions.

The nonlocal FD takes the memory of the function into account and is formed based on the integration of a kernel and the function or its standard derivative. Several kernels are used to formulate these kinds of FDs, and singular and non-singular are investigated [16, 17]. Each kernel plays a significant role in the behavior of the solution. The non-singular FD employed several kernels, including the exponential function and the Mittag-Leffler function. One of the main disadvantages of using those kinds is the lack of a solution, even for a simple one. So, the use of the Mittag-Leffler function can overcome this limitation and enrich the solution parameter [18].

Several methods are used to solve the fractional differential equations (FDEs) in the sense of the Mittag-Leffler kernel, such as pure numerical methods based on Adams Bashforth techniques [19, 20] or midpoint method [21], approximate analytic methods such as homotopy analysis method [22], Legendre polynomials [23], Bernstein polynomials [24, 25] and other collocation method based on Bell wavelets [26].

The methods that are based on orthogonal polynomials, such as Legendre or Bernstein, usually approximate the solution by polynomials with integer powers. This may lead to missing the effects of the fractional order. So, our motivations are to build a new algorithm based on the orthogonal polynomials that observes the effects of fractional power solution and compares our results with the standard one. This work will established a general framework for solving of FDE with Mittag-Leffler function of three parameters kernel based on the Bernstein polynomials in two approaches, the first one by assuming the solution as a linear combinations of the polynomial and the second one by assuming the fractional derivative is the linear combinations of the derivative which leads to the solution forms that depends on, the fractional parameters. The experimental results proved that the second approach is more accurate and gives an explicit solution based on independent parameters of powers of the fractional parameters. Moreover, clear steps for both approaches are established in an easy-to-compute manner.

The paper is organized as follows: In Section 2, we proposed definitions and theories, along with their proofs, that we needed for this paper. In Section 3, we illustrate the definition and some properties of the Bernstein polynomials, and we find the ABC fractional derivative for the Bernstein polynomials. In Section four, we presented the procedure of two approaches for solving fractional-order problems and showed the numerical results. In the fifth section, we discussed the numerical results for the Examples. In section 6, we discussed and compared the two approaches for all the presented Examples. The conclusions are presented in Section 7.

2. Basic Definitions and Theorems

This section presents the definition of the left ABC FD with kernel $E^{\gamma}_{\alpha,\mu}(\lambda, t)$, and we define the left AB fractional integral and some properties.

The R-L fractional integral of order $n-\alpha$ where $\alpha, a, t \in \mathbb{R}$ and $n-1<\alpha<n$ from the left and right respectively are defined as [27]:

$$ {}_a\mathrm{I}_t^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \tag{1}$$

$$ {}_t\mathrm{I}_b^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (t-s)^{n-\alpha-1} f(s)ds. \tag{2}$$

The R-L FD from the left and right, respectively, are defined as [28]:

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \frac{d^n}{dt^n} {}_a\mathrm{D}_t^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} ({}_a\mathrm{I}_t^{n-\alpha} f(t)), \tag{3}$$

$$ {}_t\mathrm{D}_b^{\alpha} f(t) = \frac{d^n}{dt^n} {}_t\mathrm{D}_b^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n} ({}_t\mathrm{I}_b^{n-\alpha} f(t)). \tag{4}$$

The Caputo FD for $n-1<\alpha\leq n$ is defined as [29]:

$$ {}^{C}\mathrm{D}_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds. \tag{5}$$

Definition 1. [22] The FD in the sense of generalized ABC with kernel $E^{\gamma}_{\alpha,\mu}(\lambda,t)$ is given by

$$({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma}f)(x)=\frac{M(\alpha_1)}{1-\alpha_1}\int_a^x E^{\gamma}_{\alpha_1,\mu}(\lambda,x-t)f^{(n+1)}(t)dt, \tag{6}$$

and

$$({}^{ABC}\mathrm{D}_{b}^{\alpha,\mu,\gamma}f)(x)=\frac{-M(\alpha_1)}{1-\alpha_1}\int_x^b E^{\gamma}_{\alpha_1,\mu}(\lambda,x-t)f^{(n+1)}(t)dt, \tag{7}$$

where $M(\alpha_1)$ is a function of the nationalization with M(0)=M(1)=1 and $n<\alpha\leq n+1$, $n\in\{0,1,2,...\}$,

$$\lambda=-\frac{\alpha_1}{1-\alpha_1},\quad \alpha_1=\alpha-n,\quad \mu>0 \quad and \quad \gamma\in\mathbb{R},$$

where

$$E^{\gamma}_{\alpha_1,\mu}(\lambda,x-t)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha_1 k+\mu)}\lambda^k(x-t)^{\alpha_1 k+\mu-1}, \tag{8}$$

and

$$(\gamma)_k=\gamma(\gamma+1)(\gamma+2)...(\gamma+k-1);\quad k\in\mathbb{N}.$$

Theorem 1. *[30] "The generalized FD with generalized Mittag-Leffler kernel of x^{β} $(\beta>n)$ of order α $(n<\alpha\leqq n+1)$ is given, for $\mu>0$ and $\alpha_1=\alpha-n$ $(\mathbb{N}_0)$, by*

$$\begin{aligned}{}^{\mathrm{ABC}}_{0}D^{\alpha,\mu,\gamma}x^{\beta}&=\frac{M(\alpha_1)\Gamma(\beta+1)}{1-\alpha_1}\sum_{k=0}^{\infty}\frac{\lambda^k(\gamma)_k x^{\alpha_1 k+\mu+\beta-n-1}}{k!\,\Gamma(\alpha_1 k+\beta+\mu-n)} &(9)\\ &=\frac{M(\alpha_1)\Gamma(\beta+1)}{1-\alpha_1}\,E^{\gamma}_{\alpha_1,\mu+\beta-n}(\lambda,x).\text{”} &(10)\end{aligned}$$

Theorem 2. *[18] "Let $f\in C^{n+1}[0,1]$. The Laplace Transform of the ABC-FD with kernel $E^{\gamma}_{\alpha,\mu}(\lambda,t)$ for $n<\alpha<n+1$, $\mu>0$, $\gamma=1$ and $\lambda=\frac{-\alpha_1}{1-\alpha_1}$ is given by*

$$L\left[({}_{0}^{ABC}D^{\alpha,\mu,\gamma}f)(x)\right](s)=\frac{M(\alpha_1)}{1-\alpha_1}(1-\lambda s^{-\alpha})^{-1}s^{n+1-\mu}F(s).\text{”} \tag{11}$$

Proposition 1. *[31] Let*

$$\frac{M(\alpha_1)}{1-\alpha_1}(1-\lambda s^{-\alpha})^{-1}s^{n+1-\mu}F(s)=U(s), \tag{12}$$

then

$$F(s)=\frac{1-\alpha_1}{M(\alpha_1)}s^{\mu-(n+1)}U(s)+\frac{\alpha_1}{M(\alpha_1)}s^{\mu-(n+1)-\alpha}U(s), \tag{13}$$

By applying the inverse Laplace transform to both sides, we have

$$f(t)=\frac{1-\alpha_1}{M(\alpha_1)}I^{n+1-\mu}u(t)+\frac{\alpha_1}{M(\alpha_1)}I^{n+1-\mu+\alpha}u(t). \tag{14}$$

Definition 2. The left AB fractional integral for $n < \alpha < n+1$, $\mu > 0$ and $\gamma = 1$ is given by

$$({}_{a}^{AB}\mathrm{I}^{\alpha,\mu,\gamma} f)(x) = \sum_{i=0}^{\gamma} \binom{\gamma}{i} \frac{\alpha_1^i}{M(\alpha_1)(1-\alpha_1)^{i-1}} ({}_{a}^{RL}\mathrm{I}^{\alpha i-\mu+n+1} f)(x), \tag{15}$$

where $({}_{a}^{RL}\mathrm{I}^{\alpha i-\mu+n+1} f)(x)$ is the RL fractional integral of order $\alpha i - \mu + n + 1$.

Theorem 3. *For $n < \alpha < n+1$, $\mu > 0$ and $\gamma = 1$, we conclude that*

$$\begin{aligned} {}_{a}^{AB}I^{\alpha,\mu,\gamma}\,{}_{a}^{ABC}D^{\alpha,\mu,\gamma} u(t) &= u(t) - u(a) - (t-a)u'(a) - \cdots - \\ &\frac{(t-a)^{n-1}}{(n-1)!} u^{n-1}(a) - \frac{(t-a)^n}{n!} u^n(a), \end{aligned} \tag{16}$$

where $n \in \mathbb{N}$.

Proof. By applying the operator ${}_{a}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}$ in Definition 2 on the derivative $\mathrm{D}^{\alpha,\mu,\gamma} u(t)$, we have

$$\begin{aligned} {}_{a}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}\,{}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t) &= \frac{1-\alpha_1}{M(\alpha_1)} {}_{a}^{RL}\mathrm{I}^{n+1-\mu} ({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u)(t) \\ &+ \frac{\alpha_1}{M(\alpha_1)} {}_{a}^{RL}\mathrm{I}^{n+1-\mu+\alpha} ({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u)(t), \\ &= \sum_{k=0}^{\infty} \lambda^k {}_{a}^{RL}\mathrm{I}^{\alpha_1 k+n+1} u^{(n+1)}(t) - \sum_{k=0}^{\infty} \lambda^k {}_{a}^{RL}\mathrm{I}^{\alpha_1 k+\alpha+n+1} u^{(n+1)}(t), \\ &= \sum_{k=0}^{\infty} \lambda^k \frac{1}{\Gamma(\alpha_1 k+n+1)} \cdot \int_a^x (x-t)^{\alpha_1 k+n} u^{(n+1)}(t)dt \\ &- \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{\Gamma(\alpha_1 k+\alpha+n+1)} \cdot \int_a^x (x-t)^{\alpha_1 k+\alpha+n} u^{(n+1)}(t)dt, \\ &= \int_a^x (E_{\alpha_1,n+1}(\lambda, x-t) - \lambda E_{\alpha_1,\alpha+n+1}(\lambda, x-t)) u^{(n+1)}(t)dt, \\ &= \frac{1}{\Gamma(n+1)} \int_a^x (x-t)^n u^{(n+1)}(t)dt, \\ &= u(t) - u(a) - (t-a)u'(a) - \cdots - \\ &\frac{(t-a)^{n-1}}{(n-1)!} u^{(n-1)}(a) - \frac{(t-a)^n}{n!} u^{(n)}(a). \end{aligned}$$

3. The Bernstein Polynomials and the ABC FDs

Definition 3. [32] Bernstein polynomials is a linear combination of $(l+1)$ terms which is defined as one dimension of degree l on $[0,1]$ as

$$B_{v,l}(z) = \binom{l}{v} z^i (1-z)^{l-v}, \qquad v = 0, ..., l, \tag{17}$$

or

$$\begin{aligned} B_{v,l}(z) &= \binom{l}{v} z^v \sum_{s=0}^{l-v} (-1)^s \binom{l-v}{s} z^s \\ &= \sum_{s=0}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} z^{s+v}, \quad v = 0, ..., l. \end{aligned}$$

Theorem 4. *The FD with generalized Mittag-Leffler kernel for Bernstein polynomial using for $n < \alpha \leq n+1$, $\alpha_1 = \alpha - n$ where $n \in \mathbb{N}_0$, $\mu > 0$ and $\gamma \in \mathbb{N}$ is given by*

$${}_{0}^{ABC}D^{\alpha,\mu,\gamma} B_{v,l}(z) = \sum_{s=\lceil \alpha \rceil - v}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} \frac{M(\alpha_1)}{1-\alpha_1} \Gamma(s+v+1) E^{\gamma}_{\alpha 1, \mu+s+v-n}(\lambda, z). \tag{18}$$

Proof. Using Theorem 1, we have

$$\begin{aligned} {}_{0}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} B_{v,l}(z) &= {}_{0}^{ABC}\ \mathrm{D}^{\alpha,\mu,\gamma} \left(\sum_{s=0}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} z^{s+v} \right) \\ &= \sum_{s=0}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} {}_{0}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} z^{s+v} \\ &= \sum_{s=\lceil \alpha \rceil - v}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} \frac{M(\alpha_1)}{1-\alpha_1} \Gamma(s+v+1) E^{\gamma}_{\alpha_1, \mu+s+v-n}(\lambda, z). \end{aligned}$$

4. Solutions Approach

This section shows the procedure for solving FDEs with a generalized Mittag-Leffler kernel using Bernstein polynomials. Mainly, we built two algorithms, the first one assumes the solution is a linear combination of Bernstein polynomials. In contrast, the second approach assumes that the FD is a linear combination.

Consider the FDE

$${}_{a}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) = f(u(t), t), \quad a \leq t \leq b, \tag{19}$$

with initial condition

$$u(a) = c, \tag{20}$$

where $n < \alpha \leq n+1$, $\mu > 0$ and $\gamma \in \mathbb{N}$.
Approach 1. This approach was introduced in [22], it based on the following steps.

$$u(t) = \sum_{v=0}^{l} c_v B_{v,l}(t). \tag{21}$$

Applying the ABC-FD on both sides of equation (19) to get

$$({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t)) = \sum_{v=0}^{l} c_v \sum_{s=0}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} ({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} t^{s+v}). \tag{22}$$

According to Theorem (3), we have

$$\begin{aligned} ({}_{a}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t)) = \sum_{v=0}^{l} c_v \sum_{s=\lceil \alpha \rceil - v}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} \times \\ \frac{M(\alpha_1)}{1-\alpha_1} \Gamma(s+v+1) E^{\gamma}_{\alpha_1, \mu+s+v-n}(\lambda, t), \end{aligned} \tag{23}$$

then equation (19) gives

$$\begin{aligned} \sum_{v=0}^{l} c_v \quad {}_{0}{}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} B_{v,l}(t) = f\left(\sum_{v=0}^{l} c_v B_{v,l}(t), t \right) \\ \sum_{v=0}^{l} c_v \sum_{s=\lceil \alpha \rceil - v}^{l-v} (-1)^s \binom{l}{v} \binom{l-v}{s} \frac{M(\alpha_1)}{1-\alpha_1} \Gamma(s+v+1) \times \\ E^{\gamma}_{\alpha 1, \mu+s+v-n}(\lambda, t) = f\left(\sum_{v=0}^{l} c_v B_{v,l}(t), t \right), \end{aligned} \tag{24}$$

and the initial condition gives

$$\sum_{v=0}^{l} c_v B_{v,l}(a) = c. \tag{25}$$

Now, substitute $t = t_v$ in equation (24) for $v = 1, 2, .., l$, then we have l nonlinear equations and using equation (25), then $(l+1)$ of nonlinear equations are generated. By solving those equations, we determined the c_v for $v = 0, 1, ..., l$. As a result, the approximate solution $u(t)$ can be calculated.
Approach 2. Assume

$${}_{a}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) = \sum_{v=0}^{l} c_v B_{v,l}(t). \tag{26}$$

Then, apply the ABC fractional integral operator on equation (26) to get

$$ {}_{a}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}\,{}_{a}^{ABC}\mathrm{D}_{t}^{\alpha,\mu,\gamma}u(t) = \sum_{v=0}^{l} c_v\,{}_{a}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}B_{v,l}(t). \tag{27} $$

Using Definition (2), Theorem (2), and Equation (13), we have

$$ u(t) - u(a) - \cdots - \frac{(t-a)^n}{n!}u^{(n)}(a) = \sum_{v=0}^{l} c_v \sum_{s=0}^{l-v}(-1)^s \binom{l}{v}\binom{l-v}{s} \sum_{r=0}^{\gamma}\binom{\gamma}{r}\frac{\alpha^r}{(1-\alpha)^{r-1}} \times {}_{a}^{RL}\mathrm{I}^{\alpha r-\mu+n+1}t^{s+v}, \tag{28} $$

where ${}_{a}^{RL}\mathrm{I}^{\alpha r-\mu+n+1}$ is the left RL fractional integral of order $\alpha r - \mu + n + 1$ which is defined in equation (1). we assume

$$ {}_{0}^{ABC}\mathrm{D}_{t}^{\alpha,\mu,\gamma}u(t) = \sum_{v=0}^{l} c_v B_{v,l}(t), \tag{29} $$

then (19) gives

$$ \sum_{v=0}^{l} c_v B_{v,l}(t) = f\left(\sum_{v=0}^{l} c_v{}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}B_{v,l}(t) + u(a) + \cdots + \frac{(t-a)^n}{n!}u^n(a), t\right), \tag{30} $$

and the initial condition gives

$$ \sum_{v=0}^{l} c_v{}^{AB}\mathrm{I}^{\alpha,\mu,\gamma}B_{v,l}(a) = 0. \tag{31} $$

Now, substitute $t = t_v$ in equation (30) for $v = 1, 2, .., l$, then we have l nonlinear equations and using equation (31), then $(l + 1)$ of nonlinear equations are generated. By solving those equations using Newton's Iterative Methods, we determined the c_v for $v = 0, 1, ..., l$. As a result, the approximate solution can be calculated.

5. Illustrative Examples using the two proposed Methods

In this section, we show the schemes of the three fractional order initial value problems with the help of the Bernstein polynomial using two approaches.

Example 1

Consider the following FDE [22] :

$$ {}_{0}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) = t^2, \qquad u(0) = 0, \tag{32} $$

where $\gamma, \mu, \alpha \in \mathbb{R}$, $\mu > 0$, and $0 < \alpha < 1$. By applying the operator $0^{\mathrm{AB}} I^{\alpha,\mu,\gamma}$ on the equation (32), and using Theorem 3, the exact solution is

$$ u(t)=\begin{cases} \sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{2(1-\alpha)^{1-i}\alpha^i t^{\alpha i-\mu+3}}{M(\alpha)(\alpha i-\mu+4)}+\frac{(1-\alpha)t^2}{M(\alpha)}, & \mu \geq 1 \\ \sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{2(1-\alpha)^{1-i}\alpha^i t^{\alpha i-\mu+3}}{M(\alpha)(\alpha i-\mu+4)}+\frac{2(1-\alpha)t^{3-\mu}}{M(\alpha)\Gamma(4-\mu)}, & \mu < 1. \end{cases} $$

Approach 1: This approach was introduced in [22] that is based on the following steps.

Step 1 We can approximate $u(t)$ as:

$$ u(t) = \sum_{i=0}^{m} c_i B_{i,m}(t). \tag{33} $$

Step 2 Using (18) for (32), we have

$$ \sum_{i=0}^{m} \sum_{k=\lceil\alpha\rceil-i}^{m-i} c_i(-1)^k \binom{m}{i}\binom{m-i}{k} \frac{M(\alpha_1)}{1-\alpha_1} \Gamma(k+i+1) E_{\alpha_1,\mu+k+i}^{\gamma}(\lambda, t) = t^2. \tag{34} $$

Step 3 The initial condition gives:

$$ \sum_{i=0}^{m} c_i B_{i,m}(0) = u(0). \tag{35} $$

Then we have

$$ c_0 = 0. $$

Step 4 Now, substitute collection points in equation (34) to have a system of linear equations, then by solving these equations we get c_i for $i = 1, \cdots, m$.

Step 5 Finally, substitute the values of c_i in Equation (33) to obtain the solution of (32) using the first approach. The residual error function (REF) for (32) can be defined as:

$$ REF(t) = {}_{0}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) - t^2. \tag{36} $$

In Figure 1, we plot the present solution of Example 1 using the first approach with several

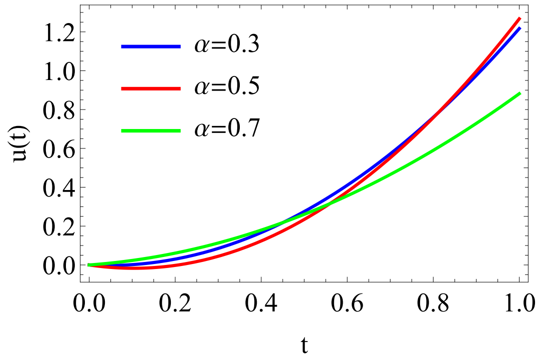

Figure 1: **The approximate solutions of Example 1 using the first approach with several values of α and μ=1 and γ=2.**

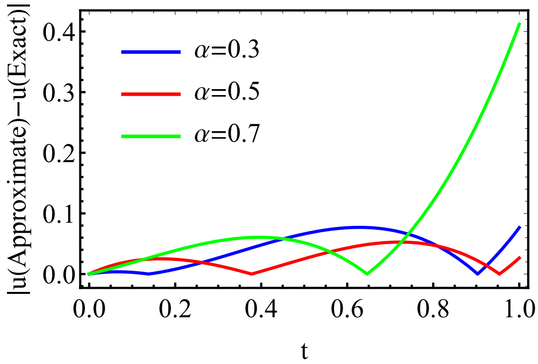

Figure 2: **Absolute error using the first approach with several values of α and fixed value of μ and γ.**

values of α and fixed $\mu = 1$, $\gamma = 1$. Figure 2 presents the absolute error of Example 1 using the first approach for several values of α and fixed $\mu = 1$, $\gamma = 2$. The solutions using Approach 1 are the same in [22].

Approach 2:

Step 1 Let

$$ {}_{0}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) = \sum_{i=0}^{m} c_i B_{i,m}(t). \tag{37} $$

Step 2 By substituting equation (37) in equation (32), we have

$$ \sum_{i=0}^{m} c_i B_{i,m}(t) = t^2. \tag{38} $$

Step 3 The initial condition

$$ \sum_{i=0}^{m} c_i {}_{0}^{AB}\mathrm{I}^{\alpha,\mu,\gamma} B_{i,m}(0) + u(0) = u(0), \tag{39} $$

gives

$$c_0 = 0. \tag{40}$$

Step 4 We apply the left AB fractional integral of equation (19) using equation (16), to have

$$\begin{aligned} u(t) - u(0) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} \sum_{n=0}^{\gamma} \binom{\gamma}{n} \frac{\alpha^n}{M(1-\alpha)^{n-1}} \times \\ \frac{\Gamma(k+i+1)}{\Gamma(\alpha n + k + i - \mu + 2)} \times t^{\alpha n + k + i - \mu + 1}. \end{aligned} \tag{41}$$

Step 5 Now, at collecation points t_r we have equation (38) in this form:

$$\sum_{i=0}^{m} c_i B_{i,m}(t_r) - t_r^2 = 0. \tag{42}$$

Step 6 From equation (42), we have a system of linear equations. By solving these equations, we get c_i for $i = 1, \cdots, m$. Finally, substitute c_i in equation (41) to obtain the approximate solution of equation (32).

In Figure 3 and Figure 4, we plot the approximate solutions and the absolute error of Example 1 using the second approach for several values of α and fixed $\mu = 1$, $\gamma = 2$ and in Table 1, we calculate the approximate solutions for several values of α at various values of t.

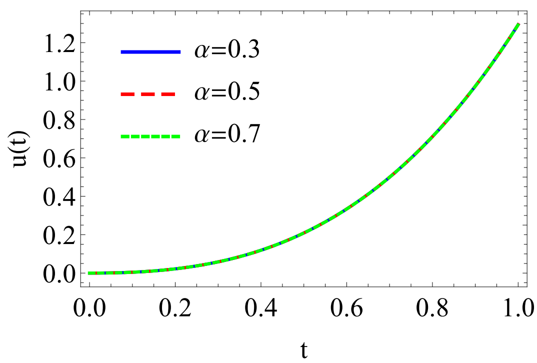

Figure 3: **The approximate solutions using the second approach with several values of α and μ=1 and γ=2.**

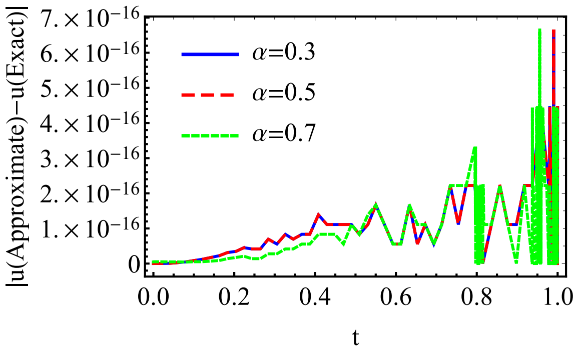

Figure 4: **The absolute error between exact and approximate solutions using the second approach with several values of α and a fixed value of μ and γ.**

Table 1: The approximate solutions for Example 1 using the second approach for several values of α.

t \ α	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
0.3	0.094069	0.0791659	0.058388	0.039047
0.5	0.277218	0.252218	0.208848	0.181213
0.7	0.567254	0.548884	0.499127	0.524649
1	1.216373	1.268468	1.293638	1.687511

Example 2

Consider the following nonlinear FDE:

$$ {}_{0}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma}u(t) = u^2(t) - u(t), \quad u(0) = 0.5, \tag{43}$$

where

$$0 < \alpha < 1 \quad and \quad \mu = \gamma = 1,$$

and the exact solution at $\alpha = 1$ is

$$u(t) = \frac{1}{e^t + 1}. \tag{44}$$

Approach 1: We solve this Example using the first approach, then equation (43) becomes

$$\sum_{i=0}^{m} c_i \sum_{k=\lceil\alpha\rceil-i}^{m-i} (-1)^k \binom{m}{i}\binom{m-i}{k} \frac{M(\alpha_1)}{1-\alpha_1}\Gamma(k+i+1)E^{\gamma}_{\alpha_1,\mu+k+i}(\lambda,t) = \left(\sum_{i=0}^{m} c_i B_{i,m}(t)\right)^2 - \sum_{i=0}^{m} c_i B_{i,m}(t), \tag{45}$$

The initial condition gives

$$c_0 = 0.5. \tag{46}$$

when we apply $m+1-\lceil\alpha\rceil$ points in equation (45), we have m nonlinear equations, then m of nonlinear equations are generated. By solving those equations, we determined the c_i for $i = 1, ..., m$. As a result, the approximate solution $u(t)$ can be calculated.

We plot the Residual Error Function (REF) for the approximate solution of Example 2 using the first approach with $\alpha \to 1$ and $\mu = \gamma = 1$ in Figure 5. The solutions using approach 1 are the same in [22].

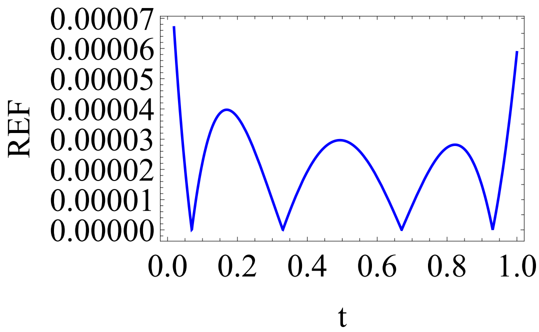

Figure 5: **The REF of Example 2 using the first approach.**

Approach 2: When we solve using the second approach, then equation (43) becomes

$$\sum_{i=0}^{m} c_i B_{i,m}(t) = \left(\sum_{i=0}^{m} c_i {}^{AB}\mathrm{I}^{\alpha,\mu,\gamma} B_{i,m}(t) + 0.5\right)^2 - \left(\sum_{i=0}^{m} c_i {}^{AB}\mathrm{I}^{\alpha,\mu,\gamma} B_{i,m}(t) + 0.5\right), \quad (47)$$

and the initial condition gives

$$c_0 = 0. \quad (48)$$

when we apply $m+1-\lceil\alpha\rceil$ points in equation (47), we have m nonlinear equations then m of nonlinear equations are generated. By solving those equations, we determined the c_i for $i = 1, ..., m$. As a result, the approximate solution $u(t)$ can be calculated.

Then the REF is as follows

$$REF(t) = {}_0^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t) - u^2(t) + u(t), \quad (49)$$

which is ploted in 6.

Now, in Figure 7, we plot the exact (for $\alpha = 1$) and approximate solution for ($\alpha = 0.5$) using the second approach that expresses that the solution does not exist. The conclusion in this Example demonstrates the results given in [33]; the FDE

$$ {}_0^{ABC}\mathrm{D}^{\alpha,1,1} u(t) = u^2(t) - u(t), \quad u(0) = 0.5,$$

has a nontrivial solution only if $u^2(0) - u(0) = 0$.

Example 3

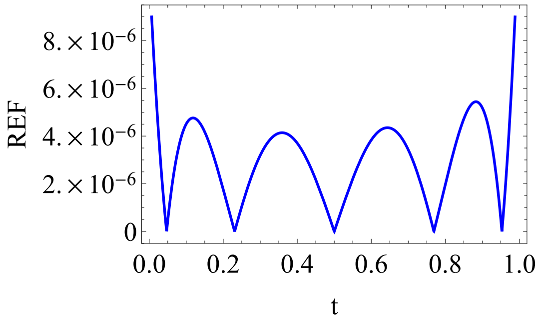

Figure 6: **The REF of Example 2 using the second approach.**

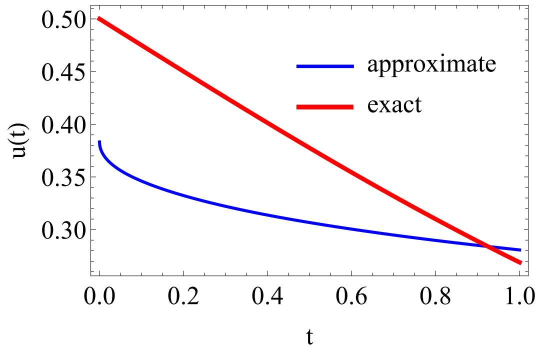

Figure 7: The exact solution and approximate solutions using the second approach with $\alpha = 0.5$, $\mu = 1$, and $\gamma = 1$.

In this Example, we investigate the FDE by applying the second approach for $1 < \alpha \leq 2$. Consider the following FDE:

$$ {}_{0}^{ABC}\mathrm{D}_t^{\alpha,\mu,\gamma} u(t) + u(t) = 0, \quad u(0) = 0 \quad and \quad u'(0) = 1, \tag{50} $$

which has an exact solution for $\alpha = 2$

$$ u(t) = \sin(t). \tag{51} $$

Let

$$ {}_{0}^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t) = \sum_{i=0}^{m} c_i B_{i,m}(t). \tag{52} $$

By applying the left AB fractional integral on equation (52), we get

$$ \begin{aligned} u(t) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} \sum_{s=0}^{\gamma} \binom{\gamma}{s} \frac{\alpha_1^s}{(1-\alpha_1)^{s-1}} \times \\ \frac{\Gamma(k+i+1)}{\Gamma(\alpha s - \mu + k + i + 3)} t^{\alpha s - \mu + k + i + 2} + u(0) + t u'(0), \end{aligned} \tag{53} $$

then

$$u(t) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} \sum_{s=0}^{\gamma} \binom{\gamma}{s} \frac{\alpha_1^s}{(1-\alpha_1)^{s-1}} \times \frac{\Gamma(k+i+1)}{\Gamma(\alpha s - \mu + k + i + 3)} t^{\alpha s - \mu + k + i + 2} + t, \tag{54}$$

and

$$u'(t) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} \sum_{s=0}^{\gamma} \binom{\gamma}{s} \frac{\alpha_1^s}{(1-\alpha_1)^{s-1}} \times \frac{\Gamma(k+i+1)}{\Gamma(\alpha s - \mu + k + i + 2)} t^{\alpha s - \mu + k + i + 1} + 1. \tag{55}$$

By substituting $t = 0$ in equation (55), we have

$$c_0 = 0. \tag{56}$$

Now, when we apply $m + 1 - \lceil \alpha \rceil$ points in equation (50), we have a system of equations, then by solving this system we get the values of c_i for $i = 1, \cdots, m$.

Then the REF is

$$REF(t) = {}_0^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} u(t) + u(t). \tag{57}$$

In Figure 8, we plot the REF for the approximate solution using the second approach of Example 3 with several values of α and fixed $\mu = \gamma = 1$. Also in Table 2, we calculate the approximate solutions with several values of α at various points of t.

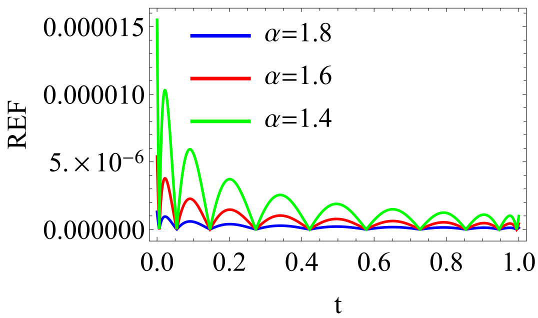

Figure 8: **The REF for the approximate solutions with several values of α and fixed μ=γ=1.**

Now, we show the REF of the approximate solution using the second approach for Example 3 with several values of μ and $\alpha = 1.8$, $\gamma = 1$ in Figure 9.

Table 2: The approximate solutions for Example 3 using the second approach for several values of α.

t \ α	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
0.2	0.19866936	0.19866934	0.19866933	0.19866933
0.5	0.47942562	0.47942557	0.47942555	0.47942554
0.8	0.71735621	0.71735615	0.71735611	0.71735609
1	0.84147113	0.84147105	0.84147101	0.84147099

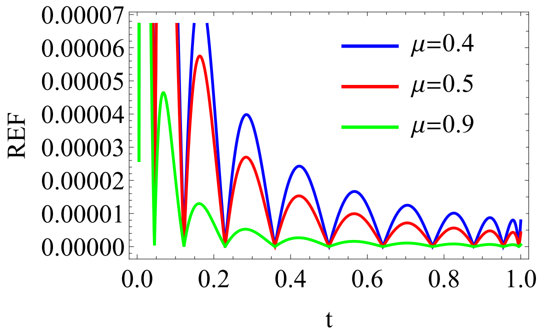

Figure 9: **The About REF of the approximate solutions with $\alpha = 1.8$, $\gamma = 1$ and several values of μ.**

6. Discussions

In Example 1, the comparison between the solutions using the first approach is shown in Figure 1. As shown, there is a significant dispersion in the results when the values of α are changed. Figure 2 illustrates the approximate solution for various values of the fractional parameters. As shown, the three curves are very close to each other, indicating that this method is more reliable and accurate. The absolute error for the first and second approaches is displayed in Figures 3 and 4, respectively. As shown, the second approach gives a better residual error than the first approach. In contrast, the second approach yields a very small error that tends to 10^{-15} compared to the first approach. For Example 2, the same trend has been observed; the residual errors are plotted in Figures 5 and 6 for the first and second approaches, respectively. As shown, the error in the second approach was significantly lower than that in the first approach, indicating that the second approach is more accurate and yields reliable results. Figure 7 shows that the solution does not exist with $\alpha = 0.5$ and $\mu = \gamma = 1$ using the second approach, which agrees with the theoretical results in [33].

In Example 3, the absolute residual error in Figure 8 decreases when the α value increases and when it becomes close to an integer number for fixing the μ and γ. Figure 9 shows the absolute residual error when the μ is varying; as indicated, the error decreases as the μ increases, and the error becomes minimum when the value of μ is close to one.

7. Conclusions

In this study, we built two frameworks for solving FDEs in the sense of ABC-FD with generalized Mittag-Leffler function by investigating the use of Bernstein Polynomials, where the first approach, suppose

$$y(t) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} t^{k+i}.$$

While, the second approach, suppose

$$({}_0^{ABC}\mathrm{D}^{\alpha,\mu,\gamma} y)(t) = \sum_{i=0}^{m} c_i \binom{m}{i} \sum_{k=0}^{m-i} (-1)^k \binom{m-i}{k} t^{k+i}.$$

When we concluded that the analytic solution using the second approach was closer to the exact solution and had better accuracy compared with the first approach, we noticed that the absolute error was very small for the second approach in all the given Examples. Also, the main feature of the second approach is capable of identifying whether an approximate solution exists or not. In contrast, approach 1 lacks this capability, as noted in Example 2. Therefore, the present approaches can be investigated for more complex problems, such as boundary value problems or integro-differential equations.

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